FNCE90056:
Investment Management

Lecture 3: Capital Asset Pricing Model

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Review

Overview

Last week:

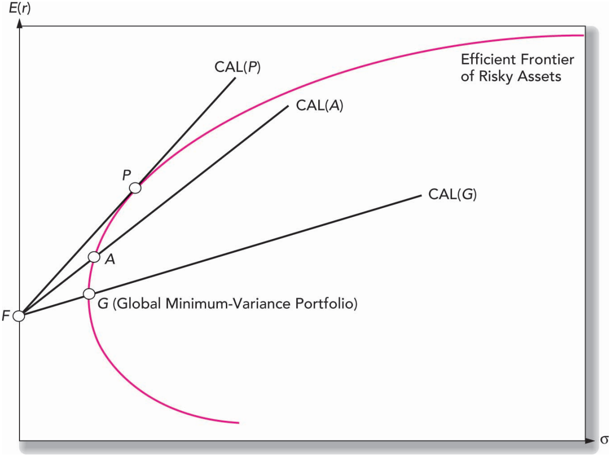
- We built optimal risky portfolios from 2+ risky assets.
- We then added in a risk-free asset to generate the optimal CAL: a line that originates at the risk-free asset and is tangent to the efficient risky frontier.

This week:

- We'll see how to use these portfolio tools to discuss the pricing of **individual assets**.
- We'll focus on the Capital Asset Pricing Model (CAPM)

Capital Allocation Line

- Recall the CAL with the efficient risky frontier:



- The line CAL(P) is the best way to invest for ANYONE who cares about expected returns and risk.

Beyond Modern Portfolio Theory

Modern Portfolio Theory (MPT)

- The **Modern Portfolio Theory (MPT)** framework:
  - simplifies choices via the efficient frontier of risky assets
  - shows how to optimally combine risky assets taking their expected returns and variance-covariance matrix as given
  - **does not** provide guidance with respect to the risk-return tradeoff for **individual** assets
- For that we need the **Capital Asset Pricing Model (CAPM)**, a theoretical model of equilibrium expected returns on risky assets
  - Extends idea of diversification under simplified assumptions

MPT plus two sets of assumptions $\Longrightarrow$ the CAPM

- Assumptions on investor behaviors
  - Investors are rational, mean-variance optimizers.
  - static choice of a single period.
  - homogeneous expectations: investors all use identical estimates of expected returns, variances, and covariances.
    - assuming all relevant information is publicly available
- Assumptions on market structure
  - All assets are publicly held and trade on public exchanges. (nontradable assets are excluded,e.g education, private firms, etc)
  - Investors can borrow or lend at a common risk-free rate, and they can take short positions on traded securities.
  - No taxes and No transaction costs.
  - There are many investors, no market power and take price as given.

Tangency portfolio = market portfolio

- If we agree on expected returns, μ, and covariances, Σ, we all want (demand) the same tangency portfolio. (Although our weights on the tangency portfolio, relative to the risk-free asset, may vary, depending on our individual risk aversions.)

- The market is the supply of all investments.

- **Equilibrium argument 1**, in equilibrium:

$$\text{Supply of risky assets} = \text{demand of risky assets}$$

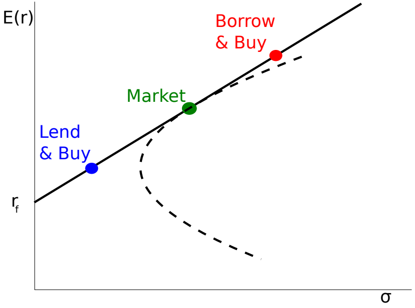
- This must mean everyone holds the **market portfolio** (as the risky component of their overall portfolio).

- Given everyone is holding market portfolio, portfolio risk is what matters to investors and is what governs the risk premiums they demand.

The Capital Market Line (CML)

While we all agree on the same tangency portfolios, our weights on the tangency portfolio, relative to the risk-free asset, may vary, depending on our individual risk aversions:

- Like risk? Borrow and buy the market
- Dislike risk? Put some money in the market, some in the risk-free asset



- Special name for this CAL: The **Capital Market Line (CML)**

Capital Asset Pricing Model

Getting beyond the CML

- Everyone wants to be on the CML. Does that tell us anything about the individual assets? Yes! And THAT is the power of the CAPM.
- Core insight of the CAPM
  - The appropriate risk premium on an individual asset has to be determined by its individual (marginal) contribution to the risk of the investors' overall portfolios.
  - This contribution to overall risk is not just given by an asset's variance, but also its covariance.

Step-by-step derivation of CAPM

- **Is there a risk-return tradeoff for individual assets? Yes!**
  - **Equilibrium argument 2**: In equilibrium, the **reward-to-risk ratio** for all asset i is the same

$$\frac{E(R_i) - R_f}{\text{Cov}(R_i, R_M)} = \frac{E(R_M) - R_f}{\sigma_M^2}, \text{for all } i \tag{1}$$

  - Thus, we obtain the **fundamental (testable) implication of the CAPM, for any asset** i:

$$\mathbb{E}[R_i] - R_f = \beta_i (\mathbb{E}[R_m] - R_f); \beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma_M^2} \tag{2}$$

Intuition of CAPM

Intuition behind CAPM

- CAPM builds on the insight that the appropriate risk premium on an asset is determined by its contribution to the risk of investors' overall portfolio.
- What is the contribution of an asset i to the overall portfolio ?
  - let's consider a 2 assets case with portfolio level returns

$$R^P = \omega_1 R_1 + \omega_2 R_2$$

  - Then the variance covariance matrix is given by

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \begin{bmatrix} \mathrm{Cov}\left(R_1, R_1\right) & \mathrm{Cov}\left(R_2, R_1\right) \\ \mathrm{Cov}\left(R_1, R_2\right) & \mathrm{Cov}\left(R_2, R_2\right) \end{bmatrix}$$

Intuition behind CAPM

- We can then calculate the portfolio level variance and decompose it into the contribution of each individual asset

$$V\left[R^P\right] = V\left[\sum_{i=1}^{2}\omega_i R_i\right] = \sum_{i=1}^{2}\sum_{j=1}^{2}\omega_i\omega_j \operatorname{Cov}\left[R_i, R_j\right]$$

$$= \underbrace{\begin{bmatrix} \omega_1\omega_1 \operatorname{Cov}[R_1, R_1] \\ + \\ \omega_1\omega_2 \operatorname{Cov}[R_1, R_2] \end{bmatrix}}_{\text{Contribution of } R_1} + \underbrace{\begin{bmatrix} \omega_2\omega_1 \operatorname{Cov}[R_2, R_1] \\ + \\ \omega_2\omega_2 \operatorname{Cov}[R_2, R_2] \end{bmatrix}}_{\text{Contribution of } R_2}$$

$$= \underbrace{\begin{bmatrix} \omega_1 \operatorname{Cov}[R_1, \omega_1 R_1] \\ + \\ \omega_1 \operatorname{Cov}[R_1, \omega_2 R_2] \end{bmatrix}}_{\text{Contribution of } R_1} + \underbrace{\begin{bmatrix} \omega_2 \operatorname{Cov}[R_2, \omega_1 R_1] \\ + \\ \omega_2 \operatorname{Cov}[R_2, \omega_2 R_2] \end{bmatrix}}_{\text{Contribution of } R_2}$$

$$= \omega_1 \operatorname{Cov}\left(R_1, \omega_1 R_1 + \omega_2 R_2\right) + \omega_2 \operatorname{Cov}\left(R_2, \omega_1 R_1 + \omega_2 R_2\right)$$

$$= \underbrace{\omega_1 \operatorname{Cov}\left(R_1, R^P\right)}_{\text{Contribution of } R_1} + \underbrace{\omega_2 \operatorname{Cov}\left(R_2, R^P\right)}_{\text{Contribution of } R_2}$$

Intuition behind CAPM

- In this two assets case, the contribution of an asset i to the overall portfolio is just weight w_i times its covariance with R_P

$$\omega_i \operatorname{Cov}\left(R_i, R^P\right)$$

- We then calculate reward of asset i as its contribution to the portfolio level risk premium, recall that $R^P = \omega_1 R_1 + \omega_2 R_2$

$$\begin{aligned} E\left(R^P\right) - R_f &= \omega_1 E(R_1) + \omega_2 E(R_2) - R_f \\ &= \underbrace{\omega_1 \left(E(R_1) - R_f\right)}_{\text{Contribution of } R_1} + \underbrace{\omega_2 \left(E(R_2) - R_f\right)}_{\text{Contribution of } R_2} \end{aligned}$$

- So the reward of asset i is just

$$\omega_i \left(E(R_i) - R_f\right)$$

- Then asset i's reward-to-risk ratio is given by

$$\frac{\omega_i \left(E(R_i) - R_f\right)}{\omega_i \operatorname{Cov}\left(R_i, R^P\right)} = \frac{E(R_i) - R_f}{\operatorname{Cov}\left(R_i, R^P\right)}$$

Intuition behind CAPM

- How about a case with 3 assets? What is the contribution of an asset i to the overall portfolio ?
  - With 3 assets, portfolio level return is given by

$$R^P = \omega_1 R_1 + \omega_2 R_2 + \omega_3 R_3$$

  - Portfolio level expected return is given by

$$E\left(R^P\right) = \omega_1 E\left(R_1\right) + \omega_2 E\left(R_2\right) + \omega_3 E\left(R_3\right)$$

  - Portfolio level variance covariance matrix is given by

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \begin{bmatrix} \mathrm{Cov}\left(R_1, R_1\right) & \mathrm{Cov}\left(R_2, R_1\right) & \mathrm{Cov}\left(R_3, R_1\right) \\ \mathrm{Cov}\left(R_1, R_2\right) & \mathrm{Cov}\left(R_2, R_2\right) & \mathrm{Cov}\left(R_3, R_2\right) \\ \mathrm{Cov}\left(R_1, R_3\right) & \mathrm{Cov}\left(R_2, R_3\right) & \mathrm{Cov}\left(R_3, R_3\right) \end{bmatrix}$$

Intuition behind CAPM

- We can then calculate the portfolio level variance and decompose it into the contribution of each individual asset

$$V\left[R^P\right] = V\left[\sum_{i=1}^{3}\omega_i R_i\right] = \sum_{i=1}^{3}\sum_{j=1}^{3}\omega_i\omega_j \operatorname{Cov}\left[R_i, R_j\right]$$

$$= \underbrace{\begin{bmatrix} \omega_1\omega_1 \operatorname{Cov}[R_1, R_1] \\ + \\ \omega_1\omega_2 \operatorname{Cov}[R_1, R_2] \\ + \\ \omega_1\omega_3 \operatorname{Cov}[R_1, R_3] \end{bmatrix}}_{\text{Contribution of} R_1} + \underbrace{\begin{bmatrix} \omega_2\omega_1 \operatorname{Cov}[R_2, R_1] \\ + \\ \omega_2\omega_2 \operatorname{Cov}[R_2, R_2] \\ + \\ \omega_2\omega_3 \operatorname{Cov}[R_2, R_3] \end{bmatrix}}_{\text{Contribution of} R_2} + \underbrace{\begin{bmatrix} \omega_3\omega_1 \operatorname{Cov}[R_3, R_1] \\ + \\ \omega_3\omega_2 \operatorname{Cov}[R_3, R_2] \\ + \\ \omega_3\omega_3 \operatorname{Cov}[R_3, R_3] \end{bmatrix}}_{\text{Contribution of} R_3}$$

$$= \underbrace{\begin{bmatrix} \omega_1 \operatorname{Cov}[R_1, \omega_1 R_1] \\ + \\ \omega_1 \operatorname{Cov}[R_1, \omega_2 R_2] \\ + \\ \omega_1 \operatorname{Cov}[R_1, \omega_3 R_3] \end{bmatrix}}_{\text{Contribution of} R_1} + \underbrace{\begin{bmatrix} \omega_2 \operatorname{Cov}[R_2, \omega_1 R_1] \\ + \\ \omega_2 \operatorname{Cov}[R_2, \omega_2 R_2] \\ + \\ \omega_2 \operatorname{Cov}[R_2, \omega_3 R_3] \end{bmatrix}}_{\text{Contribution of} R_2} + \underbrace{\begin{bmatrix} \omega_3 \operatorname{Cov}[R_3, \omega_1 R_1] \\ + \\ \omega_3 \operatorname{Cov}[R_3, \omega_2 R_2] \\ + \\ \omega_3 \operatorname{Cov}[R_3, \omega_3 R_3] \end{bmatrix}}_{\text{Contribution of} R_3}$$

$$= \omega_1 \operatorname{Cov}\left(R_1, \omega_1 R_1 + \omega_2 R_2 + \omega_3 R_3\right) + \cdots + \omega_3 \operatorname{Cov}\left(R_3, \omega_1 R_1 + \omega_2 R_2 + \omega_3 R_3\right)$$

$$= \omega_1 \operatorname{Cov}\left(R_1, R^P\right) + \omega_2 \operatorname{Cov}\left(R_2, R^P\right) + \omega_3 \operatorname{Cov}\left(R_3, R^P\right)$$

Intuition behind CAPM

- In this two assets case, the contribution of an asset i to the overall portfolio is just weight w_i times its covariance with R_P

$$\omega_i \operatorname{Cov}\left(R_i, R^P\right)$$

- We then calculate the reward of asset i as its contribution to the portfolio level risk premium

$$\begin{aligned} E\left(R^P\right) - R_f &= \omega_1 E(R_1) + \omega_2 E(R_2) + \omega_3 E(R_3) - R_f \\ &= \underbrace{\omega_1 (E(R_1) - R_f)}_{\text{Contribution of } R_1} + \underbrace{\omega_2 (E(R_2) - R_f)}_{\text{Contribution of } R_2} + + \underbrace{\omega_3 (E(R_3) - R_f)}_{\text{Contribution of } R_3} \end{aligned}$$

- So the reward of asset i is also just

$$\omega_i (E(R_i) - R_f)$$

- Then asset i's reward-to-risk ratio is given by

$$\frac{\omega_i (E(R_i) - R_f)}{\omega_i \operatorname{Cov}(R_i, R^P)} = \frac{E(R_i) - R_f}{\operatorname{Cov}(R_i, R^P)}$$

Intuition behind CAPM

- CAPM builds on the insight that the appropriate risk premium on an asset is determined by its contribution to the risk of investors' overall portfolio.
- Now, let's move to the case with n assets, the variance-covariance matrix is given by

$$\begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{i,1} & \sigma_{i,2} & \cdots & \sigma_{i,N} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_N^2 \end{bmatrix}$$

Intuition behind CAPM

- For $R^M = \sum_{i=1}^{n} \omega_i R_i$, the variance of the market portfolio $V(R^M)$ can also be calculated and decomposed into contribution of each asset

$$V\left[R^M\right] = V\left[\sum_{i=1}^{n} \omega_i R_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i \omega_j \operatorname{Cov}\left[R_i, R_j\right]$$

$$= \underbrace{\begin{bmatrix} \omega_1\omega_1 \operatorname{Cov}[R_1, R_1] \\ + \\ \omega_1\omega_2 \operatorname{Cov}[R_1, R_2] \\ + \\ \vdots \\ \omega_1\omega_n \operatorname{Cov}[R_1, R_n] \end{bmatrix}}_{\text{Contribution of} R_1} +.+ \underbrace{\begin{bmatrix} \omega_i\omega_1 \operatorname{Cov}[R_i, R_1] \\ + \\ \omega_i\omega_2 \operatorname{Cov}[R_i, R_2] \\ + \\ \vdots \\ \omega_i\omega_n \operatorname{Cov}[R_i, R_n] \end{bmatrix}}_{\text{Contribution of } R_i} + . + \underbrace{\begin{bmatrix} \omega_n\omega_1 \operatorname{Cov}[R_n, R_1] \\ + \\ \omega_n\omega_2 \operatorname{Cov}[R_n, R_2] \\ + \\ \vdots \\ \omega_n\omega_M \operatorname{Cov}[R_n, R_n] \end{bmatrix}}_{\text{Contribution of} R_n}$$

Intuition behind CAPM

- Thus, if we pick out all the terms related to asset i in above red term, we obtain the contribution of asset i's stock to $V\left[R^M\right]$

$$\underbrace{\begin{bmatrix} \omega_i\omega_1 \operatorname{Cov}[R_i, R_1] \\ + \\ \omega_i\omega_2 \operatorname{Cov}[R_i, R_2] \\ + \\ \vdots \\ \omega_i\omega_n \operatorname{Cov}[R_i, R_n] \end{bmatrix}}_{\text{Contribution of } R_i}$$

  Alternatively, we can rewrite it as

$$\begin{aligned} &\omega_{\mathrm{i}}\left[\omega_1 \operatorname{Cov}(R_{\mathrm{i}}, R_1) + \omega_2 \operatorname{Cov}(R_{\mathrm{i}}, R_2) + \cdots \right. \\ &\quad \left. + \omega_{\mathrm{i}} \operatorname{Cov}(R_{\mathrm{i}}, R_{\mathrm{i}}) + \cdots + \omega_n \operatorname{Cov}(R_{\mathrm{i}}, R_n)\right] \\ &= \omega_{\mathrm{i}} \operatorname{Cov}(R_{\mathrm{i}}, R_M) \end{aligned} \tag{3}$$

- Same result as before

Intuition behind CAPM

- The contribution of asset i to the risk premium of the market portfolio is $\omega_i \left[E\left(R_{\mathrm{i}}\right) - R_f\right]$
  - Note that $R^M = \omega_1 R_1 + \omega_2 R_2 + \cdots + \omega_i R_i + \cdots \omega_n R_n$, we have

$$E(R^M) - R_f = \cdots + \omega_i \left(E\left(R_i\right) - R_f\right) + \cdots \omega_n \left(E\left(R_n\right) - R_f\right)$$

- Therefore, the reward-to-risk ratio for investments in asset i is

$$\frac{\text{asset i's contribution to risk premium}}{\text{asset i's contribution to variance}} = \frac{\omega_{\mathrm{i}} \left[E\left(R_{\mathrm{i}}\right) - R_f\right]}{\omega_{\mathrm{i}} \operatorname{Cov}\left(R_{\mathrm{i}}, R_M\right)} = \frac{E\left(R_{\mathrm{i}}\right) - R_f}{\operatorname{Cov}\left(R_{\mathrm{i}}, R_M\right)}$$

- Market portfolio is the tangency (mean-variance efficient) portfolio, it reward-to-risk ratio is

$$\frac{\text{Market risk premium}}{\text{Market variance}} = \frac{E\left(R_M\right) - R_f}{\sigma_M^2} = \frac{E\left(R_{\mathrm{i}}\right) - R_f}{\operatorname{Cov}\left(R_{\mathrm{i}}, R_M\right)} \tag{4}$$

Intuition behind CAPM

- For any component asset of market portfolio, we measure its risk as the contribution to portfolio variance
  - depends on its covariance with the market portfolio
- For individual asset, CAPM turns the measure of risk of individual assets from its own variance to its contribution to portfolio variance.
- In contrast, for the efficient portfolio itself, variance is the appropriate measure of risk.
- **Equilibrium argument**: a basic principle of equilibrium is that all assets should offer the same reward-to-risk ratio, as the market
  - If the ratio were better for one asset than another, investors would rearrange their portfolios, tilting toward the alternative with the better trade-off and shying away from the other.
  - Such activity would impose pressure on security prices (expected return) until the ratios were equalized.

Intuition behind CAPM

- Portfolio adjustments of all investors put pressure on the price, thus lead to the equilibrium condition

$$\frac{E(R_{\mathrm{i}}) - R_f}{\mathrm{Cov}(R_i, R_M)} = \frac{E(R_M) - R_f}{\sigma_M^2} \tag{5}$$

- To determine the fair risk premium of asset i

$$E(R_{\mathrm{i}}) - R_f = \frac{\mathrm{Cov}(R_{\mathrm{i}}, R_M)}{\sigma_M^2}\left[E(R_M) - R_f\right] \tag{6}$$

- The ratio $\frac{\mathrm{Cov}(R_{\mathrm{i}}, R_M)}{\sigma_M^2}$ measures the share of total variance of M contributed by asset i (normalized by its weight ω_i).
- The ratio is denoted by β, the expected return – beta relationship

$$E(R_{\mathrm{i}}) = R_f + \beta_{\mathrm{i}}\left[E(R_M) - R_f\right] \tag{7}$$

Expected Returns on Individual Securities

- The expected return-beta relationship hold for all individual stock in the market portfolio

$$E\left(R_{\mathrm{i}}\right)=R_f+\beta_{\mathrm{i}}\left[E\left(R_M\right)-R_f\right] \tag{8}$$

$$\beta_{\mathrm{i}}=\frac{\operatorname{Cov}\left(R_{\mathrm{i}}, R_M\right)}{\sigma_M^2} \tag{9}$$

  where $\left[E\left(R_M\right)-R_f\right]$ is the market risk premium, β_{i} is asset i's risk loading.

- It tells us that the total expected rate of return is the sum of two
  - the risk-free rate (compensation for waiting, i.e., the time value of money)
  - the risk premium (compensation for worrying, specifically about investment returns)
- Moreover, it makes a very specific prediction about the size of each asset's risk premium.

Expected Returns on Individual Securities

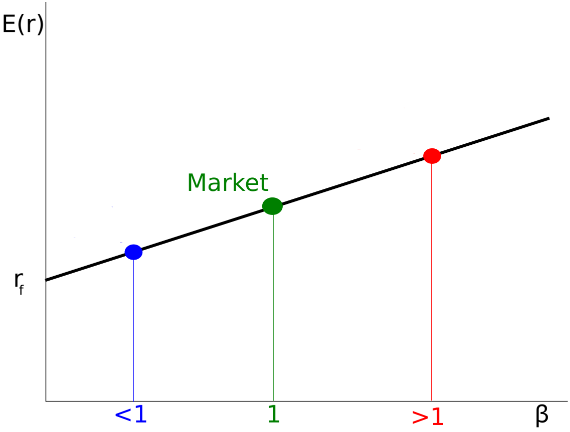
- Notice that the risk premium of asset i does not depend on its own volatility σ^i, but depends of its β_i risk loading.
- CAPM predicts that only systematic risk (loading on the market portfolio) should command a risk premium
  - while idiosyncratic (firm-specific variation uncorrelated with market portfolio) risk should not.
- If the expected return-beta relationship holds for each asset, it must hold for any combination or weighted average of assets.

$$\begin{aligned} \omega_1 E(R_1) &= \omega_1 R_f + \omega_1 \beta_1 [E(R_M) - R_f] \\ +\omega_2 E(R_2) &= \omega_2 R_f + \omega_2 \beta_2 [E(R_M) - R_f] \\ + \quad \cdots &= \cdots \\ \underline{+\omega_n E(R_n)} &\underline{= \omega_n R_f + \omega_n \beta_n [E(R_M) - R_f]} \\ E(R_P) &= R_f + \beta_P [E(R_M) - R_f] \end{aligned} \tag{10}$$

- For market portfolio, $\beta_M = 1$

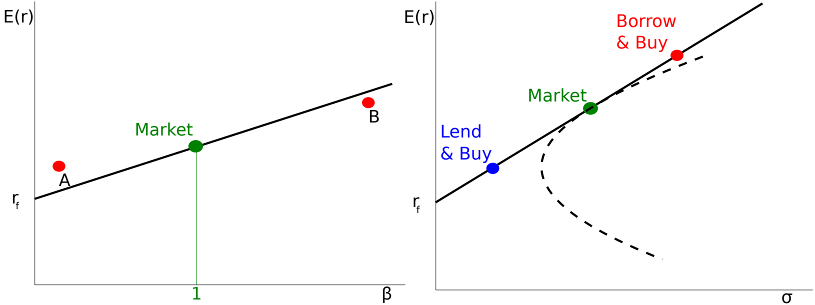
Security Market Line

A change in picture; let β be the x-axis instead of σ:



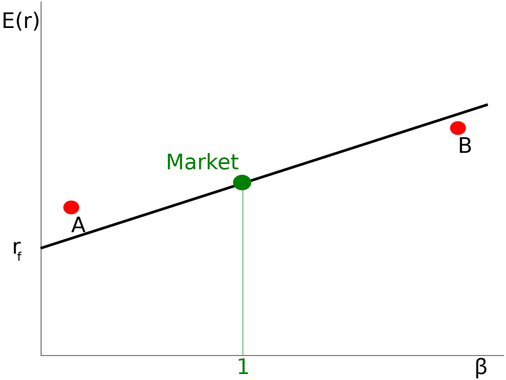
Special name for this line: The **Security Market Line (SML)**

SML versus CML



- The SML holds for EVERYTHING (if CAPM is true): assets, portfolios.
- Contrast that with the CML that holds only for EFFICIENT PORTFOLIOS.

SML: evaluating mispricing



- The SML is defined by (2): the y-intercept is r_f and the slope is the expected excess return on the market.
- If the CAPM holds:
  - Once an asset is on the SML, it is held there by market forces.
  - A's expected return is too high; B's is too low.

Alpha

CAPM alpha is the expected return in excess of that implied by the CAPM, which is also called pricing error:

$$\alpha_{CAPM} = E[r] - r_f - \beta(E[r_m] - r_f)$$

- Stock ABC has an expected return of 15% and beta of 1.2. Another stock, DEF, has expected return of 17% and a beta of 1.9. The expected return of the market is 8% and the risk-free rate is 4%.
  - Which stock should an investor buy?
  - Find the alpha of each stock.
- Required return of ABC: $4 + 1.2 \times (8 - 4) = 8.8\%$
  Alpha of ABC: 15% - 8.8% $= 6.2\%$
  Required return of DEF: $4 + 1.9 \times (8 - 4) = 11.6\%$
  Alpha of DEF: 17% - 11.6% $= 5.4\%$
- ABC is a better investment (if you must have only one of them).

SML: evaluating mispricing

- If CAMP hold (dots on SML), we have 0 α for every asset

$$\alpha_{CAPM} = E(r_i) - \widehat{E(r_i)}_{CAPM} = E(r_i) - (r_f + \beta_i (E(r_m) - r_f)) = 0$$

- For any dots above SML, like dot A, we have positive α

$$\alpha_A = \mathbb{E}[r_A] - \widehat{\mathbb{E}[r_A]}_{CAMP} = \mathbb{E}[r_A] - [r_f + \beta_A (\mathbb{E}[r_m] - r_f)] > 0$$

which implies $\mathbb{E}[r_A] - r_f > \beta_A (\mathbb{E}[r_m] - r_f)$

$$\frac{\mathbb{E}[r_A] - r_f}{\beta_A} > \frac{\mathbb{E}[r_m] - r_f}{1} \Rightarrow \frac{E(r_i) - r_f}{Cov(r_i, r_M)} > \frac{\mathbb{E}[r_m] - r_f}{\sigma_M^2}$$

- $E(r_A)$ is too high relative to CAPM, or we say the A is underpriced
    - Asset A's reward-to-risk ratio is higher than the market, investors would prefer such assets

SML: evaluating mispricing

- For any dots below SML, like dot B, we have negative α

$$\alpha_B = \mathbb{E}\left[r_B\right] - \widehat{\mathbb{E}\left[r_B\right]}_{CAMP} = \mathbb{E}\left[r_B\right] - \left[r_f + \beta_B \left(\mathbb{E}\left[r_m\right] - r_f\right)\right] < 0$$

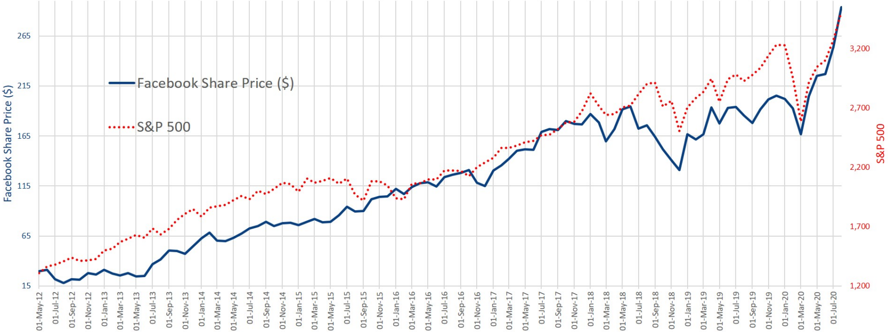
which implies $\mathbb{E}\left[r_B\right] - r_f < \beta_B \left(\mathbb{E}\left[r_m\right] - r_f\right)$

$$\frac{\mathbb{E}\left[r_B\right] - r_f}{\beta_B} < \frac{\mathbb{E}\left[r_m\right] - r_f}{1} \Rightarrow \frac{E(r_i) - r_f}{Cov(r_i, r_M)} < \frac{\mathbb{E}\left[r_m\right] - r_f}{\sigma_M^2}$$

- $E(r_B)$ is too low relative to CAPM, or we say the B is overpriced
  - Asset B's reward-to-risk ratio is lower than the market, investors would sell such assets

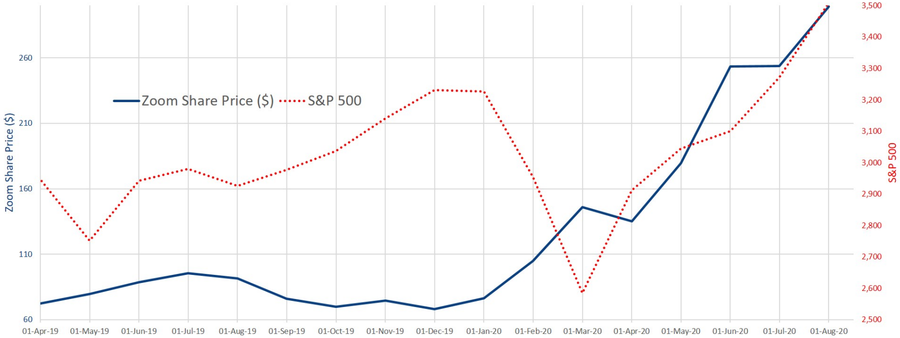
Estimating β

Facebook: $Correl[R_{FB}, R_{SP500}] = 0.41$, Beta $= 1.15$



Facebook's market cap. is over $1 trillion, making it one of the largest components of the S&P 500. So its correlation and beta with the market are positive.

Zoom: $Correl[R_{ZM}, R_{SP500}] = -0.45$, Beta = -1.25



Zoom is a countercyclical stock – doing well when the economy has been doing badly, so its correlation and beta with the market are negative.

Ordinary Least Squares (OLS) regression

- Ordinary Least Squares (OLS) regression is a method used to estimate the relationship between one dependent variable and one or more independent variables.
- We assume following linear model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik} + \varepsilon_i \tag{11}$$

  where y_i is the dependent variable, $x_{i1}, \cdots, x_{ik}$ are independent variables.
- OLS chooses the coefficients $\beta_0, \beta_1, \ldots, \beta_k$ so that the sum of squared differences between the actual y_i values and the predicted values $\hat{y}_i$ is as small as possible:

$$\min_{\beta} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

- By doing this, we are trying to get dependent variable y_i as a linear function of independent variables $x_{i1}, \cdots, x_{ik}$.

Computing Beta

Beta (β_i) is the slope from the following regression:

$$r_{i,t} - r_f = \alpha_i + \beta_i(r_{m,t} - r_f) + \epsilon_{i,t},$$

i.e. regressing $r_{i,t} - r_f$ on $r_{m,t} - r_f$ and a constant.

We can compute beta by taking covariances of both sides with $r_{m,t}$:

$$\begin{aligned} Cov[r_{m,t}, r_{i,t} - r_f] &= Cov[r_{m,t}, \alpha_i + \beta_i(r_{m,t} - r_f) + \epsilon_{i,t}] \\ Cov[r_{m,t}, r_{i,t}] &= \underbrace{Cov[r_{m,t}, \alpha_i]}_{=0} + \beta_i \underbrace{Cov[r_{m,t}, r_{m,t} - r_f]}_{Cov[r_{m,t}, r_{m,t}]} + \underbrace{Cov[r_{m,t}, \epsilon_{i,t}]}_{\text{assume } = 0} \end{aligned}$$

- $Cov[r_{m,t}, \alpha_i] = 0$ since α_i is constant.
- $Cov[r_{m,t}, r_{i,t} - r_f] = Cov[r_{m,t}, r_{i,t}]$ since r_f is constant.

Assume $Cov[r_{m,t}, \epsilon_{i,t}] = 0$ (standard regression assumption):

$$\beta_i = \frac{\sigma_{i,m}}{\sigma_m^2}$$

Regression

To estimate the β of a stock:

1. **Collect Data.**

   Let $r_{i,t}$, $r_{m,t}$, and $r_{f,t}$ denote historical individual security i, market, and risk-free returns respectively over some time period $t = 1, 2, \ldots, T$.

2. **Estimate the following regression:**

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i \left(r_{m,t} - r_{f,t} \right) + \epsilon_{i,t} \tag{12}$$

Proxies for r_m and r_f

- For the market portfolio, a market-cap weighted stock index is used, e.g. the S&P500 index, or the CRSP value-weighted (CRSP-VW) Index.
  - Remember to make sure that your market proxy includes dividends.
- For the risk-free rate, it is common to use a 1-month T-bill for portfolio applications.
  - The 1-month T-bill is used as many portfolio applications consider rebalancing every month.

Beta of a portfolio

The beta of a portfolio is the weighted average of the betas of its constituent assets, where the weights are the portfolio weights in those assets.

- With 2 assets:

$$
\begin{aligned}
r_{p,t} - r_f &= (w_a r_{a,t} + w_b r_{b,t}) - r_f \\
&= w_a(r_{a,t} - r_f) + w_b(r_{b,t} - r_f) \\
&= w_a(\alpha_a + \beta_a(r_{m,t} - r_f) + \epsilon_{a,t}) + w_b(\alpha_b + \beta_b(r_{m,t} - r_f) + \epsilon_{b,t}) \\
&= \underbrace{(w_a\alpha_a + w_b\alpha_b)}_{\alpha_p} + \underbrace{(w_a\beta_a + w_b\beta_b)}_{\beta_p}(r_{m,t} - r_f) + \underbrace{(w_a\epsilon_{a,t} + w_b\epsilon_{b,t})}_{\epsilon_t^p}
\end{aligned}
$$

- More generally: $\beta_p = \sum_{i=1}^{N} w_i\beta_i$

Example β Estimation

General Motors' β

Let's compute a β on General Motors Co (GM). To keep this transparent, we will only look at one year of data.

Month	GM Return	Market Return	T-Bill Return	GM Excess Return	Market Excess Return
Jan	6.06	7.89	0.65	5.41	7.24
Feb	-2.86	1.51	0.58	-3.44	0.93
Mar	-8.18	0.23	0.62	-8.80	-0.39
Apr	-7.36	-0.29	0.72	-8.08	-1.01
May	7.76	5.58	0.66	7.10	4.92
Jun	0.52	1.73	0.55	-0.03	1.18
Jul	-1.74	-0.21	0.62	-2.36	-0.83
Aug	-3.00	-0.36	0.55	-3.55	-0.91
Sep	-0.56	-3.58	0.60	-1.16	-4.18
Oct	-0.37	4.62	0.65	-1.02	3.97
Nov	6.93	6.85	0.61	6.32	6.24
Dec	3.08	4.55	0.65	2.43	3.90

Regression output — estimating GM's α and β

Regression Statistics	
Multiple R	0.7582
R Square	0.5749
Adjusted R Square	0.5324
Standard Error	3.5492
Observations	12

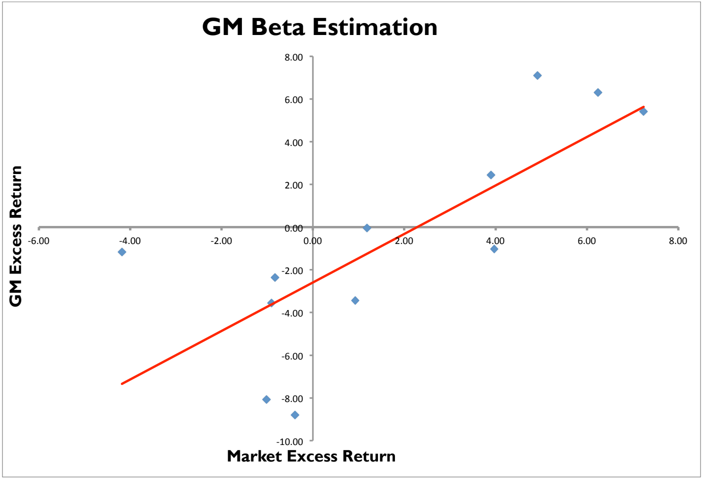
ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	170.3797	170.3797	13.5256	0.0043
Residual	10	125.9687	12.5969		
Total	11	296.3484			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	-2.5923	1.1592	-2.2363	0.0493	-5.1751	-0.0095
Beta	1.1362	0.3089	3.6777	0.0043	0.4478	1.8245

- $\hat{\beta}_{GM} = 1.14$. The p-value on that beta is $0.0043 < 0.05$, so the beta is significantly positive.
- $\hat{\alpha}_{GM} = -2.59\%$. The p-value on that alpha is $0.0493 < 0.05$, so the alpha is significantly negative.
- $R^2 = 0.57$, so the market can explain 57% of the variance of GM, but there is also a lot it can't explain.

Regression output — graphical representation



- α is the intercept, and β is the slope.
- The residuals of the regression represent the part of the returns that are not due to systematic market risk.

β for a New Firm

New firm

- With no historical data, how would you estimate β for a new company?
- The standard industry practice is to use "comparables".
  - Find a similar company, that is traded on an exchange, and use the β of that company.
  - Alternatively, use an industry β for the new firm.
- Here is an approach in the same spirit, but more useful in some applications:
  - Often it is tough to find a comparable firm.
  - Often firms differ a great deal within an industry.
  - Instead, let's compute a model-predicted β from a sample of companies with similar characteristics.

Which characteristics?

- Industry
- Firm size
- Financial leverage
- Operating leverage
- Growth/Value (book-to-market ratio)
- and many more...

Financial leverage and beta

- β is shown to be positively related to firms' financial leverage.
  - How much firm borrow to run the business?
    - ★ Financial leverage effects: higher financial leverage tends to amplify the movement of firm return over the business cycles

	High leverage	**Low leverage**
Equity	\$10	\$20
Debt	\$90	\$80
Investment today	\$100	\$100

  - Suppose loan interest rate is 1, if asset value increase by 10% tomorrow, what's the return on equity for each firm?

Methodology

1. Select a sample of companies to estimate the model.
2. Estimate $\beta_{i,t}$ for these companies using historical return data.
3. Regress estimated betas $\hat{\beta}_{i,t}$ on several characteristics that can drive betas. For example,

$$\hat{\beta}_{i,t} = a_0 + \sum_j \gamma_j IND_{j,i} + a_1 SIZE_{i,t} + a_2 FLEV_{i,t} + a_3 OLEV_{i,t} + \epsilon_{i,t},$$

   where $IND_{j,i}$ is a dummy variable that takes the value 1 if firm i belongs in industry j and 0 otherwise.
4. Suppose you are asked to find the beta for a new company XYZ in the tech industry given XYZ's characteristics. Your estimate will then be

$$\hat{\beta}_{xyz,t} = \hat{a}_0 + \hat{\gamma}_{tech} + \hat{a}_1 SIZE_{xyz,t} + \hat{a}_2 FLEV_{xyz,t} + \hat{a}_3 OLEV_{xyz,t}.$$

Rolling β Estimation

Time-varying β

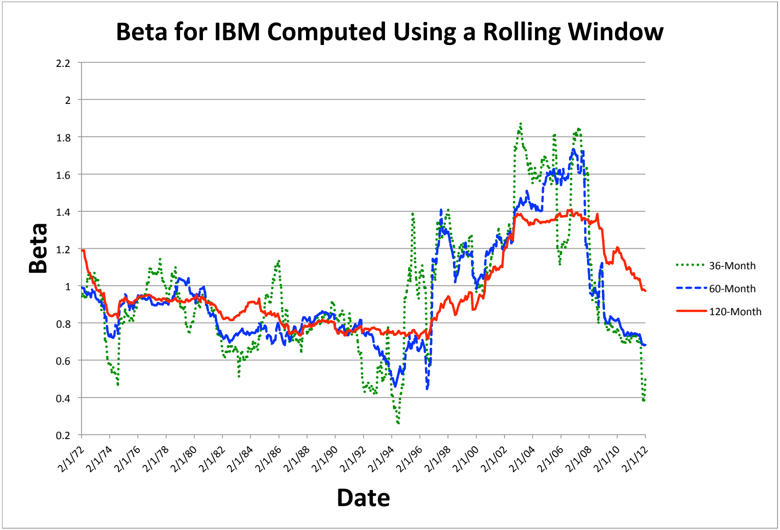
- **Betas may change over time.** That is why we typically use a 5-year window to estimate them.

- Reasons for time-varying betas:
  1. Changes in financial leverage.
  2. Changes in operating leverage.
  3. Changes in the type of a firm's operations.
  4. Mergers and acquisitions.

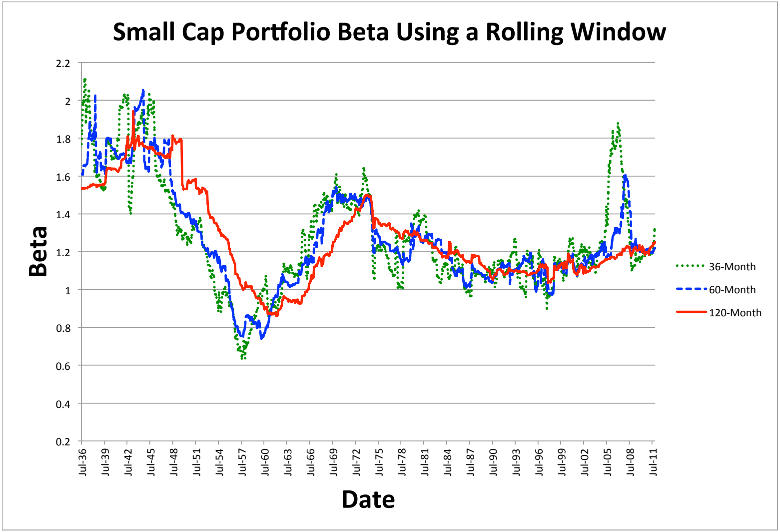
- A *rolling window* approach can be used to estimate a time-series of β:
  - At month t, estimate β using data from months $t-60$ through $t-1$,
  - At month $t+1$, estimate β using data from months $t-59$ through t.

- Alternatively, more sophisticated statistical models can be used that allow for time-variation in β.

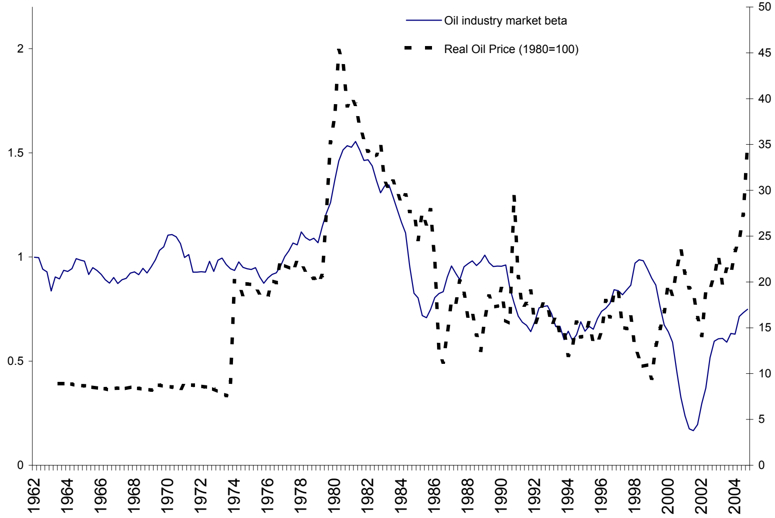
Rolling β for a stock



Rolling β for a style



Rolling β for an industry



Conclusions

Summary

1. In equilibrium, the tangency portfolio is the market portfolio, and investors hold only the market and the risk-free asset.
2. CAPM beta is a measure of systematic risk of an asset.
3. Betas are typically estimated by running a regression:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i \left(r_{m,t} - r_{f,t} \right) + \epsilon_{i,t}.$$

4. Data restrictions and time-variation in β dictate how the estimation should be done. New firm β's can be found by appealing to the characteristics of other similar firms.
5. CAPM alpha measures mis-pricing, assuming the CAPM holds.
6. Expected (required) return from CAPM can be used as a hurdle/discount rate when evaluating a firm's projects.

Appendix

Derivation of CAMP: Mean-variance analysis

- Assumptions on investor behaviors imply that investors are alike in most important ways, thus greatly simplify the demand side.

- Assumptions on market structure imply a well-functioned market in which trading behaviors ensure equilibrium to happen.

- To derive CAPM, we start from the MV analysis in the case with N risky assets and one risk-free asset
  - Investor choose weights of risky assets ($\omega = [\omega_1, \cdots, \omega_N]'$) and risk-free asset ($\omega_0 = (1 - \omega'\mathbf{1})$) to obtain

$$\begin{aligned} R^P &= \omega_0 R_f + \omega_1 R_1 + \cdots + \omega_N R_N = \omega_0 R_f + \omega' R \\ &= \omega' R + (1 - \omega'\mathbf{1}) R_f = R_f + \omega' (R - R_f \mathbf{1}) = R_f + (R^e)' \omega \end{aligned} \tag{13}$$

where ω $(N \times 1)$ is the weights of risky assets, and

$R^e = [R_1 - R_f, \cdots, R_N - R_f]'$ $(N \times 1)$ is the vector of excess returns

Derivation of CAMP: Mean-variance analysis

- The **mean-variance optimization problem** is to minimize variance

$$\min_{\omega} \frac{1}{2} V\left[R^P\right] = \min_{\omega} \frac{1}{2}\omega' \Sigma \omega \tag{14}$$

Σ $(N \times N)$ is the variance-covariance matrix. For a given level of expected portfolio return

$$E\left[R^P\right] = R_f + E\left(R^e\right)' \omega = \mu \tag{15}$$

- **Step 1**: Set up the Lagrangian with multiplier λ

$$L = \frac{1}{2}\omega' \Sigma \omega + \lambda \left(\mu - R_f - E\left(R^e\right)' \omega\right) \tag{16}$$

Derivation of CAMP: Mean-variance analysis

- **Step 2**: Derive first order conditions

$$\frac{\partial L}{\partial \omega} = 0 \Rightarrow \Sigma\omega^\star = \lambda E\left[R^e\right] \Longrightarrow \omega^\star = \lambda\Sigma^{-1}E\left[R^e\right] \tag{17}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \mu = R_f + E\left(R^e\right)'\omega^\star \tag{18}$$

- With optimal portfolio $\omega^\star$, the portfolio P's variance is

$$\begin{aligned} V\left[R^P\right] &= \omega^{\star\prime}\Sigma\omega^\star \\ &= \omega^{\star\prime}\Sigma\left(\lambda\Sigma^{-1}E\left[R^e\right]\right) \\ &= \lambda\omega^{\star\prime}\Sigma\Sigma^{-1}E\left[R^e\right] \\ &= \lambda\omega^{\star\prime}E\left[R^e\right] \\ &= \lambda\left(E\left[R^P\right] - R_f\right) \end{aligned} \tag{19}$$

which hold for all optimal P on CML, including the tangency portfolio

Derivation of CAMP: Mean-variance analysis

- We know that one optimal portfolio P is the tengency portfolio, let's denote it by M instead:

$$R^M = R'\omega^\star \tag{20}$$

- according to equation (19), we have

$$\frac{1}{\lambda} = \frac{E\left[R^M\right] - R_f}{V\left[R^M\right]} \tag{21}$$

- Now let's denote $Cov\left(R, R^M\right)$ be the $N \times 1$ vector of covariances of returns on all risky assets with the return on the tangency portfolio

$$\begin{aligned} Cov\left(R, R^M\right) &= Cov\left(R, R\right)\omega^\star \\ &= \Sigma\omega^\star \\ &= \Sigma\lambda\Sigma^{-1}E\left[R^e\right] \\ &= \lambda\Sigma\Sigma^{-1}E\left[R^e\right] \\ &= \lambda E\left[R^e\right] \end{aligned} \tag{22}$$

Derivation of CAMP: Mean-variance analysis

- For each asset i, from (22), we must have

$$Cov\left(R^i, R^M\right) = \lambda E\left[R^{i,e}\right] \Rightarrow \frac{1}{\lambda} = \frac{E(R^i) - R_f}{Cov\left(R^i, R^M\right)} \tag{23}$$

- Recall from (21), for market portfolio M

$$\frac{1}{\lambda} = \frac{E\left[R^M\right] - R_f}{V\left[R^M\right]}$$

- Equating both equations yields CAPM!

$$\frac{E(R^i) - R_f}{Cov\left(R^i, R^M\right)} = \frac{E\left[R^M\right] - R_f}{V\left[R^M\right]} \tag{24}$$

  or equivalently

$$E(R^i) - R_f = \frac{Cov\left(R^i, R^M\right)}{V\left[R^M\right]}\left[E\left(R^M\right) - R_f\right] \tag{25}$$

- This relationship is called the Sharpe-Lintner CAPM.

Derivation of CAMP: Mean-variance analysis

- One last question, what is that tangent portfolio M?
- Given above assumptions, according to MV analysis, all investors will hold (demand) the same tangent portfolio M
  - Thus, M will be the market portfolio, the value-weighted portfolio of all assets in the investable universe (current supply of assets).
  - Demand = Supply: M is market portfolio
- Individual investors with different risk averse level will choose a different position along the CAL, but same tangent portfolio
- The market portfolio is based on the common input list, which incorporates all relevant information about the universe of securities.
- Investors can skip the trouble of doing security analysis and obtain an efficient portfolio simply by holding the market portfolio.
  - the passive strategy of investing in a market portfolio is efficient.