ECON30009/90080 – Tutorial 6

This Version: Semester 2, 2025

These questions are designed to give you practice solving a 2 period RBC model.

Q1: RBC model with elastic labour supply

Assume there are N households in an economy. A household born into this economy is endowed with an initial amount of a physical asset, a_1. Each household in the economy lives for two periods and gets utility from consuming in both periods. Further the household observes disutility from supplying labour in each period. Specifically, let the household's preferences be given by

$$U(c_1, c_2, \ell_1, \ell_2) = \ln c_1 - \ell_1 + \beta (\ln c_2 - \ell_2)$$

where $0 < \beta < 1$ is the discount factor (i.e., the factor that the household discounts period 2 utility by). ℓ_t for $t \in \{1, 2\}$ is the amount of labour supplied by the household. Unlike the standard problem we solve in class, the household here can choose the amount of labour they would like to supply to the labour market.

The household is paid w_t for each unit of labour supplied in period t and the household receives rental income, $R_t a_t$, from renting out her physical asset in each period. In addition, the household receives dividend income from the firm. There is no government in this model. The representative firm produces output according to a Cobb-Douglas production function: $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$ where $0 < \alpha < 1$ and z_t is exogenous and can vary between period 1 and period 2. The firm chooses capital and labour to maximize its profits.

a) Set up the household's 2 period utility maximization problem. State what are the choice variables of the household.

Answer

The household's problem is given by:

$$\max_{c_1, c_2, \ell_1, \ell_2} \ln c_1 - \ell_1 + \beta \{\ln c_2 - \ell_2\}$$

s.t.

$$c_1 + \frac{c_2}{R_2} = R_1 a_1 + w_1 \ell_1 + \frac{w_2 \ell_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2}$$

The household chooses c_1, c_2, ℓ_1, ℓ_2 to maximize her lifetime utility. a_2 is also technically an endogenous choice variable.

b) Set up the Lagrangian to the household's problem and take first order conditions. Derive the Euler equation and LBC. Derive an equation that expresses the consumption-leisure trade-off of the household for each period t. [Hint: to derive the consumption-leisure trade-off you should arrive at an equation that relates the marginal cost of supplying one unit of labour to the marginal benefit of supplying one unit of labour.]

Answer

We can write the Lagrangian to this problem as:

$$\mathcal{L} = \ln c_1 - \ell_1 + \beta \{\ln c_2 - \ell_2\} + \lambda \left[R_1 a_1 + w_1 \ell_1 + \frac{w_2 \ell_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} - c_1 - \frac{c_2}{R_2} \right]$$

Taking FOCs:

$$(c_1): \quad \frac{1}{c_1} = \lambda$$

$$(\ell_1): \quad 1 = \lambda w_1$$

$$(c_2): \quad \frac{\beta}{c_2} = \frac{\lambda}{R_2}$$

$$(\ell_2): \quad \beta = \lambda \frac{w_2}{R_2}$$

$$(\lambda): \quad c_1 + \frac{c_2}{R_2} = R_1 a_1 + w_1 \ell_1 + \frac{w_2 \ell_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2}$$

The FOC wrt λ gives our LBC. This is one of the household's optimality conditions. Combining the FOCs wrt c_1 and c_2, we get the household's Euler equation which tells us the optimal intertemporal trade-off in consumption today and tomorrow:

$$\frac{1}{c_1} = \frac{\beta R_2}{c_2}$$

Combining the FOCs wrt c_t and ℓ_t, we get a consumption-leisure trade-off for each period $t \in \{1, 2\}$:

$$1 = \frac{w_t}{c_t}$$

The consumption-leisure trade-off describes the household's optimal allocation of labour supply. On the LHS of the equation, we have the marginal cost of supplying one unit of labour, which is represented by the marginal disutility of labour (-1). On the RHS of the equation, we have the marginal benefit of supplying one unit of labour, it is represented by the wage rate, w_t the household can earn, and with this wage the household can purchase additional consumption goods which she values at her marginal utility of consumption $\frac{1}{c_t}$.

c) Set up the representative firm's problem and solve for the firm's optimality conditions.

Answer
The firm's profit maximization problem each period is represented by:

$$\max_{K_t, L_t} z_t K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - w_t L_t$$

Taking FOCs, we can derive the firm's optimality conditions

$$(L_t): \quad (1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha} = w_t$$
$$(K_t): \quad \alpha z K_t^{\alpha-1} L_t^{1-\alpha} = R_t$$

d) Write down equations representing the market clearing conditions [For example, $N\ell_1 = L_1$ is an equation that must hold when the labour market clears in period 1. Write down the other market clearing conditions.]

Answer
The labour market clears in each period t: $N\ell_t = L_t$.
The asset market clears in each period t: $Na_t = K_t$
The goods market clears in each period t with $C_1 + K_2 = z_1 K_1^{\alpha} L_1^{1-\alpha}$ and $C_2 = z_2 K_2^{\alpha} L_2^{1-\alpha}$, where $C_t = Nc_t$.

e) Using information from the household euler equation, the household LBC and the firm's optimality conditions. Derive an expression for aggregate consumption $C_1 = Nc_1$ in terms of pre-determined K_1, choice variables L_1, K_2, exogenous variable z_1 and parameters of the model.

Answer
From the Euler equation, we have: $c_2 = \beta R_2 c_1$. We can plug this into the LBC:

$$(1+\beta)c_1 = R_1 a_1 + w_1 \ell_1 + \frac{w_2 \ell_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2}$$

Further, we can multiple on both sides by N to get C_1 on the left-hand-side of the equation

$$C_1 = \frac{1}{1+\beta}\left[R_1 a_1 N + w_1 \ell_1 N + \frac{w_2 \ell_2}{R_2} N + \pi_1 N + \frac{\pi_2}{R_2} N\right]$$

Then plugging in the firm's optimality conditions, we have:

$$C_1 = \frac{1}{(1+\beta)}\left[z_1 K_1^{\alpha} L_1^{1-\alpha} + \frac{(1-\alpha)}{\alpha} K_2\right]$$

Thus, far we have arrived at the above expression for C_1 but we note that K_2 and L_1 are endogenous.

f) Using the goods market clearing condition, derive an expression for K_2 in terms of pre-determined K_1, choice variable L_1, exogenous variable z_1 and parameters of the model.

Answer
From the goods market clearing condition in period 1, we have:

$$K_2 = zK_1^{\alpha}L_1^{1-\alpha} - C_1$$

We can plug in the expression for C_1 that we derived in part e) into the equation above and we get:

$$K_2 = \frac{\beta}{1+\beta} z_1 K_1^{\alpha} L_1^{1-\alpha} - \frac{1-\alpha}{\alpha\left(1+\beta\right)} K_2$$

Re-arranging to make K_2 the subject of the equation:

$$K_2 = \frac{\alpha\beta}{1+\alpha\beta} z_1 K_1^{\alpha} L_1^{1-\alpha}$$

Note that K_2 and L_1 are both endogenous.

g) We have one last optimality condition we have not utilized yet, the consumption leisure trade-off in period 1. Use the consumption leisure trade-off, your answer in parts e) and f) to solve for ℓ_1 in terms of pre-determined K_1, exogenous z_1 and parameters of the model.

Answer
The consumption leisure trade-off in period one can be re-arranged as:

$$c_1 = w_1$$

We can multiple both sides by N to get aggregate consumption on the LHS of the equation.

$$C_1 = w_1 N = \left(1-\alpha\right) z_1 K_1^{\alpha} L_1^{1-\alpha} \frac{N}{L_1}$$

Given our answer from part e), we know that:

$$\frac{1}{\left(1+\beta\right)}\left[z_1 K_1^{\alpha} L_1^{1-\alpha} + \frac{\left(1-\alpha\right)}{\alpha} K_2\right] = C_1 = \left(1-\alpha\right) z_1 K_1^{\alpha} L_1^{1-\alpha} \frac{N}{L_1}$$

And given our answer from part f), we can plug in for K_2

$$\left(1-\alpha\right) z_1 K_1^{\alpha} L_1^{1-\alpha} \frac{N}{L_1} = \frac{1}{\left(1+\beta\right)}\left[z_1 K_1^{\alpha} L_1^{1-\alpha} + \frac{\left(1-\alpha\right)}{\alpha} \frac{\alpha\beta}{1+\alpha\beta} z_1 K_1^{\alpha} L_1^{1-\alpha}\right]$$

We can divide the above by $z_1 K_1^{\alpha} L_1^{1-\alpha}$ and re-arrange to get the following formulation for ℓ_1

$$\ell_1 = \left(1-\alpha\right)\left(1+\alpha\beta\right)$$

h) Solve for C_1 and K_2 in terms of pre-determined K_1, exogenous variables and parameters of the model. Show that a rise in z_1 generates positive co-movement in C_1 and K_2 with GDP$= Y_1$. Show that even information about higher z_2 cannot drive an expansion in period $t = 1$.

Answer
Notably, once we know ℓ_1, this means we know L_1, and when we know L_1, we can solve for K_2 from our answer in part f), and given L_1, we know C_1 from the consumption leisure trade-off. Using our answer from part f), we have:

$$K_2 = \frac{\alpha\beta}{1+\alpha\beta} z_1 K_1^{\alpha} \left[N\ell_1\right]^{1-\alpha}$$

and thus

$$K_2 = \frac{\alpha\beta}{1+\alpha\beta} z_1 K_1^{\alpha} \left[N\left(1-\alpha\right)\left(1+\alpha\beta\right)\right]^{1-\alpha}$$

and from the consumption leisure trade-off we have:

$$C_1 = \left(1-\alpha\right) z_1 K_1^{\alpha} [N\left(1-\alpha\right)\left(1+\alpha\beta\right)]^{-\alpha} N$$

Clearly, both K_2, C_1 and Y_1 are increasing in z_1. Further we can see that K_2, C_1 and Y_1 are independent of z_2. Note that there are only 3 objects that can increase output in period 1, z_1, K_1 and L_1. K_1 is pre-determined and thus cannot change. L_1 which we solved for turned out to be a function of parameters only in this problem. As such, the only object that can increase output in this model is z_1. Without an increase in output, there are no extra additional resources to allocate to increase K_2 and C_1. As such, even if households and firms know that tomorrow's productivity is high and that MPK would be higher, they cannot increase investment today.

Q2: RBC model with government investment

Consider the RBC model we discussed in class. Suppose there is a positive exogenous government spending shock in period 1 only. Each unit of government spending $G_1 > 0$ goes towards public capital formation, i.e., building capital for period 2, K_2^G. The government runs a balanced budget and finances this government spending G_1 by collecting a lump-sum tax, τ_1, from each household in period 1 only. There are N households.

Households live 2 periods and have the following preferences: $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$ where $0 < \beta < 1$. They inelastically supply 1 unit of labour each period and receive wage income w_t for their labour where $t \in \{1, 2\}$. Households are born with initial capital a_1 and receive rental income from renting out their capital. The rental rate is given by R_t for $t \in \{1, 2\}$. Households also receive dividend income from firms.

The representative firm in this model produces according to a Cobb-Douglas production function where $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$. The firm rents capital at rate R_t and hires labour at wage rate w_t.

a) Write down the government budget constraint

Answer:
There is only positive government spending in period 1 and the government

spending is financed by collecting a lump-sum tax on all households in period 1. The government budget constraint is thus equals to:

$$G_1 = N\tau_1$$

b) Set up the firm's profit maximization problem and characterize the firm's optimality conditions

Answer:
The firm's profit maximization problem in each period t is given by:

$$\max_{K_t, L_t} z_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - R_t K_t$$

Taking FOCs, we get the firm's optimality conditions:

$$w_t = (1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha}$$

$$R_t = \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha}$$

c) Set up the household's utility maximization problem and characterize the household's optimality conditions

Answer:
Household's utility maximization problem is given by:

$$\max_{c_1, c_2} \ln c_1 + \beta \ln c_2$$

s.t.

$$c_1 + \frac{c_2}{R_2} = w_1 + \frac{w_2}{R_2} - \tau_1 + R_1 a_1 + \pi_1 + \frac{\pi_2}{R_2}$$

We can set up the Lagrangian:

$$\mathcal{L} = \ln c_1 + \beta \ln c_2 + \lambda \left[w_1 + \frac{w_2}{R_2} - \tau_1 + R_1 a_1 + \pi_1 + \frac{\pi_2}{R_2} - c_1 - \frac{c_2}{R_2} \right]$$

This problem is similar to the one we solved in lecture, the lump-sum tax only affects the LBC and not the intertemporal trade-off between consumption today vs. tomorrow. So we know after taking FOCs and combining the first order conditions, we arrive at the two optimality conditions of the household:

$$\text{Euler}: \quad \frac{1}{c_1} = \frac{\beta R_2}{c_2}$$

and

$$\text{LBC}: \quad c_1 + \frac{c_2}{R_2} = w_1 + \frac{w_2}{R_2} - \tau_1 + R_1 a_1 + \pi_1 + \frac{\pi_2}{R_2}$$

d) Solve for k_2 and c_1 in terms of exogenous variables, parameters of the model and pre-determined k_1. You may denote $g_1 = G_1/N$.

Answer:
We can plug the Euler equation into LBC:

$$c_1(1+\beta) = w_1 + \frac{w_2}{R_2} - \tau_1 + R_1 a_1 + \pi_1 + \frac{\pi_2}{R_2}$$

In equilibrium, using the information from our firm's optimality conditions and the government budget constraint, and the asset market clearing condition that $Na_1 = K_1$ (note government spending in period 1 only adds to capital in period 2) we have:

$$c_1 = \frac{1}{1+\beta}\left[z_1 k_1^{\alpha} + \frac{(1-\alpha)}{\alpha}k_2 - g_1\right]$$

where $k_2 = K_2/L_2$. Using the first period budget constraint of the household, we know:

$$\begin{aligned} a_2 &= w_1 + R_1 a_1 - c_1 - \tau_1 \\ &= \frac{\beta}{1+\beta}(z_1 k_1^{\alpha} - g_1) - \frac{1}{1+\beta}\frac{(1-\alpha)}{\alpha}k_2 \end{aligned}$$

Then since $k_2 = a_2 + k_2^g = a_2 + g_1$, we have:

$$k_2 = \frac{\alpha}{1+\alpha\beta}\left[\beta z_1 k_1^{\alpha} + g_1\right]$$

Having solved for k_2, we know what c_1 is:

$$c_1 = \frac{1}{1+\alpha\beta}z_1 k_1^{\alpha} - \frac{\alpha}{1+\alpha\beta}g_1$$

e) Show that a rise in government investment is not a candidate for driving business cycles. In particular, explain how k_2 and c_1 change with an increase in government investment.

Answer:
From the goods market clearing condition, we know that:

$$Y_1 = C_1 + K_2 + G_1$$

where $C_1 = Nc_1$ and $K_2 = Nk_2$. Clearly, output is positively related to government spending. From our answer in part d), we also saw that k_2 and consequently K_2 is increasing in government spending. So investment rises when government spending increases since government spending adds to total investment here.
However, we observe that consumption is negatively related to government spending. So a rise in G_1 actually causes aggregate consumption to fall. As such, changes in government investment expenditure cannot reconcile the positive co-movement between output and investment and consumption.