ECOM40006/90013 Econometrics 3

Week 3 Extras

Question 1: Sums and Conditional Expectations

Suppose you had a sequence of i.i.d. RVs $X_1, X_2, \dots$ with mean $\mathbb{E}(X_i) = \mu$. Suppose you define a new sequence of RVs S_n that is the sum of the RVs up to the n^{th} draw: $S_n := X_1 + X_2 + \cdots + X_n$. Calculate the following conditional expectations using the properties of conditional expectations:

(a) $\mathbb{E}(S_n|X_1)$ and $\mathbb{E}(X_n|X_{n+1})$,

(b) $\mathbb{E}(X_n|X_n, X_{n-1})$,

(c) $\mathbb{E}(X_1|S_n)$,

(d) $\mathbb{E}(S_{n+m}|S_n)$ for $m \geq 0$,

(e) $\mathbb{E}(S_n|S_{n+m})$ for $m \geq 0$.

You may find the following results useful:

- $\mathbb{E}(X_1|S_n) = \mathbb{E}(X_2|S_n) = \dots \mathbb{E}(X_n|S_n)$.
- For part (e), you might find your answer to part (c) handy.

Question 2: The Jacobian of Transformation

This question is intended to get you used to the idea of transforming probability distributions using what we call a 'change of variables.'

(a) First, consider a function $f(x)$. That is, it takes as inputs x and produces outputs $f(x)$. Now, let's change the inputs that go in there. Specifically, we're going to replace x with another variable, y. Furthermore, these two variables are related by the following relationship:

$$x = g(y), \qquad \text{where } g \text{ is another function.}$$

Suppose that $f(x)$ has a maximum at x^*. Find the corresponding value of y if we change variables and call it y^*. How are x^* and y^* related?

(b) Now consider a **probability density** $f(x)$, and suppose we have a change of variable $x = g(y)$, where $g(\cdot)$ is a strictly monotone increasing/decreasing function (so its inverse exists). Use this to show that

$$f_Y(y) = f_X(x)|g'(y)|.$$

Note: the definition of a CDF will come in handy here. Note that the absolute value may be needed to prevent cases where the density would be negative otherwise. Perhaps this might be directly related to one of those cases of monotone functions above?

Let's practice using this formula. Suppose that we had a uniformly distributed random variable $X \sim U(0,1)$. In this case we know that the PDF and CDF are, respectively,

$$f_X(x) = 1 \qquad \text{and} \qquad F_X(x) = x$$

for $x \in [0,1]$.[1] Now, consider another variable y that we create by transforming x as follows:

$$y = -\log(x).$$

(c) From before, we have $x \in [0,1]$. What values is y allowed to take?

(d) Using the expression for y above, obtain an expression $x = g(y)$, and use that to derive the probability density function of Y, $f_Y(y)$.

(e) Suppose that we wanted to find out the probability that $x \in [0.1, 0.2]$. Using the properties of the uniformly distributed RV we can write[2]

$$\mathbf{P}(x \in [0.1, 0.2]) = \int_{0.1}^{0.2} f_X(x)\, dx = 0.1.$$

The interval can be written in the form $[x_0, x_0 + dx]$ where $x_0 = 0.1$ and $dx = 0.1$. Given that $y = -\log(x)$, we can transform this interval into the interval $[y_0, y_0 + dy]$. Now, find out the corresponding values of y_0 and dy and hence calculate

$$\mathbf{P}(y \in [y_0, y_0 + dy]).$$

Do you notice any difference between this and $\mathbf{P}(x \in [0.1, 0.2])$? How about the values of dy and dx?

Now that we've gotten more familiar with how to use this formula for univariate distributions, we can naturally take a step up from here and move to bivariate distributions. Consider two normally distributed random variables $(Z_1, Z_2) \in \mathbb{R}^2$ with the following properties:

$$Z_1 \sim N(0,1), \qquad Z_2 \sim N(0,1), \qquad \text{corr}(Z_1, Z_2) = 0.$$

Their joint PDF is given as

$$f_{Z_1,Z_2}(z_1, z_2) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}(z_1^2 + z_2^2). \right\}$$

Suppose that we wanted to apply affine transformations to Z_1 and Z_2 that looked like this:

$$\begin{aligned} X_1 &= \sigma_1 Z_1 + \mu_1 \\ X_2 &= \sigma_2[\rho Z_1 + \sqrt{1-\rho^2} Z_2] + \mu_2, \end{aligned}$$

where $\mu_1, \mu_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 > 0$ and $|\rho| < 1$.

[1] These functions are zero for all other values of x to be technically correct. However we will not be relying on this property; it's just mentioned as an aside.

[2] You can use integrals if you want; this is just here to save time.

(f) Verify that $\text{Var}(X_2) = \sigma_2^2$.

(g) Rearrange Z_1 and Z_2 as functions of X_1 and X_2. Call the functions

$$Z_1 = g(X_1, X_2) \qquad \text{and} \qquad Z_2 = h(X_1, X_2).$$

Then, compute the expression $Z_1^2 + Z_2^2$. (It'll come in useful later in this question.)

(h) Compute the determinant of the Jacobian formed from $g(X_1, X_2)$ and $h(X_1, X_2)$.

(i) Compute the joint PDF

$$f_{X_1,X_2}(x_1, x_2) = f_{Z_1,Z_2}(z_1, z_2) \times \text{abs}(\det J).$$

Can you identify the joint distribution based on the PDF?

Question 3: The Partitioned Matrix Inverse

Warning: This question has a large amount of matrix algebra and is here for peace-of-mind purposes. If you want to do this quickly, refer to the solutions when you see them. Otherwise, good luck!

The derivation of the partitioned, or block matrix inverse is painful, but at the very least we are able to at least verify that the formula for the partitioned matrix inverse is valid by multiplying a block matrix with its inverse and seeing whether (i) the block diagonal elements return the identity matrix and (ii) the off-diagonal block elements are zero matrices.

Suppose that the matrix A is invertible and has the partitioned matrix representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \text{and define its inverse} \qquad A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Assume that A_{11} and A_{12} are invertible, and further define $F = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.

(a) Verify that

$$\begin{aligned} B_{11} &= A_{11}^{-1} + A_{11}^{-1}A_{12}FA_{21}A_{11}^{-1} \\ B_{22} &= F \\ B_{21} &= -FA_{21}A_{11}^{-1} \\ B_{12} &= -A_{11}^{-1}A_{12}F \end{aligned}$$

(b) Verify that

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}FA_{21}A_{11}^{-1},$$

which is the value of B_{11} above.

(c) Suppose that $A = X'X$ with $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ so that

$$A = \begin{pmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{pmatrix}.$$

Assume that $X_1'X_1$ and $X_2'X_2$ are invertible. What is A^{-1}?