Question 3.

(a) For a time series Y_t with $t = 1, \ldots, n$, and $\mathscr{Y}_{t-1} = \{Y_{t-1}, \ldots, Y_1\}$, why do we use the conditional expectation $E(Y_t|\mathscr{Y}_{t-1})$ for one-step-ahead forecasting?

The conditional expectation is the function of the data in $\mathscr{Y}_{t-1}$ than minimises the mean squared error criterion:

$$\text{MSE} = E[(Y_t - g(Y_{t-1}, Y_{t-2}, \ldots, Y_1))^2]$$

over all choices of function g.

(A proof of this statement is not hard, but has not been covered in this subject.)

(b) Define the one-step-ahead prediction error $U_t = Y_t - E(Y_t|\mathscr{Y}_{t-1})$. Show that

(i) $E(U_t|\mathscr{Y}_{t-1}) = 0$

(ii) $E(U_t) = 0$

(iii) $E(U_t U_{t-j}) = 0$ for all $j = 1, 2, \ldots$.

(i) Taking $E(\,.\,|\mathscr{Y}_{t-1})$ of both sides of the definition of U_t:

$$E(U_t|\mathscr{Y}_{t-1}) = E(Y_t|\mathscr{Y}_{t-1}) - E(Y_t|\mathscr{Y}_{t-1}) = 0$$

(ii) Apply the Law of Iterated Expectations:

$$E(U_t) \overset{\text{LIE}}{=} E[\,E(U_t|\mathscr{Y}_{t-1})\,] = 0$$

from part (i).

(iii) Again apply the LIE:

$$E(U_t U_{t-j}) \overset{\text{LIE}}{=} E[\,E(U_t U_{t-j}|\mathscr{Y}_{t-1})\,] = E[\,U_{t-j}\,E(U_t|\mathscr{Y}_{t-1})\,] = 0$$

since $U_{t-j} = (Y_{t-j} - E(Y_{t-j}|\mathscr{Y}_{t-j-1})) \in \mathscr{Y}_{t-1}$ for all $j = 1, 2, \ldots$, and from part (i).

(c) What is the implication of your answer to part (b) for practical time series model specification?

Part (iii) implies there is no autocorrelation at any lag in U_t. This is because

$$\text{cov}(U_t, U_{t-j}) = E[(U_t - E(U_t))(U_{t-j} - E(U_{t-j}))] = E[U_t U_{t-j}]$$

because of part (b)(ii).

The practical implication is that a properly specified model for $E(Y_t|\mathscr{Y}_{t-1})$ will have prediction errors U_t with no autocorrelation. Therefore if the residuals from a particular model show significant autocorrelation, we can conclude that model is not well specified for one-step-ahead forecasting.

(d) Define and compare the concepts of *recursive* and *direct* forecasting for two-step-ahead forecasting.

In general a two-step-ahead forecast has the form $E(Y_{n+2}|\mathscr{Y}_n)$.

The recursive approach specifies a one-step-ahead model $E(Y_{n+1}|\mathscr{Y}_n)$, and then applies the Law of Iterated Expectations:

$$E[Y_{n+2}|\mathscr{Y}_n] = E[E(Y_{n+2}|\mathscr{Y}_{n-1})|\mathscr{Y}_n]$$

That is the recursive two-step-ahead forecast is the one-step-ahead forecast of the one-step-ahead forecast.

The direct approach simply specifies a model directly for $E(Y_{n+2}|\mathscr{Y}_n)$, not necessarily related to the model for the one-step-ahead forecast.

The recursive approach is most common, and requires the specification of only one model, for the one-step-ahead forecast. There may be the complication of doing the derivations associated with the LIE, but for standard ARMA models these are handled by the software. The direct approach requires a separate model specification search for every forecast horizon. It requires no subsequent derivations, and allows some flexibility in different model specifications at different forecast horizons.

(e) Are the one-step-ahead prediction errors U_t defined in part (b) necessarily stationary? If so, justify this. If not, what else is required for U_t to be stationary?

Stationarity requires that $E(U_t)$, $\text{var}(U_t)$ and $\text{cov}(U_t, U_{t-j})$ are time-invariant, i.e. constant over all t.

We have already shown that $E(U_t) = 0$ and $\text{cov}(U_t, U_{t-j}) = 0$, hence constant over all t.

We have no result for the variance. It is possible that U_t is heteroskedastic, so to be stationary it is necessary to add the condition that $\text{var}(U_t) = \sigma^2 < \infty$ for all t.

(f) Suppose $Y_t = U_t + \theta_1 U_{t-1}$ is an MA(1) time series where U_t is a stationary prediction error. Derive $E(Y_t)$, $\text{var}(Y_t)$, $\text{cov}(Y_t, Y_{t-1})$ and hence the first order autocorrelation $\text{cor}(Y_t, Y_{t-1})$. Are these expressions sufficient to conclude that Y_t is stationary?

Since U_t is a stationary prediction error we can take $E(U_t) = 0$ and $\text{var}(U_t) = \sigma^2$.

Then $E(Y_t) = E(U_t) + \theta_1 E(U_{t-1}) = 0$,

$\text{var}(Y_t) = \text{var}(U_t) + 2\theta_1 \text{cov}(U_t, U_{t-1}) + \theta_1^2 \text{var}(U_{t-1}) = \sigma^2(1 + \theta_1^2)$
which also uses $\text{cov}(U_t, U_{t-1}) = 0$.

$\text{cov}(Y_t, Y_{t-1}) = \text{cov}(U_t + \theta_1 U_{t-1}, U_{t-1} + \theta_1 U_{t-2}) = \theta_1 \text{cov}(U_{t-1}, U_{t-1}) = \sigma^2 \theta_1$.

$$\text{cor}(Y_t, Y_{t-1}) = \frac{\text{cov}(Y_t, Y_{t-1})}{\sqrt{\text{var}(Y_t)\text{var}(Y_{t-1})}} = \frac{\theta_1}{1 + \theta_1^2}$$

Yes, these have verified the mean, variance and autocovariances are constant for all t, hence Y_t is stationary.

(g) Use the expression for the first order autocorrelation $\rho_1 = \mathrm{cor}(Y_t, Y_{t-1})$ in the previous part to work out the range of possible values for ρ_1 that can arise from an MA(1) model.

In case it's helpful, the quadratic formula for x that solves $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Suppose a time series produce a first order autocorrelation of 0.8. It is possible that an MA(1) model is appropriate for this time series?

The expression $\rho_1 = \dfrac{\theta_1}{1+\theta_1^2}$ can be rearranged to $\theta_1^2 - \rho_1^{-1}\theta_1 + 1 = 0$.

For any ρ_1 this quadratic would imply

$$\theta_1 = \frac{\rho_1^{-1} \pm \sqrt{\rho_1^{-2} - 4}}{2}.$$

This will be produce a (real) solution for θ_1 if $\rho_1^{-2} - 4 \geq 0$, since then the square root can be evaluated. The inequality $\rho_1^{-2} - 4 \geq 0$ implies $\rho_1^2 \leq \frac{1}{4}$, or $0.5 \leq \rho_1 \leq 0.5$.

On this basis, a time series with a first order autocorrelation of 0.8 is not consistent with an MA(1) model, since that model implies the first order autocorrelation must be in the range $[-0.5, 0.5]$.