MAST90125: Bayesian Statistical Learning

Lecture 4: Posteriors and prediction in Bayesian inference

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Introduction

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The answer is no. The inferential statements we want to make should be based on the posterior distribution, that is a distribution conditioned on the data available. This lecture will discuss various aspects of posterior based inference.

Point estimation

- When we use maximum likelihood to determine quantities of inferential interest, we aim to obtain
  - A point estimate, ideally unbiased, ($E(\hat{\theta}|\theta) = \theta$).
  - A measure of estimate precision, such as Var($\hat{\theta}$).
  - An interval based measure of uncertainty in the parameter estimate, often relying on large-sample theory.
- In Bayesian methods, inference is based on the posterior distribution, $p(\theta|y)$.
  - While it would be straight-forward to construct an interval expressing a range of possible θ from $p(\theta|y)$, what would be an appropriate point estimate? Is there even an appropriate point estimate?

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  - Does it still make sense to want an unbiased estimator?
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  - What is a point estimate supposed to be a measure of?
    - The unbiased MLE point estimate $\hat{\theta}$ is a measure of central tendency, specifically mean.
- The information given in $p(\theta|y)$ would allow for a variety of measures of central tendency to be constructed such as

$$E(\theta|y) \quad \text{Median}(\theta|y) \quad \text{Mode}(\theta|y)$$

Credible intervals

- The posterior distribution readily allows for the construction of $100(1-\alpha)$ % credible intervals:

$$\Pr(\theta \in C|y) = 1-\alpha: \ \textit{cont.} \ \int_C p(\theta|y)d\theta, \ \textit{disc.} \ \sum_{\theta' \in C} \Pr(\theta'|y),$$

where C denotes the credible region.

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    - θ is a random variable. Unlike in frequentist statistics, parameters are random variables in Bayesian statistics.
  - Question 2: If we consider the single parameter case, does C need to be a single interval?

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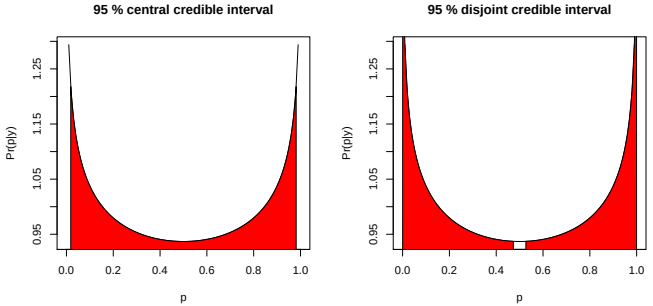
    - θ is a random variable. Unlike in frequentist statistics, parameters are random variables in Bayesian statistics.
  - Question 2: If we consider the single parameter case, does C need to be a single interval?
    - No. In fact, there may be cases where to minimise the length of the credible interval it is desirable to report a union of disjoint intervals.

Example of Credible intervals

- Consider a bi-modal Beta distribution $\alpha = \beta = 0.9$.



- While the 95 % central credible interval for p is $(0.0185, 0.9815)$ with interval width 0.963, an 95 % disjoint credible interval $(0, 0.4733) \cup (0.5267, 1)$, has interval width $2 \times 0.4733 = 0.9466$.

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  - The probability used in constructing the credible interval is conditional on the data y. This means the credible interval can be interpreted probabilistically, as in there is a $1 - \alpha$ probability that θ lies in the region C, based on the data available.

Types of credible interval

- As the beta posterior example mentioned earlier, there are multiple ways to define the credible region.
- One way would be to use the central credible region,

$$\Pr(q_{\alpha/2} \leq \theta \leq q_{1-\alpha/2}|y) = 1 - \alpha,$$

so that the limits $(q_{\alpha/2}, q_{1-\alpha/2})$ are the $\alpha/2$ and $1 - \alpha/2$ quantiles of the posterior distribution $p(\theta|y)$.

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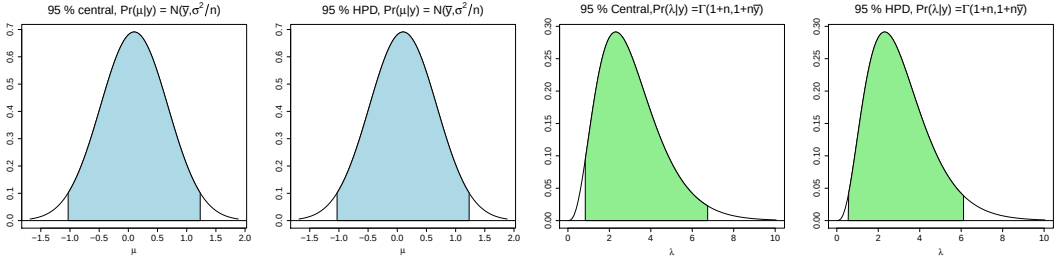
  so that the limits $(q_{\alpha/2}, q_{1-\alpha/2})$ are the $\alpha/2$ and $1 - \alpha/2$ quantiles of the posterior distribution $p(\theta|y)$.
- However, one of the main aims in inference is to maximise the precision of estimation. This suggests constructing a $100(1 - \alpha)$ % credible interval such that interval width is minimised. This is referred to as the highest posterior density (HPD) interval.

Example of credible interval

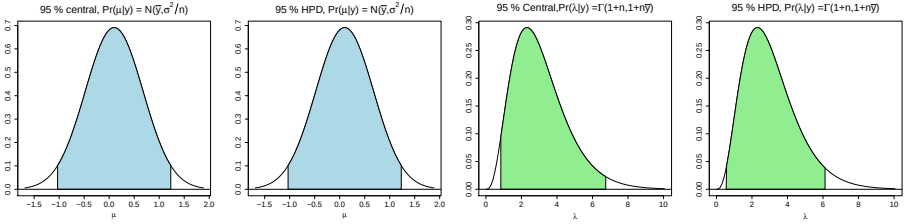


- While for the normal posterior, the central and HPD intervals are the same, $(-1.032, 1.232)$, the interval differs for the gamma posterior, with the central interval $(0.838, 6.744)$ being right shifted and about 6 % wider than the HPD interval $(0.548, 6.114)$.

Example of credible interval

- It turns out that the HPD interval is defined as follows,

$$\Pr(\theta \in C|y) = 1 - \alpha \quad \text{s.t } p(\theta_1|y) \geq p(\theta_2|y)\ \forall \theta_1 \in C, \theta_2 \notin C,$$



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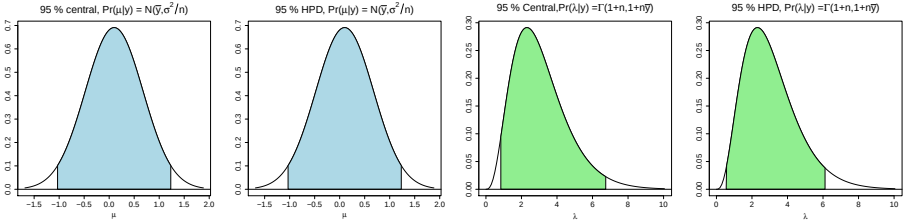

which in the case of the gamma posterior meant the limits of the HPD interval satisfy $p(\theta_L|y) = p(\theta_U|y)$ while for unimodal, symmetric distributions such as the normal, the $100(1-\alpha)$ % central and HPD intervals are equivalent.

Predictive distributions

In future lectures, we will use predictive distributions regarding the posterior distribution. Thus, we will define predictive distributions and consider an example.

- The prior predictive distribution:

$$p(y) = \int p(y, \theta) d\theta = \int p(y|\theta) p(\theta) d\theta,$$

also known as the marginal distribution of y, or the normalising constant.

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  also known as the marginal distribution of y, or the normalising constant.

- The posterior predictive distribution:

$$p(\tilde{y}|y) = \int p(\tilde{y}, \theta|y) d\theta = \int p(\tilde{y}|\theta, y) p(\theta|y) d\theta = \int p(\tilde{y}|\theta) p(\theta|y) d\theta \quad \text{as } \tilde{y} \perp y|\theta,$$

  which is the distribution of a hypothetical new observation given the data already available. This is very useful in model checking.

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Then, using the assumption $\tilde{y} \perp y|\theta$, we conclude the result.

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Predictive distribution example

Consider n conditional $i.i.d$ observations drawn from an exponential distribution with parameter λ. As the parameter $\lambda \in (0, \infty)$, a Gamma prior was chosen.

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Consider n conditional $i.i.d$ observations drawn from an exponential distribution with parameter λ. As the parameter $\lambda \in (0, \infty)$, a Gamma prior was chosen.

- The joint distribution $p(y_1, \dots y_n, \lambda)$ is

$$\prod_{i=1}^{n} p(y_i|\lambda)p(\lambda) = \lambda^n e^{-\lambda \sum_i^n y_i} \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)} = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha+n-1} e^{-\lambda(\beta+\sum_i^n y_i)} \tag{3}$$

- From (3), we can deduce that the posterior distribution $p(\lambda|y_1, \dots y_n)$ is $\text{Ga}(\alpha + n, \beta + \sum_i^n y_i)$. The prior predictive distribution can be determined by marginalising λ from (3),

$$\begin{aligned} p(y_1, \dots y_n) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int \lambda^{\alpha+n-1} e^{-\lambda(\beta+\sum_i^n y_i)} d\lambda = \frac{\beta^\alpha}{\Gamma(\alpha)} \times 1 \Big/ \left(\frac{(\beta + \sum_i^n y_i)^{\alpha+n}}{\Gamma(\alpha+n)} \right) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+n)}{(\beta+\sum_i^n y_i)^{\alpha+n}} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\beta^n} \left(1 + \frac{\sum_i^n y_i}{\beta}\right)^{-(\alpha+n)} \end{aligned}$$

Predictive distribution example

The posterior predictive distribution $p(\tilde{y}|y_1, \dots y_n)$ is

$$\begin{aligned}
\int p(\tilde{y}|\lambda)p(\lambda|y_1, \dots y_n)d\lambda &= \int \lambda e^{-\lambda\tilde{y}} \frac{(\beta + \sum_i^n y_i)^{\alpha+n}\lambda^{\alpha+n-1}e^{-(\beta+\sum_{i=1}^n y_i)\lambda}}{\Gamma(\alpha+n)} d\lambda \\
&= \frac{(\beta + \sum_i^n y_i)^{\alpha+n}}{\Gamma(\alpha+n)} \int \lambda^{\alpha+n+1-1} e^{-\lambda(\beta+\sum_i^n y_i+\tilde{y})} d\lambda \\
&= \frac{(\beta + \sum_i^n y_i)^{\alpha+n}}{\Gamma(\alpha+n)} \frac{\Gamma(\alpha+n+1)}{(\beta+\tilde{y}+\sum_i^n y_i)^{\alpha+n+1}} \\
&= \frac{\alpha+n}{\beta+\sum_i^n y_i} \left(1 + \frac{\tilde{y}}{\beta+\sum_i^n y_i}\right)^{-(\alpha+n+1)}, \\
&= \frac{a}{b}(1+\tilde{y}/b)^{-(a+1)},
\end{aligned}$$

- which is different from $p(y_1, \dots y_n)$ in the previous slide.