Topic 10. Bond Prices and the Term Structure of Interest Rates

ECON30024 Economics of Finanical Markets

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Outline

1. Introduction
2. What is a bond?
3. Zero-coupon bonds
4. Coupon-paying bonds
5. Bond valuation
6. Term structure of interest rates

Required reading: Bailey, Chapter 12, Chapter 13 (Section 13.1 and 13.4)

1. Introduction

- This topic focuses on the pricing or valuation of bonds.
- Characteristics of bonds
  - Bonds are a special form of loan/debt issued by governments, their agencies, and incorporated companies.
  - For companies, bonds provide a way to acquire capital at a known cost, without sacrificing rights of control over the company if no default.
  - Bonds are low-risk assets, in some circumstances the risks can be ignored altogether.
  - Bonds are commonly bought and sold in secondary markets, though many loans are not traded.

- Why studying bond pricing?
  - Bonds are important assets in investors' portfolios.
  - Bonds provide an important way of financing for governments and corporations.
  - The interest rates on some short-term government bonds are good candidates for the risk-free rate.
  - Monetary policy closely controls the risk-free rate, then other interest rates adjust accordingly to influence the real economy.
  - Understanding how the interest rates on different bonds are determined and related is important for policy conduct and financial decisions.

2. What is a Bond?

- A bond is a contract that commits the issuer to make a definite sequence of payments until a specified terminal date.
  - E.g.: the issuer might promise to pay $1000 per annum from now until 30/6/2030, at which time the contract will terminate with a lump sum payment of $10,000.

- The four attributes of a bond
  - **Maturity date**: in the example, 30/6/2030
  - **Face value**: the lump sum of $10,000, also called the maturity value, or principal
  - **Coupons**: the sequence of $1000 payments
  - **Default**: the rights conferred on bond-holders in the event of the issuer's default

- Maturity (redemption) date
  - Let T denote the maturity date and let t denote the present date. Then the 'life' or **time to maturity** of the bond:

$$n = T - t$$

  - The maturity date may be fixed and finite, but there are several other cases.
  - Callable bonds: issuer can terminate the contract before T.
  - Convertible bonds: holders can exchange the bond for another asset.
  - Perpetuities: promise to make a coupon payment in every time period, indefinitely into the future, $T \to \infty$.

- Coupons
  - Coupons are usually expressed at annual rate, but their payment is commonly split into installments.
  - **Coupon-paying bonds**: the simplest and most commonly encountered bond is one with **constant coupons**.
  - **Zero-coupon bonds** or purely discount bonds simply pay the bond's face value at maturity.
  - Zero-coupon bonds play an important role in bond valuation.

3. Zero-coupon Bonds

- Any ZC bond can be specified with two parameters: its face value, m, and the time to maturity date, $n = T - t$.

- Let p denote the market price today of an n-period ZC bond with face value m. Then the **yield to maturity (YTM)**, or **spot yield** on this bond, y_n, is defined as:

$$p = \frac{m}{(1 + y_n)^n} \quad \Rightarrow \; y_n = \left(\frac{m}{p}\right)^{1/n} - 1 \tag{1}$$

Example: What is the YTM on a bond with $m = 100$, $n = 4$, and market price $p = 83$?

- Note that the yield to maturity y_n only depends on the time to maturity.
  - Only one paramter, the time to maturity, is needed to define a ZC bond.
- The yield to maturity y_n is the **constant annual rate of return on the bond** if it is held to maturity.
  - Given the present price, y_n is known with **certainty**.
  - However, if the bond is sold early, its rate of return, known as **holding period yield (HPY),** is subject to **uncertainty**.

4. Coupon-paying Bonds

- In practice, coupon-paying bonds dominate bond markets.

- Consider a bond that promises to pay to its holders a coupon of c per year for n years plus the face value m at the maturity date. If the current market price of the bond is p, then its **yield to maturity**, y, is defined by:

$$p = \frac{c}{1+y} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c+m}{(1+y)^n} \tag{2}$$

  The YTM on a coupon-paying bond is defined as the constant annual interest rate that equates its market price to the present value of its coupons and face value.

- **Important difference from ZC bonds**: For a coupon-paying bond, y is not necessarily the annual rate of return from holding the bond to maturity.
  - Coupons can be reinvested (subject to uncertain rate of return) or not upon receipt.
  - This leads to an **uncertain** rate of return on the bond, even if the bond is held to maturity.
  - In fact, y is the annual rate of return only if all coupons are invested upon receipt at constant annual return y until the bond matures (see the Appendix).
- Like ZC bonds, holding period yields for coupon-paying bonds are uncertain.

5. Bond Valuation

- Earlier discussion focuses on yields on bonds given that prices of bonds are observed as the outcome of open market trading.

- However, some bonds are not traded or only traded infrequently. How to value or price such bonds?

  – We discuss a rule for calculating **'fair'** bond values based on observed prices/yields of other bonds that are actively traded.

  – The term 'fair' refers to the **absence of arbitrage opportunities** in **frictionless** markets.

  – 'Other bonds' usually refer to zero-coupon government bonds.

- First consider the valuation of a ZC bond with face value \$1 and time to maturity n years.
  - What is the YTM on this ZC bond, y_n?
    - By no arbitrage uncer **certainty**, y_n should be equal to the YTM on a **ZC government bond** with the same time to maturity.
    - So y_n can be computed from observable prices of actively traded ZC government bonds.
  - Given y_n, the ZC bond's fair value is simply determined by the NPV equation (refer to eqn. (1)):

$$p_n = \frac{1}{(1 + y_n)^n}$$

- Now, consider a coupon-paying bond that promises to pay a coupon c for the next n years, together with a face value m at maturity. Label this bond as 'B'.

  – Bond B can be regarded as equivalent to n ZC bonds: the 1st one paying c after one year, the 2nd one paying c after two years, ..., and the nth one paying $c+m$ after n years.

  – The prices of the ZC bonds are each given by

$$p_1 c, \quad p_2 c, \quad \ldots, \quad p_n(c+m),$$

  where $p_j = \frac{1}{(1+y_j)^j}$ is the price of a ZC bond paying \$1 after j years, $j = 1, 2, \ldots, n$.

  – By the arbitrage principle,

$$p_B = p_1 c + p_2 c + p_3 c + \cdots + p_n(c+m)$$

– Then we reach the following pricing equation for bond B:

$$p_B = \frac{c}{1+y_1} + \frac{c}{(1+y_2)^2} + \frac{c}{(1+y_3)^3} + \cdots + \frac{c+m}{(1+y_n)^n} \tag{3}$$

– Note that this equation is not the same as equation (2):

$$p = \frac{c}{1+y} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c+m}{(1+y)^n}$$

Here, the price of the coupon-paying bond is known and its yield to maturity y is calculated to satisfy this equation.

- In summary, the fair value of a bond is calculated by discounting its coupon payments and face value by appropriate discount rates – realised spot yields on some ZC bonds.

- Some comments on the ZC bonds.
  - ZC bonds are less commonly issued.
  - Given their importance in financial analysis, ZC bonds are often created synthetically as stripped bonds, or ‘strips’.
  - Financial institutions purchase a coupon-paying bond and ‘repackage’ it in the form of a sequence of ZC bonds.
  - The trading of stripped government bonds has become active since the 1980s.
- The bond valuation rules are justified on the basis of frictionless markets, so applications of them require caution.

6. Term Structure of Interest Rates

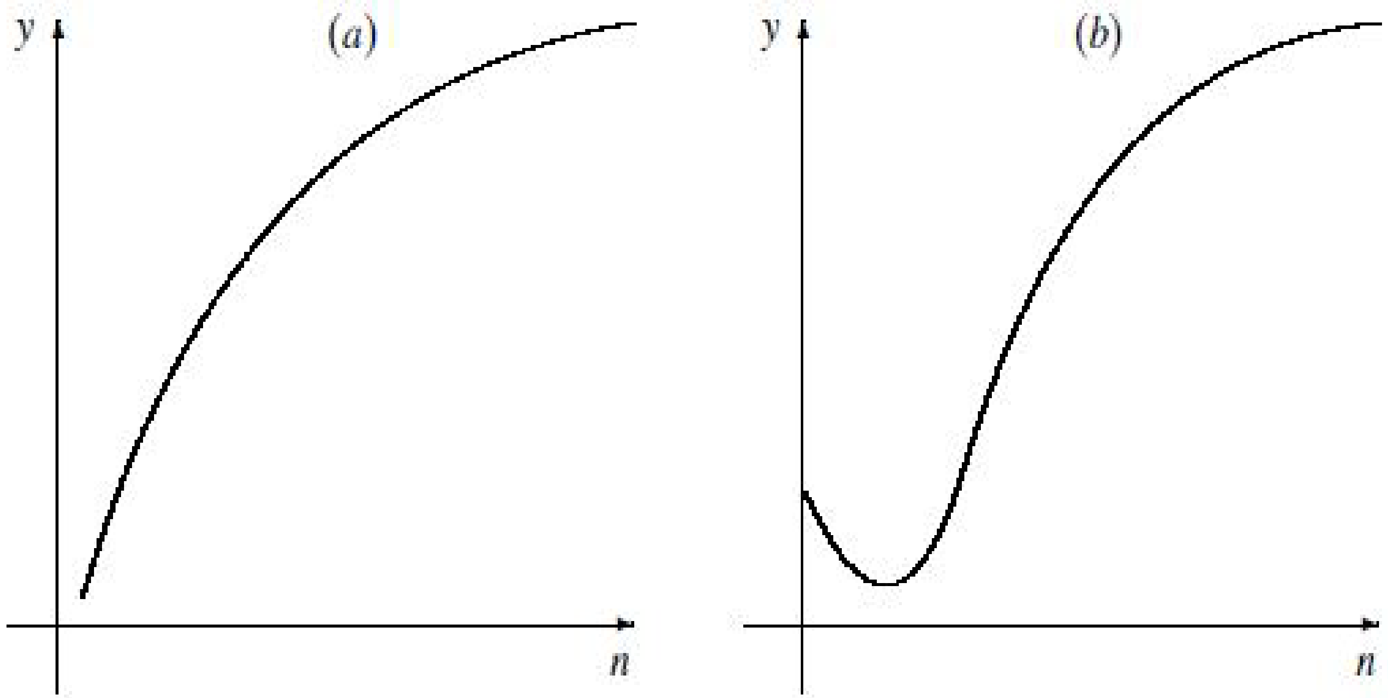
- At each date t, there are bonds with different times to maturity, and they have different yields to maturity.

- The relationship between bonds' time to maturity and their yields to maturity is the focus of the analysis of the **term structure of interest rates**.

- A common way of representing the term structure is as a **yield curve** that depicts the yields on different bonds as a function of the number of years to maturity.

  – An upward-sloping yield curve (panel (a) in Figure 2) is the conventional and most commonly observed shape.

Figure 2. Yield curves



- Yield curves are occasionally observed to slope downwards, with short-term bond yields exceeding those of long-term bonds, **inverted yield curve**.

- It is also possible that the yield curve is negatively sloped for some maturities and positively sloped for others, as in panel (b) of Figure 2.

- At each point of time, there is a yield curve. The shape of the yield curve can vary over time.

- Various theories of the term structure attempt to explain the shape of the yield curve.

- The cornerstone of term structure theories is the **expectations hypothesis of term structure**.
  - Basic idea: Long-term and short-term interest rates are linked in such a way that the expected payoff from investing in a sequence of short-term bonds equals the expected payoff from investing in a long-term bond.
  - This theory presumes **risk neutrality** for investors.
  - Mathematically, there are many ways to express the expectation hypothesis. Example: the yields on ZC bonds with different maturities satisfy

$$(1+y_{n,t})^n = (1+y_{1,t})(1+y_{1,t+1})(1+y_{1,t+2})\cdots(1+y_{1,t+n-1})$$

- We use a simple example to illustrate the prediction of the expectation hypothesis.
  - Consider a world in which one-year ('short-term') and two-year ('long-term') ZC bonds are traded.
  - The expectation hypothesis implies the following relationship:

$$(1 + y_{2,t})^2 = (1 + y_{1,t})(1 + y_{1,t+1}),$$

    i.e., in equilibrium the payoff from investing \$1 in a two-year bond must equal that from investing \$1 in a one-year bond and then reinvesting the proceeds in another one-year bond for the second year.

– Suppose that initially the yields on both bonds are 5%.

– Now, if the one-year yield is expected to increase from 5% to 8% next year, today's yield on two-year bonds will adjust to about 6.5%:

$$(1 + 6.5\%)^2 = (1 + 5\%)(1 + 8\%),$$

leading to a positively sloped yield curve.

– Conversely, if the one-year yield is expected to fall next year, the expectation hypothesis predicts that the two-year yield will fall today, leading to a negatively sloped yield curve–**inverted yield curve**.

- Many extensions of the expectations hypothesis have been proposed, stressing the relevance of risk aversion.
  - The **liquidity preference theory** argues that risk aversion of investors may imply that they prefer to hold short-term, ‘liquid’ assets unless a premium is included in the return expected from long-term assets.
  - So long-term bonds would be held only if the expected return from holding them exceeds that on short-term bonds.
  - The pattern of equilibrium bond yields should reflect such **risk, liquidity or term premia**, i.e., the yield curve should be **upward sloping**.

- An inverted yield curve rarely occurs. When it occurs, it is often viewed as indicating a looming **economic recession**, and causes a lot of concerns (Tutorial 11 discussion).

Review questions

1. Refer to Bailey, Chap. 12 for a summary of this chapter (page 300).
2. Understand the concept of a bond, in particular, the attributes of a bond and characteristics of bonds.
3. What is a zero-coupon bond? What is a coupon-paying bond? Understand why the time to maturity defines a zero-coupon bond.
4. Given market price for a ZC bond, be able to calculate its yield to maturity or spot yield. Is the relationship between p and y positive or negative?
5. Understand why the spot yield is the annual rate of return on the ZC bond only if it is held to maturity, why this rate of return is certain.
6. Given market price for a coupon-paying bond, be able to write down the equation that defines its yield to maturity. Understand why the yield to maturity is not necessarily the annual rate of return on the bond even if the bond is held to maturity.
7. Be able to define and calculate the holding period yields for zero-coupon and coupon-paying bonds. Understand why they are uncertain.

8. Given spot yields on a ZC bond, be able to calculate its fair value or fair price.
9. Using spot yields on a sequence of ZC bonds, be able to understand and calculate the fair value of a coupon-paying bonds.
10. What do we mean by the term structure of interest rate?
11. Understand the concept of yield curve, which is the graphical illustration of the term structure of interest rates. Be able to interpret the various shapes of the yield curve.
12. Briefly explain the expectation hypothesis of the term structure.
13. Understand how we can use the expectation hypothesis to predict movements in long-term interest rates from expected changes in short-term interest rates.
14. Understand why an inverted yield curve might indicate a recession is looming.
15. Understand the basic idea behind the liquidity preference theory of the term structure. According to this theory, the yield curve should be upward or downward sloping?

Appendix

Show that for a coupon-paying bond with YTM y, if the bond is held to maturity and all its coupons are invested upon receipts at annual rate y until the bond matures, then y is the annual rate of return on this bond.

- If coupon c received at $t+1$ is re-invested at rate y till the bond matures at $t+n$, then the payoff from this investment would be $c(1+y)^{n-1}$. Similarly, the re-investment of coupon c at $t+2$ would bring a payoff of $c(1+y)^{n-2}$ at $t+n$, and so on.

- The payoff of the bond at $t+n$ equals the total payoffs from re-investing the coupons in all previous years plus the coupon and face value received at $t+n$:

$$v_{t+n} = c(1+y)^{n-1} + c(1+y)^{n-2} + \cdots + c(1+y) + (c+m).$$

- So the gross rate of return on this bond is given by

$$
\begin{aligned}
\frac{v_{t+n}}{p_t} &= \frac{c(1+y)^{n-1} + c(1+y)^{n-2} + \cdots + c(1+y) + (c+m)}{\frac{c}{1+y} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c+m}{(1+y)^n}} \\
&= \frac{\Big[c(1+y)^{n-1} + c(1+y)^{n-2} + \cdots + c(1+y) + (c+m)\Big](1+y)^n}{\Big[\frac{c}{1+y} + \frac{c}{(1+y)^2} + \frac{c}{(1+y)^3} + \cdots + \frac{c+m}{(1+y)^n}\Big](1+y)^n} \\
&= \frac{\Big[c(1+y)^{n-1} + c(1+y)^{n-2} + \cdots + c(1+y) + (c+m)\Big](1+y)^n}{c(1+y)^{n-1} + c(1+y)^{n-2} + \cdots + c(1+y) + c + m} \\
&= (1+y)^n
\end{aligned}
$$

Therefore, y is the average annual rate of return on this bond.