ECON30009/90080 – Tutorial 8 Solution

This Version: Semester 2, 2025

This tutorial is designed to get you used to solving models with money in the utility function.

Question 1: Household problem only

Suppose the household lives for two periods and has the following household preferences

$$U(c_1, c_2, m_1, m_2) = (1-\beta)\ln c_1 + (1-\gamma)\ln m_1 + \beta \ln c_2 + \gamma \ln m_2$$

where c_t and m_t refer to consumption and real money balances in period t, respectively, for $t \in \{1, 2\}$. The parameters $0 < \beta < 1$ and $0 < \gamma < 1$ affect the weights households put on the utility derived from consumption and holding real money balances in each period. The household is born with nominal money balances, M_0 and nominal bonds, B_1. Each unit of B_1 has a gross nominal return of $1 + i_1$. The household receives exogenous income y_t each period for $t \in \{1, 2\}$. In period 1, households can choose how much to consume, how much money to hold, and how much to save in a bond that has a gross nominal return of $1 + i_2$. In period 2, the household can choose how much to consume and how much money to hold.

a Write down the nominal budget constraint of the household in period 1.
Answer

$$P_1c_1 + M_1 + B_2 = P_1y_1 + (1 + i_1)B_1 + M_0$$

b Write down the nominal budget constraint of the household in period 2.
Answer

$$P_2c_2 + M_2 = P_2y_2 + (1 + i_2)B_2 + M_1$$

c Write down the lifetime budget constraint of the household in nominal terms.
Answer
From the second period budget constraint, make B_2 the subject of the equation:

$$B_2 = \frac{P_2c_2 + M_2 - [P_2y_2 + M_1]}{1 + i_2}$$

And substitute this into the first period budget constraint:

$$P_1c_1 + \frac{P_2c_2}{1 + i_2} = P_1y_1 + \frac{P_2y_2}{1 + i_2} + (1 + i_1)B_1 + M_0 - M_1 + \frac{M_1 - M_2}{1 + i_2}$$

The LHS represents the present discounted value of lifetime nominal consumption spending, the RHS represents the present discounted value of lifetime nominal income less the present discounted value of changes in nominal money holdings.

d Derive an IS curve from the household problem.

Answer

We can re-write lifetime budget constraint to be real terms by dividing by P_1:

$$c_1 + \frac{\Pi_2 c_2}{1+i_2} = y_1 + \frac{\Pi_2 y_2}{1+i_2} + (1+i_1)b_1 + \frac{m_0}{\Pi_1} - m_1\frac{i_2}{1+i_2} - \frac{\Pi_2 m_2}{1+i_2}$$

where $\Pi_2 = P_2/P_1$, and $m_t = M_t/P_t$. Note that from Fisher equation $\Pi_2/(1+i_2) = 1/R_2$. Writing down the household utility maximization problem as a Lagrangian, we have:

$$\begin{aligned}\mathcal{L} &= (1-\beta)\ln c_1 + (1-\gamma)\ln m_1 + \beta \ln c_2 + \gamma \ln m_2 \\ &\quad +\lambda\left[y_1 + \frac{\Pi_2 y_2}{1+i_2} + (1+i_1)b_1 + \frac{m_0}{\Pi_1} - m_1\frac{i_2}{1+i_2} - \frac{\Pi_2 m_2}{1+i_2} - c_1 - \frac{\Pi_2 c_2}{1+i_2}\right]\end{aligned}$$

Taking FOCs wrt c_1 and c_2, we have:

$$\frac{1-\beta}{c_1} = \lambda$$

$$\frac{\beta}{c_2} = \lambda\frac{\Pi_2}{1+i_2}$$

Combining the above two FOCs, we get the Euler equation:

$$\frac{\beta}{c_2} = \frac{1-\beta}{c_1}\frac{\Pi_2}{1+i_2}$$

Taking logs of the above and re-arranging:

$$\ln c_2 - \ln c_1 = \ln\beta - \ln(1-\beta) + \ln(1+i_2) - \ln \Pi_2$$

The change in log consumption is equivalent to consumption growth. Denote this consumption growth as $\Delta \ln c$. The difference $\ln\beta - \ln(1-\beta)$ represents the growth in consumption that is affected by the weights the household puts on consumption in the future vs today. And finally the growth in consumption is affected by the gross nominal interest rate less the gross inflation rate. Since consumption growth is related to output growth, the above represents how the IS curve can be derived from the Euler equation. It shows how output growth is affected by the interest rate and the household's incentive to save for the future.

Question 2: Money in the utility function model

Consider the following money in the utility function model. Household's preferences are given by:

$$U(c_1, c_2, m_1, m_2) = c_1^{1-\beta} c_2^{\beta} + m_1^{1-\gamma} m_2^{\gamma}$$

where $m_t = M_t/P_t$, $0 < \beta < 1$ and $0 < \gamma < 1$. The household is born with physical capital a_1 *and* nominal money holdings, M_0. Price P_0 is given and exogenous. The household rents out her physical capital at the nominal rental rate $P_t R_t$, and earns the nominal wage rate $P_t w_t$ for each unit of labour supplied. Households have no disutility from labour and inelastically supply 1 unit of labour. Households also receive dividend income from firms each period as well as a nominal transfer $P_t\tau_t$ each period. There is a measure $N = 1$ of households in the economy.

Firms produce output according to a Cobb-Douglas production $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$. Firms rent capital and hire labour to produce. The monetary authority follows an exogenous money supply rule $M_{t+1}^s = \theta M_t^s$ where θ is a parameter that governs whether money supply is growing ($\theta > 1$), contracting ($\theta < 1$) or constant ($\theta = 1$). There is no other government spending in the economy. Since the government earns revenue from printing money, it runs a balanced budget by transferring the money supply created to the households, that is $P_t\tau_t = M_t^s - M_{t-1}^s$.

a Derive the household's lifetime budget constraint in real terms.
Answer
First period budget constraint in nominal terms:

$$P_1 c_1 + M_1 + P_1 a_2 = P_1 R_1 a_1 + P_1 w_1 + P_1 \pi_1 + M_0 + P_1 \tau_1$$

Second period budget constraint in nominal terms:

$$P_2 c_2 + M_2 = P_2 R_2 a_2 + P_2 w_2 + P_2 \pi_2 + P_2 \tau_2 + M_1$$

Make a_2 the subject of the equation:

$$a_2 = \frac{P_2 c_2 + M_2 - [P_2 w_2 + P_2 \pi_2 + P_2 \tau_2 + M_1]}{P_2 R_2}$$

and substitute this expression for a_2 into the first period budget constraint, and divide everywhere by P_1:

$$c_1 + \frac{M_1}{P_1} + \left\{ \frac{P_2 c_2 + M_2 - [P_2 w_2 + P_2 \pi_2 + P_2 \tau_2 + M_1]}{P_2 R_2} \right\} = R_1 a_1 + w_1 + \pi_1 + \frac{M_0}{P_0}\frac{P_0}{P_1} + \tau_1$$

Since $m_t = M_t/P_t$ and denote gross inflation as $\Pi_t = P_t/P_{t-1}$, we can re-arrange the above to get the LBC in real terms:

$$c_1 + \frac{c_2}{R_2} = R_1 a_1 + w_1 + \frac{w_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} + \left[\tau_1 - m_1 + \frac{m_0}{\Pi_1}\right] + \frac{1}{R_2}\left[\tau_2 - m_2 + \frac{m_1}{\Pi_2}\right]$$

b Set up the household's problem and derive the optimality conditions of the household.
Answer
The household's problem is:

$$\max_{c_1, c_2, m_1, m_2} c_1^{1-\beta} c_2^{\beta} + m_1^{1-\gamma} m_2^{\gamma}$$

s.t.

$$c_1 + \frac{c_2}{R_2} = R_1 a_1 + w_1 + \frac{w_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} + \left[\tau_1 - m_1 + \frac{m_0}{\Pi_1}\right] + \frac{1}{R_2}\left[\tau_2 - m_2 + \frac{m_1}{\Pi_2}\right]$$

We can write down the Lagrangian of this problem:

$$\begin{aligned}\mathcal{L} &= c_1^{1-\beta}c_2^{\beta} + m_1^{1-\gamma}m_2^{\gamma} \\ &+\lambda\left\{R_1a_1 + w_1 + \frac{w_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} + \left[\tau_1 - m_1 + \frac{m_0}{\Pi_1}\right] + \frac{1}{R_2}\left[\tau_2 - m_2 + \frac{m_1}{\Pi_2}\right] - c_1 - \frac{c_2}{R_2}\right\}\end{aligned}$$

Taking FOCs:

$$(c_1): \quad (1-\beta)\, c_1^{-\beta}c_2^{\beta} = \lambda$$

$$(c_2): \quad \beta c_1^{1-\beta}c_2^{\beta-1} = \frac{\lambda}{R_2}$$

$$(m_1): \quad (1-\gamma)\, m_1^{-\gamma}m_2^{\gamma} = \lambda\left[1 - \frac{1}{R_2\Pi_2}\right]$$

$$(m_2): \gamma m_1^{1-\gamma}m_2^{\gamma-1} = \frac{\lambda}{R_2}$$

$$(\lambda): \quad c_1 + \frac{c_2}{R_2} = R_1a_1 + w_1 + \frac{w_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} + \left[\tau_1 - m_1 + \frac{m_0}{\Pi_1}\right] + \frac{1}{R_2}\left[\tau_2 - m_2 + \frac{m_1}{\Pi_2}\right]$$

Combining the FOCs wrt c_1 and c_2, we get the Euler equation which can be written as:

$$\frac{\beta}{1-\beta}R_2c_1 = c_2$$

Combining the FOCs wrt m_2 and c_2, we get the optimal money demand in period 2 in terms of its trade-off with c_2:

$$\gamma m_1^{1-\gamma}m_2^{\gamma-1} = \beta c_1^{1-\beta}c_2^{\beta-1}$$

Combining the FOCs wrt m_1 and c_1, we get the optimal money demand in period 1 in terms of its trade-off with c_1:

$$(1-\gamma)\, m_1^{-\gamma}m_2^{\gamma} = (1-\beta)\, c_1^{-\beta}c_2^{\beta}\left[1 - \frac{1}{R_2\Pi_2}\right]$$

Finally, we have the LBC as the other household optimality condition.

c Set up the firm's problem and derive the firm's optimality conditions.
Answer
The firm's profit maximization problem is standard and similar to the one we have shown in class:

$$\max_{K_t,L_t} z_tK_t^{\alpha}L_t^{1-\alpha} - w_tL_t - R_tK_t$$

Taking FOCs, we get the firm's optimality conditions:

$$\alpha z_tK_t^{\alpha-1}L_t^{1-\alpha} = \alpha z_tk_t^{\alpha-1} = R_t$$

$$(1-\alpha)z_tK_t^{\alpha}L_t^{-\alpha} = (1-\alpha)z_tk_t^{\alpha} = w_t$$

where $k_t = K_t/L_t$

d Show that real output, real investment and real consumption do not depend on money supply.

Answer

We can plug the Euler equation into the household LBC:

$$c_1 + \frac{\beta}{1-\beta}c_1 = R_1 a_1 + w_1 + \frac{w_2}{R_2} + \pi_1 + \frac{\pi_2}{R_2} + \left[\tau_1 - m_1 + \frac{m_0}{\Pi_1}\right] + \frac{1}{R_2}\left[\tau_2 - m_2 + \frac{m_1}{\Pi_2}\right]$$

and in equilibrium, all markets clear, $L_t = N = 1$, $K_t = Na_t = a_t$, and $M_t^s = NM_t = M_t$. Substituting in the information from the firm's optimality conditions and using the fact that $\tau_t = \frac{M_t^s}{P_t} - \frac{M_{t-1}^s}{P_{t-1}}\frac{1}{\Pi_t}$, we have:

$$c_1 = (1-\beta)\left[z_1 k_1^\alpha + \frac{1-\alpha}{\alpha}k_2\right]$$

k_2 in the above is still endogenous. But from our goods market clearing condition, we know that $k_2 = y_1 - c_1$. Substituting in for c_1, we have:

$$k_2 = z_1 k_1^\alpha - (1-\beta)\left[z_1 k_1^\alpha + \frac{1-\alpha}{\alpha}k_2\right]$$

which re-arranging gives us:

$$k_2 = \frac{\alpha\beta}{\left[1-(1-\alpha)\beta\right]}z_1 k_1^\alpha$$

and

$$c_1 = \frac{(1-\beta)}{\left[1-(1-\alpha)\beta\right]}z_1 k_1^\alpha$$

and real output in period 1 is $y_1 = z_1 k_1^\alpha$. From the above, we see that k_2, c_1 and y_1 do not depend on money supply.

e Suppose $\theta = 1$. Derive an expression for i_2 in terms of parameters of the model only.

Answer

Note for $\theta = 1$, we have

$$\frac{M_t^s}{M_{t-1}^s} = 1 \implies \frac{\Pi_t m_t}{m_{t-1}} = 1$$

Dividing the FOC for m_1 by the FOC for m_2, we can expres m_2/m_1 as:

$$\frac{m_2}{m_1} = \frac{\gamma}{1-\gamma}\frac{R_2\Pi_2 - 1}{\Pi_2}$$

Substitute the above into $\Pi_2 m_2/m_1 = 1$

$$\frac{\gamma}{1-\gamma}(R_2\Pi_2 - 1) = 1$$

Using the fisher equation where $R_2\Pi_2 = 1 + i_2$, this gives us:

$$i_2 = \frac{1-\gamma}{\gamma}$$