ECON30024
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Solution to Tutorial 7

2. (1) Suppose asset returns follow a single-factor model where the factor is denoted as F_1. The variance of F_1 is 0.05. Portfolio Z is a well-diversified portfolio and its factor loading on F_1 is 1.3. What is the variance of the rate of return on Z?

   (a) 0.05
   (b) 0.0845
   (c) 0.2907
   (d) 0.065
   (e) None of the rest.

   Answer: (b)

   As asset returns follow a single-factor model, the rate of return on portfolio Z is given by

   $$r_Z = b_{Z0} + b_{Z1}F_1 + \varepsilon_Z,$$

   where the factor loading $b_{Z1} = 1.3$. Since portfolio Z is a well diversified portfolio, the idiosyncratic (specific) risk to its return is equal to zero (refer to Topic 6 slides, page 12-13):

   $$\varepsilon_Z = 0.$$

   Hence, the variance of r_Z is given by:

   $$var(r_Z) = b_{Z1}^2 var(F_1) = (1.3)^2(0.05) = 0.0845.$$

   (2) Suppose that a well-diversified portfolio Z is priced based on two factors. The factor loading for the first factor is 1.10 and for the second factor is 0.45. The expected return on the first factor is 11%, and the expected return on the second factor is 17%. The risk-free rate of return is 5.2%. According to the APT, what is the expected excess return or risk premium on portfolio Z?

   (a) 6.38%
   (b) 5.31%
   (c) 11.69%
   (d) 14%
   (e) None of the rest.

   Answer: (c)

   The risk premium on the first factor is: $\lambda_1 = 11\% - 5.2\% = 5.8\%$, and the risk premium on the second factor is: $\lambda_2 = 17\% - 5.2\% = 11.8\%$. According to the APT, the expected excess return or the risk premium on portfolio Z is given by:

   $$\mu_Z - r_0 = b_{Z1}\lambda_1 + b_{Z2}\lambda_2 = (1.1)(5.8\%) + (0.45)(11.8\%) = 6.38\% + 5.31\% = 11.69\%$$

(3) Following the question above, what is the risk premium on portfolio Z due to its exposure to the second factor?

(a) 6.38%
(b) 5.31%
(c) 11.69%
(d) 14%
(e) None of the rest.

Answer: (b)

The total risk premium on portfolio Z is composed of two parts: the risk premium on portfolio Z due to its exposure to the first factor, which is given by $b_{Z1}\lambda_1 = 6.38\%$, and the risk premium on portfolio Z due to its exposure to the second factor, which is given by $b_{Z2}\lambda_2 = 5.31\%$.

(4) What is an underlying assumption in the APT?

(a) Asset markets are frictionless.
(b) Asset returns can be explained linearly by a finite number of systematic factors.
(c) Investors can build a well diversified portfolio of assets where unsystematic risk can be eliminated through diversification.
(d) The arbitrage principle holds.
(e) All of the rest.

Answer: (e)

(5) Which of the following statements is TRUE?

(a) There is a theory of investor behaviour underlying the choices of factors in the APT.
(b) As the CAPM, the APT is underpinned by a portfolio selection theory for individual investors.
(c) As the CAPM, the APT assumes all asset markets clear.
(d) Both the CAPM and APT predict the risk premiums on an asset is a linear function of the asset's beta(s), which measure the asset's risk exposure to some systematic factors.
(e) The CAPM prediction is identical to the APT prediction in a single factor model where the single factor is the rate of return on the market portfolio.

Answer: (d)

3. (a) The APT predicts:

$$\mu_j - r_0 = \lambda_1 b_{j1} + \lambda_2 b_{j2} \quad \text{for } j = A, B$$

where λ_1, λ_2 are the two risk premia corresponding to factors 1 and 2, respectively. Hence:

$$\mu_A - r_0 = \lambda_1 b_{A1} + \lambda_2 b_{A2}$$
$$\mu_B - r_0 = \lambda_1 b_{B1} + \lambda_2 b_{B2}$$

Substituting into the relevant values:

$$0.16 - 0.06 = \lambda_1(1.2) + \lambda_2(0.4)$$
$$0.26 - 0.06 = \lambda_1(0.8) + \lambda_2(1.6)$$

Solving the two equations gives:

$$\lambda_1 = 0.05, \quad \lambda_2 = 0.10.$$

(b) The expected rate of return on asset C predicted by the APT:

$$\mu_C = r_0 + \lambda_1 b_{C1} + \lambda_2 b_{C2} = 0.06 + (0.05)(1) + (0.10)(0.5) = 0.16.$$

But asset C is observed to yield $0.12 < 0.16$. Hence, according to the APT, asset C is over-priced – it yields less than predicted. This result can be interpreted as either (i) that there are profitable investment opportunities as a consequence of asset market disequilibrium or (ii) that the specified APT is not an appropriate model for these markets, or both.

4. (a) Consider a portfolio consisting of the risk-free asset and portfolio A such that it has the same factor loading or systematic risk as portfolio B.[1] Call such a portfolio Z, and denote it by $(1-q, q)$, then the rate of return on portfolio Z is given by

$$r_Z = (1-q)r_0 + qr_A,$$

where r_A follows the single-factor model:

$$r_A = b_{A0} + b_{A1}F_1 + \varepsilon_A = b_{A0} + b_{A1}F_1,$$

where the last equality follows the fact that a well-diversified portfolio is only subject to systematic risk. Hence

$$r_Z = [(1-q)r_0 + qb_{A0}] + (qb_{A1})F_1.$$

To make portfolio Z have the same factor loading as portfolio B, q must satisfy

$$qb_{A1} = b_{B1} \quad \Rightarrow q = \frac{b_{B1}}{b_{A1}} = \frac{0.7}{1.2}$$

Note that the expected return on portfolio Z is given by

$$\mu_Z = (1-q)r_0 + q(14\%) = 10.25\%,$$

which is strictly higher than the expected return on portfolio B.

[1] Alternatively, you can consider a portfolio consisting of the risk-free asset and portfolio B such that it has the same factor loading or systematic risk as portfolio A. I encourage to try to do this and check your answer with me.

Note that portfolio Z and B are both well-diversified portfolios such that they are only subject to systematic risk which is captured by the single factor. As their factor loadings are the same, these two portfolios have the same level of risk exposure to the systematic factor, i.e., they have the same risk. Since portfolio Z yields a strictly higher expected return, an arbitrage opportunity exists.

An arbitrage strategy is to short sell portfolio B and use the proceeds to buy portfolio Z.

(b) Based on this arbitrage investment strategy, we can construct an arbitrage portfolio. Assume the arbitrage portfolio is given by (y_B, y_Z) where y_B and y_Z are investment in portfolio B and Z respectively. Because the investment strategy is to short sell portfolio B and use the proceeds to buy portfolio Z, we have $y_B < 0$ and $y_Z > 0$. Then zero outlay implies that $y_B + y_Z = 0$ such that $y_B = -y_Z$, i.e., the portfolio is rewritten as $(-y_Z, y_Z)$.

The payoff on this portfolio is given by $r_B(-y_Z) + r_Z y_Z = (r_Z - r_B)y_Z$, where

$$
\begin{aligned}
r_Z &= (1-q)r_0 + qr_A = (1-q)r_0 + q(b_{A0} + b_{A1}F_1) = ((1-q)r_0 + qb_{A0}) + qb_{A1}F_1 \\
r_B &= b_{B0} + b_{B1}F_1
\end{aligned}
$$

Recall from part (a) that $\mu_Z > \mu_B$, where

$$
\begin{aligned}
\mu_Z &= ((1-q)r_0 + qb_{A0}) + qb_{A1}E(F_1) \\
\mu_B &= b_{B0} + b_{B1}E(F_1)
\end{aligned}
$$

With $qb_{A1} = b_{B1}$ (also from part (a)), this implies that

$$((1-q)r_0 + qb_{A0}) > b_{B0}.$$

Then we have

$$r_Z - r_B = (1-q)r_0 + qb_{A0} - b_{B0} > 0.$$

As a result, the payoff on the portfolio that we constructed earlier is strictly positive in every state of the world. This portfolio is indeed an arbitrage portfolio.