ECOM90024 - Assignment 3 - Q2

Suppose that the innovations ε_t of a time series are governed by an ARCH(2) process:

$$\varepsilon_t = \sigma_t v_t$$

$$v_t \sim iid\ N(0,1)$$

$$\sigma^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$

a) Derive the unconditional variance of the process & the associated restrictions it imposes on the ARCH coefficients.

Starting with the conditional variance:

$$E[\varepsilon^2 | \Omega_{t-1}] = E[\sigma^2 v^2 | \Omega_{t-1}]$$

$$= \sigma^2 [v^2 | \Omega_{t-1}]$$

iid $\sim N(0,1)$

$$= \sigma_t^2$$

Using the law of iterated expectations, we then derive unconditional variance

$$E[\varepsilon_t^2] = E[E[\sigma_t^2 v_t^2 | \Omega_{t-1}]] = E[\sigma_t^2]$$

$$E[\sigma_t^2] = E[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2]$$

$$= \alpha_0 + \alpha_1 E[\varepsilon_{t-1}^2] + \alpha_2 E[\varepsilon_{t-2}^2]$$

By assuming this process is covariance stationary, it must be that: $E[\varepsilon_t^2] = E[\varepsilon_{t-1}^2] = E[\varepsilon_{t-2}^2]$.

$$E[\varepsilon_t^2] = \alpha_0 + \alpha_1 E[\varepsilon_t^2] + \alpha_2 E[\varepsilon_{t-2}^2]$$

$$E[\varepsilon_t^2](1 - \alpha_1 - \alpha_2) = \alpha_0$$

$$E[\varepsilon_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2} = \frac{\alpha_0}{1 - (\alpha_1 + \alpha_2)}$$

Therefore, for positive unconditional variance:

$$0 \leq (\alpha_1 + \alpha_2) < 1$$

6) Given the information set Ω_t, derive expressions for the 1-step and 2-step ahead forecasts of the conditional variance in terms of the variable in the conditioning set.

The 1-step ahead forecast is given by:

$$\hat{\sigma}^2_{t+1|t} = \alpha_0 + \alpha_1 \varepsilon_t^2 + \alpha_2 \varepsilon_{t-1}^2$$

The 2-step ahead forecast is given by:

$$\hat{\sigma}^2_{t+2|t} = \alpha_0 + \alpha_1 E[\varepsilon^2_{t+1} | \Omega_t] + \alpha_2 \varepsilon_t^2$$

Since $E[\varepsilon^2_{t+1} | \Omega_t] = \hat{\sigma}^2_{t+1|t}$

$$\hat{\sigma}^2_{t+2|t} = \alpha_0 + \alpha_1 (\alpha_0 + \alpha_1 \varepsilon_t^2 + \alpha_2 \varepsilon_{t-1}^2) + \alpha_2 \varepsilon_t^2$$

$$\hat{\sigma}^2_{t+2|t} = \alpha_0 (1 + \alpha_1) + \alpha_1^2 \varepsilon_t^2 + \alpha_2 (\varepsilon_{t-1}^2 + \varepsilon_t^2)$$