ECOM40006/ECOM90013 Econometrics 3
Department of Economics
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Assignment 3 Solutions

Semester 1, 2025

Let $Y_1, Y_2, \ldots, Y_n$ denote a simple random sample from a population with probability density function

$$f(y) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1, \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. Show that the sample mean $\overline{Y}$ is a consistent estimator of $\theta/(\theta+1)$. [7 marks]

   Hint: First derive the mean of the population and then remember that laws of large numbers are your friends.

   Solution:
   As an aside, this distribution is a Beta with parameters $\alpha = \theta$ and $\beta = 1$, where if $X \sim B(\alpha, \beta)$ then

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I_{(0,1)}(x), \quad \alpha > 0, \beta > 0.$$

   Now,

$$\mathrm{E}[Y] = \int_0^1 \theta y^{\theta-1} \times y \,\mathrm{d}y = \theta \int_0^1 y^{\theta} \,\mathrm{d}y = \left.\frac{\theta y^{\theta+1}}{\theta+1}\right|_0^1 = \frac{\theta}{\theta+1}.$$

$$\mathrm{E}\left[Y^2\right] = \int_0^1 \theta y^{\theta-1} \times y^2 \,\mathrm{d}y = \theta \int_0^1 y^{\theta+1} \,\mathrm{d}y = \left.\frac{\theta y^{\theta+2}}{\theta+2}\right|_0^1 = \frac{\theta}{\theta+2}.$$

   Therefore,

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = \frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2 = \frac{\theta}{(\theta+2)(\theta+1)^2}.$$

   For the sample mean of a simple random sample we know that

$$\mathrm{E}\left[\overline{Y}\right] = \frac{1}{n}\sum_{j=1}^{n} \mathrm{E}[Y] = \frac{\theta}{\theta+1}$$

   and

$$\mathrm{Var}\left[\overline{Y}\right] = \mathrm{E}\left[\frac{1}{n}\sum_{j=1}^{n} \mathrm{E}[Y]\right] = \frac{1}{n^2}\sum_{j=1}^{n} \mathrm{Var}[Y] = \frac{\theta}{n(\theta+2)(\theta+1)^2}.$$

We can establish mean square convergence, as

$$\lim_{n\to\infty} \frac{\theta}{\theta+1} = \frac{\theta}{\theta+1}$$

and

$$\lim_{n\to\infty} \frac{\theta}{n(\theta+2)(\theta+1)^2} = 0.$$

Hence, consistency is established as

$$\overline{Y} \xrightarrow{m.s.} \frac{\theta}{\theta+1} \Rightarrow \overline{Y} \xrightarrow{p} \frac{\theta}{\theta+1}.$$

2. Derive a consistent method of moments estimator, $\tilde{\theta}$ say, for θ. [1 mark]

   Solution:
   Let $\mathrm{E}[Y] \equiv \mu = \theta/(\theta+1)$. Then $\mu(\theta+1) = \theta \implies \theta = \mu/(1-\mu)$. As $\overline{Y}$ is consistent for μ, a consistent method of moments estimator for θ is $\tilde{\theta} = \overline{Y}/(1-\overline{Y})$.

3. Specify the log-likelihood function for this sample. [1 mark]

   Solution:
   By iid'ness

$$\mathcal{L}(\theta \mid y_1,\ldots,y_n) = \prod_{i=1}^{n} \theta y_i^{\theta-1} = \theta^n \prod_{i=1}^{n} y_i^{\theta-1}$$

$$\implies \ln \mathcal{L}(\theta \mid y_1,\ldots,y_n) = \sum_{i=1}^{n} [\ln\theta + (\theta-1)\ln y_i] = n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln y_i.$$

4. Derive the maximum likelihood estimator, $\hat{\theta}$ say, for θ and prove that it is, indeed a *maximum* likelihood estimator. [3 marks]

   Solution:
   The score is

$$\mathcal{S}(\theta) = \frac{\mathrm{d}\mathcal{L}(\theta)}{\mathrm{d}\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln y_i.$$

   Then the first-order condition is

$$0 = \mathcal{S}(\hat{\theta}) = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n}\ln y_i \implies \frac{n}{\hat{\theta}} = -\sum_{i=1}^{n}\ln y_i \implies \hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\ln y_i}.$$

   Observe that, as $0 < y_i < 1$, $\ln y_i < 0$ for all $i = 1,\ldots,n$. Therefore, $-\sum_{i=1}^{n}\ln y_i > 0$, for all samples $y_1,\ldots,y_n$, and so all possible $\hat{\theta} > 0$. In particular, $\hat{\theta}$ cannot equal zero.

   To check the second-order condition, observe that

$$H(\hat{\theta}) = \left.\frac{\mathrm{d}^2\mathcal{L}(\theta)}{\mathrm{d}\theta^2}\right|_{\theta=\hat{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{n}{\theta} + \sum_{i=1}^{n}\ln y_i\right)_{\theta=\hat{\theta}} = -\frac{n}{\hat{\theta}^2} < 0,$$

   as $\hat{\theta}^2 > 0$ for all $n > 0$, because $\hat{\theta}$ cannot equal zero.

5. Derive the Fisher information for the sample. [2 marks]

   Solution:
   By definition, the Fisher information is defined to be $\mathcal{I}(\theta) = -\operatorname{E}[H(\theta)]$, where $H(\theta)$ denotes the Hessian which, as this is a scalar problem, is just the second derivative. From the previous question we see that

$$H(\theta) = -\frac{n}{\theta^2} \implies \mathcal{I}(\theta) = -\operatorname{E}\left[-\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}.$$

6. Suppose that someone wishes to test the null hypothesis null hypothesis $H_0 : \theta = 1$ against the alternative that $H_1 : \theta \neq 1$. State the true population density function and describe in words the implication for the population when this null hypothesis is true. [2 marks]

   Solution:
   If H_0 is true then the population distribution reduces to

$$f(y) = \begin{cases} \theta y^{\theta-1} = 1, & 0 < y < 1, \theta = 1, \\ 0, & \text{otherwise.} \end{cases}$$

   That is, the population is uniformly distributed over the unit interval $0 < y < 1$.

7. Derive likelihood ratio, Lagrange multiplier and Wald tests for the hypotheses of Question 6. In each case provide the decision rule that you would use in practice to apply the test, including any critical value(s) you may need. [12 marks]

   Solution:
   The likelihood ratio test is

$$\lambda = -2(\ln \mathcal{L}(1) - \ln \mathcal{L}(\hat{\theta})) = 2(\ln \mathcal{L}(\hat{\theta}) - \ln \mathcal{L}(1)) \underset{a}{\overset{H_0}{\sim}} \chi^2_1.$$

   At the 5% level the relevant critical value is approximately 3.84 and the decision ru is that any value of $\lambda > 3.84$ falls in the rejection region. From our answer to Question 3 we have an expression for the log-likelihood, whence we see that

$$\begin{aligned} \lambda &= 2\left\{\left[n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln y_i\right]_{\theta=\hat{\theta}} - \left[n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln y_i\right]_{\theta=1}\right\} \\ &= 2n\left[\ln\hat{\theta} + (\hat{\theta}-1)\frac{1}{n}\sum_{i=1}^{n}\ln y_i\right] \\ &= \frac{2n}{\hat{\theta}}\left[\hat{\theta}\ln\hat{\theta} + 1 - \hat{\theta}\right]. \end{aligned}$$

   The Wald test statistic is given by

$$W = \mathcal{I}(\hat{\theta})(\hat{\theta}-\theta_0)^2 \underset{a}{\overset{H_0}{\sim}} \chi^2_1.$$

   Again, at the 5% level, the relevant critical value is approximately 3.84 and the decision rule is that any value of $W > 3.84$ falls in the rejection region. From our answer to Question 5 we have an expression for the Fisher information and so

$$W = \frac{n}{\hat{\theta}^2}(\hat{\theta}-1)^2 = n\left(1 - \frac{1}{\hat{\theta}}\right)^2.$$

Finally, the Lagrange multiplier, or efficient score, test statistic is given by

$$LM = [\mathcal{I}(\theta_0)]^{-1}[\mathcal{S}(\theta_0)]^2 \overset{H_0}{\underset{a}{\sim}} \chi^2_1.$$

We see that once again, at the 5% level, the relevant critical value is approximately 3.84 and the decision rule is that any value of $LM > 3.84$ falls in the rejection region. From our answers to Question 4 and Question 5 we have expressions for both the score and the Fisher information which we are then able to evaluate at $\theta_0 = 1$. Thus,

$$\begin{aligned} LM &= \left(\frac{n}{\theta_0^2}\right)^{-1}\left[\frac{n}{\theta_0} + \sum_{i=1}^{n} \ln y_i\right]^2 = \frac{\theta_0^2}{n} \times n^2 \left[\frac{1}{\theta_0} + \frac{1}{n}\sum_{i=1}^{n} \ln y_i\right]^2 \\ &= n\left[1 + \frac{1}{n}\sum_{i=1}^{n} \ln y_i\right]^2 = n\left(1 - \frac{1}{\hat\theta}\right)^2 \end{aligned}$$

In each case $\hat\theta$ is as defined in Question 4. Note that all three statistics are close to zero when $\hat\theta$ is close to unity, which it should be, for sufficiently large samples, when H_0 is true.

8. Without appeal to the generic properties of maximum likelihood estimators, prove that $\hat\theta$ is consistent for θ. [6 marks]

Solution:
We need to show that

$$\operatorname{plim} \hat\theta = -\frac{1}{\operatorname{plim} \frac{1}{n}\sum_{i=1}^{n} \ln y_i} = \theta \implies \operatorname{plim} \frac{1}{n}\sum_{i=1}^{n} \ln y_i = -\frac{1}{\theta}.$$

Consider

$$\mathcal{J}_1 = \mathrm{E}\,[\ln y] = \int_0^\infty (\ln y)\theta y^{\theta-1}\,\mathrm{d}y.$$

Make the change of variables $u = -\ln y \implies y = \exp\{-u\}$. Then $\mathrm{d}y/\,\mathrm{d}u = -\exp\{-u\} \implies \mathrm{d}y = -\exp\{-u\}\,\mathrm{d}u$. Moreover, as $y \to 1$, $u \to 0$ and as $y \to 0$, $u \to -\infty$. Combining these results yields

$$\begin{aligned} \mathcal{J}_1 &= \int_{-\infty}^{0} (-u)\theta(e^{-u})^{\theta-1}(-\exp\{-u\}\,\mathrm{d}u) \\ &= \theta\int_{-\infty}^{0} e^{-\theta u} u\,\mathrm{d}u \\ &= -\theta\int_{0}^{\infty} e^{-\theta u} u^{2-1}\,\mathrm{d}u \\ &= -\theta\Gamma(2)\theta^{-2} = -\frac{1}{\theta}, \end{aligned}$$

as required. Next observe that

$$\mathcal{J}_2 = \mathrm{E}\left[(\ln y)^2\right] = \int_0^\infty (\ln y)^2\theta y^{\theta-1}\,\mathrm{d}y.$$

Making the same change of variables as before yields

$$\begin{aligned}
\mathcal{J}_2 &= \int_{-\infty}^{0} (-u)^2 \theta (e^{-u})^{\theta-1} (-\exp\{-u\}\,\mathrm{d}u) \\
&= -\theta \int_{-\infty}^{0} e^{-\theta u} u^2 \,\mathrm{d}u \\
&= \theta \int_{0}^{\infty} e^{-\theta u} u^{3-1} \,\mathrm{d}u \\
&= \theta \Gamma(3) \theta^{-3} = \frac{2}{\theta^2}.
\end{aligned}$$

Therefore,

$$\mathrm{Var}\,[\ln y] = \frac{2}{\theta^2} - \left(-\frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}.$$

In light of these results we observe that

$$\lim_{n\to\infty} \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} \ln y_i\right] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\,[\ln y_i] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} -\frac{1}{\theta} = -\frac{1}{\theta}.$$

Similarly,

$$\lim_{n\to\infty} \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} \ln y_i\right] = \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}\,[\ln y_i] = \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{\theta^2} = \lim_{n\to\infty} \frac{1}{n\theta^2} = 0.$$

Consequently, we have established, via mean square convergence, that $\operatorname{plim} \hat{\theta} = \theta$, as required.

Your answers to the Assignment should be submitted via the LMS no later than 4:30pm, Thursday 22 May. Your mark for this assignment may contribute up to 10% towards your final mark in the subject.

No late assignments will be accepted and an incomplete exercise is better than nothing.