Assignment 1 Q3 working

a) Set-up household problem & derive optimality conditions

$$\max_{c^y_t,\, c^o_{t+1} \geq 0} (c^y_t)^{\beta}(c^o_{t+1})^{1-\beta}$$

$$\text{s.t.} \quad c^y_t + \frac{c^o_{t+1}}{R_{t+1}} = w_t$$

To derive the optimality conditions, we must solving the Lagrangian by solving all of its endogenous partial derivatives.

$$\mathcal{L} = (c^y_t)^{\beta}(c^o_{t+1})^{1-\beta} + \lambda\left(w_t - c^y_t - \frac{c^o_{t+1}}{R_{t+1}}\right)$$

$$\frac{\partial \mathcal{L}}{\partial c^y_t} = \beta(c^y_t)^{\beta-1}(c^o_{t+1})^{1-\beta} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c^o_{t+1}} = (1-\beta)(c^y_t)(c^o_{t+1})^{-\beta} - \frac{\lambda}{R_{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w_t - c^y_t - \frac{c^o_{t+1}}{R_{t+1}} = 0$$

By dividing the first two FOCs we get the Euler equation:

$$\frac{\beta (c_t^y)^{\beta-1} (c_{t+1}^o)^{1-\beta}}{(1-\beta)(c_t^y)^{\beta}(c_{t+1}^o)^{-\beta}} = \frac{\lambda}{\lambda / R_{t+1}}$$

$$\frac{\beta}{1-\beta} \cdot \frac{c_{t+1}^o}{c_t^y} = R_{t+1}$$

$$\frac{c_{t+1}^o}{c_t^y} = \frac{1-\beta}{\beta} R_{t+1}$$

Note: the LBC (already given above) is the other optimality condition

6) Do the same for the firm

The representative firm has the following Cobb-Douglas production function:

$$Y_t = z K_t^{\alpha} L_t^{1-\alpha}, \quad 0 < \alpha < 1$$

This firm is also profit maximising:

$$\max_{K_t, L_t} \Pi_t = zK^{\alpha}L^{1-\alpha} - R_t K_t - w_t L_t$$

For the firm, optimality involves deriving expressions for optimal labour and capital demand.

Practically, this means equating the respective marginal products to w & R. If this wasn't the case, the firm would change it's production inputs.

$$MPL = \frac{\partial \Pi_t}{\partial L} = (1-\alpha) z \left(\frac{K}{L}\right)^{\alpha} = w$$

$$MPK = \frac{\partial \Pi_t}{\partial K} = \alpha z \left(\frac{K}{L}\right)^{\alpha-1} = R$$

MPL can also be rearranged to derive optimal labour demand:

$$L = \left[\frac{(1-\alpha) z K^{\alpha}}{w}\right]^{\frac{1}{\alpha}}$$

c) Using these conditions, derive an expression for c^y in terms of k_t & other exog.

Doing this takes a few steps. First, we need to insert the Euler equation into the LBC:

$$c^y_t + \frac{1}{R_{t+1}}\left(\frac{1-\beta}{\beta} R_{t+1} c^y_t\right) = w_t$$

$$c^y_t + \frac{1-\beta}{\beta} c^y_t = w_t$$

$$c^y_t + \frac{1}{\beta} c^y_t = w_t$$

$$c^y_t = \beta w_t$$

Now we need to sub this the MPL expression:

$$w_t = \frac{\partial \pi}{\partial L} = (1-\alpha) z K^\alpha L^{-\alpha}$$

$$= (1-\alpha) z \left(\frac{K}{L}\right)^\alpha$$

$$= (1-\alpha) z k_t^\alpha$$

Now we combine

$$c_t^y = \beta w_t = \beta(1-\alpha)z k_t^\alpha$$

d) Define the transition eqn & provide some intuition for it.

Again, we define k_t as capital per worker:

$$k_t \equiv \frac{K_t}{L_t}$$

We then define the MPL in per capita terms, as above:

$$w_t = (1-\alpha)z k_t^\alpha$$

We also know households consume a fraction β of their wage & save the rest:

$$c_t^y = \beta w_t$$
$$a_{t+1} = (1-\beta) w_t$$

And with full depreciation ($\delta = 1$) tomorrow's capital stock equals today's saving:

$$k_{t+1} = a_{t+1}$$

Therefore, we can define k_{t+1} in terms of savings & wages:

$$k_{t+1} = (1-\beta) w_t$$

$$k_{t+1} = (1-\beta)(1-\alpha) z k_t^{\alpha}$$

Today's capital stock determines wages, which determine savings, where savings determine next period's capital. Thus all other macroeconomic variables are a function of k_t & it's path determines everything.

e) Define steady state capital per person, $\bar{k}$.

At the steady state capital per worker doesn't change. Therefore:

$$k_{t+1} = k_t = \bar{k}$$

We plug this $\bar{k}$ term into ~~both~~ both sides of our transition eqn:

$$\bar{k} = (1-\beta)(1-\alpha)z\bar{k}^{\alpha}$$

And solve for $\bar{k}$:

$$\bar{k}^{1-\alpha} = (1-\beta)(1-\alpha)z$$

$$\bar{k} = \left[(1-\beta)(1-\alpha)z\right]^{\frac{1}{1-\alpha}}$$

f) what happens when TFP becomes a positive production externality

→ Optimality conditions remain unchanged. The agents in this model are still representative/atomistic and too small to affect TFP individually.

→ However, at the aggregate level this changes the functional form of output as TFP now depends on total capital

$$Y_t = Z_t K^{\alpha} L^{1-\alpha}$$
$$= \bar{Z} K_t^{1-\alpha} K^{\alpha} L^{1-\alpha}$$
$$= \bar{Z} K_t L_t^{1-\alpha}$$

Turning this into a per capita form gives:

$$y_t = \frac{Y_t}{L_t} = \bar{Z} N^{1-\alpha} k_t$$

Production is now linear in k_t and no longer concave: the externality removes diminishing returns to capital in aggregate.

This has implications for the transition equation: savings are also linear to k_t & no longer suffer diminishing returns.

$$w_t = (1-\alpha)\bar{z}N^{1-\alpha}k_t$$

$$k_{t+1} = (1-\beta)(1-\alpha)\bar{z}N^{1-\alpha}k_t$$

There are three different steady states: coeff. ([insert coeff.])

- If the slope of k_{t+1} is less than one capital per worker will shrink over time
- If equal to exactly 1 any value of k_t will be indefinitely sustainable
- If greater than one, growth becomes explosive.