Week 3 Solutions

Semester 1, 2025

Tutorial Questions

Given that it is a while since we had a lecture, and the previous lecture was mostly about defining concepts (which doesn't generate much by way of tutorial questions), I thought that this would be a good time to do a little bit of matrix revision. (I say revision because, if nothing else, much of it will have been covered in Daniel's Math Camp). Failing that, take a look at the Matrices handout that is now in the Handouts folder on the LMS.

The following questions will be based around the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ -1 & 4 & 5 & 3 \end{bmatrix}$$

Using these matrices, answer the following questions.

1. Find the sum $\mathbf{A} + \mathbf{B}$ and the difference $\mathbf{A} - \mathbf{B}$.

   Solution:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+2 \\ 3+2 & 4+4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}.$$
$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1-1 & 2-2 \\ 3-2 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

2. Find the determinant of both $\mathbf{A}$ and $\mathbf{B}$.

   Solution:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2.$$
$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 2 = 0.$$

3. Use elementary row and column operations to reduce each of $\mathbf{A}$ and $\mathbf{B}$ to equilalent canonical form. (See Section 8.1 of the Matrices handout.) Determine the matrices $\mathbf{P}$ and $\mathbf{Q}$ required to achieve the final outcome.

   Solution:
   Steps:

(a) Reduce 1,1 element of each matrix to unity. (It is already unity in each case, so we can ignore this step in these cases.)

(b) Sweep out the rest of column 1.

(c) If possible, reduce the 2,2 element to unity.

(d) Sweep out the rest of column 2

For $\mathbf{A}$ we implement the steps using the collowing row operations:

(a) Step 2: Premultiply by $\mathbf{R}_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

(b) Step 3: Scale the 2,2, element to be unity by premultiplying by $\mathbf{R}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$.

(c) Step 4: Subtract twice the second row from the first by premultiplying by $\mathbf{R}_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Thus, $\mathbf{C}_A = \mathbf{P}_A\mathbf{A}\mathbf{Q}_A = \mathbf{R}_4\mathbf{R}_3\mathbf{R}_2\mathbf{A} = \mathbf{I}_2$, where $\mathbf{Q}_A = \mathbf{I}_2$ and

$$\mathbf{P}_A = \mathbf{R}_4\mathbf{R}_3\mathbf{R}_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

For $\mathbf{B}$ we have the following steps:

(a) Step 1: Sweep out the second row by premultiplying by $\mathbf{P}_B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(b) Step 2: Noting that the second row is now all 0s we need to sweep out the second column using a column operation, which involves post-multiplying by the matrix $\mathbf{Q}_B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

Thus,

$$\mathbf{C}_B = \mathbf{P}_B\mathbf{A}\mathbf{Q}_B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Using your answer to 3, determine the rank of both $\mathbf{A}$ and $\mathbf{B}$.

   Solution:
   Clearly $\mathbf{A}$ has full rank, i.e., $\rho_A = 2$, whereas the rank of $\mathbf{B}$ is 1, i.e., $\rho_B = 1$.

5. Using your answer to 3, determine the inverse of $\mathbf{A}$.

   Solution:
   We showed in 3 that $\mathbf{P}_A\mathbf{A} = \mathbf{I}_2$. Therefore, $\mathbf{A}^{-1} = \mathbf{P}_A$.

6. Find a full rank factorization for $\mathbf{B}$ and use it to construct a Moore-Penrose generalised inverse for $\mathbf{B}$. Show that the conditions of a Moore-Penrose generalised inverse are actually satisfied.

   Solution:
   First observe that we can write

$$\mathbf{C}_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \ 0] = \mathbf{e}_1\mathbf{e}_1' \quad \text{say,}$$

and so $\mathbf{B} = \mathbf{P}_B^{-1}\mathbf{C}_B\mathbf{Q}_B^{-1} = \mathbf{K}\mathbf{L}$, where $\mathbf{K} = \mathbf{P}_B^{-1}\mathbf{e}_1$ and $\mathbf{L} = \mathbf{e}_1'\mathbf{Q}_B^{-1}$. We have

$$\mathbf{P}_B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \implies \mathbf{P}_B^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow \mathbf{K} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\mathbf{Q}_B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \implies \mathbf{Q}_B^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{L} = [1, 2].$$

You should check that $\mathbf{B} = \mathbf{K}\mathbf{L}$. (It does.) Note that $\mathbf{L} = \mathbf{K}'$ because $\mathbf{B}$ is symmetric. The Moore-Penrose inverse here is given by[1]

$$\begin{aligned}\mathbf{B}^+ &= \mathbf{L}'(\mathbf{K}'\mathbf{B}\mathbf{L}')^{-1}\mathbf{K}' = \mathbf{K}(\mathbf{L}\mathbf{B}\mathbf{K})^{-1}\mathbf{L} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left([1, 2] \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{-1} [1, 2] \\ &= \frac{1}{25}\mathbf{K}\mathbf{L} = \frac{1}{25}\mathbf{B}.\end{aligned}$$

The conditions required of a Moore-Penrose inverse are

(a) $\mathbf{B}\mathbf{B}^+\mathbf{B} = \mathbf{B}$;

(b) $\mathbf{B}^+\mathbf{B}\mathbf{B}^+ = \mathbf{B}^+$;

(c) $\mathbf{B}\mathbf{B}^+$ symmetric; and

(d) $\mathbf{B}^+\mathbf{B}$ symmetric.

As $\mathbf{B}^+ = c\mathbf{B}$, for $c = 1/25$, we can resolve both conditions 6(c) and 6(d) by demonstrating that $\mathbf{B}\mathbf{B}$ is symmetric.

$$\mathbf{B}\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

which is symmetric, as required. If we then cube $\mathbf{B}$ we can address conditions 6(a) and 6(b). Thus

$$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 50 \\ 50 & 100 \end{bmatrix}.$$

Condition 6(a) requires $c\mathbf{B}^3 = \mathbf{B}$ and condition 6(b) requires $c^2\mathbf{B}^3 = c\mathbf{B}$, or simply $c\mathbf{B}^3 = \mathbf{B}$, as for condition 6(a). We observe that

$$c\mathbf{B}^3 = \frac{1}{25} \begin{bmatrix} 25 & 50 \\ 50 & 100 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

as required.

7. Find eigenvalues and eigenvectors for $\mathbf{A}$ and $\mathbf{B}$.

Solution:

For the eigenvalues of $\mathbf{A}$ we need to find those λ that satisfy the characteristic equation $\det(\mathbf{W}) = 0$, where $\mathbf{W} = \mathbf{A} - \lambda\mathbf{I}_2$. That is, we need to solve

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2 = 0.$$

[1]As an aside, because $\mathbf{B} = \mathbf{K}\mathbf{K}'$, whenever $\mathbf{B}$ is symmetric, an alternative expression for the Moore-Penrose generalised inverse is $\mathbf{B}^+ = \mathbf{L}'(\mathbf{K}'\mathbf{B}\mathbf{L}')^{-1}\mathbf{K}' = \mathbf{K}(\mathbf{K}'\mathbf{K})^{-2}\mathbf{K}'$.

The solutions are given by

$$\lambda = \frac{5 \pm \sqrt{25 - 4(1)(-2)}}{2} = \frac{5 \pm \sqrt{33}}{2}$$

We can obtain eigenvectors as solutions to $\mathbf{Wx} = \mathbf{0}$. We can use elementary row operations to reduce $\mathbf{W}$, so that we can simply read $\mathbf{x}$ off directly. That is, $\mathbf{RWx} = \mathbf{0}$. Setting

$$\mathbf{R} = \begin{bmatrix} 1/(1-\lambda) & 0 \\ -3/(1-\lambda) & 1 \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & \frac{2}{1-\lambda} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $x_1 = -2x_2/(1-\lambda)$, with x_2 arbitrary, for either value of λ. To check that this is so, observe that

$$\begin{aligned}
\mathbf{Ax} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2x_2/(1-\lambda) \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2/(1-\lambda) + 2x_2 \\ -6x_2/(1-\lambda) + 4x_2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_2[(1-\lambda) - 1]/(1-\lambda) \\ x_2[-6 + 4(1-\lambda)]/(1-\lambda) \end{bmatrix} = \begin{bmatrix} -2\lambda x_2/(1-\lambda) \\ x_2(-2-4\lambda)/(1-\lambda) \end{bmatrix} \\
&= \begin{bmatrix} -2\lambda x_2/(1-\lambda) \\ x_2[-2 - 4\lambda - (1-\lambda) + (1-\lambda)]/(1-\lambda) \end{bmatrix} \\
&= \begin{bmatrix} -2\lambda x_2/(1-\lambda) \\ x_2[\underbrace{-2 - 5\lambda + \lambda^2}_{=0} + (1-\lambda)\lambda]/(1-\lambda) \end{bmatrix} = \begin{bmatrix} -2\lambda x_2/(1-\lambda) \\ \lambda x_2 \end{bmatrix} = \lambda\mathbf{x}.
\end{aligned}$$

thus, we have two eigenvectors depending upon our choice of λ. As specified, there are infinie sets of eigenvectors as x_2 is arbitrary. A common way around this problem is to normalize the eigenvectors $\mathbf{v}$ so that $\mathbf{v}'\mathbf{v} = 1$. We can do this directly be defining the eigenvectors to take the form $\mathbf{v} = (\mathbf{x}'\mathbf{x})^{-1/2}\mathbf{x}$. Observe that

$$\mathbf{x}'\mathbf{x} = \left(\frac{-2x_2}{1-\lambda}\right)^2 + x_2^2 = \frac{(\lambda^2 - 2\lambda + 5)x_2^2}{(1-\lambda)^2}$$

and so

$$\mathbf{v}_\lambda = \begin{bmatrix} -\dfrac{2x_2}{1-\lambda} \div \sqrt{\dfrac{(\lambda^2 - 2\lambda + 5)x_2^2}{(1-\lambda)^2}} \\ x_2 \div \sqrt{\dfrac{(\lambda^2 - 2\lambda + 5)x_2^2}{(1-\lambda)^2}} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{\sqrt{\lambda^2 - 2\lambda + 5}} \\ \dfrac{1-\lambda}{\sqrt{\lambda^2 - 2\lambda + 5}} \end{bmatrix}$$

Thus, for $\lambda_1 = (5 + \sqrt{33})/2$ and $\lambda_2 = (5 - \sqrt{33})/2$

$$[\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}] = \begin{bmatrix} -\dfrac{2}{\sqrt{\lambda_1^2 - 2\lambda_1 + 5}} & -\dfrac{2}{\sqrt{\lambda_2^2 - 2\lambda_2 + 5}} \\ \dfrac{1-\lambda_1}{\sqrt{\lambda_1^2 - 2\lambda_1 + 5}} & \dfrac{1-\lambda_2}{\sqrt{\lambda_2^2 - 2\lambda_2 + 5}} \end{bmatrix} = \begin{bmatrix} -0.4160 & -0.8246 \\ -0.9094 & 0.5658 \end{bmatrix}.$$

We can do similar things for $\mathbf{B}$ as we have just done for $\mathbf{A}$. Thus, the eigenvalues are solutions to the polynomial $\det(\mathbf{B} - \lambda\mathbf{I}_2) = (1-\lambda)(4-\lambda) - 4 = \lambda(\lambda - 5) = 0$. That is, $\lambda_1 = 5$ and $\lambda_2 = 0$. The eigenvectors must satisfy

$$(\mathbf{B} - \lambda\mathbf{I}_2)\mathbf{v}_\lambda = \mathbf{0},$$

with $\mathbf{v}_\lambda'\mathbf{v}_\lambda = 1$ and the first element positive. We note that the unnormalized eigenvectors have the same structure as we had before when $\lambda \neq 0$ so that $\mathbf{x}_{\lambda_1} = [-2x_2/(1-\lambda_1), x_2]'$, with x_2 arbitrary. But when $\lambda = 0$ we have simply $\mathbf{Bx} = \mathbf{0} \implies x_1 = -2x_2$m with x_2 arbitrary. Hence, normalizing the vectors to have unit length yields

$$[\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}] = \begin{bmatrix} -\dfrac{2}{\sqrt{\lambda_1^2 - 2\lambda_1 + 5}} & -\dfrac{2}{\sqrt{5}} \\ \dfrac{1-\lambda_1}{\sqrt{\lambda_1^2 - 2\lambda_1 + 5}} & \dfrac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \\ -\dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -0.4472 & -0.8944 \\ -0.8944 & 0.4472 \end{bmatrix}.$$

8. Confirm the spectral decomposition for $\mathbf{B}$.

   Solution:
   We need to show that $\mathbf{B} = [\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}] \operatorname{diag}(\lambda_1, \lambda_2)[\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}]'$. Thus,

$$\begin{bmatrix} -\dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \\ -\dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\dfrac{1}{\sqrt{5}} & -\dfrac{2}{\sqrt{5}} \\ -\dfrac{2}{\sqrt{5}} & \dfrac{1}{\sqrt{5}} \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \mathbf{B},$$

   as required. Observe that the matrix $[\mathbf{v}_{\lambda_1}, \mathbf{v}_{\lambda_2}]$ is orthonormal, which was not the case when working with an asymmetric matrix.

9. Use elementary matrix operations to find the inverse of $\mathbf{C}$. Show all workings. (You may wish to check your answer via computer but you need to do the calculations by hand.)

   Solution:
   To begin, construct the augmented matrix $\mathbf{A} = [\mathbf{C} \mid \mathbf{I}_3]$. Thus,

$$\mathbf{A} = \left[\begin{array}{ccc|ccc} 2 & -2 & -1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right].$$

   These are the steps that I chose:

   (a) Interchange the first and last rows. $\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

   (b) Sweep out the first column: $\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.

   (c) We have a zero at the 1,2 element and a 1 and the 2,2 element, so we simply need to sweep out the second column. $\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

   (d) We need to scale the 3,3 element of $\mathbf{A}$ by a factor of minus 1 and then sweep out the rest of the third column of $\mathbf{A}$. $\mathbf{E}_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

After applying these operations to the augmented matrix, we are able to read off directly that

$$\mathbf{C}^{-1} = \begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}$$

You should check that $\mathbf{C}\mathbf{C}^{-1} = \mathbf{I}_3$. (It does!) Finally, if you wish to combine all of the row operations into a single matrix we find that $\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \mathbf{C}^{-1}$ and this must be the final outcome, no matter how you chose to do the operations.

10. Use elementary matrix operations to construct the equivalent canonical form of $\mathbf{D}$ and thereby determine its rank. Construct a Moore-Penrose inverse of $\mathbf{D}$ based on a full rank decomposition.

Solution:
To begin,

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ -1 & 4 & 5 & 3 \end{bmatrix}.$$

Observe that $\mathbf{D}$ is of dimension 3×4 and so its rank can be at most 3. Here is my set of elementary row operations:

(a) Add row 1 to row 3: $\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Divide row 2 by 2 and, at the same time subtract twice row 2 from row 3:

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

(c) We have reduced the rows but the equivalent canonical form will also require some column operations. The first step is to sweep out the first row by adding column 1 to column 3 and also subtracting column 1 from column 4:

$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(d) Our final step is to sweep out the rest of the second row of $\mathbf{D}$. We do this by subtracting the second column from the third and fourth, respectively. These column operations are contained in

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, we find that the equivalent canonical form of $\mathbf{D}$ is given by

$$\mathbf{PDQ} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{G} \quad \text{(say)},$$

where

$$\mathbf{P} = \mathbf{R}_2\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \mathbf{C}_1\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We conclude that $\mathbf{D}$ is of rnk 2. Observe that we can write

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \mathbf{HJ},$$

say, and so a full rank factorization of $\mathbf{D}$ is $\mathbf{D} = \mathbf{KL}$, where $\mathbf{K} = \mathbf{P}^{-1}\mathbf{H}$ (the first two columns of $\mathbf{P}^{-1}$) and $\mathbf{L} = \mathbf{JQ}^{-1}$ (the first two rows of $\mathbf{Q}^{-1}$). This means that we need $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$. Again starting from the relevant augmented matrices $\mathbf{A}_P$ and $\mathbf{A}_Q$ the relevant row operations are:

(a) First scale the 2nd row by a factor of 2 and cincurrently subtract the first row from the third

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{A}_P = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right] = \mathbf{A}_P^1 \quad \text{(say)}.$$

(b) Add twice the second row of $\mathbf{A}_P^1$ to its third row and we are done

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{A}_P^1 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{array}\right] = \mathbf{A}_P^2 \quad \text{(say)}.$$

Hence,

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix} \implies \mathbf{K} = \mathbf{P}^{-1}\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}.$$

(c) The structure of $\mathbf{Q}$ is such that we first need to sweep out the upper part of the 3rd column and then the upperpart of the fourth column. To sweep out the 3rd column:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}_Q = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \mathbf{A}_Q^1 \quad \text{(say)}.$$

(d) To sweep out the 4th column, add the fourth row to each of the first two:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}_Q^1 = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Hence,

$$\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \mathbf{L} = \mathbf{J}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Let's just check this:

$$\mathbf{KL} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ -1 & 4 & 5 & 3 \end{bmatrix} = \mathbf{D},$$

as required. Having found $\mathbf{K}$ and $\mathbf{L}$ we can now find the Moore-Penrose inverse as $\mathbf{D}^{+} = \mathbf{L}'(\mathbf{K}'\mathbf{D}\mathbf{L}')^{-1}\mathbf{K}'$. First,

$$\mathbf{K}'\mathbf{D}\mathbf{L}' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 2 & 2 \\ -1 & 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -12 \\ -12 & 60 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{D}^{+} &= \frac{6}{360 - 144} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} 5 & 2 & -1 \\ 1 & 1 & 1 \\ -4 & -1 & 2 \\ 6 & 3 & 0 \end{bmatrix} \end{aligned}$$

If we would the 4 Penrose conditions using this statement of $\mathbf{D}^{+}$ then one can show that all four conditions are satisfied. Specifically, (i) $\mathbf{D}\mathbf{D}^{+}\mathbf{D} = \mathbf{D}$, (ii) $\mathbf{D}^{+}\mathbf{D}\mathbf{D}^{+} = \mathbf{D}^{+}$,

$$\text{(iii)}\ \ \mathbf{D}\mathbf{D}^{+} = \frac{1}{18} \begin{bmatrix} 15 & 6 & -3 \\ 6 & 6 & 6 \\ -3 & 6 & 15 \end{bmatrix}$$

$$\text{(iv)}\ \ \mathbf{D}^{+}\mathbf{D} = \frac{1}{18} \begin{bmatrix} 6 & 0 & -6 & 6 \\ 0 & 6 & 6 & 6 \\ 6 & 6 & 12 & 0 \\ 6 & 6 & 0 & 12 \end{bmatrix}$$

Interestingly, if one uses Matlab to perform the calculations then (iv) becomes

$$\mathbf{D}^{+}\mathbf{D} = \begin{bmatrix} 0.3333 & -0.0000 & -0.3333 & 0.3333 \\ -0.0000 & 0.3333 & 0.3333 & 0.3333 \\ -0.3333 & 0.3333 & 0.6667 & 0 \\ 0.3333 & 0.3333 & -0.0000 & 0.6667 \end{bmatrix}$$

What one sees is that the (3,4) and (4,3) elements have different signs, although both values are esseentially zero, which has no sign. If one digs a little deeper then the 14th decomial place of the (4,3) element is -4. Why is this? It illustrates the sorts of numerical errors that can creep in in even seemingly simple calculations like this one. In this case, the calculation is sufficiently close to the machine zero that truncation and/or rounding errors have convinced Matlab that this term is something other than zero, even though it is not. You can get some spectacular results trying to invert near singular matrices. In short, it comes about because computers convert fractions to decimal numbers and those conversions often only yield approximations, to the level of accuracy that the computer works to. If you perform the same calculation in Matlab, but keep out the scale factor of 1/18 then you obtain the correct answer.