ECOM40006/90013 Econometrics 3

Week 2 Extras

Question 1: Revisiting basic definitions

Explain the roles of the following functions:

(a) The *distribution function* (DF), also known as the *cumulative distribution function* (CDF).

(b) The *probability function* (PF), or *probability mass function* (PMF).

(c) The *density function*, or *probability density function* (PDF).

Question 2: The bivariate normal distribution

Before dealing with the whole multivariate normal distribution, let's visit its younger cousin, the bivariate normal distribution. This question is more aimed at getting you to experiment around with some of the features of this function and seeing what manual calculations may give.

Suppose that X_1 and X_2 are jointly distributed as bivariate normal with mean

$$\mu = \mathbb{E}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \mathbb{E}(X_1) \\ \mathbb{E}(X_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix},$$

and covariance matrix

$$\Sigma = \text{Var}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

In this case the probability density function of x_1 and x_2 can be written as (ignoring parameters in the function notation for clarity):

$$f_{X_1,X_2}(x_1,x_2) = (2\pi)^{-1} \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{vmatrix}^{-1/2} \exp\left\{ -\frac{1}{2}\begin{pmatrix} x_1-\mu_1 & x_2-\mu_2 \end{pmatrix}' \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}^{-1} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix} \right\}$$

(a) Let's deal with this bit by bit. First, show that

$$\begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{vmatrix}^{-1/2} = \det(\Sigma)^{-1/2} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

where $\rho \equiv \sigma_{12}/\sigma_1\sigma_2$ is the correlation between x_1 and x_2.

(b) Now, let's look at the stuff inside the large brackets. Show that

$$\begin{pmatrix} x_1-\mu_1 & x_2-\mu_2 \end{pmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}^{-1} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$
$$= \frac{1}{1-\rho^2}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]$$

and hence the PDF for the bivariate normal can be written as

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right].\right\}$$

(c) Take as given that the marginal density of X_1 is

$$f(x_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{(x_1-\mu)^2}{2\sigma_1^2}\right\}.$$

Show that the conditional density of X_2 given X_1 is

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_2-\rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1)}{\sigma_2\sqrt{1-\rho^2}}\right)^2\right\}.$$

Looking at the density of X_2 given X_1, can you tell what $X_2|X_1$ is distributed as?

Question 3: A definite problem

A symmetric and idempotent matrix A can be decomposed into $A = Q\Lambda Q'$ with $Q' = Q^{-1}$. For matrices like these, we may sometimes be interested in the properties of the diagonal matrix of eigenvalues Λ. In particular, here are a number of statements for you to verify.[1]

(a) If A is positive semi-definite then so is Λ.

(b) If A is negative definite then so is Λ.

(c) If A is idempotent, then Λ is also idempotent with all diagonal elements being either zero or one. *Hint: $AA = A$, right? An eigenvector should also satisfy $Av = \lambda v$ as well. This should imply something about what λ has to be.*

(d) If A is full rank, then Λ is also full rank, implying that all diagonal elements of Λ are non-zero.

[1]For questions involving definiteness, a general method is: (i) diagonalize A then (ii) define a new vector $y = Q'x$. After that, (iii) use the definition of a quadratic form to achieve the desired result, ensuring that you confirm any necessary properties midway (e.g. the inverse of a matrix actually exists).