Tutorial 11 Answers

```
library(rugarch)
library(moments)
data("sp500ret")
Y <- sp500ret$SP500RET*100

h <- 5
n <- length(Y)-h
Yf <- ts(Y[(n+1):length(Y)], start=n+1)
Y <- ts(Y[1:n], start=1, end=n)

AR4GARCH11 <- ugarchfit(data=Y,
    ugarchspec(variance.model=list(garchOrder=c(1,1)),
               mean.model=list(armaOrder=c(4,0))))
```

1. The estimated coefficients can be obtained as follows:

```
print(round(coef(AR4GARCH11),3))
```

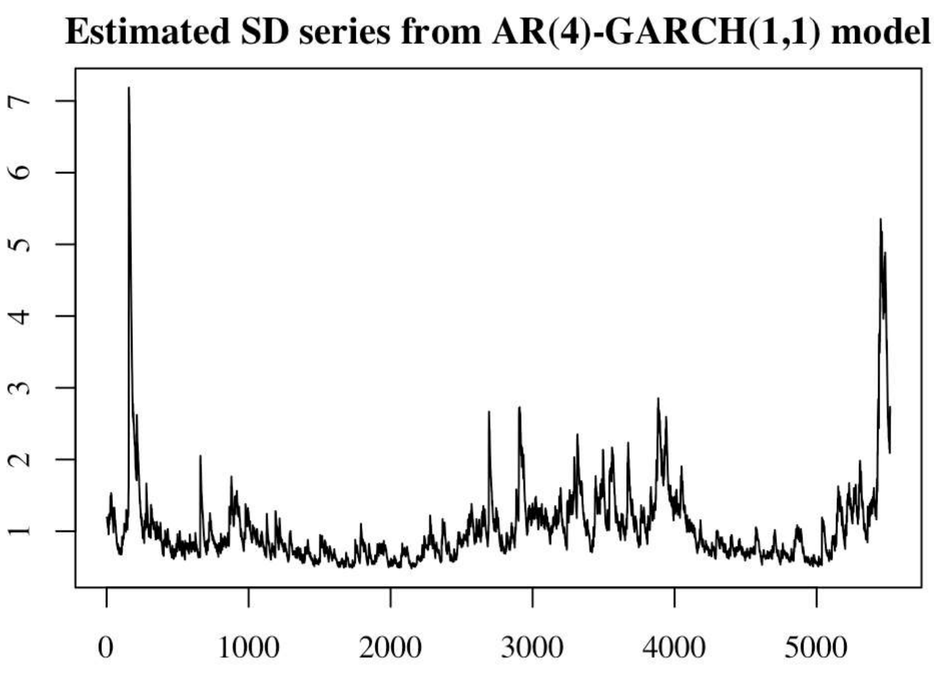
```
    mu    ar1    ar2    ar3    ar4  omega alpha1  beta1
 0.052 -0.010 -0.015 -0.019 -0.043  0.014  0.089  0.904
```

The estimated AR(4)-GARCH(1,1) model therefore consists of the following equations.

$$
\begin{aligned}
Y_t &= 0.052 + \widehat{Z}_t \\
Z_t &= -0.010\, Z_{t-1} - 0.015\, Z_{t-2} - 0.019\, Z_{t-3} - 0.043\, Z_{t-4} + \widehat{U}_t \\
\widehat{\sigma}_t^2 &= 0.014 + 0.089 U_{t-1}^2 + 0.904 \sigma_{t-1}^2
\end{aligned}
$$

2. The fitted standard deviation series is computed using

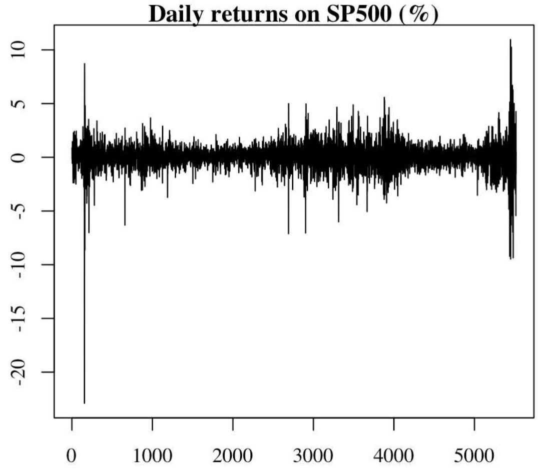
```
AR4GARCH11_sigma <- ts(sigma(AR4GARCH11), start=1, end=n)
```

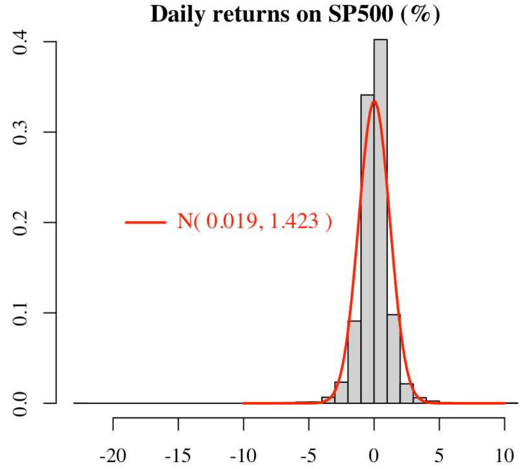


The estimated standard deviations are highest near the beginning and near the end of the sample period. The large spike in variation at the start of the sample corresponds to the 1987 stock market crash. The sample ends in early 2009, so the increased variation near the end of the sample corresponds to the GFC.

3. a. The descriptive statistics and plots are shown below. The mean return is slightly positive (share prices having a long term overall tendency to increase over this period). The skewness is negative and kurtosis of ~36 very much greater than 3, implying non-normality. Both skewness and kurtosis will be inflated in magnitude by the large negative outlier for the 1987 crash, as well as some of the other outliers. The large kurtosis is evident in the long tails of the histogram, counterbalanced by the taller than normal peak around the mean.

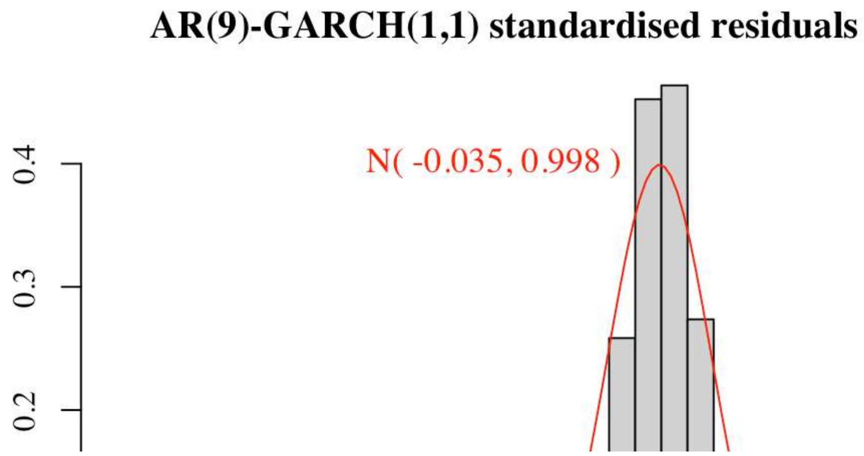
Mean	Std.Dev.	Skewness	Kurtosis
0.019	1.193	-1.541	36.185

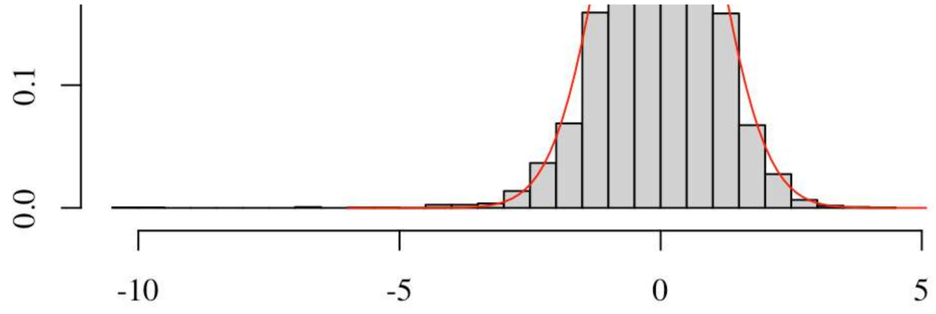




b. The descriptive statistics and histogram below reveal that the residuals retain clear signs of non-normality (some negative skewness and substantial excess skewness) although these are reduced relative to the same measures for the original data series. It is commonly the case with GARCH models that the standardised residuals are closely to normality than the original data, but still not close to approximately normal.

Mean	Std.Dev.	Skewness	Kurtosis
-0.035	0.999	-0.75	8.332





4. The forecasts, intervals and errors are as follows.

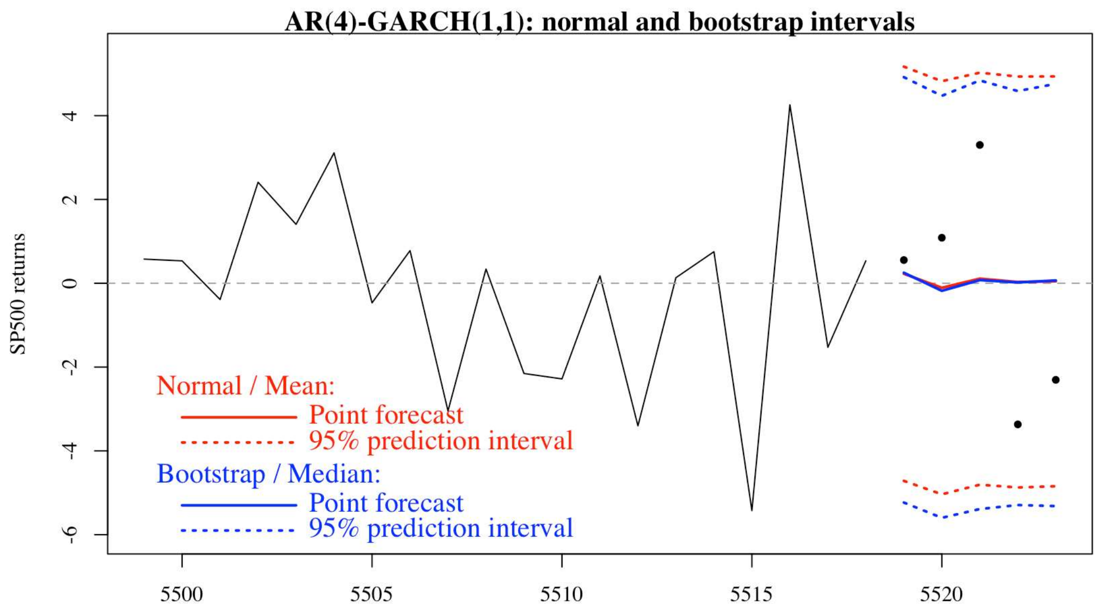
```
  Lower   Point Upper Actual  Error Sigma
 -4.716   0.227 5.170  0.554  0.327 2.522
 -5.037  -0.108 4.822  1.087  1.194 2.515
 -4.807   0.110 5.026  3.301  3.191 2.508
 -4.874   0.030 4.933 -3.368 -3.398 2.502
 -4.844   0.047 4.937 -2.305 -2.352 2.495
```

In this case the actual value turns out to lie within the prediction interval at all of the 5 forecast days. The RMSE is as follows.

```
 RMSE
2.400
```

5. The bootstrap results are as follows.

```
  Lower Median Upper
 -5.233  0.252 4.919
 -5.598 -0.180 4.473
 -5.388  0.076 4.840
 -5.293  0.020 4.587
 -5.318  0.067 4.760
```



Visually there are not huge differences between the bootstrap and normal results. Both upper and lower bootstrap prediction intervals lie below the corresponding normal intervals, a downwards shift that would be consistent with the negative skewness seen in the standardised residuals.

There would be little difference in using the median as point forecasts instead of the mean. The RMSE of the median forecasts is given by

```
MedianErrors <- Yf-AR4GARCH11BSForecasts$Median
RMSE_Median <- sqrt(mean(MedianErrors^2))
```

```
RMSE Median
      2.417
```

The RMSE for the median is actually slightly less than that for the mean, although this should not be over-stated since the practical differences between the two sets of forecasts are clearly very small.

It might be wondered why we would consider the median instead of mean for forecasting when we know the conditional mean is the optimal forecasting function. The theoretical reasoning is that the conditional mean minimises the mean *squared* error, while the median minimises the mean *absolute* error. It should be noted this use of the bootstrap is not a full model for the median (which would involve the estimation of conditional quantiles), but it is an interesting sensitivity analysis for the importance of non-normality for the forecasts and intervals from the model.

Consideration of the median and its optimality with respect to the mean *absolute* error criterion suggests the Mean Absolute Error as a (commonly used) alternative point forecast evaluation method. For any generic forecast errors $\widehat{U}_t$, the Mean Absolute Error is

$$MAE = \frac{1}{h}\sum_{t=n+1}^{n+h} |\widehat{U}_t|.$$

We can compare the MAE for the mean and median forecasts as follows, but there is little difference here because the forecasts themselves are so similar.

```
MAE_mean <- mean(abs(AR4GARCH11Forecasts$Error))
MAE_median <- mean(abs(MedianErrors))
```

```
     Mean Median
MAE 2.092  2.111
```

In this case it might seem surprising that there is not a great difference between the bootstrap and normal prediction intervals, given that the standardised residuals are clearly non-normal (3(b)). While the skewness and kurtosis in 3(b) do not align with the values for a normal distribution, these moments are not directly relevant to the construction of prediction intervals. Of direct relevance are the quantiles of the standardised residuals, computed here alongside the quantiles of a normal distribution with matching mean and standard deviation for comparison.

```
Q_BS <- quantile(Us, probs=c(0.025,0.975))
Q_Norm <- qnorm(c(0.025,0.975), mean=mean(Us), sd=sd(Us))
```

```
        2.5% 97.5%
Q_BS   -2.138 1.865
Q_Norm -1.993 1.924
```

This shows the quantiles of the model's standardised residuals are shifted downwards a small amount relative to the quantiles implied by the normal distribution. This downward shift results in the downward shift in the prediction intervals seen in the plot above. So even though the kurtosis of 8 is well above the normal value of 3, this doesn not result in such a dramatic difference in the quantiles.

It is quite common to see formal hypothesis tests for normality (eg Jarque-Bera tests) being applied in this type of data analysis. These tests are not very informative about practical outcomes, whether applied to the standardised residuals or (especially) to the original data. For a start it is certain that no data is ever *exactly* normally distributed, so in such a formal testing scenario we know for certain the null hypothesis of normality is false. That is not interesting. What is interesting is to assess the extent to which the distributional properties of the standardised residuals affect the forecasts and prediction intervals arising from the model, for which quantiles and bootstrapping as shown are directly relevant (while skewness, kurtosis, and normality testing are not directly informative).