ECON30009/90080 – Tutorial 4 Solutions

This Version: Semester 2, 2025

Note: these questions are designed to give you some practice solving the social planner's problem.

Answers in red

Question 1: The social planner's problem with exogenous government spending

The following question asks you to consider a variation of the social planner's problem. Unlike the example we did in class, in this question the economy is faced with some level of exogenous government spending, G and the associated government spending in per capita terms is denoted as $G/N = g$. Government spending goes towards financing a public good that households get utility from. We will assume that households have preferences given by

$$U(c_t^y, c_{t+1}^o) = \ln c_t^y + \beta \ln c_{t+1}^o + \eta \ln g$$

and firms have the following production function:

$$Y_t = \left[\theta K_t^\gamma + (1-\theta)(z_L L_t)^\gamma\right]^{1/\gamma}$$

where z_L represents exogenous labour-augmenting technology. Capital depreciates fully after use in production in a period, i.e., $\delta = 1$. The population in each generation is constant and is equal to N households.

a) Write an expression for output per capita
   Output is given by:

$$Y_t = \left[\theta K_t^\gamma + (1-\theta)\left(z_L L_t\right)^\gamma\right]^{\frac{1}{\gamma}}$$

   Dividing by L_t, we have:

$$\begin{aligned} y_t &= \frac{1}{L_t}\left[L_t^\gamma\{\theta k_t^\gamma + (1-\theta)\, z_L^\gamma\}\right]^{\frac{1}{\gamma}} \\ &= \left[\theta k_t^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1}{\gamma}} \end{aligned}$$

b) Consider steady state. Write down what the resource constraint is in per capita terms in steady state.
   The resource constraint in steady state is given by:

$$\overline{c}^y + \overline{c}^o + \overline{k} + g = \left[\theta \bar{k}^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1}{\gamma}}$$

c) Suppose the planner wants to maximize welfare in steady state. Set up the planner's problem. State what are the endogenous choice variables of the social planner.

The planner's problem in steady state is given by:

$$\max \ln \overline{c}^y + \beta \ln \overline{c}^o + \eta \ln g$$

s.t.

$$\overline{c}^y + \overline{c}^o + \overline{k} + g = \left[\theta \bar{k}^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1}{\gamma}}$$

The planner's choice variables are $\overline{c}^y, \overline{c}^o$ and $\overline{k}$.

d) Solve for the planner's choice of steady state $\bar{k}$ $\overline{c}^y$, and $\overline{c}^o$ in terms of parameters of the model and exogenous variables.

We can set up the Lagrangian to this problem as:

$$\mathcal{L} = \max \ln \overline{c}^y + \beta \ln \overline{c}^o + \gamma \ln g + \lambda \left\{ \left[\theta \bar{k}^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1}{\gamma}} - \overline{c}^y - \overline{c}^o - \overline{k} - g \right\}$$

Taking FOC

$$(\overline{c}^y): \quad \frac{1}{\overline{c}^y} = \lambda$$

$$(\overline{c}^o): \quad \frac{\beta}{\overline{c}^o} = \lambda$$

$$(\overline{k}): \quad 1 = \theta \overline{\bar{k}}^{\gamma-1} \left[\theta \bar{k}^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1-\gamma}{\gamma}}$$

$$(\lambda): \quad \left[\theta \bar{k}^\gamma + (1-\theta)\, z_L^\gamma\right]^{\frac{1}{\gamma}} - \overline{c}^y - \overline{c}^o - \overline{k} - g = 0$$

Focusing first on the FOC wrt $\overline{k}$, we can re-arrange the terms to make $\overline{k}$ the subject of the equation:

$$\boxed{\bar{k} = \left[\frac{(1-\theta)}{\theta\left(\theta^{-\frac{1}{1-\gamma}} - 1\right)}\right]^{1/\gamma} z_L}$$

Combining the first order conditions wrt $\overline{c}^y$ and $\overline{c}^o$, we get the pareto optimal intertemporal trade-off between consumption of the young and old:

$$\frac{1}{\overline{c}^y} = \frac{\beta}{\overline{c}^o}$$

Plug the above into the resource constraint and re-arrange to make $\overline{c}^y$ the subject of the equation:

$$\boxed{\overline{c}^y = \frac{1}{(1+\beta)}\left[\left(1 - \theta^{\frac{1}{1-\gamma}}\right)^{\frac{\gamma-1}{\gamma}} (1-\theta)^{1/\gamma} z_L - g\right]}$$

Using the planner's intertemporal trade-off between $\bar{c}^y$ and $\bar{c}^o$, we can solve for $\bar{c}^o$:

$$\boxed{\bar{c}^o = \frac{\beta}{(1+\beta)}\left[\left(1-\theta^{\frac{1}{1-\gamma}}\right)^{\frac{\gamma-1}{\gamma}}(1-\theta)^{1/\gamma} z_L - g\right]}$$

e) Explain what happens to the planner's choice of steady state investment and consumption when exogenous government spending G rises to finance a higher level of public goods.
When exogenous government spending increases, steady state $\bar{k}$ and thus investment is unchanged. Instead $\bar{c}^y$ and $\bar{c}^o$ is lower when government spending increases. Intuitively, the planner must devote more resources to government spending when G increases. The planner optimally chooses to lower consumption of private goods from the young and old so as to allocate more resources to the rise in g.

Question 2: Government budget constraints

This question is meant to get you used to writing down the government budget constraint under the different ways the government may use tax instruments and debt to finance its government spending.

For each part of this question, assume that the government spends G_t in every period t. It either finance this government spending by collecting tax revenue or by issuing debt B_{t+1} or by doing both. If the government issues debt, it must pay off its debt as well as any interest that it owes on the debt. We will assume that r_t is the interest rate. You may assume that there are N households in each generation.

a) Suppose the government spends G_t and completely finances this spending within period by only levying a lump-sum tax on young households. Write down what the government budget constraint looks like in per-capita terms in period t. You may denote government spending per capita in period t as $G_t/N = g_t$.
The government spending is completely financed within period by the lump-sum tax. This implies the government budget constraint is given by:

$$g_t = \tau_t$$

where τ_t represents the lump-sum tax.

b) Now suppose the government spends G_t and completely finances this spending by issuing a proportional tax on the wage income of young households. Write down what the government budget constraint looks like in per-capita terms in period t.
The government spending is completely financed within period by the

proportional tax on wage income. This implies the government budget constraint is given by:

$$g_t = \tau_t^w w_t$$

where τ_t^w represents the proportional tax on wage income.

c) Now suppose the government spends G_t. In each period, the government issues debt $B_{t+1} = G_t$ and completely repays this debt with a proportional tax on the consumption of old households. Write down what the government budget constraint looks like in per-capita terms in period t.
The government spending is financed in the current period by debt B_{t+1}. This implies $G_t = B_{t+1}$. The debt is completely repaid with a proportional tax on the consumption of the old, this implies $(1 + r_t)B_t = N\tau_t^c c_t^o$. This implies the government budget constraint is given by:

$$g_t + (1 + r_t)b_t = \tau_t^c c_t^o + b_{t+1}$$

where τ_t^c represents the proportional tax on the consumption of the old.

d) Now suppose the government spends G_t. Each period the government finances its spending and repays its debt by levying both a proportional tax on the revenue of firms and by issuing new debt. That is, the government can roll over its debt across periods. Write down what the government budget constraint looks like in per-capita terms. What other condition needs to be satisfied to ensure the government remains solvent?
The government's total expenditure in a period is partially financed in the current period by debt B_{t+1} and partially by a proportional tax on firms' revenues: $\tau_t^y Y_t$. The government budget constraint is given by:

$$g_t + (1 + r_t)b_t = \tau_t^y y_t + b_{t+1}$$

We require that the government remains solvent (repays all its debt in the limit) which means that the transversality equation must apply:

$$\lim_{s\to\infty} \frac{B_{t+s}}{R_t R_{t+1} \dots R_{t+s}} = 0$$

where $R_t = (1 + r_t)$