Quantitative Analysis of Finance I

ECON90033

WEEK 12

COINTEGRATION TESTING (cont.)

<u>Reference</u>:

HMPY: § 6.4-6.5

Dr László Kónya

January 2023

JOHANSEN'S COINTEGRATION RANK TESTS

- Johansen's (*J*) cointegration rank tests are nowadays the standard procedure of determining the existence of cointegrated relationships between two or more variables.

  The *J* tests are based on the simultaneous estimation of all possible cointegrating equations.

  Compared to the *EG* test, the advantage of the *J* tests is that they can be used to verify the cointegration rank, i.e., the number of cointegrating relationships in a multivariate system of *n* variables.

- The key to the *J* tests is the relationship among the number of endogenous variables (n), the number of different stochastic trends these variables have (k), and the cointegration rank (r).

  Suppose that $n = 2$ variables, y_t and z_t, are (pure) random walks, i.e., they are $I(1)$ and each has a stochastic trend.

$\longrightarrow$ $\boxed{y_t = \sum_i \varepsilon_{1,t-i}}$ and $\boxed{z_t = \sum_i \varepsilon_{2,t-i}}$

If there are two stochastic trends then there's no CI!

There are two possible cases:

i. The two stochastic trends are different, $k = 2$.

$\longrightarrow$ y_t and z_t are not $CI(1,1)$ $\longrightarrow$ $r = 0 = 2 - 2 = n - k$.

ii. The two stochastic trends are the same (up to a scalar).

$\longrightarrow$ y_t and z_t are $CI(1,1)$ $\longrightarrow$ $r = 1 = 2 - 1 = n - k$.

Suppose now that there is a third variable ($n = 3$), v_t, which is also a (pure) random walk,

$$\boxed{v_t = \sum_i \varepsilon_{3,t-i}}$$

In this case there are three possibilities:

i. The three stochastic trends are different, $k = 3$.

$\longrightarrow$ y_t, z_t and v_t are not $CI(1,1)$, i.e., $r = 0 = 3 - 3 = n - k$.

ii. There are two different stochastic trends, $k = 2$.

For example, y_t and z_t have the same and v_t has a different stochastic trend, so the system of y_t, z_t and v_t is driven by two different stochastic trends.

$$y_t = \sum_i \varepsilon_{1,t-i}$$
$$z_t = \lambda \sum_i \varepsilon_{1,t-i} \quad , \quad \lambda \neq 0$$

and

$$v_t = \sum_i \varepsilon_{3,t-i}$$
$$\neq \omega \sum_i \varepsilon_{1,t-i} \quad , \quad \omega \neq 0$$

$\longrightarrow$ y_t and z_t are $CI(1,1)$, but none of them is cointegrated with v_t individually.

However, together they are 'cointegrated' with v_t, although the 3rd element of the cointegration vector is zero.

$\longrightarrow$ y_t, z_t and v_t are $CI(1,1)$, and $r = 1 = 3 - 2 = n - k$.

iii. There is only one stochastic trend, $k = 1$.

In this case y_t, z_t and v_t have the same stochastic trend, which drives the whole system.

Namely,

$$y_t = \sum_i \varepsilon_{1,t-i}$$
$$z_t = \lambda \sum_i \varepsilon_{1,t-i} \quad , \quad \lambda \neq 0$$
$$v_t = \omega \sum_i \varepsilon_{1,t-i} \quad , \quad \omega \neq 0$$

⟶ Any two of these three variables are *CI*(1,1) and the cointegration vectors are

$$\begin{bmatrix}\alpha_1 & \alpha_2 & 0\end{bmatrix} \qquad \begin{bmatrix}\beta_1 & 0 & \beta_3\end{bmatrix} \qquad \begin{bmatrix}0 & \gamma_2 & \gamma_3\end{bmatrix}$$

Any two of these cointegration vectors are linearly independent of each other, but not of the third vector.

⟶ y_t, z_t and v_t are *CI*(1,1), and $r = 2 = 3 - 1 = n - k$.

- In general, $n \geq 2$ number of $I(1)$ variables can have $1 \leq k \leq n$ different stochastic trends and their cointegration rank is $0 \leq r = n - k \leq n - 1$.

  There are three interesting cases:

  i. If $r = 0$, the variables are not cointegrated. Thus, there are $k = n$ different stochastic trends but no error correction in the system, so the appropriate model is a *VAR* in the first differences.

  ii. If $r = n$, there are n linearly independent stationary combinations of the variables, so there is not any stochastic trend, $k = 0$. In this case the appropriate model is a *VAR* in the levels.

  iii. If $1 \leq r \leq n - 1$, there are r independent cointegration relations and $k = n - r$ different stochastic trends in the system. In this case the appropriate model is a *VECM*.

  In the Johansen methodology the key is to determine r.

r = cointegration rank for a group of variables

- Suppose that y_t and z_t are $CI(1,1)$ and their error correction model has only one lag:

$$
\begin{aligned}
\Delta y_t &= a_{10} + \alpha_1 \left(y_{t-1} - \beta z_{t-1} \right) + a_{11} \Delta y_{t-1} + a_{12} \Delta z_{t-1} + \varepsilon_{1t} \\
\Delta z_t &= a_{20} + \alpha_2 \left(y_{t-1} - \beta z_{t-1} \right) + a_{21} \Delta y_{t-1} + a_{22} \Delta z_{t-1} + \varepsilon_{2t}
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix}}_{\Delta \mathbf{x}_t} = \underbrace{\begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix}}_{\boldsymbol{\pi}_0} + \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\boldsymbol{\alpha}} \underbrace{\left(y_{t-1} - \beta z_{t-1} \right)}_{\boldsymbol{\beta}' \mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\boldsymbol{\pi}_1} \underbrace{\begin{bmatrix} \Delta y_{t-1} \\ \Delta z_{t-1} \end{bmatrix}}_{\Delta \mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t}
$$

$$
\longrightarrow \quad \Delta \mathbf{x}_t = \boldsymbol{\pi}_0 + \boldsymbol{\pi} \mathbf{x}_{t-1} + \boldsymbol{\pi}_1 \Delta \mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t \quad , \quad \boldsymbol{\pi} = \boldsymbol{\alpha} \boldsymbol{\beta}'
$$

Since every element of $\Delta \mathbf{x}_t$, $\Delta \mathbf{x}_{t-1}$ and ε_t are $I(0)$,

need to make sure ε_t are serially uncorrelated

$$
\boldsymbol{\pi} \mathbf{x}_{t-1} = \Delta \mathbf{x}_t - \boldsymbol{\pi}_0 - \boldsymbol{\pi}_1 \Delta \mathbf{x}_{t-1} - \boldsymbol{\varepsilon}_t
$$

must be also stationary. Consequently, the rows of matrix π are cointegration vectors for $\mathbf{x}_t$.

critical for test statistics

- In general, for n number of $I(1)$ variables $VECM(p)$ takes the form

$$\boxed{\Delta \mathbf{x}_t = \boldsymbol{\pi}_0 + \boldsymbol{\pi}\mathbf{x}_{t-1} + \boldsymbol{\pi}_1 \Delta \mathbf{x}_{t-1} + \boldsymbol{\pi}_2 \Delta \mathbf{x}_{t-2} + \ldots + \boldsymbol{\pi}_p \Delta \mathbf{x}_{t-p} + \boldsymbol{\varepsilon}_t}$$

where π_0 is an $(n \times 1)$ vector of intercept terms;
π and π_i $(i = 1, 2,\ldots,p)$ are $(n \times n)$ coefficient matrices;
$\boldsymbol{\varepsilon}_t$ is an $(n \times 1)$ vector of white noise error terms that might be contemporaneously correlated.

Since every element of $\Delta \mathbf{x}_t$, $\Delta \mathbf{x}_{t-1}$ and ε_t are $I(0)$, so must be the rows of

$$\boxed{\boldsymbol{\pi}\mathbf{x}_{t-1} = \Delta \mathbf{x}_t - \boldsymbol{\pi}_0 - \boldsymbol{\pi}_1 \Delta \mathbf{x}_{t-1} - \boldsymbol{\pi}_2 \Delta \mathbf{x}_{t-2} - \ldots - \boldsymbol{\pi}_p \Delta \mathbf{x}_{t-p} - \boldsymbol{\varepsilon}_t}$$

Each row of $\pi \mathbf{x}_{t-1}$ is a stationary linear combination of $I(1)$ variables. Hence, each row of π is a cointegration vector and represents a cointegrating linear combination of the components of $\mathbf{x}_t$.

Matrix π is the key term in *VECM*, and there are three instances when $\pi \mathbf{x}_{t-1} : I(0)$ is satisfied.

a) If $r = rank(\pi) = 0$, $\pi = \mathbf{0}$ and the variables are not cointegrated.

Consequently, there are $k = n$ different stochastic trends but no error correction in the system, so the appropriate model is a *VAR* in the first differences.

b) If $r = n$, there are n linearly independent stationary combinations of the variables, so there is not any stochastic trend ($k = 0$).

Consequently, all variables are $I(0)$ and the appropriate model is a *VAR* in levels.

c) Otherwise, if $1 \leq r \leq n - 1$, there are r independent cointegration relations and $k = n - r$ different stochastic trends in the system.

In this case the appropriate model is a *VECM*.

It can be shown, that r is equal to the number of non-zero characteristic roots, also called eigenvalues, of matrix π.

π is an $n \times n$ matrix and it has n characteristic roots. Let's denote these roots, ordered from largest to smallest, as $\lambda_1, \lambda_2, \ldots, \lambda_n$.

In practice, these characteristic roots are unknown, but they can be estimated using the ML method, and the Johansen procedure aims to find r by checking the significance of the estimated characteristic roots.

This can be done using any of the following two tests:

i. Trace test

H_0: the cointegrating rank is $\leq r_0$

H_A: the cointegrating rank is $> r_0$,

ii. Maximum eigenvalue test

H_0: the cointegrating rank is r_0

H_A: the cointegrating rank is $r_0 + 1$,

for $r_0 = 0, 1, \ldots, n-1$ in both cases.

The test statistics are

$$\boxed{\lambda_{trace}(r) = -T \sum_{i=r+1}^{n} \ln(1-\hat{\lambda}_i)}$$

and

$$\boxed{\begin{aligned}\lambda_{eigen}(r, r+1) &= -T \ln(1-\hat{\lambda}_{r+1}) \\ &= \lambda_{trace}(r) - \lambda_{trace}(r+1)\end{aligned}}$$

… and they have non-standard sampling distributions.

These tests are to be ~~performed~~ evaluated sequentially moving from $r_0 = 0$ to $r_0 = n - 1$, until we first fail to reject H_0.

- So far, we have tacitly assumed that the equations in the *VEC* model have a constant, a_{i0}. This is only one possibility.

  In general, as far as the deterministic terms in *VEC* models are concerned, there are three possible cases:

  (1) no deterministic term at all,
  (2) a constant (c_0),
  (3) a linear deterministic term ($c_1 t$).

  The deterministic terms raise two important questions.

  i. How to incorporate them in the *VEC* model?

     The deterministic term(s) can be either outside the *EC* term (i.e., in the short-run part of the model), or inside the *EC* term (i.e., in the long-run part of the model), or in both.

     These options, however, create an identification issue.

For example, suppose that there is a constant term but no trend in the *EC* term of a bivariate *VECM*(0) model, so that

$$\Delta y_t = \alpha_1 \left(y_{t-1} - \beta_0 - \beta_1 z_{t-1} \right) + \varepsilon_{1t}$$
$$\Delta z_t = \alpha_2 \left(y_{t-1} - \beta_0 - \beta_1 z_{t-1} \right) + \varepsilon_{2t}$$

This model is equivalent with

$$\Delta y_t = -\alpha_1 \beta_0 + \alpha_1 \left(y_{t-1} - \beta_1 z_{t-1} \right) + \varepsilon_{1t}$$
$$\Delta z_t = -\alpha_2 \beta_0 + \alpha_2 \left(y_{t-1} - \beta_1 z_{t-1} \right) + \varepsilon_{2t}$$

and also, with

$$\Delta y_t = -\alpha_1 (\beta_0 - \beta_0^*) + \alpha_1 \left(y_{t-1} - \beta_0^* - \beta_1 z_{t-1} \right) + \varepsilon_{1t}$$
$$\Delta z_t = -\alpha_2 (\beta_0 - \beta_0^*) + \alpha_2 \left(y_{t-1} - \beta_0^* - \beta_1 z_{t-1} \right) + \varepsilon_{2t}$$

where β_0* is an arbitrary constant.

→ This model is not uniquely identified, unless some restriction is imposed on it, like e.g., having a constant either only inside or only outside of the *EC* term, or setting the sample mean of the *EC* term equal to zero.

Likewise, if there is a linear trend in *VECM*, it cannot be uniquely identified without some restriction(s).

We perform the Johansen tests with the *ca.jo()* function of the *urca* package. To avoid ambiguity, the *ecdet* argument of this function allows for the following three possibilities:

syntax for dealing with determ. trends

ecdet = "none" : no deterministic term in *EC*,
a constant outside *EC*.

ecdet = "const" : a constant in *EC*,
no deterministic term outside *EC*.

ecdet = "trend" : a trend variable but no constant in *EC*,
a constant outside *EC*.

ii. What do the deterministic terms imply?

Suppose again that there is a constant term but no trend in a *VECM*(0). It can be either in the *EC* term or outside it.

$$y_{t-1} - c_0 - \beta z_{t-1} = \varepsilon_{t-1}$$

$$y_{t-1} - \beta z_{t-1} = c_0 + \varepsilon_{t-1}$$

The equilibrium error is stationary around the constant.

$$\Delta y_t = c_0 + \dots$$

$$y_t = c_0 t + \dots$$

The dependent variable of an *ECM* is a first-differenced variable,

and a constant in the first differences generates a linear trend in the levels.

Likewise, a linear trend in the *EC* term would imply that the equilibrium error trend stationary,

while a linear trend outside the *EC* term would imply a quadratic trend in the level series.

Returning to *ca.jo()* function, in the light of these implications of the deterministic terms,

ecdet = *“none”* is appropriate for (linearly) trending series, granted that all trends are stochastic,

ecdet = *“const”* is proper only if none of the variables appears to have a sustained tendency to increase or decrease,

ecdet = *“trend”* is reasonable when there is some long-run linear growth that the cointegrating relation cannot account for.

How to decide which ecdet arg. to use

Ex 3: (HMPY, pp. 118-126, 139-143)

In Ex 3 of the week 11 lectures, we considered the logarithms of monthly S&P500 equity prices and dividends (*LP* and *LD*) from January 1871 to September 2016, and performing *ADF*, *KPSS* and *EG* tests we concluded that these two variables are cointegrated, CI(1,1).

a) Test for cointegration using the Johansen (*J*) tests.

Both variables are trending, so in the *ca.jo()* function let *ecdet* = *“trend”*.

Another important argument of the *ca.jo()* function is K (≥ 2), the lag length in the corresponding level *VAR*.

In general, it can be determined by relying on *VARselect()* and checking whether the *VAR*(*p*) model selected by the preferred model selection criterion passes the *BG* test for autocorrelation. If it does not, the lag length should be increased gradually till the null hypothesis of no autocorrelation is maintained (if ever).

In this case, the model selection criteria are in favour of 6, 7 or 10 lags, but even at 10 lags there is some higher order autocorrelation. Still, for the sake of illustration, let $K = 10$.

Let’s perform the trace test first.

```
library(urca)
data = cbind(LP, LD)
j.trace = ca.jo(data, type = "trace", K = 10, ecdet = "trend")
summary(j.trace)
```

The printout has four parts.

The first presents the eigenvalues.

```
######################
# Johansen-Procedure #
######################

Test type: trace statistic , with linear trend in cointegration

Eigenvalues (lambda):
[1] 1.101360e-02 4.020647e-03 8.673617e-19
```

The two largest reported eigenvalues approximately equal to 0.0110 and 0.0040.

The second part shows the trace test statistics for the null hypotheses of $r_0 = 0$ and $r_0 \leq 1$, and the 1%, 5%, 10% critical values.

```
Values of teststatistic and critical values of test:

          test 10pct  5pct  1pct
r <= 1 |  7.01 10.49 12.25 16.26
r = 0  | 26.26 22.76 25.32 30.45
```

These tests must be evaluated sequentially starting with $r_0 = 0$, until H_0 is first maintained (if ever).

→ At the 5% significance level H_0: $r_0 = 0$ is rejected, but H_0: $r_0 \leq 1$ is maintained even at the 10% level, implying $r = 1$ and $k = 1$.

The third part shows three estimated cointegration relations that correspond to the three eigenvalues in the first table.

```
Eigenvectors, normalised to first column:
(These are the cointegration relations)

                  LP.l10        LD.l10     trend.l10
LP.l10      1.0000000000  1.000000000  1.000000000
LD.l10     -1.4623625413  0.844255485  0.944926290
trend.l10   0.0008542031 -0.008095354 -0.004482069
```

Since we concluded (at the 5% level) that $r = 1$, only the largest, i.e., the first, eigenvalue and the corresponding cointegration relation are relevant.

⟶ The estimate of the *EC* term (normalized with respect to the first endogenous variable, *LP*) is

$$LP_t - 1.4624LD_t + 0.0009t$$

By definition, this linear combination of the two endogenous variables (not including *t*) is (trend) stationary, and indeed it is, as shown by the *ADF* test:

*lc = LP - 1.4624*LD*
summary(ur.df(lc, type = "trend",
selectlags = "BIC"))

```
Value of test-statistic is: -4.3196

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.96 -3.41 -3.12
```

H_0: unit root is rejected even at the 1% significance level.

Returning to the *ca.jo()* printout,

```
Weights W:
(This is the loading matrix)

          LP.l10       LD.l10     trend.l10
LP.d -0.007192742 -0.0015890167 -9.187486e-18
LD.d  0.001610846 -0.0001059895  8.684491e-18
```

The last part shows the estimated speed of adjustment coefficients.

Recall matrix $\pi = \alpha'\beta$ (see slide #8), where π is an $n \times n$ matrix, α' is an $n \times r$ matrix of the speed of adjustment coefficients, and β is an $r \times n$ matrix of the *EC* coefficients.

Since in this example $r = 1$, only the first column vector is relevant on this printout, i.e., $\alpha' = [-0.0072, 0.0016]$.

Let's now perform the maximum eigenvalue test.

j.eigen = ca.jo(data, type = “eigen", K = 2, ecdet = "trend“)
summary(j.eigen)

```
Values of teststatistic and critical values of test:

          test 10pct  5pct  1pct
r <= 1 |  7.01 10.49 12.25 16.26
r = 0  | 19.26 16.85 18.96 23.65
```

At the 5% significance level the conclusion is like before, i.e., $r = 1$.

The rest of the printout is the same as before.

b) Estimate a *VECM* assuming that the cointegration rank is 1.

After having performed the *J* test with the *ca.jo()* function, the *cajorls()* function can return the corresponding *VECM*.

cajorls(j.trace, r = 1)

```
$rlm

Call:
lm(formula = substitute(form1), data = data.mat)

Coefficients:
           LP.d        LD.d
ect1       -0.0071927   0.0016108
constant    0.0296165  -0.0058712
LP.dl1      0.2973886   0.0008547
LD.dl1     -0.0268980   0.8847662
LP.dl2     -0.0708358   0.0068419
LD.dl2      0.4853772   0.0085178
LP.dl3     -0.0328905   0.0074241
LD.dl3     -0.2319479  -0.2433649
LP.dl4      0.0241510   0.0016053
LD.dl4      0.2513937   0.2127609
LP.dl5      0.0559615   0.0129672
LD.dl5     -0.0344848   0.0152697
LP.dl6     -0.0145133   0.0146134
LD.dl6     -0.4387474   0.0208300
LP.dl7      0.0118546   0.0019950
LD.dl7     -0.3515010  -0.0292004
LP.dl8      0.0229382   0.0119698
LD.dl8      0.8216825   0.0107654
LP.dl9      0.0079719   0.0074620
LD.dl9     -0.4459201  -0.0334139


$beta
                   ect1
LP.l10     1.0000000000
LD.l10    -1.4623625413
trend.l10  0.0008542031
```

The coefficients of *ect1* are the elements of vector α (see slide #21).

The terms below *ect1* are outside the *EC* term.

ect1 refers to the *EC* term and these three coefficients are inside *EC*, i.e., they are the elements of vector β (see slide #20).

→ The estimated error correction term (the same as on slide #20) is

$$ect_t = LP_t - 1.4624LD_t + 0.0009t$$

… and the two equations of the estimated *VECM* without the augmenting terms are:

$$\widehat{\Delta LP_t} = 0.0296 - 0.0072ect_{t-1} + \ldots$$

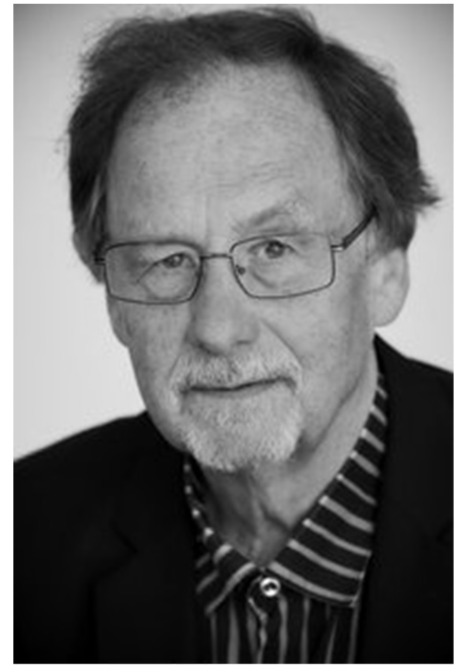
$$\widehat{\Delta LD_t} = -0.0059 + 0.0016ect_{t-1} + \ldots$$

WHAT SHOULD YOU KNOW?

- Johansen’s cointegration tests
- Error correction
- Vector error correction model

BOARD OF FAME

Søren Johansen (1939-):



Danish statistician and econometrician

Professor of economics at University of Copenhagen and at Aarhus University

The world’s most cited researcher within the field of economics in the 1990s

Cointegration

ECON90033

SEMESTER 2 FINAL EXAM:

Date: Wednesday 8 November 2023

Time: 12:30 pm

Venue: on campus

Before the exam double-check the details on

https://students.unimelb.edu.au/admin/exams

FINAL EXAM INFORMATION

- On the final exam you will have 15 minutes reading time and 120 minutes writing time.

  It will be an open-book exam, so you will be allowed to bring any printed or hand-written material.

  Note, however, that neither a formula sheet nor statistical tables will be provided. Bring your own standard normal, *t*, *F*, chi-square tables and *Casio FX82* calculator.

- You are supposed to know everything that you were taught in this and the prerequisite subjects. Hence, every topic discussed on the review and lecture slides and in the tutorial handouts can be tested in the final exam.

  A sample exam paper will be uploaded to the subject website.

  This sample paper is like the final exam paper in terms of style, length, number of questions and difficulty.

Note, however, that the sample exam paper is only for illustration. A two-hour exam is not long enough to test all important issues we studied during the semester and neither the sample exam questions, nor the final exam questions are meant to cover all examinable topics.

There will be four questions in the sample exam paper and in the final exam paper alike, each of them consisting of several tasks. Every question and task is compulsory.

- In general, there will be five types of questions and tasks.

  i. You might be asked about some statistical / econometric concept, definition, theorem etc.

  ii. You might need to explain the meaning of some formulas and/or manipulate them.

  iii. You might have to perform some calculations ‘manually’.

  iv. You will have to be able to evaluate *R* printouts.

  v. You will have to answer the questions precisely but concisely.

- The answers for the sample exam paper will be made available during the Swot Vac week.

- To maximize your chance to pass the exam, let alone to do so with a high mark, try to pursue the following strategy:

  i. Review the lecture notes, the corresponding sections in the prescribed textbook, tutorial exercises, assignment questions and solutions.

     Make sure that you really understand every little detail. If you need assistance, attend the pre-exam consultations (times and venues will be published on the LMS), and/or contact me.

  ii. For each topic, try to complete a few exercises from the textbook or some similar book(s), maybe from the internet.

     You should choose exercises for which the solutions are provided so that you can check your answers. You might even rely on illustrative examples from similar textbooks, granted that you are disciplined enough and do not peep at the solutions before making some genuine effort.

     If possible, do some calculations with *R* as well and study the details on the printouts.

iii. Once you think that you are well prepared for the exam, try to complete the sample exam paper in 2 hours.

If you do not manage, do not give up. Forget the time limit, look up similar exercises in the lecture notes, in the tutorial handouts or in the recommended text and try to follow the patterns.

Thanks for your attention during the semester and good luck for the final exam!

I wish you best of luck for your exams, a happy festive season, a safe and pleasant break, and lots of success in your further studies and future professional life.