Topic 5. The Capital Asset Pricing Model (CAPM)

ECON30024 Economics of Finanical Markets

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Outline

1. Assumptions of the CAPM
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4. CAPM in asset prices
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Required reading: Chap. 6 of Bailey

1. Assumptions of the CAPM

- A core objective of asset pricing theory is to explain the risk premium, $\mu_j - r_0$, for each asset. One of the most widely discussed explanations is provided by the CAPM.

- The CAPM extends the mean-variance analysis of portfolio selection for an individual investor to the market as a whole.

- It addresses the question: if all investors behave according to a mean-variance objective and if they all have the same beliefs, then what can be inferred about the pattern of expected returns on assets when asset markets are in equilibrium?

- The CAPM is based on a long list of assumptions, condensed into assumptions on markets and assumptions on investors.

- Assumptions on markets
  - Markets are in equilibrium: prices of assets adjust so that aggregate demand for each asset equals its supply.
  - Frictionless markets
    - Neutral transaction costs and taxes
    - No institutional restrictions on assets trade
    - Divisible assets
  - Unlimited risk-free borrowing and lending
  - Competitive asset markets (investors are price takers)

- Assumptions on investors
  - All investors behave according to a single-period investment horizon.
  - All investors select their portfolios according to a mean-variance objective.
  - All investors have **homogeneous beliefs**, i.e., use the same estimates of the expectations, variances and covariances of asset returns
    (μ_j, σ_j, ρ_{ij} values are the same for all investors)
- Why these assumptions are needed in the derivation of CAPM? (Tutorial 5 discussion)

2. Derivation of the CAPM

- To derive the CAPM prediction, we first characterise an individual investor's portfolio selection problem, then impose market clearing.
  - Recall the basic analytical framework in Topic 1.

2.1 A necessary condition for investors' efficient portfolios

- Recall how the PF with n risky assets and a risk-free asset for an individual investor is defined (Topic 4 slide 34):

$$(*) \qquad \min_{(a_0, a_1, a_2, \ldots, a_n)} \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}$$

$$\text{s.t.} \quad \mu_P = a_0 r_0 + \sum_{j=1}^{n} a_j \mu_j, \qquad a_0 + \sum_{j=1}^{n} a_j = 1$$

An efficient portfolio is a solution to problem $(*)$.

– Note that $a_0 = 1 - \sum_{j=1}^{n} a_j$ so that

$$\mu_P = \left(1 - \sum_{j=1}^{n} a_j\right) r_0 + \sum_{j=1}^{n} a_j \mu_j = r_0 + \sum_{j=1}^{n} (\mu_j - r_0) a_j \qquad (1)$$

– So problem $(*)$ can be rewritten as

$$\min_{(a_1, a_2, \ldots, a_n)} \sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij}, \quad \text{s.t. } (1)$$

- Form the Lagrangian for this minimisation problem:

$$\mathcal{L} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma_{ij} + \gamma \left(\mu_P - r_0 - \sum_{j=1}^{n} (\mu_j - r_0) a_j \right),$$

where γ is the Lagrange multiplier on the constraint.

– The first-order conditions: partially differentiate $\mathcal{L}$ with respect to a_1, a_2, …, a_n, γ and set the expressions to zero.

$$\frac{\partial L}{\partial a_j} = 2\sum_{i=1}^{n} a_i \sigma_{ij} - \gamma(\mu_j - r_0) = 0,\ j = 1, 2, \ldots, n \quad (2)$$

$$\frac{\partial L}{\partial \gamma} = \mu_P - r_0 - \sum_{j=1}^{n} (\mu_j - r_0) a_j = 0 \quad \text{(this is (1))}$$

– Eqn. (2) implies that

$$\sum_{i=1}^{n} a_i \sigma_{ij} = \frac{\gamma}{2}(\mu_j - r_0),\ j = 1, 2, \ldots, n$$

Note that $\sum_{i=1}^{n} a_i \sigma_{ij}$ is the co-variance of r_P and r_j:

$$\begin{aligned} cov(r_P, r_j) &= cov(a_0 r_0 + a_1 r_1 + \ldots + a_n r_n, r_j) \\ &= a_1 cov(r_1, r_j) + a_2 cov(r_2, r_j) + \ldots + a_n cov(r_n, r_j) \\ &= \sum_{i=1}^{n} a_i \sigma_{ij} \equiv \sigma_{jP} \end{aligned}$$

So we have

$$\sigma_{jP} \equiv \sum_{i=1}^{n} a_i \sigma_{ij} = \frac{\gamma}{2}(\mu_j - r_0), \; j = 1, 2, \ldots, n \tag{3}$$

– Now, multiply (3) by a_j on both sides:

$$\sum_{i=1}^{n} a_i a_j \sigma_{ij} = \frac{\gamma}{2}(\mu_j - r_0) a_j$$

Sum over j:

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_i a_j \sigma_{ij} = \frac{\gamma}{2} \sum_{j=1}^{n} (\mu_j - r_0) a_j,$$

which is equivalent to, by the definition of σ_P^2 and (1),

$$\sigma_P^2 = \frac{\gamma}{2}(\mu_P - r_0) \tag{4}$$

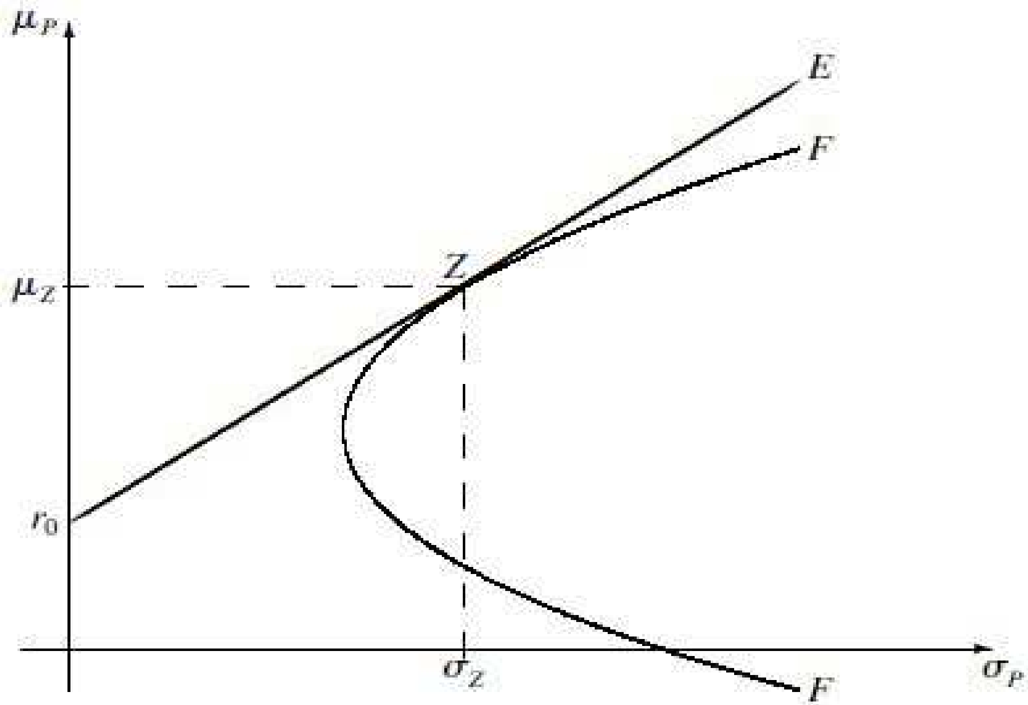
– Combining (3) and (4) yields

$$\frac{\mu_j - r_0}{\sigma_{jP}} = \frac{\mu_P - r_0}{\sigma_P^2}, \quad \text{for } j = 1, 2, \ldots, n. \tag{5}$$

- Any **efficient portfolio** P must satisfy (5).
- See the Appendix for the economic intuition behind (5).
  - For P to be an efficient portfolio, the marginal increase in the portfolio's expected return per unit of risk resulting from a small change in a_j must be the same for all assets.
  - Eqn. (5) can also be interpreted as a **no arbitrage** condition: the risk and return tradeoffs must be equalised across assets.

2.2 Market clearing and capital market line

- Recall the EF with n risky assets and a risk-free asset (Figure 10 in Topic 4). The tangent portfolio Z is an **efficient portfolio consisting of only n risky assets**.



- Hence, the risky asset portfolio Z must satisfy (5):

$$\frac{\mu_j - r_0}{\sigma_{jZ}} = \frac{\mu_Z - r_0}{\sigma_Z^2}, \quad \text{for } j = 1, 2, \ldots, n.$$

  Define $\beta_{jZ} = \frac{\sigma_{jZ}}{\sigma_Z^2}$, the equation above can be rewritten as

$$\mu_j - r_0 = \beta_{jZ}(\mu_Z - r_0), \quad \text{for } j = 1, 2, \ldots, n \tag{6}$$

- Next, we impose market clearing and show that Z is the **market portfolio**.

- The assumption of homogeneous beliefs implies that all investors have the same EF.
  - All investors' **optimum portfolios are portfolios of the risk-free asset and** Z.
  - Because investors can differ by their mean-variance objectives, their optimum portfolios can be different.

- Let $i = 1, 2, \ldots, m$ index the investors in the market, each with initial wealth A^i and holding an optimum portfolio given by

$$(1 - a^i_Z,\, a^i_Z),$$

  where a^i_Z is the proportion of A^i invested in portfolio Z.

- Let $Z \equiv (z_1, z_2, \ldots, z_n)$, where z_j is the proportion of asset j's value in the total value of portfolio Z, $\sum_{j=1}^n z_j = 1$

- What is investor i's demand for risky asset j, $j = 1, 2, \ldots, n$?
  - Investor i invests $A^i a^i_Z$ in portfolio Z.
  - Hence, investor i invests $A^i a^i_Z z_j$ in asset j.
  - Her holding of asset j, i.e., her demand for asset j is

$$x^i_j = \frac{A^i a^i_Z z_j}{p_j},$$

- What is the total demand for risky asset j by all investors?

$$\sum_{i=1}^{m} x_j^i = \sum_{i=1}^{m} \left[\frac{A^i a_Z^i z_j}{p_j} \right] = \left(\sum_{i=1}^{m} A^i a_Z^i \right) \frac{z_j}{p_j} \equiv B \frac{z_j}{p_j},$$

where $B \equiv \sum_{i=1}^{m} A^i a_Z^i$ is the total investment in risky assets portfolio Z, i.e., total market value of risky assets.

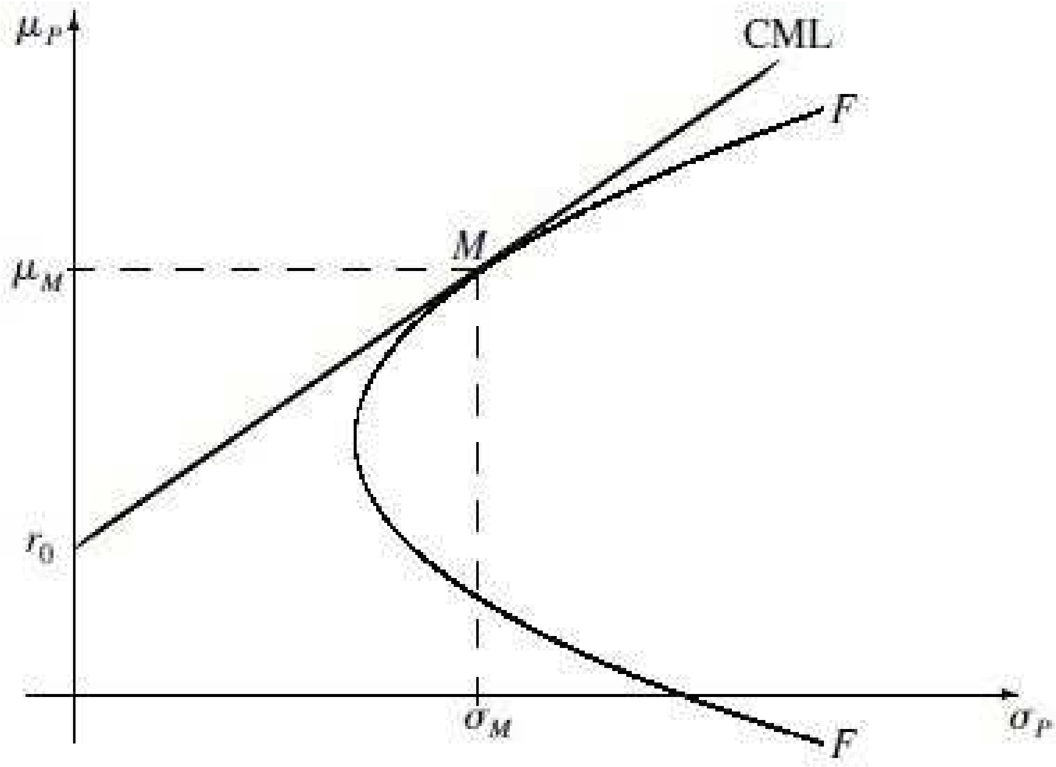
- Suppose the exogenous fixed supply of risky asset j is X_j, then **market clearing** for asset j implies:

$$X_j = \sum_{i=1}^{m} x_j^i = B \frac{z_j}{p_j}$$

$$\Rightarrow \quad z_j = \frac{p_j X_j}{B}, \qquad \text{for } j = 1, 2, \ldots, n, \tag{7}$$

- Note that $\frac{p_j X_j}{B}$ is the market value of asset j relative to the total market value of all risky assets, i.e., the **market share of asset** j, denoted as m_j
  - In equilibrium, the tangent portfolio Z is the **market portfolio**, $M \equiv (m_1, m_2, \ldots, m_n)$.
  - The market portfolio is a portfolio of risky assets, with the proportion of each risky asset given by its market share in the risky assets market.
- Replacing Z with M in the previous figure, we obtain the EF for all investors in equilibrium, which is called the **capital market line** (CML), as shown in Figure 1.

Figure 1. The capital market line



– The CML is a straight line represented by

$$\mu_P = r_0 + \left(\frac{\mu_M - r_0}{\sigma_M} \right) \sigma_P. \tag{8}$$

– In equilibrium, all investors have the same EF, which is the CML.

– All investors' optimum portfolios are located on the CML, and hence are portfolios of the risk-free asset and the market portfolio.

– However, the optimum portfolios of investors are not necessarily the same; different investors can choose to hold different efficient portfolios on the CML.

2.3 The CAPM and its implications

- In **equilibrium**, equation (6) becomes

$$\mu_j - r_0 = \beta_j(\mu_M - r_0), \quad \text{for } j = 1, 2, \ldots, n$$

where

$$\beta_j \equiv \beta_{jM} = \frac{\sigma_{jM}}{\sigma_M^2}.$$

Equivalently,

$$\mu_j = r_0 + \beta_j(\mu_M - r_0), \quad \text{for } j = 1, 2, \ldots, n. \tag{9}$$

This is the **basic form of the CAPM**!

- Implications of the CAPM
  - The expected return on an asset equals the risk-free rate plus a **risk premium** which depends on the asset's **beta** coefficient.

– The risk premium on asset j, commonly defined as the expected excess return $\mu_j - r_0$, is given by:

$$\mu_j - r_0 = \beta_j(\mu_M - r_0),$$

where $\mu_M - r_0$ is the **market risk premium**.

– Recall that $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$, so β_j measures the comovement of asset j's return with the market return.

– β_j measures the **systematic risk** of asset j, arising from fluctuations in the market return which is a **systematic** and undiversifiable risk inherent to the market as a whole.

– The CAPM implies that asset j's risk premium is determined by β_j rather than σ_j (the std. of r_j).

2.4 The security market line (SML)

- The CAPM equation (9) implies a linear relationship between an asset's expected return and its beta:

$$\mu_j = r_0 + (\mu_M - r_0)\beta_j, \quad \text{for all } j$$

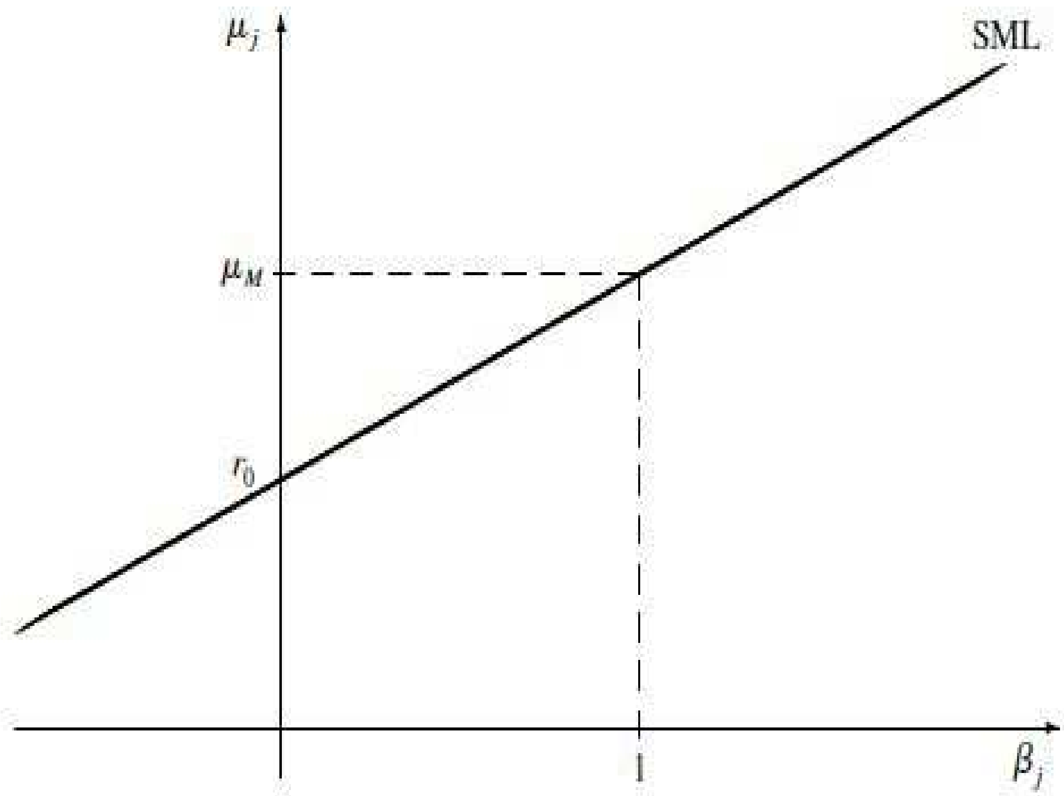
  Plotting μ_j against β_j in the (β_j, μ_j) space, we get the SML.

- The intercept of SML equals the risk-free rate, r_0, and the slope equals $\mu_M - r_0$.

- The CAPM predicts that the expected returns and betas of all assets and portfolios of assets lie on the SML.

- Difference between CML and SML (Tutorial 5 Discussion)

Figure 2. The security market line



Interim Summary

- The CAPM assumes a world in which (a) asset markets are in equilibrium; (b) investors choose portfolios to maximise mean-variance objectives according to common values of the means and variances of asset returns.
- The CAPM predicts that in equilibrium all investors have the same EF, which is the CML, and they hold efficient portfolios of the risk-free asset and the market portfolio.
- The CAPM implies that expected asset returns can be predicted from their betas. The linear relationship forms the security market line.
- The CAPM implies that each asset's beta is a more appropriate measure of its risk than its standard deviation of return.

3. Properties of beta

- Recall the CAPM as summarised in (9):

$$\mu_j = r_0 + \beta_j(\mu_M - r_0),$$

where $\beta_j \equiv \frac{\sigma_{jM}}{\sigma_M^2}$ measures the systematic risk of asset j.

- Rewrite β, using $\rho_{jM} \equiv \frac{\sigma_{jM}}{\sigma_j \sigma_M}$:

$$\beta_j = \frac{\rho_{jM}\, \sigma_j \sigma_M}{\sigma_M^2} = \rho_{jM} \frac{\sigma_j}{\sigma_M}.$$

  - The market portfolio M has a beta equal to 1, $\beta_M = 1$
  - What is the beta of an efficient portfolio E on the CML?

$$\beta_E = \frac{\sigma_E}{\sigma_M}, \quad \text{because } \rho_{EM} = 1$$

– An asset with $\beta_j = 0$ is called a **zero-beta** asset:

$$\beta_j = 0 \quad \Rightarrow \mu_j = r_0 + \beta_j(\mu_M - r_0) = r_0.$$

Where is this asset on the SML? Is it risk free?

- We can classify all assets in terms of their betas:
  - $\beta_j = 1$: its risk premium equals the market risk premium.
  - $\beta_j > 1$: "aggressive asset" with $\sigma_j > \sigma_M$
  - $\beta_j < 1$: "defensive asset"
    - Assets with negative betas offer a form of insurance against volatility in the market return, and hence their expected reutrns are lower than the risk-free rate.

- Beta is "linearly additive".
  - Suppose asset 1 and asset 2 have betas of β_1 and β_2.
  - Consider a portfolio P with a proportion a in asset 1 and $1-a$ in asset 2, then

$$\begin{aligned}
\beta_P &= \frac{\sigma_{PM}}{\sigma_M^2} = \frac{cov(r_P, r_M)}{\sigma_M^2} = \frac{cov(ar_1 + (1-a)r_2, r_M)}{\sigma_M^2} \\
&= \frac{a\, cov(r_1, r_M) + (1-a)cov(r_2, r_M)}{\sigma_M^2} \\
&= \frac{a\, \sigma_{1M} + (1-a)\sigma_{2M}}{\sigma_M^2} \\
&= a\left(\frac{\sigma_{1M}}{\sigma_M^2}\right) + (1-a)\left(\frac{\sigma_{2M}}{\sigma_M^2}\right) \\
&= a\beta_1 + (1-a)\beta_2
\end{aligned}$$

4. The CAPM in asset prices

- The CAPM is a model of asset prices, though it is typically expressed in terms of expected rates of return rather than prices.

- To see this, recall the definition of the rate of return on asset j:

$$r_j = \frac{v_j - p_j}{p_j} = \frac{v_j}{p_j} - 1.$$

Taking expectations on both sides:

$$\mu_j = \frac{E(v_j)}{p_j} - 1.$$

- The CAPM prediction for μ_j is given by (9):

$$\mu_j = r_0 + \beta_j(\mu_M - r_0)$$

- Combining the two equations above gives

$$p_j = \frac{E(v_j)}{1+\mu_j} = \frac{E(v_j)}{1+r_0+\beta_j(\mu_M - r_0)}, \qquad j = 1, 2, \ldots, n. \quad (10)$$

  This is the CAPM prediction for asset prices.

  – In the absence of uncertainty, no arbitrage implies that $\mu_M = r_0$, so price is simply given by the present value of the payoff: $p_j = \frac{v_j}{1+r_0}$.

  – With uncertainty, v_j is replaced by $E(v_j)$; and the **discount factor** includes the risk premium $\beta_j(\mu_M - r_0)$ to capture the risk aversion of investors.

- Back to the SML, suppose an asset locates above the SML. Is this asset over-priced or under-priced, according to the CAPM?

5. Extensions of the CAPM

5.1 The Black CAPM

- Thus far it has been assumed that investors can borrow or lend without restriction.
- Now remove this opportunity and assume that all assets are risky. All other assumptions remain. This defines what is called the 'Black CAPM' (Black, 1972).
- CML in the Black CAPM
  - As investors have homogeneous beliefs, they all have the same portfolio frontier (a hyperbola).
  - The market portfolio M is efficient, lying on the upward sloping arm of the portfolio frontier, as shown in Figure 3.

CML in Black CAPM

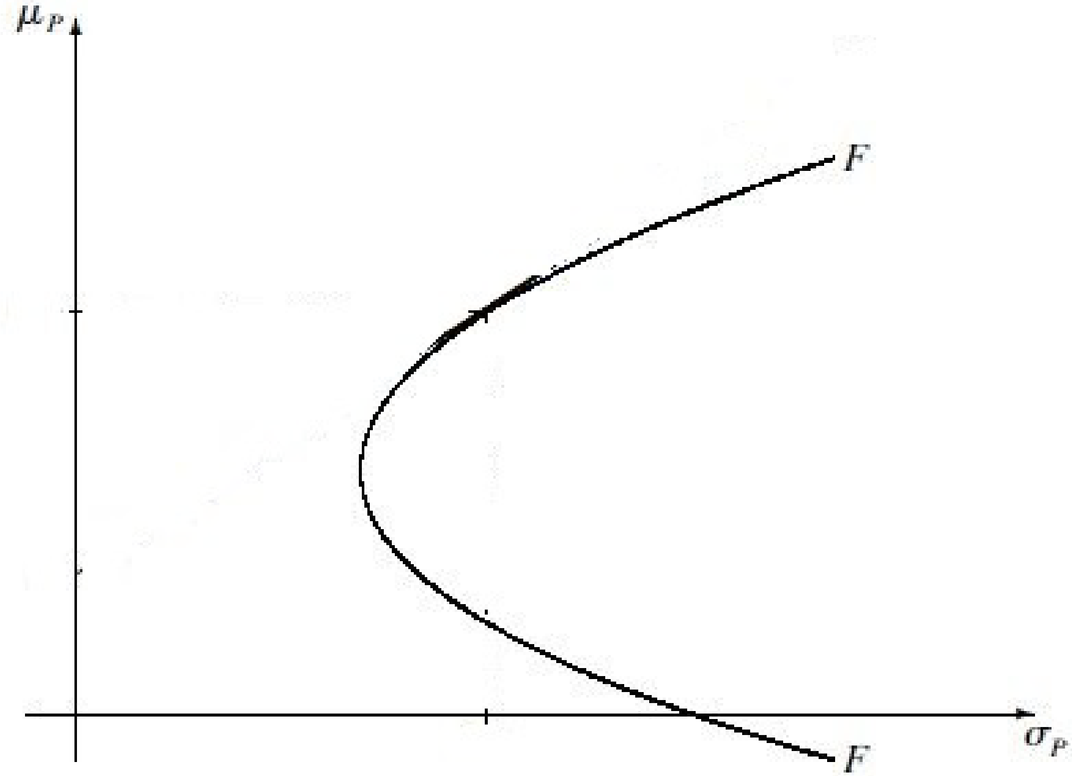
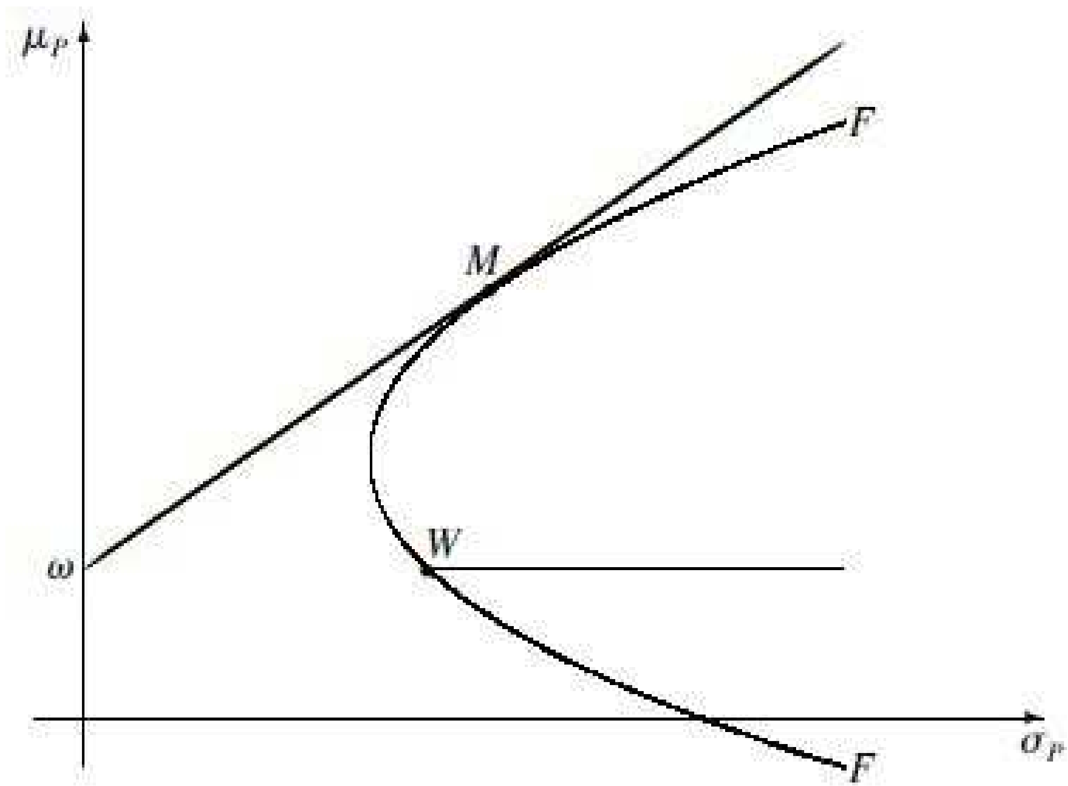


Figure 3. Zero-beta portfolio



  - Now draw a line tangent to M, which meets the vertical axis at some point, ω.
  - The ray from ω passing through M is the capital market line in the Black CAPM model (is CML the EF of investors?)

- Zero-beta portfolios
  - All the portfolios or assets located on the horizontal line starting at W have an expected return equal to ω.
  - It can be shown that for any portfolio on this line, its rate of return has zero correlation with the market return.
  - All portfolios on this line are **'zero-beta portfolios'**.

- The Black CAPM:

$$\mu_j = \omega + \beta_j(\mu_M - \omega), \quad \text{for } j = 1, 2, \ldots, n. \tag{11}$$

  - Intuitively, ω plays the same role as the risk-free rate r_0 in the standard CAPM.
  - A security market line can be constructed from (11) in the same way as when a risk-free asset is available.

- Applications of the model are much the same as when a risk-free asset is available, except that now ω must be estimated or calculated separately.
  - ω can be estimated by constructing a zero-beta portfolio.
  - We delay the testing and application of CAPM to Topic 7.

5.2 Other extensions

- Following the introduction of the CAPM, a range of extensions were proposed and they continue to be.
- They all boil down to relaxing one or more of the assumptions listed at the outset.
- Some of the modifications are routine - for example, allowing for different tax rates on dividend income and capital gains.
- Others, such as allowing for heterogeneous beliefs, involve much more complexity.
- One important extension involves generalising the single-period mean-variance framework to allow for **intertemporal decision** making with expected utility. We'll briefly discuss it in Topic 9.

Review questions

1. What are the assumptions of the CAPM?
2. Understand the procedure to derive the CAPM. What does the necessary condition capture and what does market clearing do?
3. Understand how we showed portfolio Z is the market portfolio M in equilibrium. Why all investors hold the same portfolio of risky assets?
4. Be able to draw the capital market line, and understand why it is the same for all investors, why all efficient portfolios of investors locate on the CML, and why the optimal portfolios of investors can be different points on the CML.
5. Why all efficient portfolios have the same Sharpe ratio as the market portfolio? Why all efficient portfolios are perfectly correlated?
6. Write down the CAPM in its basic form. Understand each notation in the equation.
7. Understand the definition of the security market line and understand why all assets or portfolios should lie on the SML if CAPM is correct. Be able to draw the SML.

8. Understand the definition and meaning of beta in the CAPM. What kind of risk it measures? Be able to calculate an asset's beta, using given information on mean, variance and covariance.
9. According to the CAPM, how is the risk premium of an asset determined?
10. How to use the linearly additive property of beta to calculate the beta of a portfolio, given betas of individual assets in the portfolio?
11. Be able to derive the CAPM prediction for asset prices and understand how the expected payoff is discounted to yield today's price under the CAPM.
12. If you observe an asset located above or below the SML, whether this asset is over-priced or under-priced by the CAPM?
13. Represent a zero-beta portfolio on the (β_j, μ_j) space and on the (σ_P, μ_P) space. In the standard CAPM model, is the risk-free asset the only zero-beta asset?
14. Understand how the Black CAPM is constructed, in particular, the interpretation of ω.
15. Write down the Black CAPM equation.

Appendix: the derivation of eqn. (2)

- When we differentiate $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}$ with respect to a particular a_j, we only need to consider items that contain a_j.
- Take a_1 as an example. Items in $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}$ that contain a_1:

$$
\begin{aligned}
& a_1 a_1 \sigma_{11} + a_2 a_1 \sigma_{21} + a_3 a_1 \sigma_{31} \cdots + a_n a_1 \sigma_{n1} \\
& \qquad a_1 a_2 \sigma_{12} + a_1 a_3 \sigma_{13} \cdots + a_1 a_n \sigma_{1n} \\
&= a_1^2 \sigma_{11} + 2[a_2 a_1 \sigma_{21} + a_3 a_1 \sigma_{31} \cdots + a_n a_1 \sigma_{n1}]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \frac{\partial\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}\right]}{\partial a_1} &= 2[a_1\sigma_{11}+a_2\sigma_{21}+a_3\sigma_{31}\cdots+a_n\sigma_{n1}] \\
&= 2\sum_{i=1}^{n} a_i \sigma_{i1}
\end{aligned}
$$

- Similarly, $\frac{\partial\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}\right]}{\partial a_j} = 2\sum_{i=1}^{n} a_i \sigma_{ij}$, as in eqn. (2).

Appendix: Economic intuition behind (5)

- Suppose an investor holds a feasible portfolio P. Denote its expected return and risk by μ_P and σ_P, respectively.
- Now suppose that a small increase is made in the holding of asset j and the necessary funds needed is borrowed at the risk-free interest rate. The holding of other risky assets remains unchanged.
  - Denote the increase in the proportion of j in the portfolio by Δa_j (and $\Delta a_0 = -\Delta a_j$ by construction).
- The change in the portfolio's expected return:

$$\mu_P = a_0 r_0 + \sum_{i=1}^{n} a_i \mu_i$$

$$\Rightarrow \ (\Delta \mu_P)^j = \Delta a_0 r_0 + \Delta a_j \mu_j = \Delta a_j (\mu_j - r_0)$$

- The change in the portfolio's risk:

$$\sigma_P = \left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_{ij}\right]^{\frac{1}{2}}$$

$$\begin{aligned}\Rightarrow \frac{\partial \sigma_P}{\partial a_j} &= \frac{1}{2}\left[\sum_{i=1}^n \sum_{j=1}^n a_i a_j \sigma_{ij}\right]^{-\frac{1}{2}} \left(2\sum_{i=1}^n a_i \sigma_{ij}\right) \\ &= \left[\sigma_P^2\right]^{-\frac{1}{2}} \sigma_{jP} = \frac{\sigma_{jP}}{\sigma_P}\end{aligned}$$

  So

$$(\Delta\sigma_P)^j \approx \left(\frac{\partial \sigma_P}{\partial a_j}\right) \Delta a_j = \frac{\sigma_{jP}}{\sigma_P} \Delta a_j$$

- For P to be an efficient portfolio, it must be the case that a small change in the portfolio proportion of any asset must disturb the expected return per unit of risk by the same amount for all assets.

- That is, $\frac{(\Delta\mu_P)^j}{(\Delta\sigma_P)^j} = \frac{\mu_j - r_0}{\frac{\sigma_{jP}}{\sigma_P}}$ must be equal for all assets, i.e., $\frac{\mu_j - r_0}{\sigma_{jP}}$ must be equal for all assets $j = 1, 2, \ldots, n$:

$$\frac{\mu_1 - r_0}{\sigma_{1P}} = \frac{\mu_2 - r_0}{\sigma_{2P}} = \ldots = \frac{\mu_n - r_0}{\sigma_{nP}}$$

- The portfolio P can itself be interpreted as a single composite asset, so

$$\frac{\mu_P - r_0}{\sigma_P^2} = \frac{\mu_j - r_0}{\sigma_{jP}}$$

for all assets $j = 1, 2, \ldots, n$. This is equation (5).

- When we differentiate $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}$ with respect to a particular a_j, we only need to consider items that contain a_j.
- Take a_1 as an example. Items in $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}$ that contain a_1:

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&\qquad a_1 a_2 \sigma_{12} + a_1 a_3 \sigma_{13} \cdots + a_1 a_n \sigma_{1n} \\
&= a_1^2 \sigma_{11} + 2[a_2 a_1 \sigma_{21} + a_3 a_1 \sigma_{31} \cdots + a_n a_1 \sigma_{n1}]
\end{aligned}
$$

$$
\Rightarrow \quad \frac{\partial\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}\right]}{\partial a_1} = 2[a_1\sigma_{11}+a_2\sigma_{21}+a_3\sigma_{31}\cdots+a_n\sigma_{n1}]
$$

$$
= 2\sum_{i=1}^{n} a_i \sigma_{i1}
$$

- Similarly, $\frac{\partial\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_{ij}\right]}{\partial a_j} = 2\sum_{i=1}^{n} a_i \sigma_{ij}$, as in eqn. (2).