ECOM40006/ECOM90013 Econometrics 3
Department of Economics
University of Melbourne

Assignment 1 Solutions

Semester 1, 2025

1. In this question we will assume that $x \sim \mathrm{N}(\mu, \Sigma)$ is a p-vector, as is μ, and that the $p \times p$ matrix $\Sigma > 0$.

   (a) If v is any fixed p-vector, show that

   $$g = \frac{v'(x-\mu)}{\sqrt{v'\Sigma v}} \sim \mathrm{N}(0,1). \qquad \text{(1 mark)}$$

   Solution:
   From the properties of the Normal distribution

   $$x-\mu \sim \mathrm{N}(0,\Sigma) \Rightarrow v'(x-\mu) \sim \mathrm{N}(0, v'\Sigma v) \Rightarrow g \sim \mathrm{N}\left(0, \frac{v'\Sigma v}{\sqrt{(v'\Sigma v)^2}}\right) = \mathrm{N}(0,1),$$

   as required.

   (b) If v is now a random vector independent of x for which $P(v'\Sigma v = 0) = 0$, show that $g \sim \mathrm{N}(0,1)$ and is independent of v. Why have we assumed $P(v'\Sigma v = 0) = 0$? Provide an equivalent statement of this assumption not involving Σ. (4 marks)

   Solution:
   We know that the joint distribution of two random variates f and v can be written as the product of the conditional distribution of one given the other, multiplied by the marginal distribution of the conditioning variate. In particular, we can write $f(g,v) = f(g \mid v)f(v)$. We can obtain $f(f \mid v)$ exactly as before, except now we condition on v. Thus, $g = v'(x-\mu) \mid v \sim \mathrm{N}(0, v'\Sigma v) \Rightarrow g \mid v \sim \mathrm{N}(0,1)$. As the conditional distribution is not a function of v then it is also the unconditional (or marginal) distribution of g. Hence, we see that $f(g,v) = f(g)f(v)$ and we conclude that g is independent of v. The assumption is made to ensure that, in the construction of g we don't divide by zero. (Strictly, we are assuming that the set of v that would lead to a division by zero is a set of measure zero.) An alternative statement of this assumption would be simply $P(v = 0) = 0$.

   (c) Hence show that if $y = [y_1, y_2, y_3]' \sim \mathrm{N}(0, I_3)$ then

   $$h = \frac{y_1 e^{y_3} + y_2 \log|y_3|}{[e^{2y_3} + (\log|y_3|)^2]^{1/2}} \sim \mathrm{N}(0,1). \qquad \text{(2 marks)}$$

Solution:
First, in the notation of the previous parts, set $x = [y_1, y_2]' \sim \mathrm{N}(0, I_2)$ and $v = [e^{y_3}, \log|y_3|]'$. (We don't know the distribution of v but we do know that it is independent of x, given that it is a function of y_3 alone, which is independent of the elements of x given that the elements of y are jointly normally distributed, with covariance matrix I_3, making them mutually independent.) Then we see that

$$v'x \mid v \sim \mathrm{N}(0, v'I_2v = v'v) \Rightarrow \frac{v'x}{(v'v)^{1/2}} \mid v \sim \mathrm{N}(0,1).$$

Noting that $v'v = e^{2y_3} + (\log|y_3|)^2$, we see that $h \mid v \sim \mathrm{N}(0,1)$. As the conditional distribution does not depend on the conditioning variable, v, it is also the unconditional or marginal distribution of h. That is, $h \sim \mathrm{N}(0,1)$, as required.

2. Suppose that $x \sim \mathrm{N}(\mu, \Sigma)$, where the $p \times p$ matrix $\Sigma > 0$, and that v is a fixed p-vector. If r_i, the ith element of the vector r, is the correlation between x_i and $v'x$, show that $r = (cD)^{-1/2}\Sigma v$, where $c = v'\Sigma v$ and $D = \mathrm{diag}(\Sigma)$. (3 marks)

   Bonus question: When does $r = \Sigma v$? (1 mark)

   Solution:
   First, observe that $v'x \sim \mathrm{N}(v'\mu, v'\Sigma v)$ is a scalar quantity. Second, observe that the vector of covariances between the elements of x and $v'x$ is defined to be

$$\begin{aligned}\mathrm{Cov}\,[x, v'x] &= \mathrm{E}\,[(x-\mu)v'(x-\mu)] = \mathrm{E}\,[(x-\mu)(x-\mu)'v] \\ &= \mathrm{E}\,[(x-\mu)(x-\mu)']\,v = \Sigma v.\end{aligned}$$

   To obtain correlations from covariances we need to divide the covariances by the standard deviations of each of the variables. The standard deviation of $v'x$ is simply $c = \sqrt{v'\Sigma v}$. The standard deviations of the elements of x can be obtained as the square roots of the diagonal elements of Σ. That is the square roots of the diagonal matrix D. The correlations, obtained as the division of the covariances by the standard deviations can, therefore, be represented as $r = (cD)^{-1/2}\Sigma v$, as required.

   For the Bonus Question: In order for $r = \Sigma v$, we need $cD = I$, an identity matrix, so that $D = c^{-1}I$. That is, all the elements of x must have the same variance, namely $c^{-1} = (v'\Sigma v)^{-1}$.

Your answers to the Assignment should be submitted via Canvas no later than 4:30pm, Thursday 27 March.

No late assignments will be accepted and an incomplete exercise is better than nothing.

Your mark for this assignment may contribute up to 10% towards your final mark in the subject.