ECOM90024 - Assignment 2 - Question 1

Consider the following ARMA(1,1) model:

$$Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$\varepsilon_t \sim iid(0, \sigma^2)$$

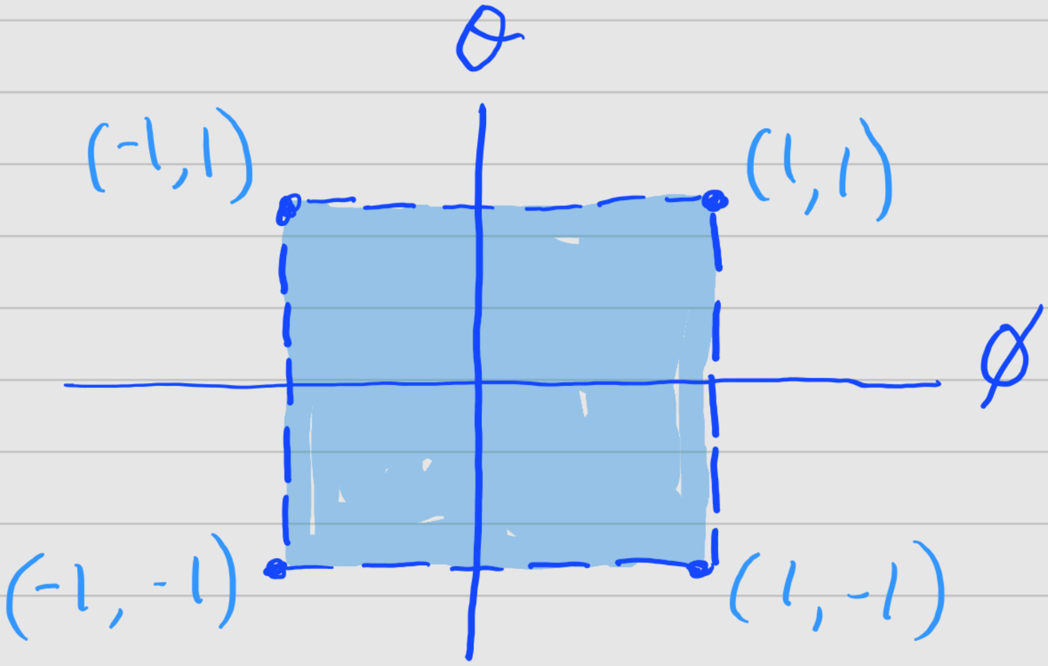
a) Using an appropriate diagram, depict the set of values for which Y_t will be investible AND covariance stationary

Rewrite the model in terms of lag operators:

$$(1 - \phi L) Y_t = (1 + \theta L) \varepsilon_t$$

$\therefore$ ARMA(1,1) is investible if $|\theta| < 1$ and stationary if $|\phi| < 1$.

All points in the shaded area below satisfy these constraints for investibility and stationarity



b) On the same diagram, depict the set of values of the parameters ϕ and θ for which Y_t will be a white noise process

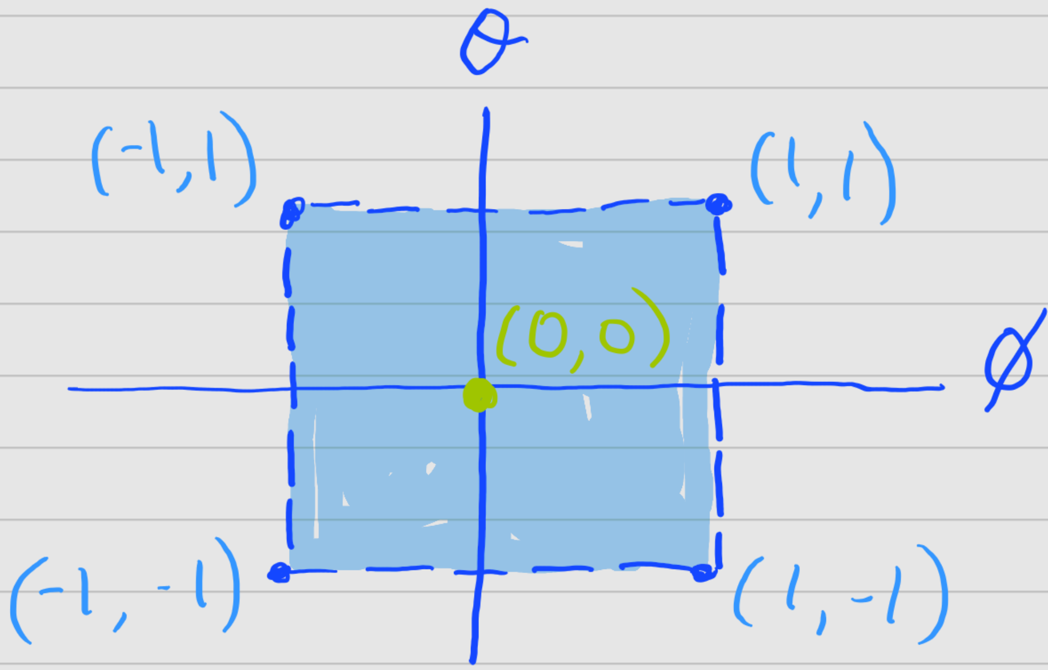
A white noise process is defined as:

$$Y_t = \varepsilon_t \, , \quad \varepsilon_t \sim iid\, N(0, \sigma^2)$$

In order for the mean equation of the ARMA(1,1) defined in part (a) to be a white noise process, parameter values would need to be zero:

$$Y_t = 0 \cdot Y_{t-1} + \varepsilon_t + 0 \cdot \varepsilon_{t-1} = \varepsilon_t$$

Graphically, this is represented by the yellow point below. It is only a point because this area only captures the origin.



c) Assuming invertibility and autocovariance stationarity, derive the autocovariance and autocorrelation functions for Y_t.

Before calculating the ACVF and ACF, we first need to define the expected mean.

$$E[Y_t] = \mu = E[\phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]$$

$$E[Y_t] = \mu = \phi E[Y_{t-1}] + E[\varepsilon_t] + \theta E[\varepsilon_{t-1}]$$

$$\mu = 0$$

Now we can derive the variance equation

$$Var(Y_t) = E[(Y_t - \mu)^2]$$

$$Var(Y_t) = E\left[\left(\phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}\right)^2\right]$$

$$= E\left[\left(\phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}\right)\left(\phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}\right)\right]$$

$$= E\Big[\underline{\phi^2 Y_{t-1}^2} + \phi Y_{t-1}\varepsilon_t + \underline{\phi\theta Y_{t-1}\varepsilon_{t-1}} +$$

$$\phi Y_{t-1}\varepsilon_t + \underline{\varepsilon_t^2} + \varepsilon_t \theta \varepsilon_{t-1} +$$

$$\underline{\phi\theta Y_{t-1}\varepsilon_{t-1}} + \theta\varepsilon_{t-1}\varepsilon_t + \underline{\theta^2 \varepsilon_{t-1}^2}\Big]$$

Because we have assumed covariance stationarity, all errors are iid. Therefore all cross product error terms fall out, as do any other terms that have mismatched lags.

$$Var(Y_t) = E\left[\phi Y_{t-1}^3 + \varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^3 + 2\phi\theta Y_{t-1}\varepsilon_{t-1}\right]$$

$$Var(Y_t) = E\left[\phi^2 Y_{t-1}^2 + \varepsilon^2(1 + \theta^2 + 2\phi\theta)\right]$$

$$Var(Y_t) = \phi^2 Var(Y_{t-1}) + \sigma^2(1 + \theta^3 + 2\phi\theta)$$

$$Var(Y_t) - \phi^2 Var(Y_{t-1}) = \sigma^2(1 + \theta^2 + 2\phi\theta)$$

Again, because we have assumed covariance stationarity: $Var(Y_t) = Var(Y_{t-1})$

$$Var(Y_t)(1-\phi^2) = \sigma^2(1+\theta^2+2\phi\theta)$$

$$Var(Y_t) = \frac{\sigma^2(1+\theta^2+2\phi\theta)}{1-\phi^2} = \gamma_0$$

Now we can define the autocovariance function. This can be done more easily by rearranging the equation

$$Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$Y_t - \phi Y_{t-1} = \varepsilon_t + \theta \varepsilon_{t-1}$$

Multiply both sides by $Y_{t-\tau}$:

$$Y_t \cdot Y_{t-\tau} - \phi Y_{t-1} \cdot Y_{t-\tau} = \varepsilon_t \cdot Y_{t-\tau} + \theta \varepsilon_{t-\tau} \cdot Y_{t-\tau}$$

For $\tau = 1$, we get the following assuming $\gamma_\tau = Cov(Y_t, Y_{t-\tau})$ and taking expectations:

$$E[Y_t \cdot Y_{t-1} - \phi Y_{t-1} \cdot Y_{t-1}] = E[\varepsilon_t \cdot Y_{t-1} + \theta \varepsilon_{t-1} \cdot Y_{t-1}]$$

$$\gamma_1 - \phi \gamma_0 = \theta \sigma^2$$

Doing the same for $\tau = 2$ gets us the following:

$$E[Y_t \cdot Y_{t-2} - \phi Y_{t-1} \cdot Y_{t-2}] = E[\varepsilon_t \cdot Y_{t-2} + \theta \varepsilon_{t-1} \cdot Y_{t-2}]$$

$$\gamma_2 - \phi \gamma_1 = 0$$

This solution will apply for all $\tau > 1$

therefore:

$$\gamma(\tau) = \begin{cases} \theta\sigma^2, & \tau \leq 1 \\ 0, & \tau > 1 \end{cases}$$

or,

$$\gamma_\tau - \phi\gamma_{\tau-1} = \theta\sigma^2 \text{ for } \tau \leq 1$$

$$\gamma_\tau - \phi\gamma_{\tau-1} = 0, \text{ for } \tau > 1$$

Now we can define the autocorrelation function:

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0}$$

Simply divide both sides of the autocorrelation function by γ_0

$$\frac{\gamma_\tau}{\gamma_0} + \phi \frac{\gamma_{\tau-1}}{\gamma_0} = \frac{\theta \sigma^2}{\gamma_0}$$

For $\tau = 1$:

$$\rho_1 + \phi \rho_0 = \frac{\theta \sigma^2}{\gamma_0}$$

Because $\rho_0 = 1$ as observations are perfectly correlated with themselves:

$$\rho_1 = \frac{\theta \sigma^2}{\gamma_0} - \phi$$

However, for $\tau > 1$:

$$\rho_\tau + \phi \rho_{\tau-1} = 0$$

$$\rho_\tau = -\phi \rho_{\tau-1}$$

Therefore:

$$\rho_\tau = \begin{cases} \dfrac{\sigma \sigma^2}{\gamma_0} - \phi, & \text{for } \tau \leq 1 \\ -\phi \rho_{\tau-1}, & \text{for } \tau > 1 \end{cases}$$