Quantitative Analysis of Finance I

ECON90033

WEEK 4

DETERMINISTIC AND STOCHASTIC TRENDS

SPURIOUS REGRESSION

TESTING FOR A UNIT ROOT / STATIONARITY

ASSET PRICE BUBBLES

<u>Reference</u>:

HMPY: § 5.1-5.3, 5.5

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DETERMINISTIC AND STOCHASTIC TRENDS

- Recall from week 3 that a stochastic process is (weakly or covariance) stationary if it has constant and finite unconditional means and autocovariances that do not depend on time.

  $\longrightarrow$ A stochastic process is (weakly or covariance) nonstationary if its mean and/or autocovariances change in time.

  Let's illustrate nonstationarity in the mean with two simple stochastic processes.

  (a) Suppose that y_t is the sum of its mean, μ_t, which is a polynomial of order d in the time variable t, and a random error, ε_t, which is a white noise, i.e.,

$$y_t = \mu_t + \varepsilon_t \quad , \quad \mu_t = \underbrace{\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_v t^v}_{\substack{\text{This is a deterministic trend, and} \\ \text{it is predictable from its own past} \\ \text{without error at any point in time.}}} \quad , \quad \underbrace{\varepsilon_t : WN(0, \sigma^2)}_{\substack{\text{This is} \\ \text{stationary.}}}$$

Because of its time-dependent mean (μ_t), this stochastic process is nonstationary. It is, however, stationary around the deterministic trend because $y_t - \mu_t = \varepsilon_t$.

→ $\{y_t\}$ is called a trend-stationary (*TS*) process as it can be made stationary by subtracting the deterministic trend.

(b) Alternatively, nonstationarity in the mean might be the result of a nonstationary *AR* component in the data generating process.

Consider, for example, a *pure* *AR*(1) process, i.e., an *AR* process that does not have any deterministic term, with y_0 initial value

$$y_t = \varphi_1 y_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t : WN(0, \sigma^2)$$

It can be shown with backward iteration, that

$$y_t = \varphi_1^t y_0 + \sum_{i=0}^{t-1} \varphi_1^i \varepsilon_{t-i}$$

…, and that y_t has the following first and second moments ($t \to \infty$):

mean station

$$E(y_t) = 0 \text{ if } |\varphi_1| < 1; \quad \pm y_0 \text{ if } |\varphi_1| = 1; \quad \pm\infty \text{ if } |\varphi_1| > 1$$

$$Var(y_t) = \frac{\sigma^2}{1-\varphi_1^2} \text{ if } |\varphi_1| < 1; \quad \sigma^2 t \text{ if } |\varphi_1| = 1; \quad \infty \text{ if } |\varphi_1| > 1$$

$$Cov(y_t, y_{t-k}) = \varphi_1^k \frac{\sigma^2}{1-\varphi_1^2} \text{ if } |\varphi_1| < 1; \quad \sigma^2 (t-k) \text{ if } |\varphi_1| = 1; \quad \pm\infty \text{ if } |\varphi_1| > 1$$

outcomes from different φ values

i. The mean, variance and autocovariances are asymptotically time independent, and hence *AR*(1) is stationary, if and only if $|\varphi_1| < 1$.

ii. The mean, variance and autocovariances all approach infinity (in absolute value), so *AR*(1) is non-stationary (explosive), if $|\varphi_1| > 1$.

iii. The mean is constant (in absolute value), but the variance and autocovariances approach infinity (in absolute value), so *AR*(1) is non-stationary, if $|\varphi_1| = 1$.

As a sidenote, let's drop the error term from a pure $AR(1)$ model:

$$y_t = \varphi_1 y_{t-1} \quad , \quad \varphi_1 \neq 0$$

This is a homogeneous first-order linear difference equation, and its general solution takes the form $A\alpha^t$, where $A \neq 0$ is an arbitrary real number.

Plugging the general solution in the difference equation, we get the characteristic equation, whose solution is called characteristic root.

$$A\alpha^t = \varphi_1 A\alpha^{t-1} \longrightarrow \alpha = \varphi_1 \longrightarrow$$

$\varphi_1 = 1$ implies $\alpha = 1$, i.e., a unit (characteristic) root.

Consider now a pure $AR(2)$ model without the error term:

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} \quad , \quad \varphi_2 \neq 0$$

This is a homogeneous second-order linear difference equation.

The general solution is again $A\alpha^t$, where $A \neq 0$ is an arbitrary real constant and α is a real or complex number, i.e., $\alpha = a + bi$, where a and b are real numbers and i is an imaginary number that satisfies $i^2 = -1$, $|i| = 1$.

Plugging again the general solution in the difference equation, we get …

$$A\alpha^{t} = \varphi_1 A\alpha^{t-1} + \varphi_2 A\alpha^{t-2} \longrightarrow \alpha^2 = \varphi_1 \alpha + \varphi_2$$

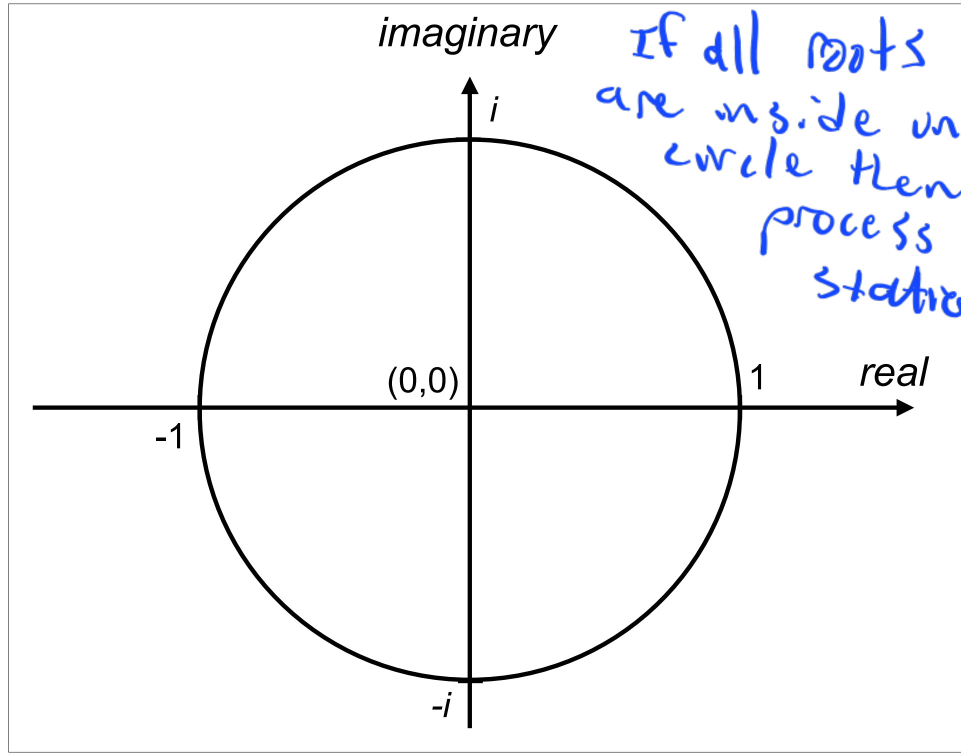
any point on circle represents a poss. unit roots

This is a quadratic equation, and it has either two (not necessarily different) real solutions, or two complex solutions: α_1, α_2.

In either case, the α_1 and α_2 characteristic roots can be illustrated in the real-imaginary coordinate system.

The circle of unit radius around the origin is the unit circle.

Any root that falls on the unit circle, real or complex, is a unit autoregressive characteristic root, briefly referred to as a unit root.



If all roots are inside unit circle then the process is stationary

In general, an $AR(p)$ model without its error term and deterministic terms is a p^{th} order homogeneous linear difference equation and the corresponding characteristic equation has p real and/or complex characteristic roots.

Let's now return to the *AR*(1) process and consider the special case of $\varphi_1 = 1$:

$$y_t = y_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t : WN(0, \sigma^2)$$

This *AR*(1) process is called a random walk or a unit-root process.

From the variance and autocovariance of *AR*(1) processes (slide #4), the autocorrelation coefficients (k > 1) of a random walk are

$$\rho_{y_t, y_{t-k}} = \sqrt{1 - \frac{k}{t}}$$

at 1, it becomes non-stationary instantaneously, unlike all previous real nos

i. $\rho_{t,t-k}$ depends both on k and t.

ii. For a given t, $\rho_{t,t-k}$ is a monotonously decreasing function of k.

iii. $\rho_{t,t-k} < 1$, but if k is small compared to t, $\rho_{t,t-k} \approx 1$.

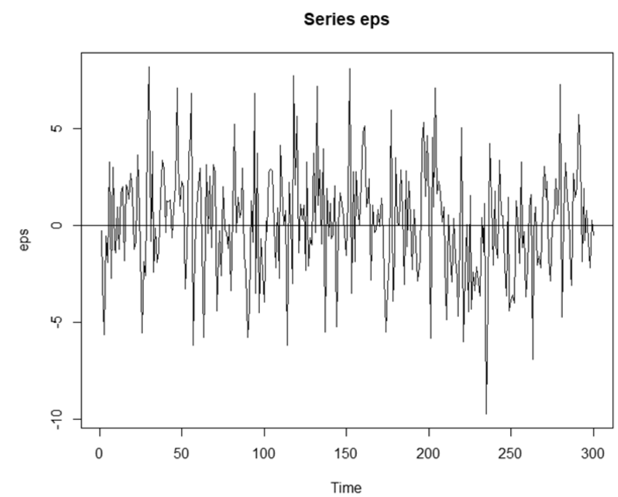
In practice *ACF* is estimated for the last sample period (for $t = T$) and, given that T is reasonably large, *SACF* will display a very slow decay.

Consequently, it is practically impossible to distinguish the *SACF* of a random walk process from the *SACF* of a stationary *AR*(1) process with $|\varphi_1| < 1$ but $\varphi_1 \approx 1$ – called near-unit-root process.

Ex 1:

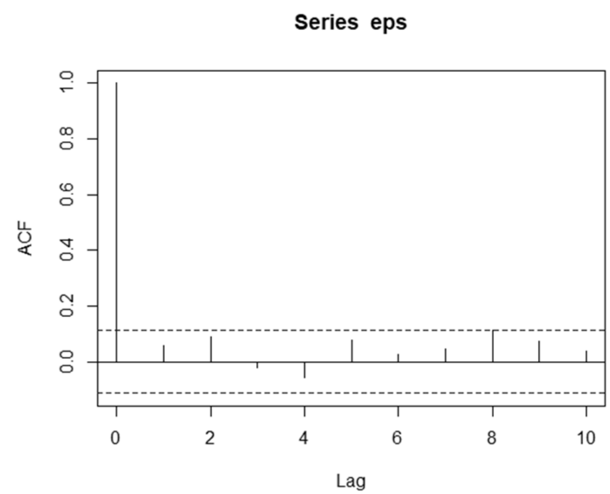
To illustrate the difference between stationary and nonstationary $AR(1)$ processes, draw 300 random numbers from ε_t: $N(0, 3)$ and simulate three $AR(1)$ processes with $\varphi_1 = 0.6$, $\varphi_1 = 0.95$ and $\varphi_1 = 1$, respectively, with $y_0 = 10$ initial value. Plot each series and the corresponding correlogram.

```
eps = ts(rnorm(300, mean = 0, sd = 3), start = 1)
ts.plot(eps, col = "blue")
abline(h = 0)
```



As expected, *eps* appears to randomly fluctuate around zero,

```
ka = floor(min(10, length(eps)/5))
acf(y1, lag.max = ka, plot = TRUE)
```

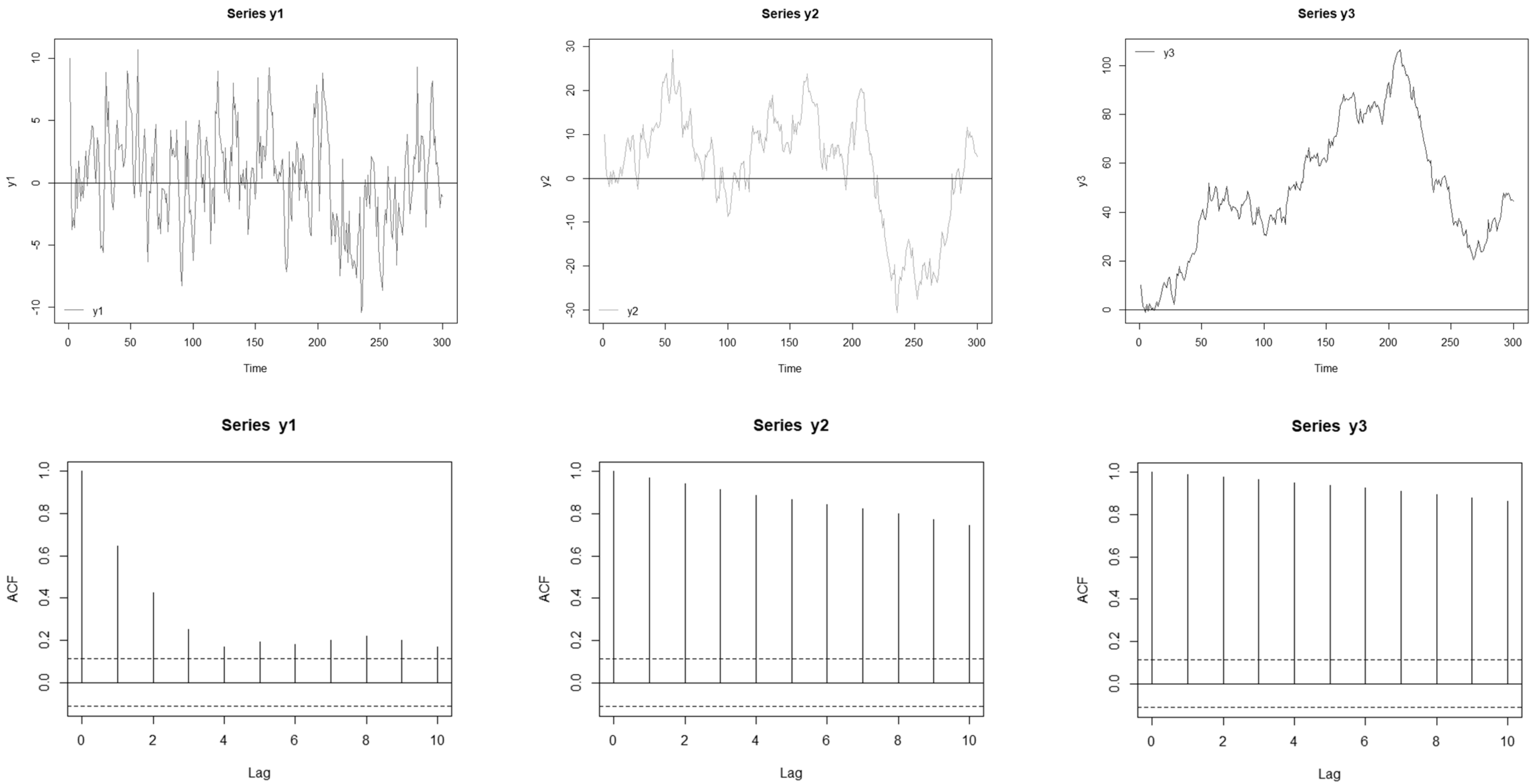


… and does not have any significant autocorrelation coefficient ($0 < k \leq 10$).

y1 = ts(0, start = 1, end = 300)
y1[1] = 10
for (t in 2:300)
*{y1[t] = ts(0.6*y1[t-1] + eps[t])} …*

y2 = ts(0, start = 1, end = 300)
y2[1] = 10
for (t in 2:300)
*{y2[t] = ts(0.95*y2[t-1] + eps[t])} …*

y3 = ts(0, start = 1, end = 300)
y3[1] = 10
for (t in 2:300)
{y3[t] = ts(y3[t-1] + eps[t])} …



The time series plots and the correlograms alike illustrate that a near-unit-root series looks more like a random walk, rather than a stationary *AR* series.

- From the general solution of the pure *AR*(1) process (see slide #3), the general solution of a pure random walk is

i.e. error/var increases over time

$$y_t = y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i} = y_0 + \sum_{i=1}^{t} \varepsilon_i \longrightarrow \boxed{E(y_t) = y_0} \;,\; Var(y_t) = \sigma^2 t$$

$\longrightarrow$ The solution of a pure random walk process is the sum of two terms, the initial value and the cumulative sum of a relatively long sequence of shocks or innovations (ε).

Although the (unconditional) expected value is constant, each shock or innovation imparts a random but permanent change in the conditional expected value of the series, $E(y_t \mid \varepsilon_{t-1}, \varepsilon_{t-2},..., \varepsilon_0)$.

Since these changes are driven by stochastic shocks, the resulting seemingly systematic but still unpredictable long-term movement in $\{y_t\}$ is called stochastic trend.

The main feature of a stochastic trend is that both its level and slope changes randomly.

- A pure random walk can be augmented with deterministic terms, like a constant or a time trend. They alter the expected value, but not the variance. Consequently, every random walk is nonstationary.

Their first differences, however, are stationary; or at least the deviations of their first differences from the linear trend component are stationary, i.e., $\Delta y_t = y_t - y_{t-1}$ is trend-stationary.

in every period there's a determ. trend i.e. constant

Pure random walk: $y_t = y_{t-1} + \varepsilon_t \longrightarrow \Delta y_t = \varepsilon_t$

Random walk with drift: $y_t = a_0 + y_{t-1} + \varepsilon_t \longrightarrow \Delta y_t = a_0 + \varepsilon_t$

Random walk with drift and linear trend:

$$y_t = a_0 + y_{t-1} + a_2 t + \varepsilon_t \longrightarrow \Delta y_t = a_0 + a_2 t + \varepsilon_t$$

Random walks are difference-stationary (*DS*) processes.

Since they need to be differenced once to achieve stationarity, they are said to be integrated of order one, *I*(1). → 1 = number of times differencing needed for stationarity

In general, a process (or series) that is non-stationary due to *d* number of (real) unit roots, becomes stationary after differencing *d* times, and it is called integrated of order *d*, *I*(*d*).

this number tells you the number of unit roots (real)

- So far, we have focused on $AR(1)$ processes, but the concept of integration can be generalized to arbitrary $ARMA(p,q)$ processes.

  Namely, an invertible $ARMA(p,q)$ process in the levels of y_t, which is invertible but non-stationary because it has d number of real AR unit roots, is equivalent to a stationary $ARMA(p-d,q)$ process in the dth differences, i.e., in $\Delta^d y_t$, and $y_t : I(\mathrm{d})$.

$\longrightarrow$ Autoregressive-Integrated-Moving Average model,

$$ARIMA(p,d,q)$$

Order of the AR component

Order of the MA component

Number of times the series has to be differenced before it becomes stationary (if ever)

Since stationary series do not need to be differenced at all, they are said to be integrated of zero order, $I(0)$.

not all stochastic processes are ARIMA processes.

SPURIOUS REGRESSION

- The standard estimation techniques and hypothesis test procedures typically used in time-series regression analysis are based, among others, on the assumption that the random error variable is stationary.

  This requirement is certainly satisfied when every variable in the model (dependent and independent alike) is stationary.

  However, the regression results can be completely misleading when this requirement is violated because some of the variables are nonstationary.

Spurious regression: When a random walk is regressed onto another independent random walk, the OLS sample regression equation might look good (high R^2, significant F- and t-statistics), but it is in fact dubious without any real meaning because there is no direct relationship between the variables.

∴ You need to do all the relevant tests!

To see why, suppose that two independent random walks without drift and trend are regressed on each other, and that $y_0 = x_0 = 0$.

$$\rightarrow \quad \boxed{y_t = \sum_{i=1}^{t} \varepsilon_{yi}} \qquad \boxed{x_t = \sum_{i=1}^{t} \varepsilon_{xi}} \qquad \text{and} \qquad \boxed{y_t = \alpha + \beta x_t + \xi_t}$$

$$\rightarrow \quad \boxed{\xi_t = y_t - \alpha - \beta x_t = \sum_{i=1}^{t} \varepsilon_{yi} - \beta \sum_{i=1}^{t} \varepsilon_{xi} - \alpha = \sum_{i=1}^{t} \underbrace{(\varepsilon_{yi} - \beta \varepsilon_{xi})} - \alpha}$$

This is a white noise, like ε_y, $\varMathvarepsilon_x$.

Because it's a cumulative sum (ε)

→ In general, ξ_t is also a random walk, and the stochastic trends in $\{y_t\}$, $\{x_t\}$ are likely to cause OLS to find a significant correlation between these series, unless these stochastic trends 'neutralize' each other.

tacit assumption of OLS

A random walk error term, however, is inconsistent with the assumptions underlying OLS, and this problem will not disappear no matter how large the sample is.

In fact, the problem becomes even worse as the sample size increases because the larger the sample, the more likely that in the regression of y_t on x_t the $\beta = 0$ hypothesis is falsely rejected.

TESTING FOR A UNIT ROOT / STATIONARITY

→ In practice, you should always use several test

- The non-stationary characteristic of a time series is often visible on the time-series plot or on the sample correlogram. Still, formal hypothesis testing might be required to confirm the first impression.

  To this end, numerous tests are available, but we discuss only the Dickey-Fuller (*DF*) τ tests (*tau*) for a unit root in the lectures, and the Kwiatkowski-Phillips-Schmidt-Shin (*KPSS*) test for stationarity in the tutorial next week.

- Consider three *AR*(1) models without and with constant and time trend.

These are equiv.

Model 1: $y_t = \varphi_1 y_{t-1} + \varepsilon_t$ $\longrightarrow$ $\Delta y_t = (\varphi_1 - 1) y_{t-1} + \varepsilon_t = \gamma y_{t-1} + \varepsilon_t$

Model 2: $y_t = a_0 + \varphi_1 y_{t-1} + \varepsilon_t$ $\longrightarrow$ $\Delta y_t = a_0 + \gamma y_{t-1} + \varepsilon_t$

Model 3: $y_t = a_0 + \varphi_1 y_{t-1} + a_2 t + \varepsilon_t$ $\longrightarrow$ $\Delta y_t = a_0 + \gamma y_{t-1} + a_2 t + \varepsilon_t$

In each case ε_t is a white noise error, $\gamma = \varphi_1 - 1$, and the test for a unit root can be based either on φ_1 or on γ.

→ This is just 1 deterministic trend: linear.

Accordingly, there are two possible but equivalent pairs of hypotheses:

$$H_0 : \varphi_1 = 1 \quad vs \quad H_A : |\varphi_1| < 1 \qquad \text{and} \qquad H_0 : \gamma = 0 \quad vs \quad H_A : (-2 <)\, \gamma < 0$$

Note:

a) Under H_0 there is a unit root and y_t is $I(1)$, while under H_A there is no unit root and y_t is $I(0)$.

b) Since $I(1)$ processes are non-stationary, unit-root testing is sometimes referred to as testing for non-stationarity, though $I(1)$ processes, or $I(d)$ processes in general ($d > 0$), are not the only non-stationary processes.

c) These null and alternative hypotheses are not exhaustive, they ignore the possibilities of $\varphi_1 > 1$ ($\gamma > 0$) and $\varphi_1 \leq -1$ ($\gamma \leq -2$).

   In most applications this fact does not really narrow down the applicability of unit root tests because $|\varphi_1| > 1$ means that the process is explosive, while $\varphi_1 = -1$ results in a 'random walk' oscillation, clearly unrealistic scenarios in most applications in economics and finance.

   Still, there are a few exceptions, like modelling asset price bubbles (the last topic of this lecture), where explosive data pattern is a reasonable scenario.

- Dickey-Fuller τ-test: Estimate the appropriate model with OLS, calculate the conventional *t*-ratio and perform a left-tail test. The τ test statistic, however, does not follow a *t* distribution.

Depending on the deterministic term(s) in the *DF* regression, there are three τ test statistics: τ for Model 1, τ_μ for Model 2 and τ_τ for Model 3.

Each of these *DF* τ test statistics has the same form,

which rand. walk model to use

$$\tau = \frac{\hat{\varphi}_1 - 1}{s_{\hat{\varphi}_1}} = \frac{\hat{\gamma}}{s_{\hat{\varphi}_1}}$$

but the sampling distributions are different, and they also depend on the sample size.

In practice the true data generating process is unknown. How can we decide which *AR*(1) model to use in the *DF* test?

As a simple rule, use Model 1 only if the data series fluctuates around zero, use Model 2 if the data series has a nonzero mean, and Model 3 if it has some trend.

Otherwise, the *DF* τ and τ_μ tests on a (trend-) stationary series might fail to reject the null hypothesis, i.e., they might attribute a non-zero sample mean or a trend to a non-existent random walk.

- The sampling distributions, and thus the critical values, of the $DF\tau$ test statistics assume that the ε_t error variable in the test regression is not autocorrelated.

  This assumption is violated when the data generating process is a higher order *AR*(*p*) process or has a higher order *AR* component.

  As a remedy, in this case, the *DF* test regression must be augmented with lagged first-differences of y_t.

  $\longleftarrow$ Augmented Dickey-Fuller (*ADF*) τ -test:

  For *AR*(*p*), *p* > 2 processes the 'basic' *ADF* test regression is

$$\Delta y_t = \gamma y_{t-1} + \sum_{i=2}^{p} \beta_i \Delta y_{t-i+1} + \varepsilon_t$$

  … while for invertible *ARMA*(*p,q*) processes it is

$$\Delta y_t = \gamma y_{t-1} + \sum_{i=2}^{k} \beta_i \Delta y_{t-i+1} + \varepsilon_t \quad , \quad k > p$$

  p and *q* are unknown in practice, but *k* must increase with the sample size (*T*), e.g., at a controlled rate of $T^{1/3}$ or $T^{1/4}$.

We can add a constant or a constant and a trend to this basic model, estimate it with OLS, and test for a unit root using the appropriate *DF* τ-test.

Ex 2:

In part (f) of Ex 2 of week 3 we used the *auto.arima()* function to find the 'best' fitting *ARIMA* model for *PM*. It turned out to be *ARMA*(2,1), i.e., *ARIMA*(2,0,1). This function relies on unit root testing (by default, on the *KPSS* test) to find the order of integration and concluded that d = 0. Nevertheless,

a) For the sake of illustration, let's perform the *ADF* test on *PM*.

Since *PM* is trending, the τ_τ test (Model 3) is the appropriate version.

There are several *R* functions for the *ADF* test. Let's use this time *adf.test()* from the *tseries* package, which always uses a constant and a linear trend in the *ADF* test regression.

↳ i.e. alway model 3!

library(tseries)
adf.test(PM)

```
        Augmented Dickey-Fuller Test

data:  PM
Dickey-Fuller = -4.071, Lag order = 7, p-value = 0.01
alternative hypothesis: stationary

Warning message:
In adf.test(PM) : p-value smaller than printed p-value
```

⟶ The *ADF* τ_τ test rejects the unit root null hypothesis at the 1% significance level, confirming that *PM* is likely (trend) stationary.

- Most non-seasonal economic time series can be modeled as $I(0)$ and $I(1)$ processes. Some variables, like prices, wages, money balances etc., however, might be integrated of order 2 or even higher (very rare), and hence must be differenced more than once to achieve stationarity.

→ ADF test assumes $I(1) \to$ can't detect stationarity for $d = c(2,3)$

Yet, this possibility is often ignored in practice, or even if it is considered, it is dealt with the repeated application of some test for a single unit root, first on the original level series and, *if a unit root is detected*, then on the differenced series, etc.

Although this idea seems logical and appealing, it is actually invalid.

resolution

← Standard unit root tests assume that the DGP has at most one unit root, so the first (few) test(s) in this sequential procedure might be misleading when there are more unit roots.

In addition, simulation studies demonstrated that following this procedure it is more likely to conclude that the DGP is $I(0)$ when it is actually $I(2)$ than when it is actually $I(1)$. check for inconsistency

Consequently, it is recommended to perform both stages irrespectively of the outcome of the first stage. (Alternatively, one can implement the Dickey-Pantula test for multiple unit roots - but we do not discuss it in this course).

Sol. for ADF: always perform test on level & first difference

(Ex 2)

b) Granted that the data generating process behind *PM* is indeed an *ARMA*(2,1) process, *PM* might have at most two *AR* unit roots. Hence, repeat the *ADF* test on the first difference of *PM*.

Testing for a unit root in the level of *PM* in part (a) we had a trend in the test regression (Model 3). Differencing, however, eliminates this trend.

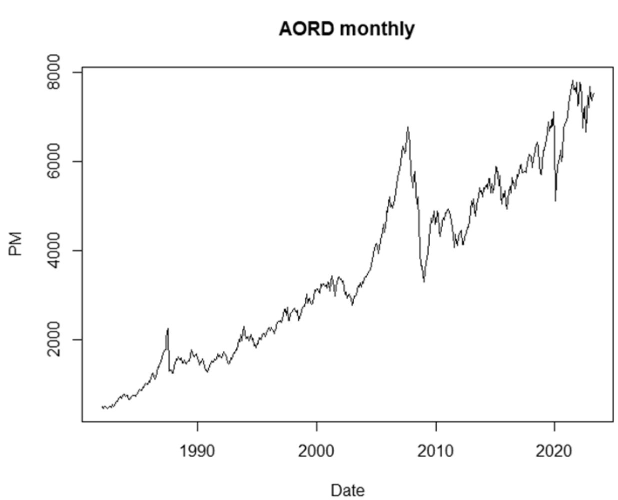
$$\longleftarrow \quad y_t = a_0 + \varphi_1 y_{t-1} + a_2 t + \varepsilon_t \qquad y_{t-1} = a_0 + \varphi_1 y_{t-2} + a_2 (t-1) + \varepsilon_{t-1}$$

$$y_t - y_{t-1} = a_2 + \varphi_1 (y_{t-1} - y_{t-2}) + \varepsilon_t - \varepsilon_{t-1}$$

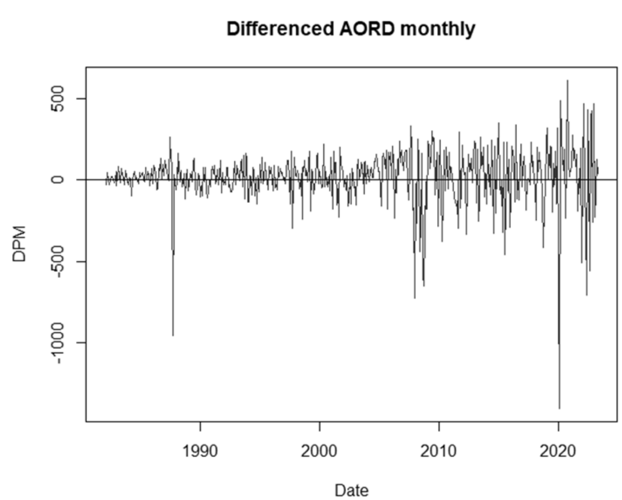
$$\Delta y_t = a_2 + \varphi_1 \Delta y_{t-1} + \Delta \varepsilon_t \quad \longleftarrow$$

Model 2

plot.ts(PM,…



plot.ts(diff(PM, 1),…



Similarly, a second round of differencing (needless this time) would eliminate the constant from Model 2, leading to Model 1.

In part (a) we used *adf.test()* from the *tseries* package. Since it always uses a constant and a linear trend in the *ADF* test regression, let's rely on a different function this time, *ur.df()* from the *urca* package.

library(urca)
summary(ur.df(PM, type = "trend", lags = 7))

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression trend
```

The *ur.df* printout is very detailed, only the top and bottom parts are shown here.

```
Value of test-statistic is: -4.071 6.5727 8.3485

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.98 -3.42 -3.13
phi2  6.15  4.71  4.05
phi3  8.34  6.30  5.36
```

τ_τ

The results of three tests are reported on this printout, but we consider only the first.

It is the same test as the one on slide #19.

note:
2 ADFs

summary(ur.df(lag(PM, -1), type = "drift", lags = 6)) Differentiation decreases the lag order by 1.

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################

Test regression drift

Value of test-statistic is: -7.8287 30.645

Critical values for test statistics:
      1pct  5pct 10pct
tau2 -3.44 -2.87 -2.57
phi1  6.47  4.61  3.79
```

τ_μ

This time two tests were run by *ur.df*, but again we consider only the first. It rejects the unit root null hypothesis.

Based on the two *ADF* tests on the level and on the differenced series, we can safely conclude that *PM* is (trend) stationary.

ASSET PRICE BUBBLES

- A financial bubble in an asset price occurs when the actual price of an asset moves in excess of its market fundamental price.
  It is characterized by the rapid increase of the asset price to a point that is unsustainable, causing the asset to burst or contract in value.

  For example, the Asian financial crisis (1997), the US dot-com bubble (1995–2000), or the Melbourne real estate bubble in 2021–2022.

Source: https://www.housing-fever.com/statistic/

- Consider the following simple model of the price of some risk-free asset, which states that the payoff from buying a risk-free asset in *t* and then selling it in $t+1$ is equal to the expected payoff from holding the asset from *t* to $t+1$:

$$P_t(1+R) = E_t[P_{t+1} + D_{t+1}]$$

where P_t: asset price in period *t*,
D_t: dividend payment in period *t*,
E_t: conditional expected value based on the set of all available information at time *t* (Ω_t),
R: constant risk-free interest rate.

$$P_t = \frac{E_t[P_{t+1} + D_{t+1}]}{1+R} = \beta E_t[P_{t+1} + D_{t+1}] \quad , \quad \beta = \frac{1}{1+R}$$

← Discount rate

$$= \beta E_t[\beta E_t[P_{t+2} + D_{t+2}] + D_{t+1}]$$

$$= \beta E_t(D_{t+1}) + \beta^2 E_t(D_{t+2}) + \beta^2 E_t(P_{t+2})$$

$$= \ldots = \underbrace{\sum_{i=1}^{k} \beta^i E_t(D_{t+i})}_{\text{Present value of the asset}} + \underbrace{\beta^k E_t(P_{t+k})}_{\text{“Bubble” } (B_t)}$$

→ i.e bubble is any part of price in excess of the PV

Present value of the asset + “Bubble” (B_t)

If there is no bubble, $B_t = 0$.

Otherwise, for $k = 1$

$$B_t = \beta E_t(P_{t+1})$$

… and if the future price is determined by a bubble (i.e., $P_{t+1} = B_{t+1}$),

$$B_t = \beta E_t(B_{t+1}) \longrightarrow E_t(B_{t+1}) = \frac{B_t}{\beta} = (1+R)B_t$$

Moving forward $h > 1$ periods,

$$E_t(B_{t+h}) = (1+R)^h B_t$$

This exponential function is explosive if $R > 0$.

$\longrightarrow$ One can test for bubbles by performing a right-tail *ADF* (*ADFrt*) test for a unit root with

$$H_0 : \gamma = 0 \ \ (\text{i.e. } \varphi_1 = 1) \quad vs \quad H_A : \ \gamma > 0 \ \ (\text{i.e. } \varphi_1 > 1)$$

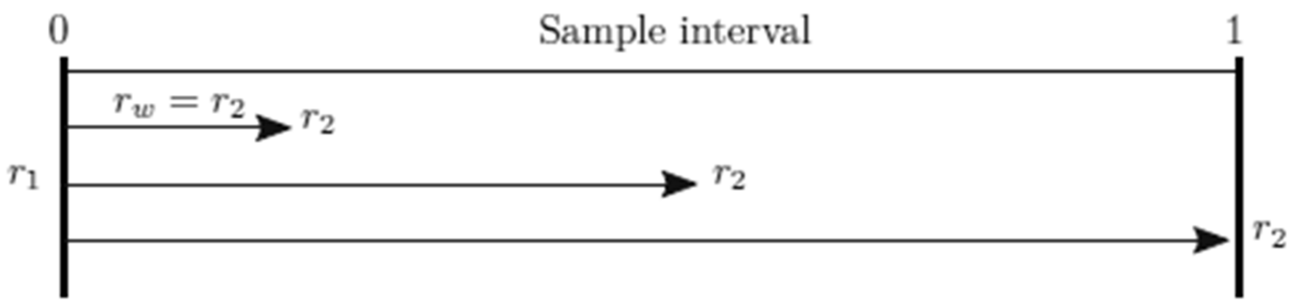
Unit *AR* characteristic root i.e., no price bubble *vs* ‘explosive’ *AR* characteristic root, i.e., price bubble

The *ADFrt* test statistic is the same as the usual *ADF* test statistic, but it has different sampling distributions.

- When the bubble collapses during the sample period, the *ADFrt* test may lack power (low probability of rejecting a false H_0).

  To increase power, Phillips, Wu and Yu (2011) advocated the supremum *ADF* (*SADF*) test for H_A: single collapsing bubble.

  It is based on the recursive calculations of the *ADF* test statistics with an expanding sample window (interval) starting at t = 0 and with an ending point that gradually increases by one from $t = 0.1T$ to $t = T$:

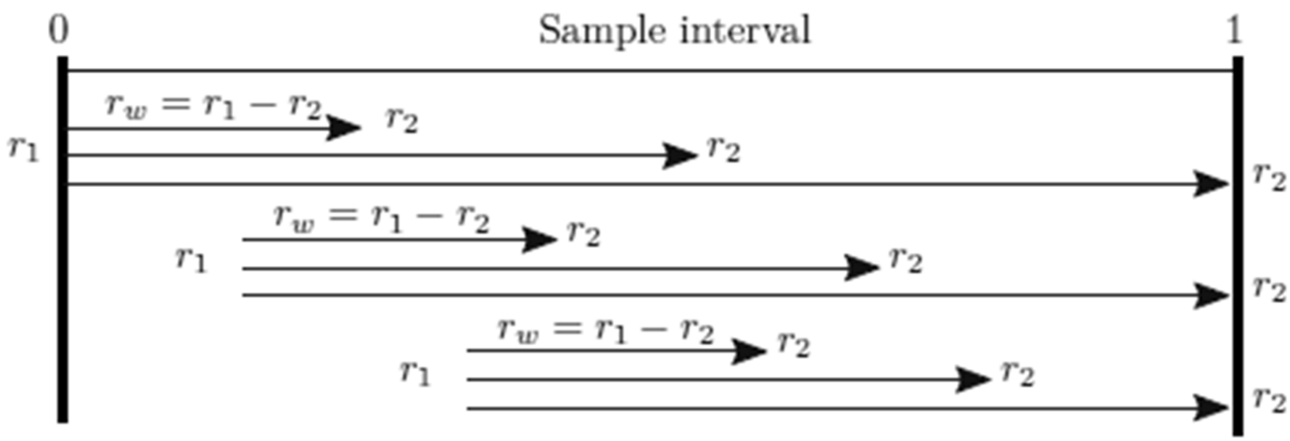


r_1, r_2 and r_w denote the starting point, the ending point and the width of the sample window over the sample period [1,T] normalized to [0, 1].

  The *ADF* test statistic is calculated from each of these sub-samples and the *SADF* test statistic is the largest (supremum) of them.

Phillips, Shi and Yu (2015) suggested a further extension of the *ADFrt* test, the generalized supremum *ADF* (*GSADF*) test, for H_A: multiple periodically collapsing bubbles.

In this case the starting point of the sample window is increased gradually from t = 0 by one at a time, and the *SADF* test is performed with each of these starting points:



Again, the *ADF* test statistic is calculated from each sub-sample and the *GSADF* test statistic is the largest (supremum) of them.

The sampling distributions of the *SADF* and *GSADF* test statistics are not standard, the critical values are obtained by simulation.

- If any of these tests rejects the null hypothesis of no bubble in favour of the alternative hypothesis of a single or multiple bubbles, one can use date-stamping to obtain the starting and ending points of the bubble(s).

  The estimated starting point of a bubble is the first time where the test statistic exceeds the corresponding critical value, while its ending point is the first time after the starting point where the test statistic drops below the corresponding critical value.

Ex 3:

Consider the non-fundamental component of the log price-to-rent ratio in Melbourne (*NFCM*).

a) Perform the *ADF* and *ADFrt* tests on *NFCM* with *adf.test()* by setting its *alternative* argument to *stationary* and *explosive*, respectively.

library(tseries)
adf.test(NFCM, alternative = "stationary")

```
	Augmented Dickey-Fuller Test

data:  NFCM
Dickey-Fuller = -1.8918, Lag order = 4, p-value = 0.6215
alternative hypothesis: stationary
```

adf.test(NFCM, alternative = “explosive")

```
	Augmented Dickey-Fuller Test

data:  NFCM
Dickey-Fuller = -1.8918, Lag order = 4, p-value = 0.3785
alternative hypothesis: explosive
```

Both tests maintain the unit root null hypothesis.

b) Perform the *SADF* and *GSADF* tests on *NFCM*.

We rely on the *radf()* function of the *exuber* package. It performs the *ADFrt*, *SADF* and *GSADF* tests.

library(exuber)
NFCM_radf = radf(NFCM, lag = 4)
summary(NFCM_radf)

```
── Summary (minw = 21, lag = 4) ───

series1 :
# A tibble: 3 × 5
  stat   tstat   `90`    `95`    `99`
  <fct>  <dbl>   <dbl>   <dbl>   <dbl>
1 adf    -1.32  -0.451  -0.0582  0.661
2 sadf    2.01   1.00    1.32    1.93
3 gsadf   2.07   1.78    2.05    2.71
```

The shortest sample window used in the *SADF* and *GSADF* tests was 21 months.

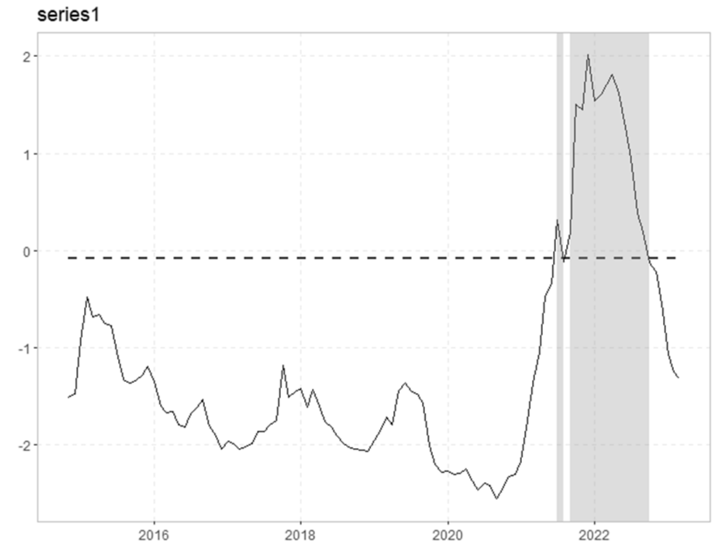
The critical values were simulated from 2000 replications (this information is on the printout, but it is not shown here).

The *ADFrt* test fails to reject the null hypothesis, but the *SADF* and *GSADF* tests reject it at the 1% and at the 5% significance level, respectively.

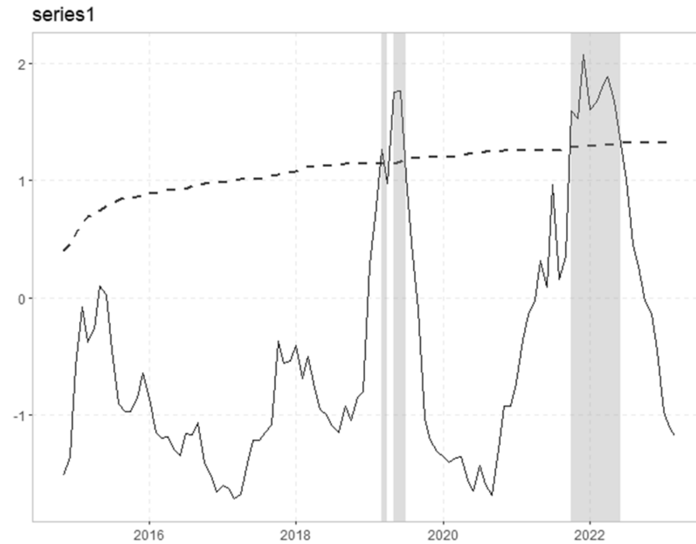
Note: This *ADFrt* test statistic is different from the one on the previous slide because *adf.test()* uses a constant and trend in the test regression, while *radf()* uses only a constant.

To see the details of the alleged bubble(s), we date-stamp them next.

autoplot(NFCM_radf, option = "sadf")



autoplot(NFCM_radf, option = “gsadf")



The blue lines illustrate the test statistics and the dashed red lines the 5% critical values.

The shaded areas represent the detected bubbles.

NFCM_dst_sadf = datestamp(NFCM_radf, option = "sadf")
NFCM_dst_sadf

```
── Datestamp (min_duration = 0) ────────────

series1 :
       Start       Peak        End Duration
1 2021-07-01 2021-07-01 2021-08-01        1
2 2021-09-01 2021-12-01 2022-10-01       13
```

NFCM_dst_gsadf = datestamp(NFCM_radf, option = “gsadf")
NFCM_dst_gsadf

```
── Datestamp (min_duration = 0) ────────────

series1 :
       Start       Peak        End Duration
1 2019-03-01 2019-03-01 2019-04-01        1
2 2019-05-01 2019-06-01 2019-07-01        2
3 2021-10-01 2021-12-01 2022-06-01        8
```

We ignore detected bubbles that have very short duration.

Both tests revealed a ‘real’ but slightly different bubble peaking in Dec 2021.

WHAT SHOULD YOU KNOW?

- Deterministic and stochastic trends
- Trend stationary and difference stationary series
- Spurious regression
- Dickey-Fuller (*DF*, *ADF*) unit root tests
- Asset price bubbles
- Supremum *ADF* (*SADF*) and generalized supremum *ADF* (*GSADF*) tests

BOARD OF FAME

David Alan Dickey (1945-):

American statistician

William Neal Reynolds Professor in the Department of Statistics at North Carolina State University

Time series econometrics, Dickey-Fuller test

Wayne Arthur Fuller (1931-):

American statistician

Professor at Iowa State University

Fellow of the American Statistical Association, the Econometric Society, the Institute of Mathematical Statistics

Time series econometrics, Dickey-Fuller test, survey sampling