Topic 4. Portfolio Selection with Mean-Variance Objective

ECON30024 Economics of Finanical Markets

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Outline

1. A simple case: one risky asset with a risk-free asset
   - The portfolio frontier
   - The optimum portfolio
2. Two risky assets with no risk-free asset
   - The efficient frontier and efficient portfolios
   - The minimum risk portfolio
3. Many risky assets with no risk-free asset
4. Many risky assets with a risk-free asset

Required reading: Chap. 5 of Bailey

1. A Simple Case: One Risky Asset with a Risk-free Asset

- This topic focuses on a single investor's portfolio decision with mean-variance objective.
  - Provides a theory of asset demand, forming the foundation of the CAPM.
  - Provides a method for the practical construction of optimal portfolios.

- First consider a **simple case**: an investor has initial wealth A to divide between:
  - a risk-free asset, asset 0: p_0, r_0
  - a risky asset, asset 1: p_1, r_1

$$\mu_1 = E(r_1) > r_0,\ \sigma_1^2 = var(r_1)$$

- The investor's problem: chooses a **portfolio** of asset 0 and asset 1 to maximise a mean-variance **objective** given by

$$G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2, \quad \alpha > 0,$$

subject to her budget **constraint**, taking r_0, μ_1, σ_1 as given.

(1) A portfolio is defined as

- a vector of asset holdings, (x_0, x_1);
- or a vector of proportions of wealth invested in each asset, (a_0, a_1), $a_0 \equiv \frac{p_0 x_0}{A}$, $a_1 \equiv \frac{p_1 x_1}{A}$.

(2) The investor's budget or wealth constraint:

$$p_0 x_0 + p_1 x_1 = A, \quad \text{or equivalently}$$

$$a_0 + a_1 = 1.$$

(3) What are μ_P and σ_P^2 in the objective function?

- From Topic 3 (and Tutorial 3, Q3), the rate of return on the portfolio is given by

$$r_P = a_0 r_0 + a_1 r_1$$

- Then,

$$\mu_P = E(r_P) = a_0 r_0 + a_1 E(r_1) = a_0 r_0 + a_1 \mu_1$$

$$= r_0 + a_1(\mu_1 - r_0)$$

$$\sigma_P^2 = var(r_P) = var(a_1 r_1) = a_1^2 var(r_1) = a_1^2\,\sigma_1^2$$

- Given (r_0, μ_1, σ_1), the portfolio selection problem simplifies to

$$\max_{a_1} \left\{\mu_P - \alpha\sigma_P^2\right\}, \quad \text{where}$$

$$\mu_P = r_0 + a_1(\mu_1 - r_0)$$

$$\sigma_P^2 = a_1^2\,\sigma_1^2$$

- Solving the problem:
  - Rewrite the problem using the expressions for μ_P and σ_P^2:

$$\max_{a_1} \left\{r_0 + a_1(\mu_1 - r_0) - \alpha a_1^2\,\sigma_1^2\right\},$$

  - The first-order condition:

$$\mu_1 - r_0 - \alpha\sigma_1^2(2a_1) = 0$$

$$\Rightarrow a_1^* = \frac{\mu_1 - r_0}{2\alpha\sigma_1^2}$$

- The **optimum portfolio** is given by $(1 - a_1^*, a_1^*)$.

- Properties of the optimum portfolio:
  - Is a_1^* always greater than 0 (no short sale of asset 1)?
  - How does a_1^* depend on $\mu_1 - r_0$?
  - Can a_1^* be greater than 1? What does this mean?
  - How does a_1^* depend on σ_1?
  - How does a_1^* depend on α?

- The optimum portfolio's return r_P has mean and std:

$$\mu_E = r_0 + a_1^*(\mu_1 - r_0) = r_0 + \frac{(\mu_1 - r_0)^2}{2\alpha\sigma_1^2}, \quad \sigma_E = a_1^*\sigma_1 = \frac{\mu_1 - r_0}{2\alpha\sigma_1}$$

- An alternative way to formulate the portfolio selection problem
  - The expression for σ_P^2 implies that $\sigma_P = a_1\sigma_1$, i.e.,

$$a_1 = \frac{\sigma_P}{\sigma_1}.$$

  - Plugging this expression into the expression for μ_P gives

$$\mu_P = r_0 + \frac{\sigma_P}{\sigma_1}(\mu_1 - r_0), \quad \text{i.e.}$$

$$\mu_P = r_0 + \left(\frac{\mu_1 - r_0}{\sigma_1}\right)\sigma_P, \tag{1}$$

where

$$\frac{\mu_1 - r_0}{\sigma_1} \equiv s_1$$

is called the **Sharpe ratio** of asset 1: its expected excess return normalised by its risk.

– The portfolio selection problem can be re-formulated as

$$\max_{(\sigma_P,\mu_P)} \left\{\mu_P - \alpha\sigma_P^2\right\}$$

$$\text{s.t.} \quad \mu_P = r_0 + \left(\frac{\mu_1 - r_0}{\sigma_1}\right)\sigma_P \tag{1}$$

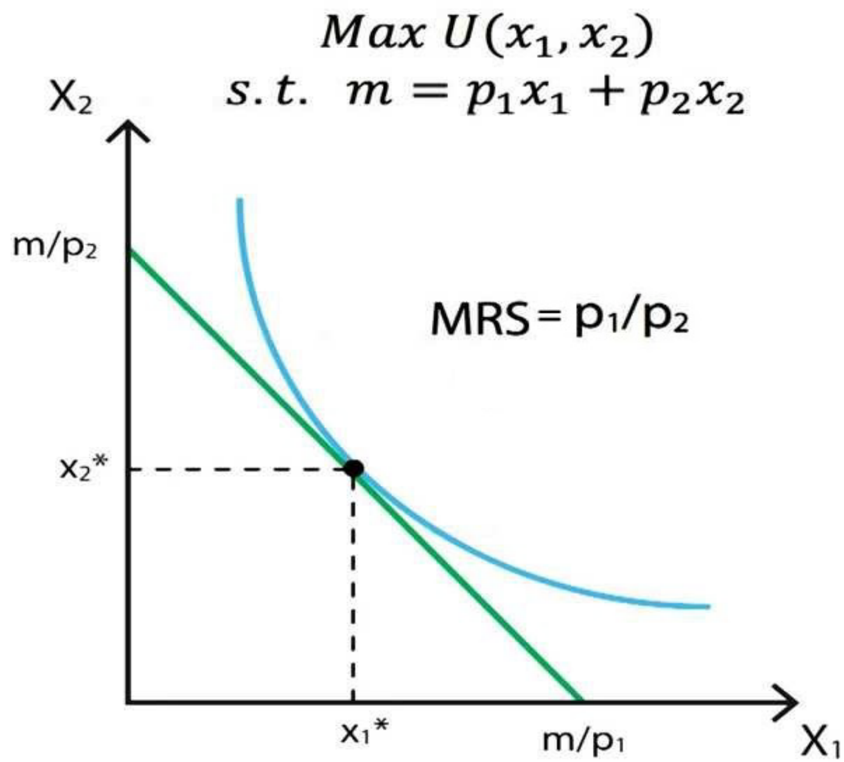
The solution to this problem is exactly (σ_E, μ_E) on slide 7 (see Exercise_Topic4).

- A graphical illustration

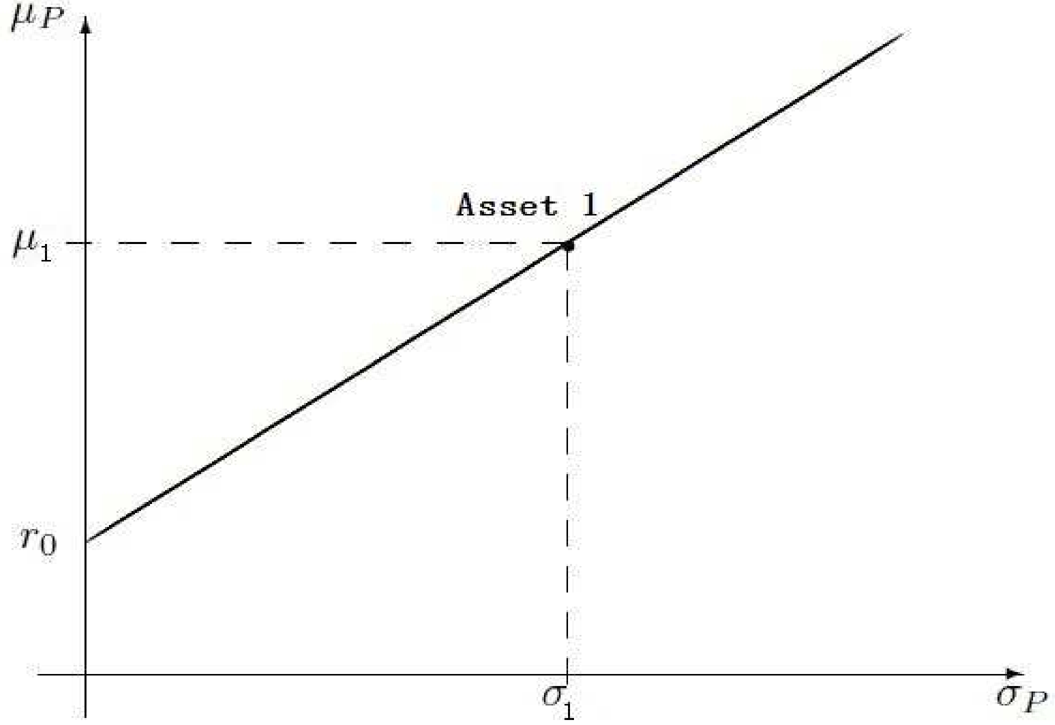
  – Eqn. (1) defines a straight line in the (σ_P, μ_P) space, which is called the **portfolio frontier (PF)**.

  – In this case, the PF represents the set of feasible portfolios from which the investor can choose.

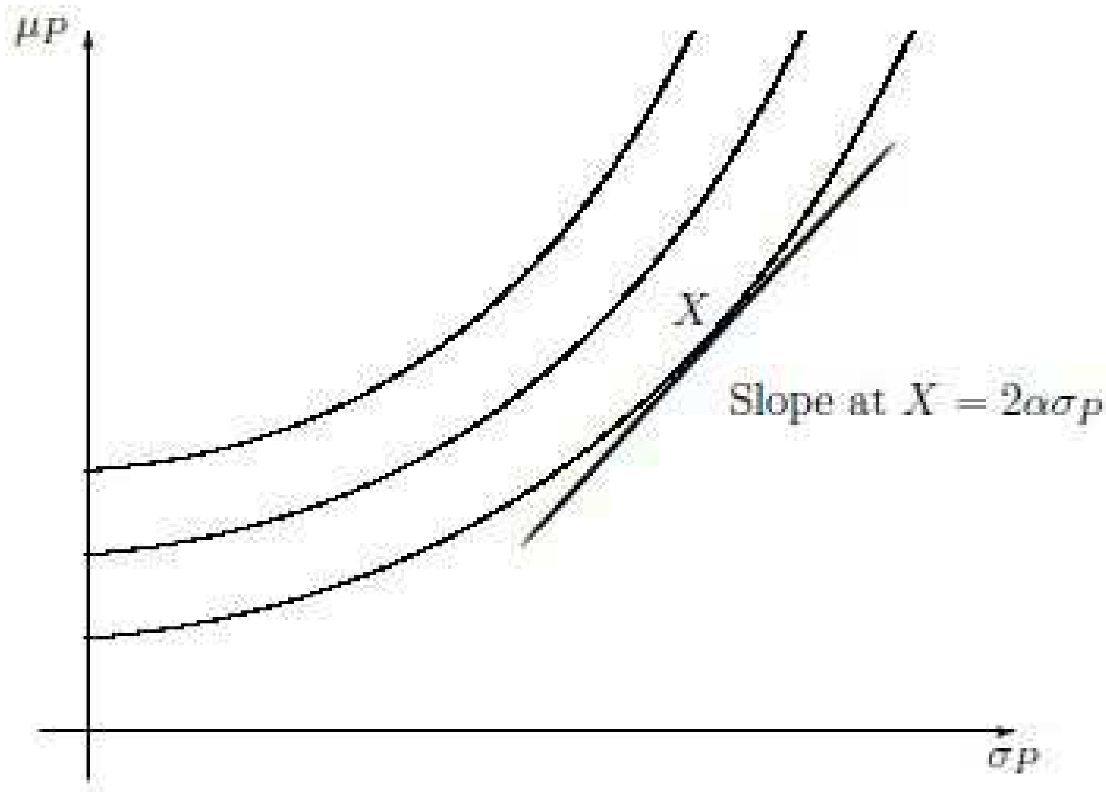
A utility maximisation problem in Micro



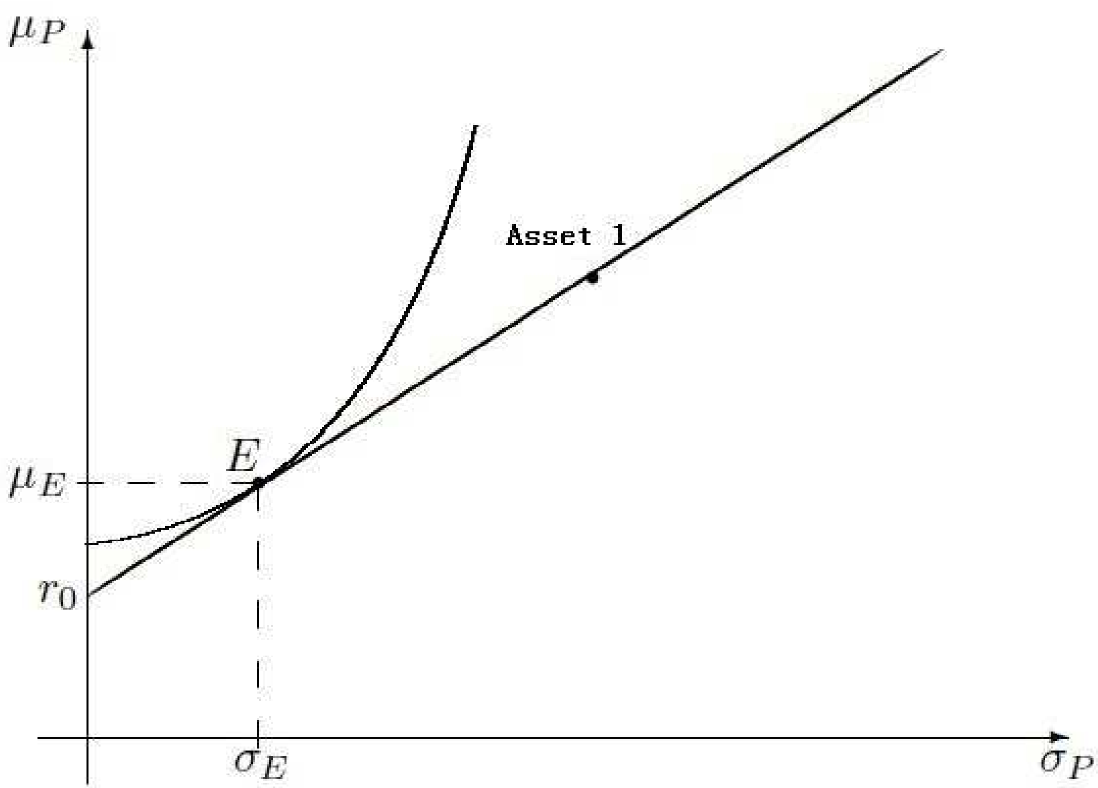
– Figure 1: Portfolio frontier with a risky asset and a risk-free asset



– Figure 2: The indifference curves for $G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$

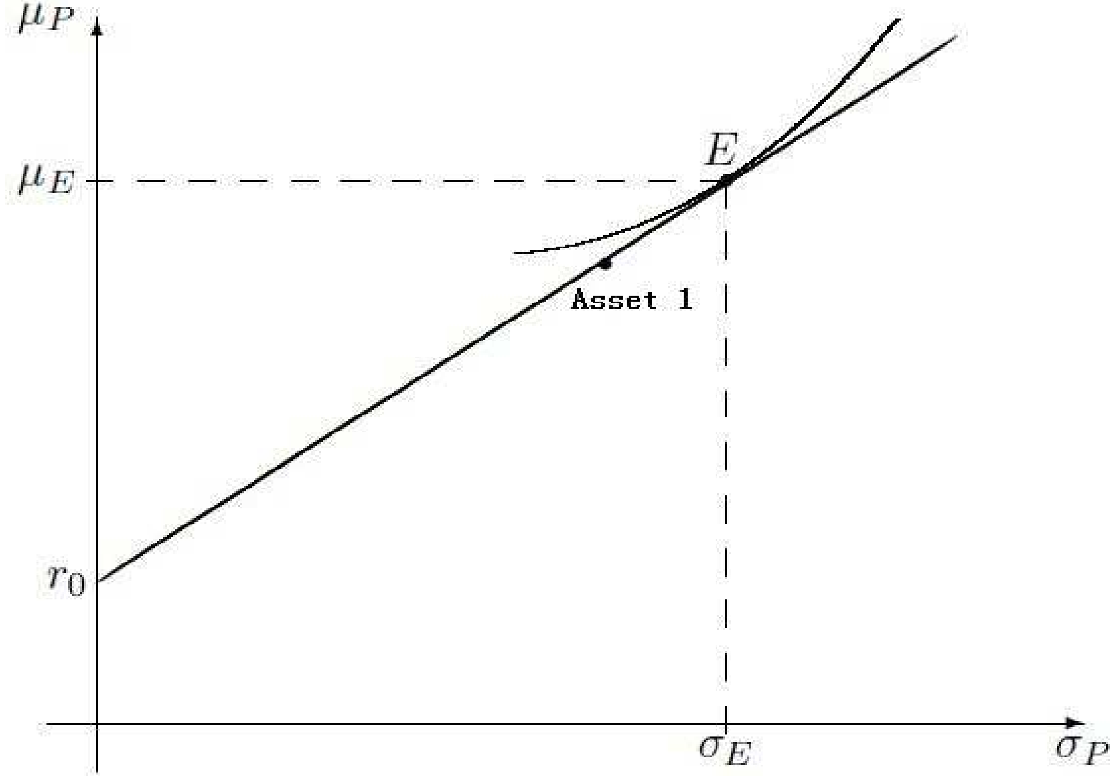


– Figure 3: The optimum portfolio: a lender, $a_1^* < 1$



Since $a_1^* = \frac{\mu_1 - r_0}{2\alpha\sigma_1^2}$, $a_1^* < 1$ if $\alpha > \frac{\mu_1 - r_0}{2\sigma_1^2}$.

– Figure 4: The optimum portfolio: a borrower, $a_1^* > 1$



$a_1^* > 1$ if $\alpha < \frac{\mu_1 - r_0}{2\sigma_1^2}$

– In Figure 3 and 4, at the optimum portfolio, represented by point E, the PF is tangent to an indifference curve.

– So their slopes should be equal:

$$\frac{\mu_1 - r_0}{\sigma_1} = 2\alpha\sigma_P$$

$$\Rightarrow \sigma_P = \frac{\mu_1 - r_0}{2\alpha\sigma_1}, \quad \mu_P = r_0 + \frac{(\mu_1 - r_0)^2}{2\alpha\sigma_1^2}$$

This is exactly the optimum portfolio (σ_E, μ_E) we found earlier.

- Think about how an increase in r_0 will change the PF and the optimum portfolio (Tutorial 4).

Interim Summary

- An investor's portfolio selection problem is to choose a portfolio of assets to maximise her mean-variance objective, subject to her budget constraint.

- A portfolio can be represented by a vector of proportions of initial wealth invested in each asset, or by a point in the (σ_P, μ_P) space.

- The PF with a risky asset and a risk-free asset is a straight line, with intercept at the risk-free asset and passing through the risky asset.

- The optimum portfolio is the point at which the PF is tangent to an indifference curve of $G(\mu_P, \sigma_P^2)$.

2. Two Risky Assets with No Risk-free Asset

- Now suppose the investor divides her wealth between two risky assets: asset 1 (μ_1, σ_1^2), and asset 2 (μ_2, σ_2^2). Assume that

$$\mu_1 > \mu_2, \quad \sigma_1 > \sigma_2$$

  - The covariance between r_1 and r_2:

$$\sigma_{12} \equiv cov(r_1, r_2)$$

  - The correlation coefficient between r_1 and r_2:

$$\rho_{12} \equiv \frac{cov(r_1, r_2)}{std(r_1) \cdot std(r_2)} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \in [-1, 1].$$

    · $\rho_{12} = 1$: perfect positive correlation between r_1 and r_2

    · $\rho_{12} = -1$: perfect negative correlation between r_1 and r_2

2.1 Portfolio frontier with two risky assets

- The return on a portfolio $(a_1, 1-a_1)$ is given by

$$r_P = a_1 r_1 + (1-a_1) r_2, \quad \text{so}$$

$$\begin{aligned}
\mu_P &\equiv E(r_P) = a_1\mu_1 + (1-a_1)\mu_2 = \mu_2 + a_1(\mu_1 - \mu_2) && (2)\\
\sigma_P^2 &\equiv var(r_P) = var(a_1 r_1 + (1-a_1) r_2)\\
&= var(a_1 r_1) + var[(1-a_1) r_2)] + 2cov(a_1 r_1, (1-a_1) r_2)\\
&= a_1^2 var(r_1) + (1-a_1)^2 var(r_2) + 2a_1(1-a_1)cov(r_1, r_2)\\
&= a_1^2\sigma_1^2 + (1-a_1)^2\sigma_2^2 + 2a_1(1-a_1)\sigma_{12}\\
&= a_1^2\sigma_1^2 + 2a_1(1-a_1)\rho_{12}\sigma_1\sigma_2 + (1-a_1)^2\sigma_2^2 && (3)
\end{aligned}$$

  - Let a_1 move between $[0, 1]$ and use (3) and (2) to trace out every point $(\sigma_P(a_1), \mu_P(a_1))$ in the (σ_P, μ_P) space, we can obtain the **portfolio frontier**.

- The restriction $a_1 \in [0,1]$ implies that short sale of either risky asset is not permitted.

- Alternatively, we can get a direct relationship between μ_P and σ_P by using (2) to express

$$a_1 = \frac{\mu_P - \mu_2}{\mu_1 - \mu_2}$$

and plugging it in (3) to get (see Topic4_extraderivation)

$$\sigma_P^2(\mu_P) = (\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12})\left(\frac{\mu_P - \mu_2}{\mu_1 - \mu_2} - \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}\right)^2 + \frac{\sigma_1^2\sigma_2^2\left(1 - \rho_{12}^2\right)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}} \tag{4}$$

  - Another way to plot the PF: let μ_P vary between 0 and a big positive number and find corresponding σ_P values, and trace out the points in (σ_P, μ_P) space.

2.2 Some special cases of the PF

- Perfect positive correlation ($\rho_{12} = 1$)
  - Recall that σ_P^2 is given by (3):

$$\sigma_P^2 = a_1^2\sigma_1^2 + 2a_1(1-a_1)\rho_{12}\sigma_1\sigma_2 + (1-a_1)^2\sigma_2^2$$

With $\rho_{12} = 1$, (3) simplifies to

$$\begin{aligned} \sigma_P^2 &= a_1^2\sigma_1^2 + 2a_1(1-a_1)\sigma_1\sigma_2 + (1-a_1)^2\sigma_2^2 \\ &= [a_1\sigma_1 + (1-a_1)\sigma_2]^2 \end{aligned}$$

$$\Rightarrow \ \sigma_P = a_1\sigma_1 + (1-a_1)\sigma_2 = \sigma_2 + a_1(\sigma_1 - \sigma_2) \qquad (5)$$

Recall $\quad \mu_P = \mu_2 + a_1(\mu_1 - \mu_2) \qquad (2)$

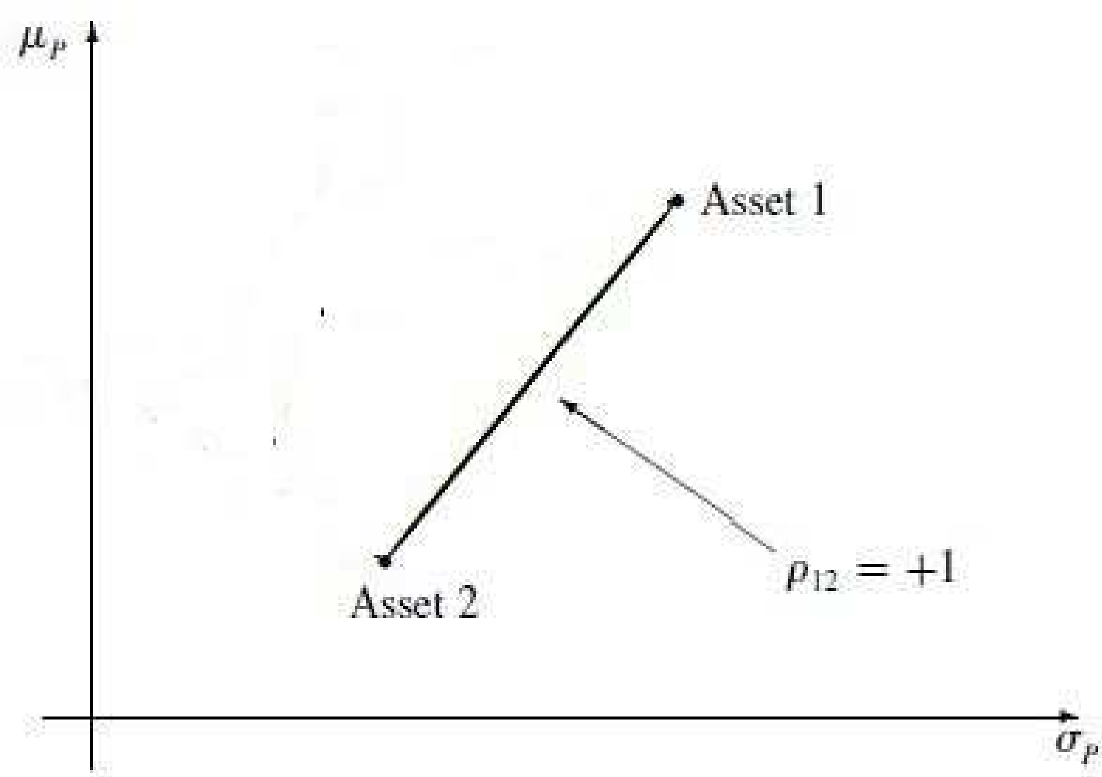
– So (2) and (5) imply that the relationship between μ_P and σ_P is linear.

– The PF is a straight line connecting (σ_1, μ_1) and (σ_2, μ_2), as shown in Figure 5.

– Note that $\sigma_P \geq \min(\sigma_1, \sigma_2)$: if two assets' returns are perfectly positively correlated, investors cannot achieve risk reduction by choosing a combination of the two assets.

– In this case, there is **no diversification** of risk.

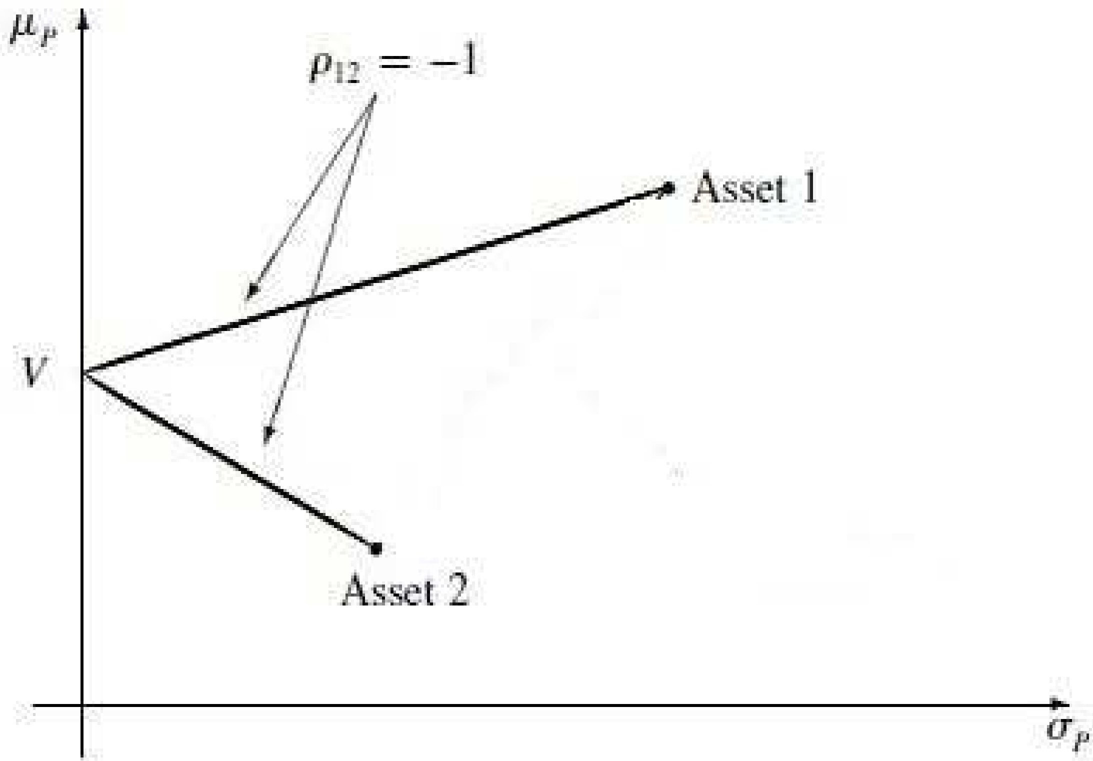
– Figure 5: Portfolio frontier with two risky assets ($\rho_{12} = 1$)



$(\mu_1 > \mu_2,\ \sigma_1 > \sigma_2)$

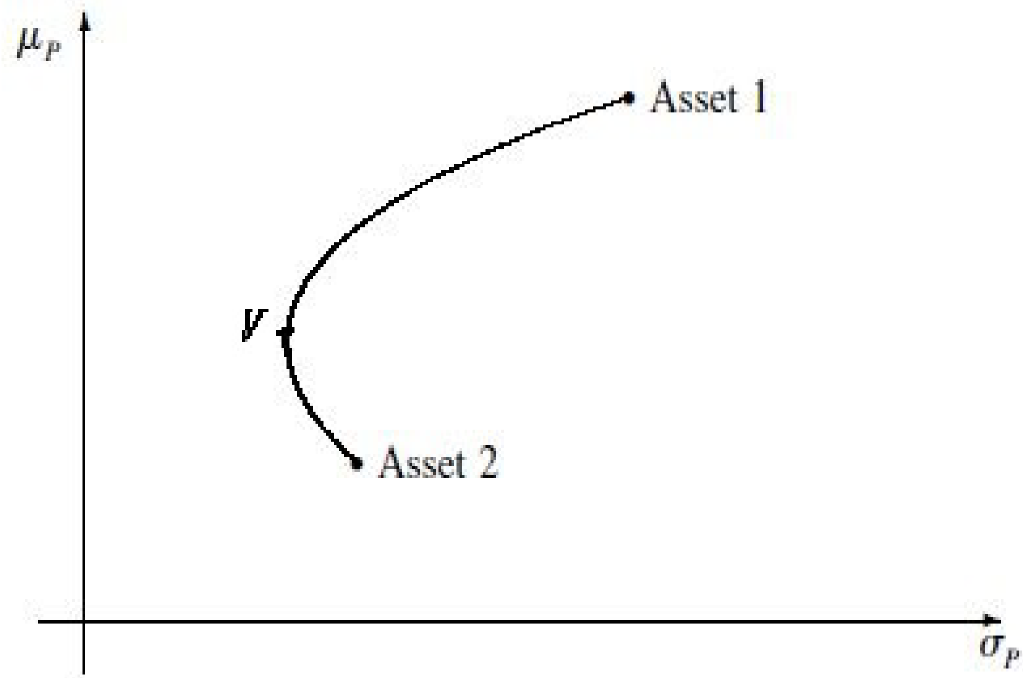
- Perfect negative correlation ($\rho_{12} = -1$)
  - With $\rho_{12} = -1$, (3) implies that
$$\sigma_P^2 = [a_1\sigma_1 - (1-a_1)\sigma_2]^2 = [(\sigma_1+\sigma_2)a_1 - \sigma_2]^2, \quad \text{i.e.}$$
$$\sigma_P = \begin{cases} (\sigma_1+\sigma_2)a_1 - \sigma_2 & \text{if } a_1 > \frac{\sigma_2}{\sigma_1+\sigma_2} \\ 0 & \text{if } a_1 = \frac{\sigma_2}{\sigma_1+\sigma_2} \\ \sigma_2 - (\sigma_1+\sigma_2)a_1 & \text{if } a_1 < \frac{\sigma_2}{\sigma_1+\sigma_2} \end{cases} \tag{6}$$
  - There exists a portfolio that has zero risk – **perfect diversification** of risk.
  - The PF is piece-wise linear (Figure 6)
  - The **efficient frontier (EF)**: the upward-sloping arm of the PF; the set of (mean-variance) **efficient portfolios** at which μ_P is maximised for a given σ_P.

– Figure 6: Portfolio frontier with two risky assets ($\rho_{12} = -1$)



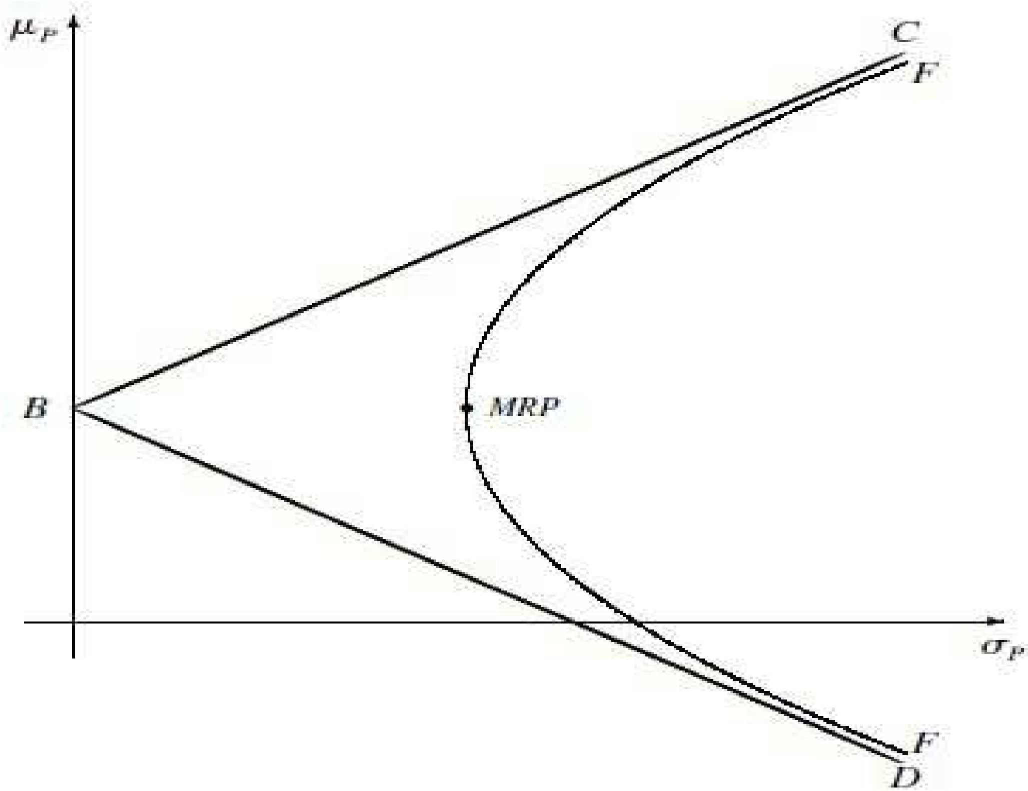
($\mu_1 > \mu_2$, $\sigma_1 > \sigma_2$)

- General correlation ($-1 < \rho_{12} < 1$)
  - The PF defined by equation (2)&(3) or by equation (4) is a hyperbola.
  - Figure 7: PF with two risky assets ($-1 < \rho_{12} < 1$)



- Again, the EF is the upward-sloping arm of the PF.
- Note that there is a **minimum risk portfolio** (MRP) represented by V.
- With $\rho < 1$, there is diversification of risk.
- What is the MRP in the perfect positive/negative correlation case?
- Find the MRP when $\rho_{12} = 0$ (see Exercise_Topic4).

– If we allow for $a_1 < 0$ and $a_1 > 1$, i.e., if we allow for short sales, the PF with two risky assets $(-1 < \rho_{12} < 1)$ is illustrated in Figure 8.



2.3 The optimum portfolio

- The portfolio selection problem can be formulated as

$$\max_{a_1} G(\mu_P, \sigma_P^2), \text{ subject to (2) and (3)}, \quad \text{or}$$

$$\max_{(\sigma_P, \mu_P)} G(\mu_P, \sigma_P^2), \text{ subject to (4)}$$

- Only portfolios on the **efficient frontier** will be chosen.
- The optimum portfolio is the tangent point of the EF to an indifference curve of $G(\mu_P, \sigma_P)$.
- Putting indifference curves on Figure 5-7 for yourself. Is the optimum portfolio the minimum risk portfolio?
- See Exercise_Topic 4 for an example of finding the minimum risk portfolio and optimum portfolio of two risky assets.

Interim Summary

- The EF is the set of efficient portfolios for which expected return is maximised for a given level of risk. It is the upward-sloping part of the PF.
- With $\rho_{12} = 1$, the EF is a straight line connecting the two risky assets.
- With $\rho_{12} = -1$, the EF is a straight line connecting the risk-free MRP and the risky asset with a higher expected return.
- With general $-1 < \rho_{12} < 1$, the PF is a hyperbola in the (σ_P, μ_P) space.
- With $\rho_{12} < 1$, there is reduction of risk from diversification.

3. Many Risky Assets with No Risk-free Asset

- Suppose there are $n > 2$ 'genuinely different' risky assets for the investor to choose.
  - Portfolio: $(a_1, \ldots, a_n)$, $\sum_{j=1}^{n} a_j = 1$
  - The return on the portfolio: $r_P = \sum_{j=1}^{n} a_j r_j$
  - The expected return on the portfolio:

$$\mu_P \equiv E(r_P) = \sum_{j=1}^{n} a_j \mu_j \tag{7}$$

  - The variance of the return on the portfolio:

$$\sigma_P^2 = var(r_P) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \, cov(r_i, r_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}, \tag{8}$$

    where $\sigma_{jj} = var(r_j) = \sigma_j^2$.

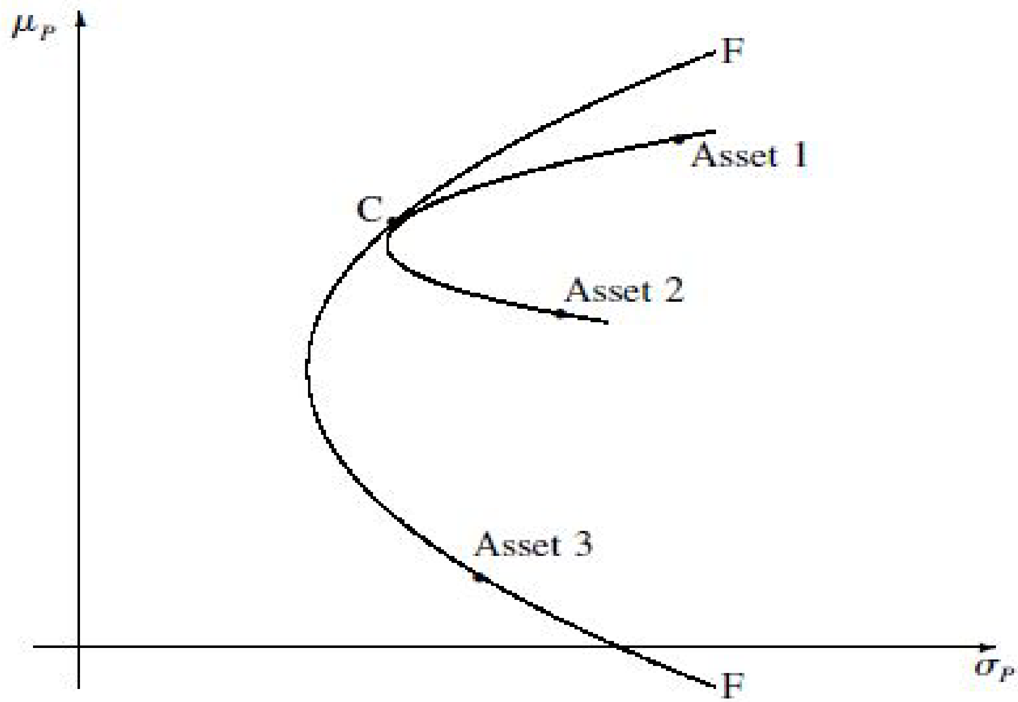
- Difference from the two assets case
  - With two risky assets: given μ_P, a portfolio on the PF is uniquely determined (recall equation (2)).
  - With more than two assets: many portfolios can have the same μ_P, then which one should be on the PF?
- The PF is defined as follows: for a given value of μ_P, choose a portfolio that has minimum variance of return:

$$\min_{(a_1, a_2, \ldots, a_n)} \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij},$$

$$\text{s.t.} \quad \mu_P = \sum_{j=1}^{n} a_j \mu_j, \quad \sum_{j=1}^{n} a_j = 1$$

  - A separate minimisation is carried out for each given value of μ_P. As μ_P varies, the PF is traced out.

- A graphical illustration
  - Figure 9: Portfolio frontier with three risky assets



Point C represents an efficient portfolio of assets 1 and 2.

  - The PF with a larger number of assets is located **to the left** of the PF with fewer assets. Why?

- As illustrated in Figure 9, the PF with $n > 2$ risky assets can be constructed by two "composite" assets, where each "composite" asset is a portfolio of assets or a mutual fund.
  - This is formally established in the **first mutual fund theorem** of portfolio analysis.
- The efficient frontier (EF) is again the upward sloping part of the PF. A portfolio on the EF is mean-variance efficient.
  - It has minimum variance of return among portfolios that have the same expected return as itself.
  - It has maximum expected return among portfolios that have the same variance of return as itself.

4. Many Risky Assets with A Risk-free Asset

- Now suppose the investor has access to n risky assets and a risk-free asset with return r_0.

- The PF: for a given value of μ_P, choose portfolio proportions to minimise σ_P^2:

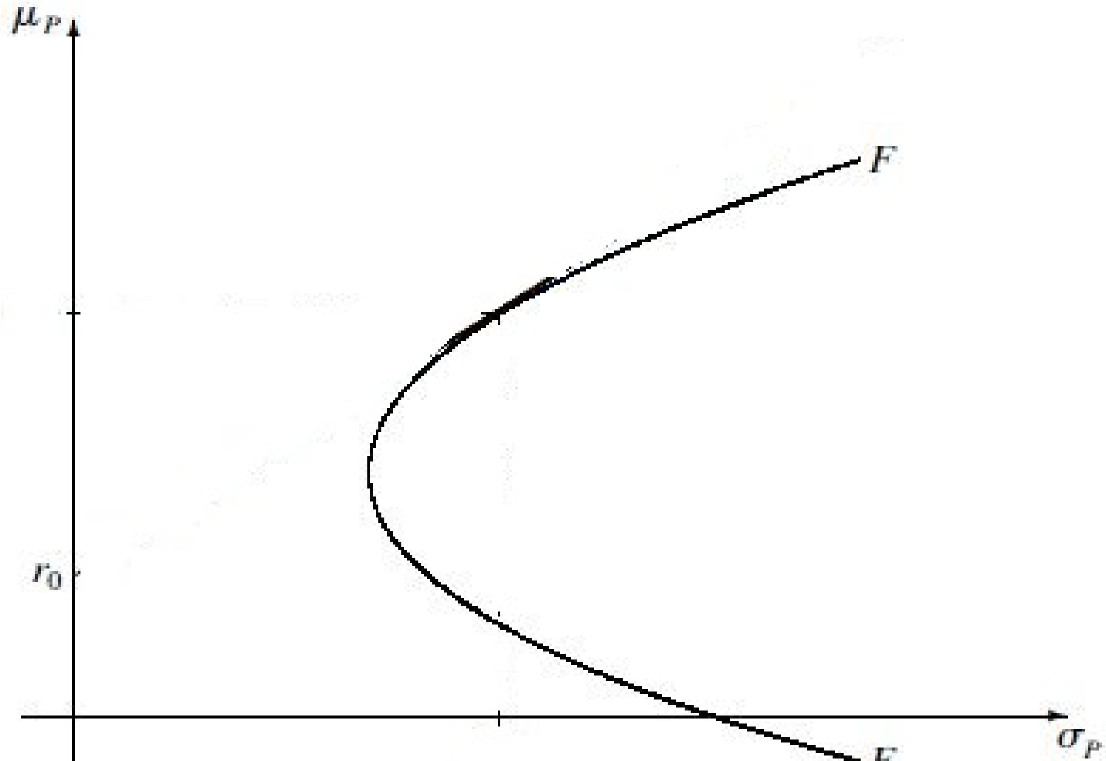
$$\min_{(a_0, a_1, a_2, \ldots, a_n)} \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}$$

$$\text{s.t} \quad \mu_P = a_0 r_0 + \sum_{j=1}^{n} a_j \mu_j, \qquad a_0 + \sum_{j=1}^{n} a_j = 1$$

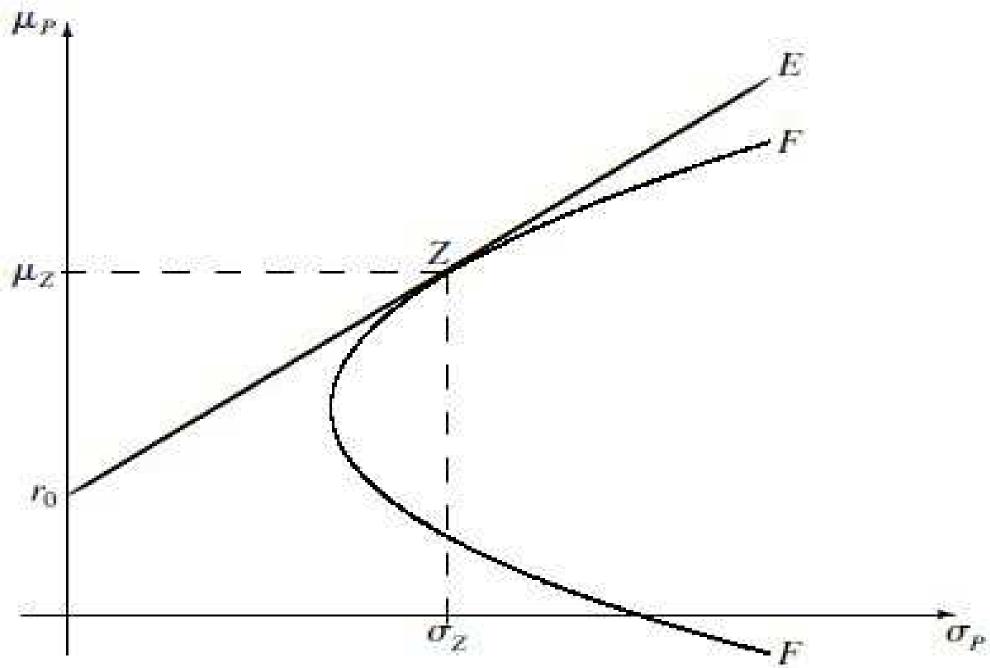
For each μ_P, find the corresponding σ_P. Tracing out the obtained (σ_P, μ_P) gives the PF.

- **Second mutual fund theorem** of portfolio analysis: Any efficient portfolio of n risky assets and a risk-free asset can be constructed as a combination of the risk-free asset and a mutual fund of the risky assets.
  - So the EF can be constructed as the PF with a risk-free asset and a composite asset of the n risky assets.
  - Recall that the PF with a risk-free asset and a risky asset is a straight line.
  - There can be many such straight lines, each corresponding to a portfolio of the n risky assets. Which one is the EF?

- Figure 10: Efficient frontier with $n \geq 2$ risky assets and a risk-free asset



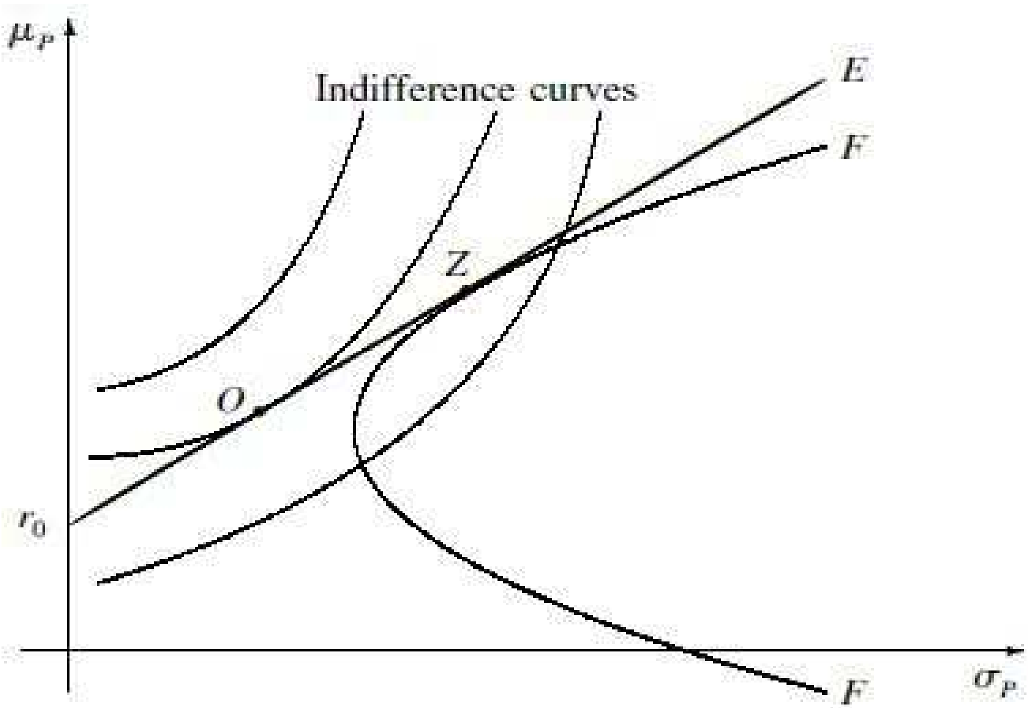
- Figure 10: Efficient frontier with $n \geq 2$ risky assets and a risk-free asset



FZF represents the portfolio frontier with the n risky assets. Z is an efficient portfolio of the n risky assets, known as the **tangent portfolio**, which has the highest Sharpe ratio among all portfolios of the n risky assets.

- Then the portfolio selection problem goes back to the simplest case: one risky asset with a risk-free asset.

- The optimum portfolio is illustrated in Figure 11.



Summary

- In all cases, the portfolio selection problem is to choose a portfolio on the **portfolio frontier** to maximise the mean-variance objective $\max_{(\sigma_P, \mu_P)} G(\mu_P, \sigma_P^2)$.
- The **efficient frontier** (EF) is the upward-sloping part of the PF, representing the set of efficient portfolios.
- The optimum portfolio is the point at which the EF is tangent to an indifference curve of G.
- However, the PF and hence EF are different in different cases.

- The PF/EF with one risky asset and a risk-free asset is a straight line, with intercept at the risk-free rate and passing through the risky asset.
- The PF/EF with $n \geq 2$ risky assets and a risk-free asset is also a straight line, with intercept at the risk-free rate and tangent to the EF with n risky assets.
- The PF with $n \geq 2$ genuinely different risky assets is a hyperbola in the (σ_P, μ_P) space.
  - It is obtained by minimising risk, σ_P^2, for each level of expected return, μ_P.
  - When $n = 2$, σ_P^2 is uniquely determined by μ_P.

Review questions

1. Describe the portfolio selection problem for an investor with mean-variance objective in words.
2. What is a portfolio? Name several different ways to represent a portfolio.
3. How is the portfolio frontier defined in each case? In particular, understand the difference between 2 assets case and $n > 2$ assets case.
4. What is the difference between the efficient frontier and the portfolio frontier? What is an efficient portfolio? What is the efficient frontier in each case?
5. Understand why the optimum portfolio is the tangent point of the efficient frontier to an indifference curve of the mean-variance objective.
6. Be able to solve the portfolio selection problem with a risky asset and a risk-free asset, using several alternative ways.
7. Be able to draw the portfolio frontier with a risky asset and a risk-free asset, and illustrate the optimum portfolio graphically.

8. Be able to draw the portfolio frontier with two risky assets for the perfect positive correlation and perfect negative correlation case.

9. For other cases ($n \geq 2$ risky assets with or without risk free asset), understand how the portfolio frontier is defined and its shape.

10. Understand the first and second mutual fund theorem, and how they are applied to construct the portfolio frontier in the many assets case.

11. Understand the gains from diversification. Why diversification doesn't work for the perfect positive correlation case?

12. What is a minimum risk portfolio? Understand how to find the minimum risk portfolio and the optimum portfolio of two risky assets.

13. Understand the concept of Sharpe ratio, and understand why the tangent portfolio Z in Figure 10 has the highest Sharpe ratio.