Week 3 lab MAST90125: Bayesian Statistical Learning

Question One

In class, we consider the case where observations followed $y_i|\lambda \sim \text{Exp}(\lambda)$, and assumed the prior distribution of λ was $\text{Ga}(\alpha, \beta)$. You are told the sample mean, $\bar{y}$, was 1.21.

a) Determine the posterior distribution of λ.

b) Calculate the 95 % central and HPD credible intervals for λ for $n = 2, 5, 10, 20, 50$. Please discuss the patterns observed.

Question Two

Prove: Assume $\tilde{y}$ and y are independent given θ. Prove that

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta \quad \text{as } \tilde{y} \perp y|\theta,$$

Question Three

Prove: Assume the posterior density function $p(\theta|y_1, ..., y_n)$ is infinitely differentiable at its MAP estimator $\hat{\theta}_{\text{MAP}}$, and the derivatives of any order of $p(\theta|y_1, ..., y_n)$ is upper bounded by a positive constant M, where $y_1, ..., y_n$ are independent given θ. Prove that, as $n \to \infty$, the posterior should converge to a normal distribution whose variance is $(\frac{-d^2 \log(p(\theta|y_1,...y_n))}{d\theta^2}|_{\theta=\hat{\theta}_{\text{MAP}}})^{-1}$. Namely, we have

$$p(\theta|y_1, \ldots y_n) \to \mathcal{N}(\hat{\theta}_{\text{MAP}}, I(\hat{\theta}_{\text{MAP}})^{-1}),$$

where $I(\hat{\theta}_{\text{MAP}}) = \frac{-d^2 \log(p(\theta|y_1,...y_n))}{d\theta^2}|_{\theta=\hat{\theta}_{\text{MAP}}}$.