ECOM40006/ECOM90013 Econometrics 3
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Assignment 2: Elasticity of Substitution in Production Functions

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1 Introduction

By definition,[1] the elasticity of substitution of input L for K in a production function $Q(K, L)$ is

$$\sigma_{LK} = \frac{\mathrm{d}\ln[L/K]}{\mathrm{d}\ln\left[\frac{Q_K}{Q_L}\right]} = \frac{\left(\frac{L}{K}\right)^{-1}\mathrm{d}\left(\frac{L}{K}\right)}{\left(\frac{Q_K}{Q_L}\right)^{-1}\mathrm{d}\left[\frac{Q_K}{Q_L}\right]} = \frac{KQ_K}{LQ_L}\left[\frac{\mathrm{d}\left(\frac{Q_K}{Q_L}\right)}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1}, \tag{1}$$

where $Q_K \equiv \partial Q/\partial K$ and $Q_K \equiv \partial Q/\partial L$.

2 The Cobb-Douglas Production Function

The Cobb-Douglas production function is defined to be $Q(K, L) = \alpha K^\beta L^\delta$. Hence,

$$Q_K = \frac{\partial \alpha K^\beta L^\delta}{\partial K} = \beta\alpha K^{\beta-1}L^\delta = \frac{\beta\alpha K^\beta L^\delta}{K} = \frac{\beta Q(K,L)}{K}$$

and

$$Q_L = \frac{\partial \alpha K^\beta L^\delta}{\partial L} = \delta\alpha K^\beta L^{\delta-1} = \frac{\delta\alpha K^\beta L^\delta}{L} = \frac{\delta Q(K,L)}{L}.$$

Making the relevant substitutions in (1) we obtain

$$\begin{aligned}\sigma_{LK} &= \frac{KQ_K}{LQ_L}\left[\frac{\mathrm{d}\left(\frac{Q_K}{Q_L}\right)}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1} = \frac{K\beta Q(K,L)/K}{L\delta Q(K,L)/L}\left[\frac{\mathrm{d}\left(\frac{\beta Q(K,L)/K}{\delta Q(K,L)/L}\right)}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1} \\ &= \frac{\beta}{\delta}\left[\frac{\beta}{\delta}\frac{\mathrm{d}\left(\frac{L}{K}\right)}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1} = \frac{\beta}{\delta}\frac{\delta}{\beta} = 1.\end{aligned}$$

[1]See, for example, https://en.wikipedia.org/wiki/Elasticity_oQ_substitution, or any good microeconomics textbook.

3 The CES Production Function

The CES production function can be thought of as a generalisation of the Cobb-Douglas production function.[2] We shall use the formulation

$$Q(K,L) = A\left(\delta K^{-\rho} + (1-\delta)L^{-\rho}\right)^{-1/\rho},$$

where K and L are the factor inputs, capital and labour say, and the parameters of the function are $A > 0$, $0 < \delta < 1$, and $-1 < \rho \neq 0$. defining $Q_\rho = \delta K^{-\rho} + (1-\delta)L^{-\rho}$, so that

$$Q(K,L) = \exp\left\{\ln\left[AQ_\rho\right]\right\},$$

we obtain

$$\begin{aligned}
Q_K \equiv \frac{\partial Q(K,L)}{\partial K} &= \exp\left\{\ln\left[AQ_\rho\right]\right\} \times \frac{\partial \ln\left[AQ_\rho^{-1/\rho}\right]}{\partial K} \\
&= Q(K,L) \times \left[\frac{\partial \ln A}{\partial K} - \left(\frac{1}{\rho}\right)\frac{\partial \ln Q_\rho}{\partial K}\right] \\
&= -\frac{Q(K,L)}{\rho Q_\rho} \times \frac{\partial Q_\rho}{\partial K} \\
&= \frac{Q(K,L)K^{-(\rho+1)}}{Q_\rho}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
Q_L \equiv \frac{\partial Q(K,L)}{\partial L} &= \exp\left\{\ln\left[AQ_\rho^{-1/\rho}\right]\right\} \times \frac{\partial \ln\left[AQ_\rho^{-1/\rho}\right]}{\partial L} \\
&= Q(K,L) \times \left[\frac{\partial \ln A}{\partial L} - \left(\frac{1}{\rho}\right)\frac{\partial \ln Q_\rho}{\partial L}\right] \\
&= -\frac{Q(K,L)}{\rho Q_\rho} \times \frac{\partial Q_\rho}{\partial L} \\
&= \frac{Q(K,L)L^{-(\rho+1)}}{Q_\rho}.
\end{aligned}$$

Armed with these derivatives we can proceed as before. For simplification we first note that

$$\frac{Q_K}{Q_L} = \frac{\frac{Q(K,L)K^{-(\rho+1)}}{Q_\rho}}{\frac{Q(K,L)L^{-(\rho+1)}}{Q_\rho}} = \left(\frac{L}{K}\right)^{\rho+1}.$$

Then

$$\begin{aligned}
\sigma_{LK} &= \frac{KQ_K}{LQ_L}\left[\frac{\mathrm{d}\left(\frac{Q_K}{Q_L}\right)}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1} = \left(\frac{L}{K}\right)^{\rho}\left[\frac{\mathrm{d}\left(\frac{L}{K}\right)^{\rho+1}}{\mathrm{d}\left(\frac{L}{K}\right)}\right]^{-1} = \left(\frac{L}{K}\right)^{\rho}\left[(\rho+1)\left(\frac{L}{K}\right)^{\rho}\right]^{-1} \\
&= \frac{1}{1+\rho}, \quad \text{as required.}
\end{aligned}$$

[2]See Appendix.

A Obtaining the Cobb-Douglas From CES

Consider the limit

$$\lim_{\rho\to 0} Q = A \lim_{\rho\to 0} Q_\rho^{-1/\rho} = A \lim_{\rho\to 0} \exp\left\{-\frac{\ln Q_\rho}{\rho}\right\} = A \exp\left\{\lim_{\rho\to 0}\left[-\frac{\ln Q_\rho}{\rho}\right]\right\}.$$

Observe that both numerator and denominator of this final limit approach 0 as $\rho \to 0$. Applying L'Hôpital's rule gives

$$\lim_{\rho\to 0}\left[-\frac{\ln Q_\rho}{\rho}\right] = \lim_{\rho\to 0}\left[-\frac{\frac{d}{d\rho}\ln Q_\rho}{\frac{d}{d\rho}\rho}\right] = \lim_{\rho\to 0}\left[-\frac{Q'_\rho}{Q_\rho}\right].$$

Observe that because

$$\lim_{\rho\to 0} Q_\rho = \lim_{\rho\to 0}\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right] = 1 \neq 0,$$

and[3]

$$\lim_{\rho\to 0} Q'_\rho = -\lim_{\rho\to 0}\left[\delta K^{-\rho}\ln K + (1-\delta)L^{-\rho}\ln L\right] = -\left[\delta\ln K + (1-\delta)\ln L\right],$$

we have

$$-\lim_{\rho\to 0}\left[\frac{Q'_\rho}{Q_\rho}\right] = -\frac{\lim_{\rho\to 0} Q'_\rho}{\lim_{\rho\to 0} Q_\rho} = \delta\ln K + (1-\delta)\ln L.$$

Combining these results reveals

$$\lim_{\rho\to 0} Q = A\exp\left\{-\lim_{\rho\to 0}\left[\frac{\ln Q_\rho}{\rho}\right]\right\} = A\exp\left\{\delta\ln K + (1-\delta)\ln L\right\} = AK^\delta L^{(1-\delta)},$$

which is of Cobb-Douglas form. That is, the Cobb-Douglas production function can be thought of as a limiting case of a CES production function or, conversely, the CES production function generalizes the Cobb-Douglas production function.[4] In general, the Cobb-Douglas function has the form $Q = \alpha K^\beta L^\delta$ where α and β need not sum to unity. The restriction $\beta + \delta = 1$ is economically important because it implies constant returns to scale (e.g. double quantities of all inputs and output also doubles, regardless of level of production). Its validity can be tested, and frequently is, because it is often viewed as an unrealistic characteristic of a production function.

[3]Here we have used the result that, for g not a function of x,

$$\frac{\mathrm{d}g^{f(x)}}{\mathrm{d}x} = \frac{\mathrm{d}\exp\{f(x)\ln g\}}{\mathrm{d}(f(x)\ln g)} \times \frac{\mathrm{d}(f(x)\ln g)}{\mathrm{d}f(x)} \times \frac{\mathrm{d}f(x)}{\mathrm{d}x} = g^{f(x)} \times \ln g \times \frac{\mathrm{d}f(x)}{\mathrm{d}x}.$$

In particular,

$$\frac{\mathrm{d}Z^{-\rho}}{\mathrm{d}\rho} = -Z^{-\rho}\ln Z.$$

[4]This latter interpretation in more in keeping with the fact that the Cobb-Douglas function has a much longer history than does the CES function.