Lecture 5
Stationarity

Stationarity

A time series Z_t is defined to be (weakly or covariance) stationary if:

1. $E(Z_t)$ is constant across all t.

2. $\text{var}(Z_t)$ is finite and constant across all t.

3. For each $j = 1, 2, \ldots,$
$\text{cov}(Z_t, Z_{t-j})$ is constant across all t.

Various behaviours of time series, models and forecasts can depend on this.

Are prediction errors stationary?

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$$U_t = Y_t - E(Y_t | \mathscr{Y}_{t-1})$$

Recall $E(U_t | \mathscr{Y}_{t-1}) = 0$

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t|\mathscr{Y}_{t-1})$$

Recall $E(U_t|\mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t.

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t|\mathscr{Y}_{t-1})$$

Recall $E(U_t|\mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

Are prediction errors stationary?

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Recall $E(U_t | \mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

2.

3. For each $j = 1, 2, \ldots,$
$\text{cov}(U_t, U_{t-j}) = 0$ across all t.

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t|\mathscr{Y}_{t-1})$$

Recall $E(U_t|\mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

2.

3. For each $j = 1, 2, \ldots,$
$\operatorname{cov}(U_t, U_{t-j}) = 0$ across all t. ✓

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t | \mathscr{Y}_{t-1})$$

Recall $E(U_t | \mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

2. $\mathrm{var}(U_t) = ?$

3. For each $j = 1, 2, \ldots,$
$\mathrm{cov}(U_t, U_{t-j}) = 0$ across all t. ✓

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t | \mathscr{Y}_{t-1})$$

Recall $E(U_t | \mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

2. U_t might be heteroskedastic.

3. For each $j = 1, 2, \ldots,$
$\text{cov}(U_t, U_{t-j}) = 0$ across all t. ✓

Are prediction errors stationary?

$$U_t = Y_t - E(Y_t | \mathscr{Y}_{t-1})$$

Recall $E(U_t | \mathscr{Y}_{t-1}) = 0$, and this implies

1. $E(U_t) = 0$ across all t. ✓

2. Can assume $\text{var}(U_t) = \sigma^2$ across all t.

3. For each $j = 1, 2, \ldots,$
$\text{cov}(U_t, U_{t-j}) = 0$ across all t. ✓

MA(1) Model

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $$\begin{aligned} E(Y_t) &= E(U_t) + \theta_1 \, E(U_{t-1}) \\ &= \quad 0 \quad + \theta_1 \quad 0 \end{aligned}$$

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \ 0$ ✓

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \ 0$ ✓
2. $\mathrm{var}(Y_t) = \mathrm{var}(U_t) + 2\theta_1 \mathrm{cov}(U_t, U_{t-1}) + \theta_1^2 \ \mathrm{var}(U_{t-}$

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \text{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = 0$ ✓
2. $\text{var}(Y_t) = \text{var}(U_t) + 2\theta_1 \text{cov}(U_t, U_{t-1}) + \theta_1^2 \text{var}(U_t$

$$= \sigma^2 + 2\theta_1 \quad 0 \quad + \theta_1^2 \quad \sigma^2$$

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \ 0$ ✓
2. $\mathrm{var}(Y_t) = \sigma^2(1 + \theta_1^2)$ for every t ✓

MA(1) Model

$$Y_t = U_t + \theta_1 U_{t-1}, \quad \mathrm{var}(U_t) = \sigma^2$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = 0$ ✓
2. $\mathrm{var}(Y_t) = \sigma^2(1 + \theta_1^2)$ for every t ✓
3. $\mathrm{cov}(Y_t, Y_{t-1}) = \sigma^2\theta_1$ ✓

   $\mathrm{cov}(Y_t, Y_{t-j}) = 0 \quad (j = 2, 3, \ldots)$ ✓

Linear Trend Model

Linear Trend Model

$$Y_t = \beta_0 + \beta_1 \mathrm{Time}_t + U_t$$

where U_t is a *stationary* prediction error.

Linear Trend Model

$$Y_t = \beta_0 + \beta_1 \text{Time}_t + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \beta_0 + \beta_1 \text{Time}_t + E(U_t)$

$$(\text{Time}_t \text{ is deterministic})$$

Linear Trend Model

$$Y_t = \beta_0 + \beta_1 \text{Time}_t + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \beta_0 + \beta_1 \text{Time}_t + \ 0$

Linear Trend Model

$$Y_t = \beta_0 + \beta_1 \mathrm{Time}_t + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \beta_0 + \beta_1 \mathrm{Time}_t$

Mean depends on time $\Rightarrow$ **not** stationary.

AR(1) Model

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

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$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1}) + E(U_t)$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1}) + \ 0$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1})$

Is it possible for $E(Y_t) = E(Y_{t-1})$?
(i.e. mean not changing from one time to the next?)

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1})$

$E(Y_t) = E(Y_{t-1})$ requires $E(Y_t) = 0$ for every t.

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1})$

Formally, by induction:

- if we *assume* $E(Y_1) = 0$
- then $E(Y_t) = 0$ for every t.

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = \phi_1 E(Y_{t-1})$

If we assume $E(Y_1) = \mu_1 \neq 0$ then

$$E(Y_2) = \phi_1 \mu_1$$
$$E(Y_3) = \phi_1^2 \mu_1$$

i.e. $E(Y_t)$ changes with t

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

1. $E(Y_t) = 0$ for every t

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $$\begin{aligned} \mathrm{var}(Y_t) = {} & \phi_1^2 \, \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ & + 2\, \phi_1 \, \mathrm{cov}(Y_{t-1}, U_t) \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $$\mathrm{var}(Y_t) = \phi_1^2\,\mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ +2\,\phi_1\,\mathrm{cov}(Y_{t-1}, U_t)$$

$$\mathrm{cov}(Y_{t-1}, U_t) = E[\,Y_{t-1}\,U_t\,] \quad (\text{since } E(U_t) = 0)$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $$\mathrm{var}(Y_t) = \phi_1^2\, \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ +2\,\phi_1\, \mathrm{cov}(Y_{t-1}, U_t)$$

$$\begin{aligned} \mathrm{cov}(Y_{t-1}, U_t) &= E[\, Y_{t-1}\, U_t\,] \quad (\text{since } E(U_t) = 0) \\ &= E[\, E(Y_{t-1}\, U_t\, |\mathscr{Y}_{t-1})] \quad (\text{LIE}) \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $$\mathrm{var}(Y_t) = \phi_1^2 \,\mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ +2\,\phi_1\, \mathrm{cov}(Y_{t-1}, U_t)$$

$$\begin{aligned}
\mathrm{cov}(Y_{t-1}, U_t) &= E[\,Y_{t-1}\,U_t\,] \quad (\text{since } E(U_t) = 0) \\
&= E[\,E(Y_{t-1}\,U_t\,|\mathscr{Y}_{t-1})] \quad (\text{LIE}) \\
&= E[\,Y_{t-1}\,E(U_t\,|\mathscr{Y}_{t-1})] \quad (Y_{t-1} \in \mathscr{Y}_{t-1})
\end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $$\begin{aligned} \mathrm{var}(Y_t) = {} & \phi_1^2 \, \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ & + 2\, \phi_1 \, \mathrm{cov}(Y_{t-1}, U_t) \end{aligned}$$

$$\begin{aligned} \mathrm{cov}(Y_{t-1}, U_t) &= E[\, Y_{t-1} \, U_t \,] \quad (\text{since } E(U_t) = 0) \\ &= E[\, E(Y_{t-1} \, U_t \, | \mathscr{Y}_{t-1})] \quad (\text{LIE}) \\ &= E[\, Y_{t-1} \, E(U_t \, | \mathscr{Y}_{t-1})] \quad (Y_{t-1} \in \mathscr{Y}_{t-1}) \\ &= 0 \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\quad \mathbf{var}(Y_t) = \phi_1^2 \, \mathbf{var}(Y_{t-1}) + \mathbf{var}(U_t)$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\mathrm{var}(Y_t) = \phi_1^2 \, \mathrm{var}(Y_{t-1}) + \sigma^2$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\quad \mathrm{var}(Y_t) = \phi_1^2 \, \mathrm{var}(Y_{t-1}) + \sigma^2$

Is it possible for $\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1})$?

(i.e. var not changing from one time to the next?)

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\mathbf{var}(Y_t) = \phi_1^2\, \mathbf{var}(Y_{t-1}) + \sigma^2$

Is it possible for $\mathbf{var}(Y_t) = \mathbf{var}(Y_{t-1})$?

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\quad \mathbf{var}(Y_t) = \phi_1^2 \, \mathbf{var}(Y_t) \quad + \; \sigma^2$

$$\Rightarrow \quad \mathbf{var}(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

where U_t is a *stationary* prediction error.

2. $\mathbf{var}(Y_t) = \phi_1^2\, \mathbf{var}(Y_t) \quad + \sigma^2$

$$\Rightarrow \quad \mathbf{var}(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}$$

Only valid for $|\phi_1| < 1$.

AR(1) Model

By induction, if we assume $|\phi_1| < 1$ and

$$E(Y_1) = 0 \quad \text{and} \quad \text{var}(Y_1) = \frac{\sigma^2}{1 - \phi_1^2}$$

then for all $t = 1, 2, 3, \ldots$

$$E(Y_t) = 0 \quad \text{and} \quad \text{var}(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$\begin{aligned} 3. \quad & \text{Multiply by } Y_{t-1}, \text{ take cov :} \\ & \text{cov}(Y_t,\ Y_{t-1}) = \phi_1 \text{cov}(Y_{t-1},\ Y_{t-1}) + \text{cov}(U_t, Y_{t-1} \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$\begin{aligned}
&\text{3.} \quad \text{Multiply by } Y_{t-1}, \text{ take cov :} \\
&\quad \text{cov}(Y_t,\ Y_{t-1}) = \phi_1 \text{cov}(Y_{t-1},\ Y_{t-1}) + \text{cov}(U_t, Y_{t-1}) \\
&\quad \qquad\qquad\qquad\ \ = \phi_1 \quad \text{var}(Y_{t-1}) \qquad + \qquad 0
\end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$\begin{aligned}
&3. \quad \text{Multiply by } Y_{t-1}, \text{ take cov}: \\
&\quad \operatorname{cov}(Y_t,\, Y_{t-1}) = \phi_1 \operatorname{cov}(Y_{t-1},\, Y_{t-1}) + \operatorname{cov}(U_t, Y_{t-1}) \\
&\quad\quad = \phi_1 \quad \operatorname{var}(Y_{t-1}) \quad + \quad 0 \\
&\quad\quad = \phi_1 \quad \frac{\sigma^2}{1-\phi_1^2}
\end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

3. Multiply by Y_{t-1}, take cov :

$$\begin{aligned} \mathrm{cov}(Y_t,\ Y_{t-1}) &= \phi_1 \mathrm{cov}(Y_{t-1},\ Y_{t-1}) + \mathrm{cov}(U_t, Y_{t-1}) \\ &= \phi_1 \quad \mathrm{var}(Y_{t-1}) \qquad + \qquad 0 \\ &= \phi_1 \quad \frac{\sigma^2}{1-\phi_1^2} \quad \text{constant across all} \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$\begin{aligned} &3. \quad \text{Multiply by } Y_{t-2}, \text{ take cov :} \\ &\quad \text{cov}(Y_t,\ Y_{t-2}) = \phi_1 \text{cov}(Y_{t-1},\ Y_{t-2}) + \text{cov}(U_t, Y_{t-2}) \end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$
\begin{aligned}
&\text{3.} \quad \text{Multiply by } Y_{t-2}, \text{ take cov}: \\
&\quad \operatorname{cov}(Y_t,\ Y_{t-2}) = \phi_1 \operatorname{cov}(Y_{t-1},\ Y_{t-2}) + \operatorname{cov}(U_t, Y_{t-2}) \\
&\quad \phantom{\operatorname{cov}(Y_t,\ Y_{t-2})} = \phi_1 \quad \phi_1 \frac{\sigma^2}{1-\phi_1^2} \quad + \quad 0
\end{aligned}
$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

$$\begin{aligned}
3. \quad & \text{Multiply by } Y_{t-2},\ \text{take cov}: \\
& \text{cov}(Y_t,\ Y_{t-2}) = \phi_1 \text{cov}(Y_{t-1},\ Y_{t-2}) + \text{cov}(U_t, Y_{t-2}) \\
& \qquad = \phi_1^2 \ \frac{\sigma^2}{1-\phi_1^2}
\end{aligned}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

3. In general :

$$\text{cov}(Y_t,\ Y_{t-j}) = \phi_1^{j}\ \frac{\sigma^2}{1-\phi_1^2}$$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

So the AR(1) model *can be* stationary:

1. $E(Y_t) = 0$

2,3. $\text{cov}(Y_t, Y_{t-j}) = \phi_1^j \dfrac{\sigma^2}{1-\phi_1^2}, \quad j = 0, 1, 2, \ldots$

AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $|\phi_1| < 1$.

So the AR(1) model *can be* stationary:

1. $E(Y_t) = 0$ These are constant for all t.

2,3. $\mathrm{cov}(Y_t, Y_{t-j}) = \phi_1^j \frac{\sigma^2}{1-\phi_1^2}, \quad j = 0, 1, 2, \ldots$

Random Walk

AR(1) Model: Random Walk

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $\phi_1 = 1$.

As before, if we assume $E(Y_1) = 0$ then

$$E(Y_t) = 0 \quad \text{for all } t.$$

(Does not depend on ϕ_1.)

AR(1) Model: Random Walk

$$Y_t = \quad Y_{t-1} + U_t$$

U_t is a *stationary* prediction error.

$$\begin{aligned}\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ +2\,\mathrm{cov}(Y_{t-1}, U_t)\end{aligned}$$

AR(1) Model: Random Walk

$$Y_t = \quad Y_{t-1} + U_t$$

U_t is a *stationary* prediction error.

$$\begin{aligned} \mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) \\ +2\,\mathrm{cov}(Y_{t-1}, U_t) \\ = 0 \end{aligned}$$

AR(1) Model: Random Walk

$$Y_t = \quad Y_{t-1} + U_t$$

U_t is a *stationary* prediction error.

$$\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1}) + \sigma^2$$

- $\mathrm{var}(Y_t)$ *increases* by σ^2 every time period
- i.e. $\mathrm{var}(Y_t)$ increases *linearly* with time.

AR(1) Model: Random Walk

$$Y_t = \quad Y_{t-1} + U_t$$

U_t is a *stationary* prediction error.

$$\text{var}(Y_t) = \text{var}(Y_{t-1}) + \sigma^2$$

- i.e. $\text{var}(Y_t)$ **cannot be constant** when $\phi_1 = 1$.
- $\Rightarrow$ the random walk is non-stationary

Explosive AR(1) Model

"Explosive" AR(1) Model

$$Y_t = \phi_1 Y_{t-1} + U_t$$

U_t is a *stationary* prediction error and $\phi_1 > 1$.

$$\mathrm{var}(Y_t) = \phi_1^2 \, \mathrm{var}(Y_{t-1}) + \sigma^2$$

$\phi_1 > 1 : \mathrm{var}(Y_t)$ increases *exponentially* with time.

"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2 \, \mathrm{var}(Y_{t-1}) + 1$

$$\mathrm{var}(Y_1) = \qquad 2 \qquad \text{(assumption)}$$

"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$

$$\mathrm{var}(Y_1) = 2 \qquad \text{(assumption)}$$

$$\mathrm{var}(Y_2) = 1.1^2 \times 2 + 1$$

"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2 \, \mathrm{var}(Y_{t-1}) + 1$

$$\mathrm{var}(Y_1) = \qquad 2 \qquad \text{(assumption)}$$

$$\mathrm{var}(Y_2) = \qquad 3.42$$

"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$

$$\begin{aligned}
\mathrm{var}(Y_1) &= 2 && \text{(assumption)} \\
\mathrm{var}(Y_2) &= 3.42 \\
\mathrm{var}(Y_3) &= 1.1^2 \times 3.42 + 1
\end{aligned}$$

"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$

$$\begin{aligned} \mathrm{var}(Y_1) &= 2 && \text{(assumption)} \\ \mathrm{var}(Y_2) &= 3.42 \\ \mathrm{var}(Y_3) &= 5.14 \end{aligned}$$

"Explosive" AR(1) Model

Eg. $\quad \text{var}(Y_t) = 1.1^2 \, \text{var}(Y_{t-1}) + 1$

$$\begin{aligned}
\text{var}(Y_1) &= 2 \qquad \text{(assumption)} \\
\text{var}(Y_2) &= 3.42 \\
\text{var}(Y_3) &= 5.14 \\
\text{var}(Y_4) &= 1.1^2 \times 5.14 + 1
\end{aligned}$$

"Explosive" AR(1) Model

Eg. $\quad \text{var}(Y_t) = 1.1^2 \, \text{var}(Y_{t-1}) + 1$

$$\begin{aligned}
\text{var}(Y_1) &= 2 && \text{(assumption)} \\
\text{var}(Y_2) &= 3.42 \\
\text{var}(Y_3) &= 5.14 \\
\text{var}(Y_4) &= 7.22
\end{aligned}$$

"Explosive" AR(1) Model

Eg. $\mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$

$$
\begin{aligned}
\mathrm{var}(Y_1) &= 2 && \text{(assumption)} \\
\mathrm{var}(Y_2) &= 3.42 \\
\mathrm{var}(Y_3) &= 5.14 \\
\mathrm{var}(Y_4) &= 7.22 \\
\mathrm{var}(Y_5) &= 1.1^2 \times 7.22 + 1
\end{aligned}
$$

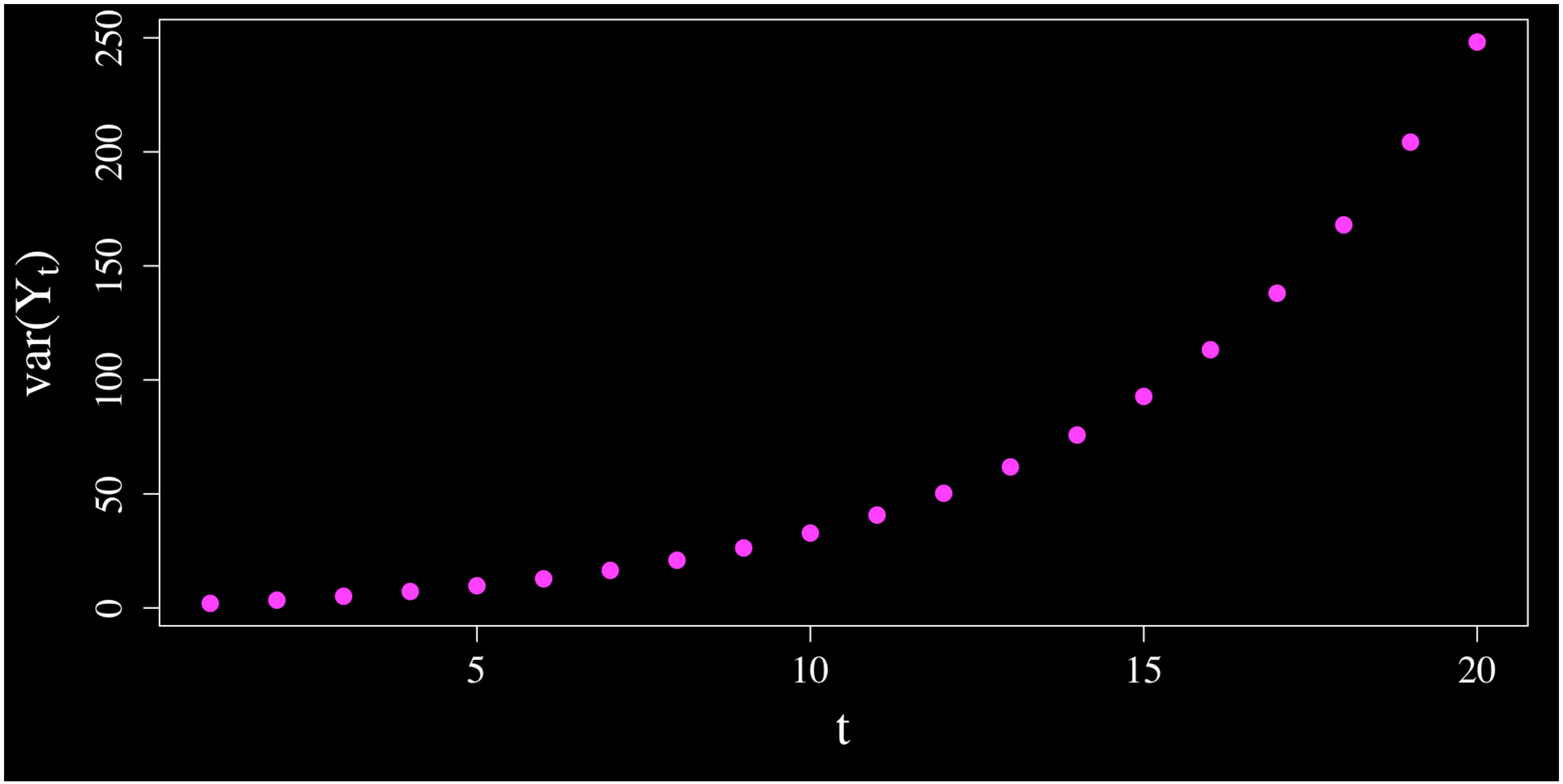
"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$

$$\begin{aligned}
\mathrm{var}(Y_1) &= 2 \qquad \text{(assumption)} \\
\mathrm{var}(Y_2) &= 3.42 \\
\mathrm{var}(Y_3) &= 5.14 \\
\mathrm{var}(Y_4) &= 7.22 \\
\mathrm{var}(Y_5) &= 9.74
\end{aligned}$$

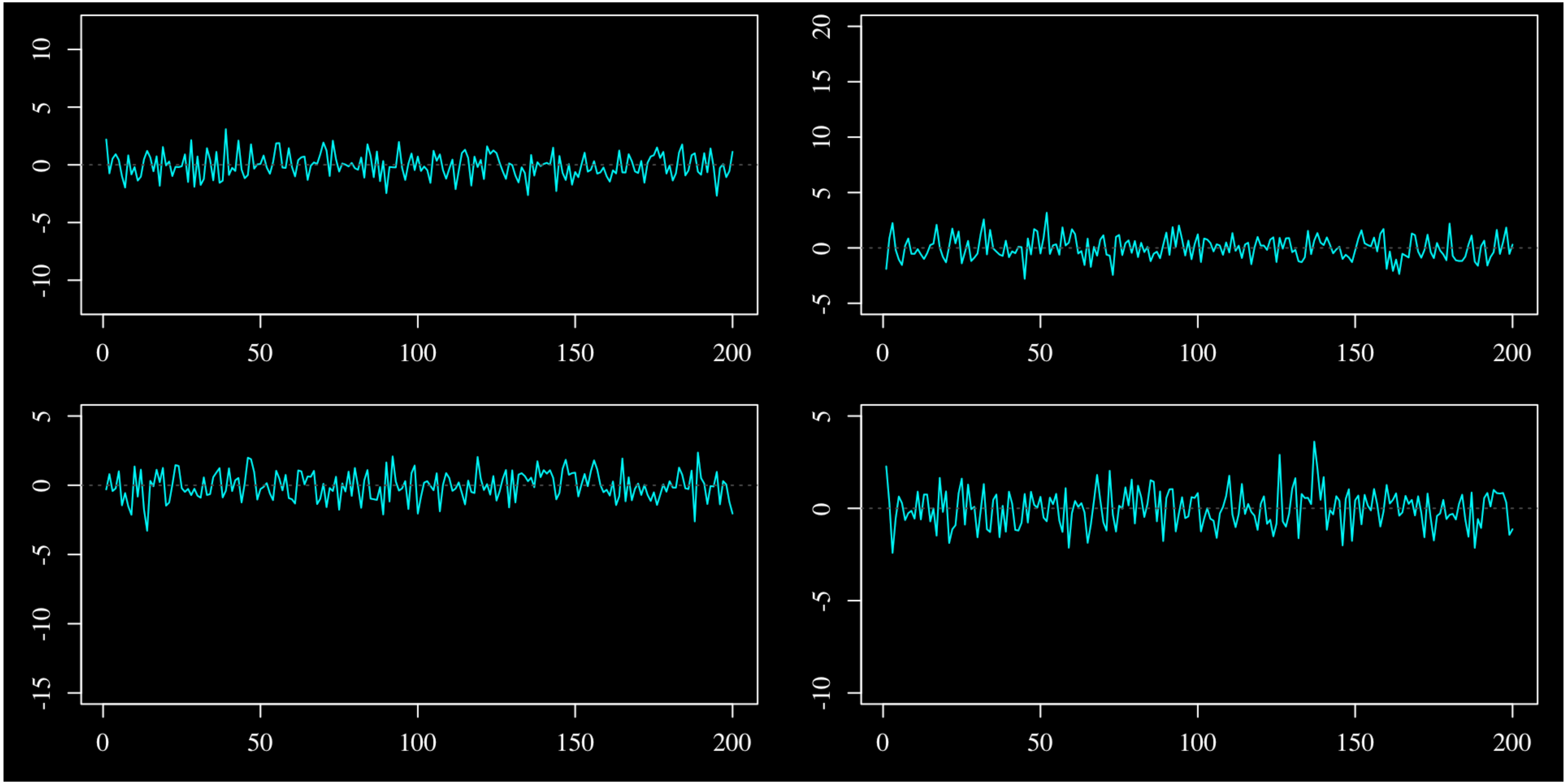
"Explosive" AR(1) Model

Eg. $\quad \mathrm{var}(Y_t) = 1.1^2\,\mathrm{var}(Y_{t-1}) + 1$



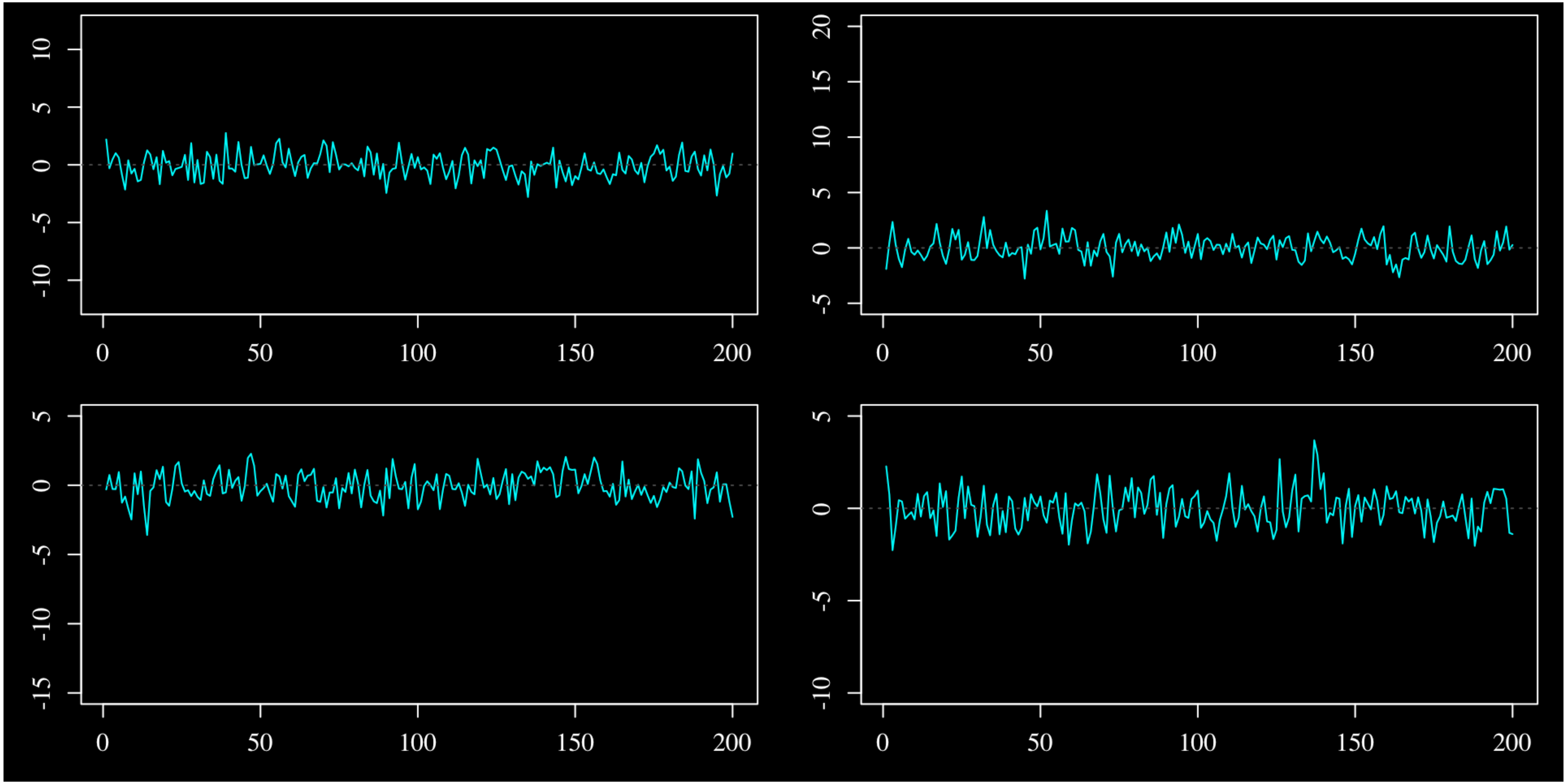
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.0$$



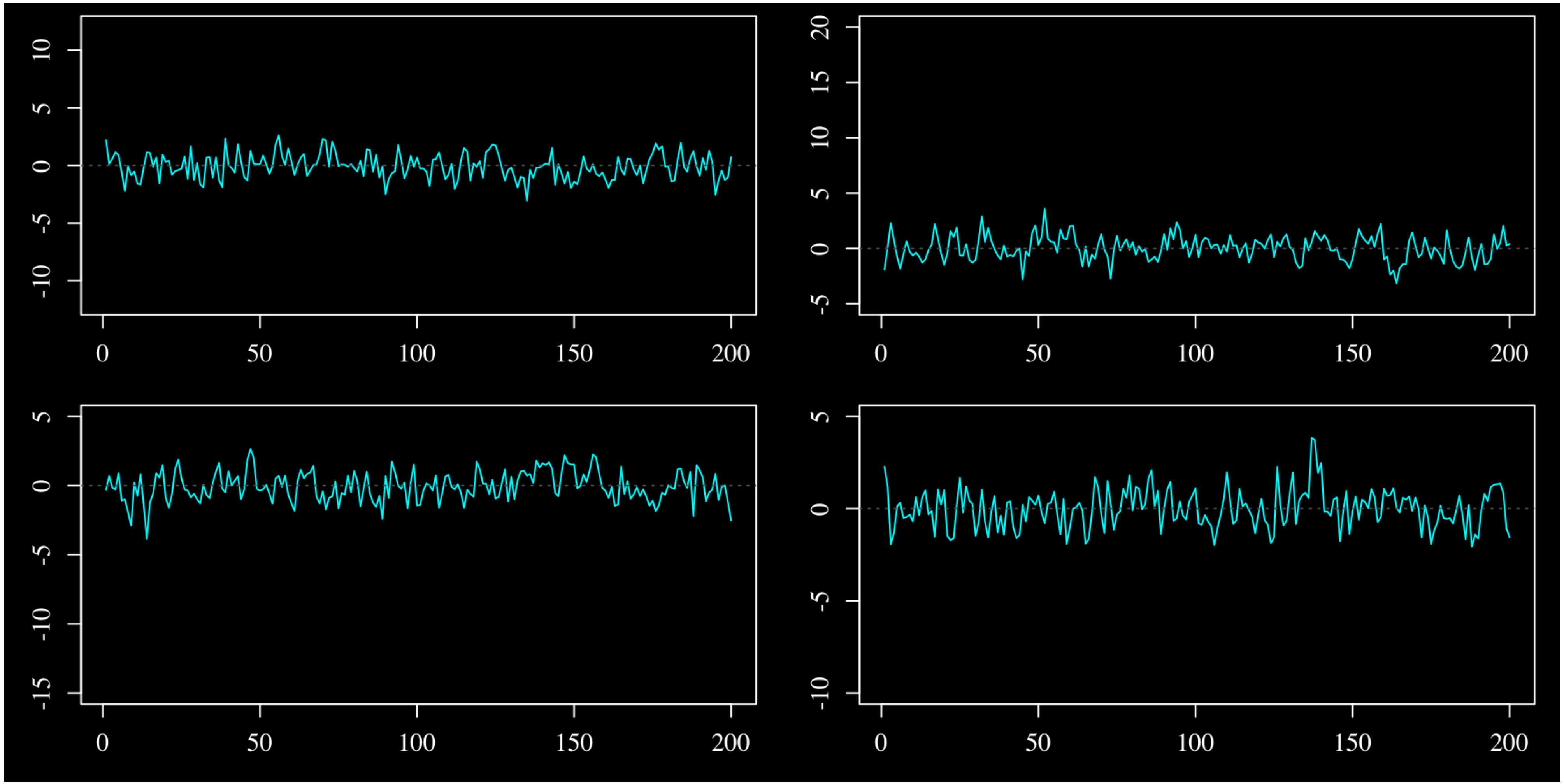
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.2$$



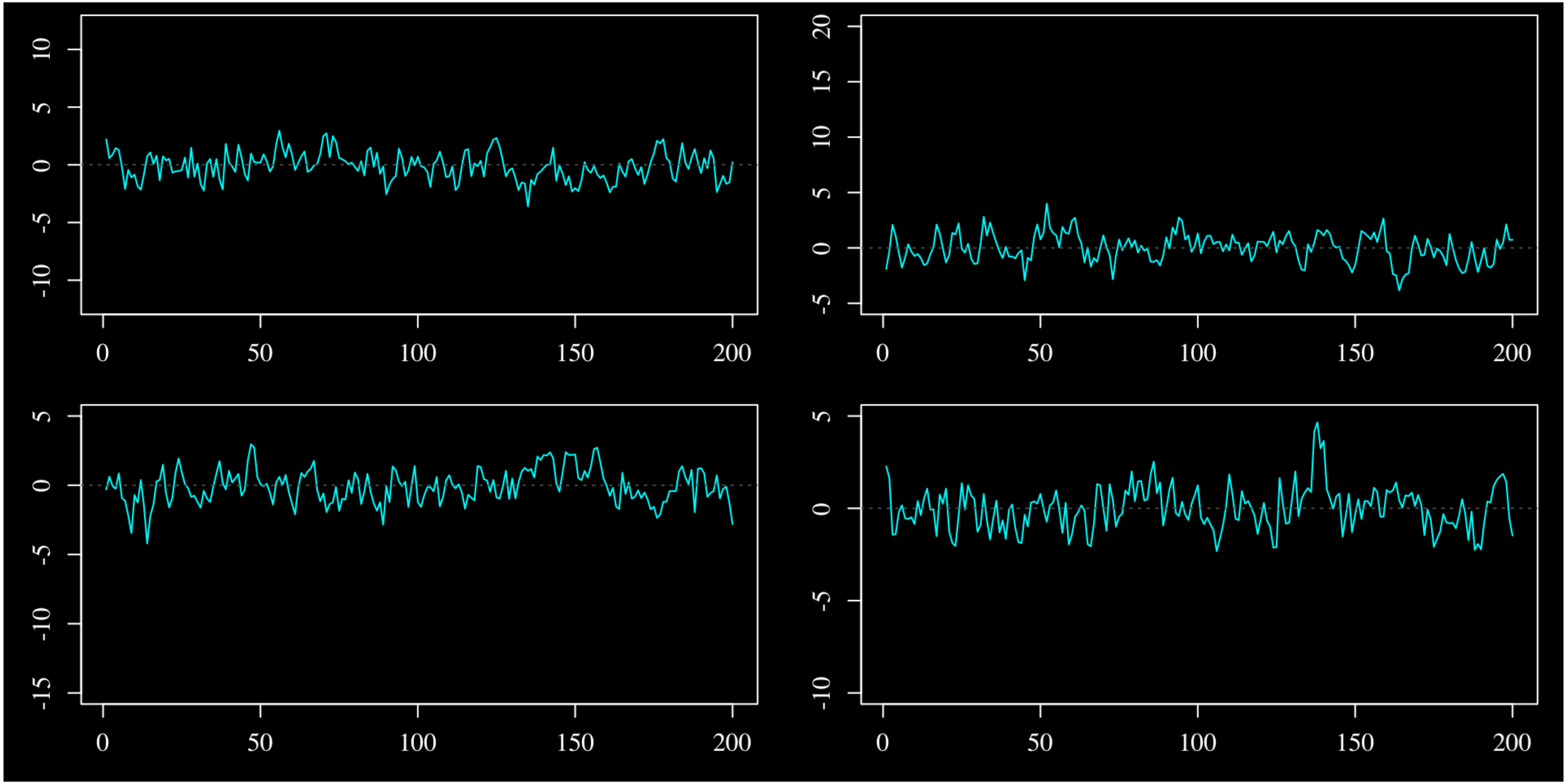
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.4$$



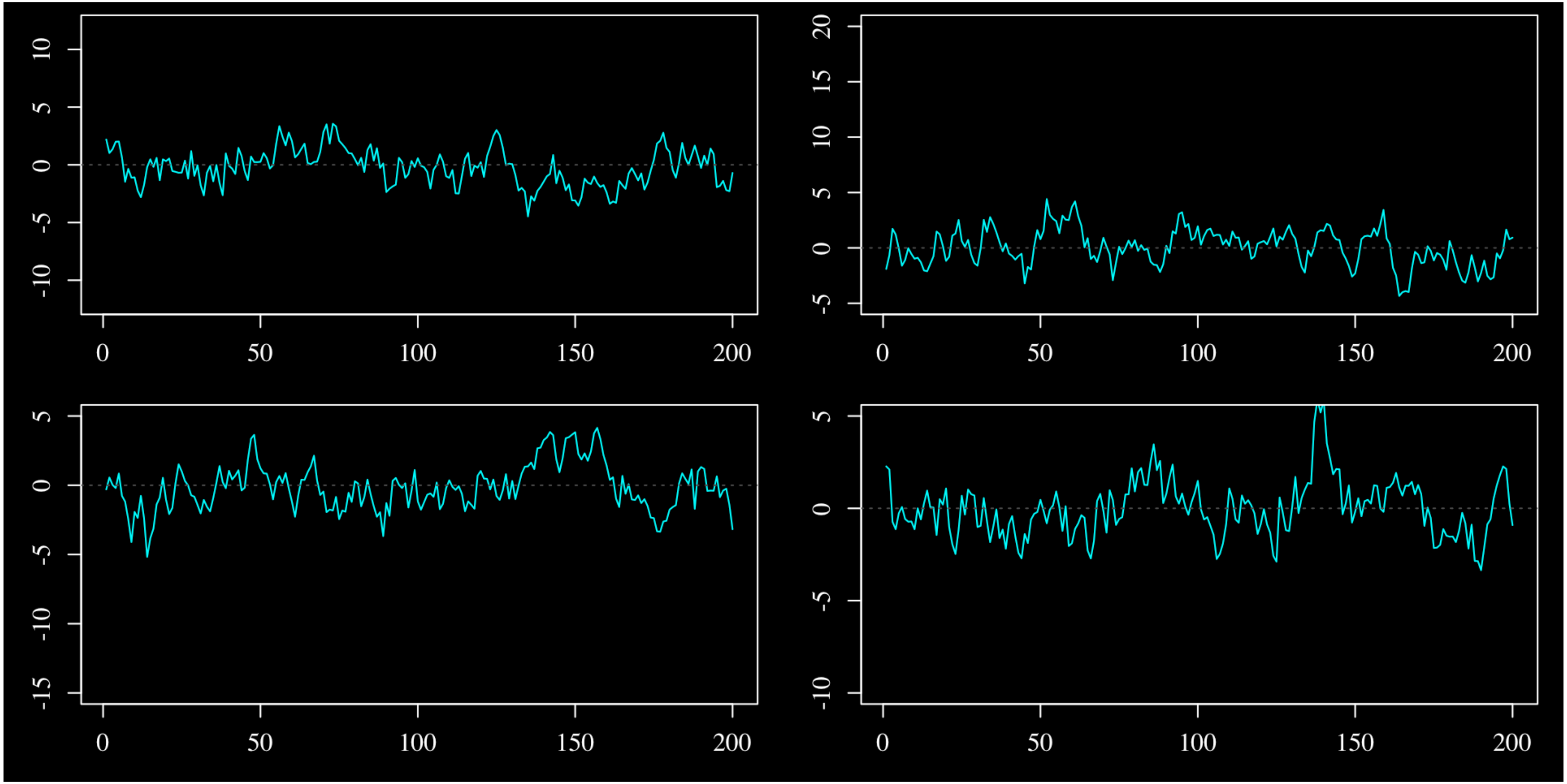
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.6$$



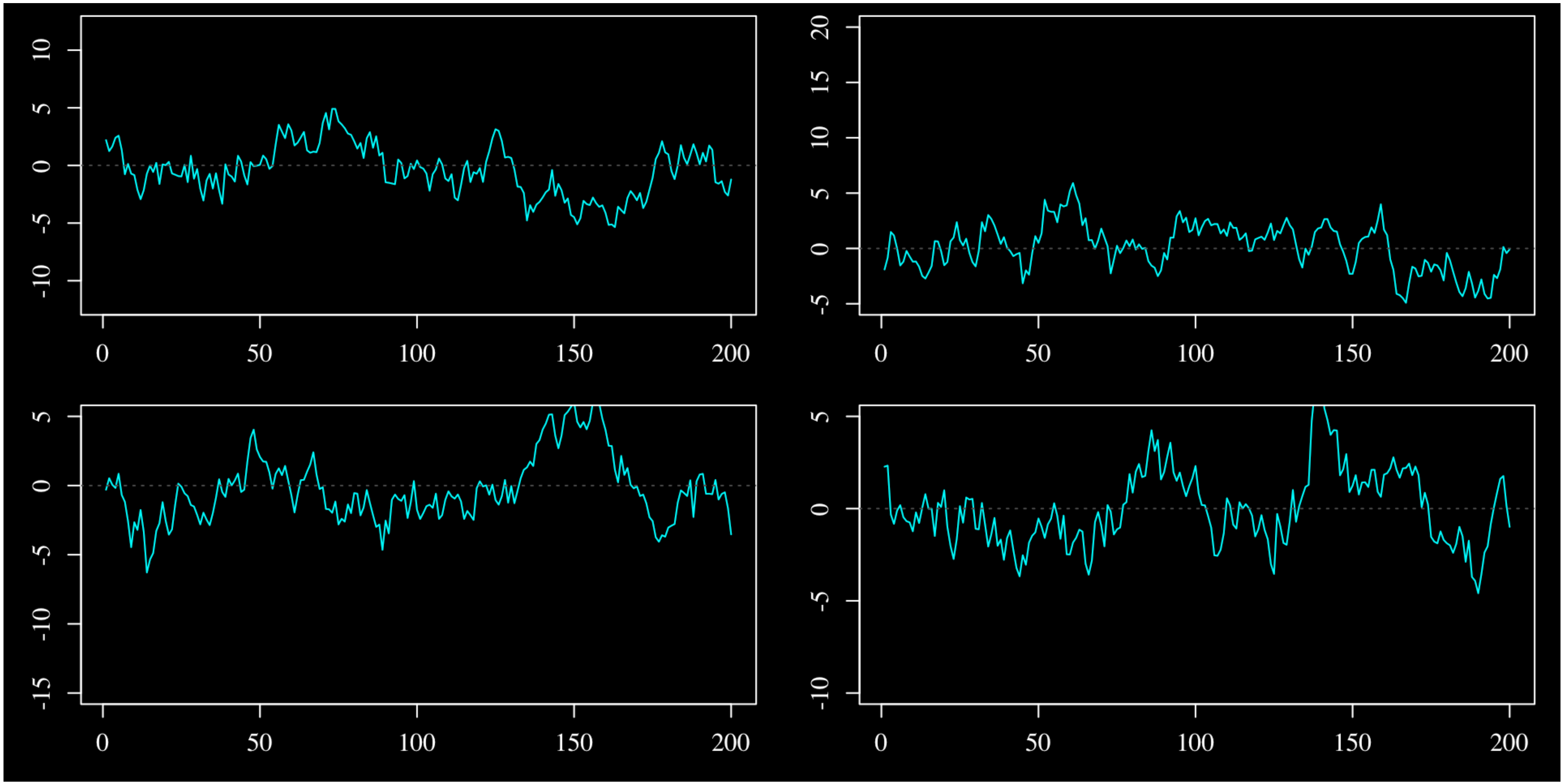
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.8$$



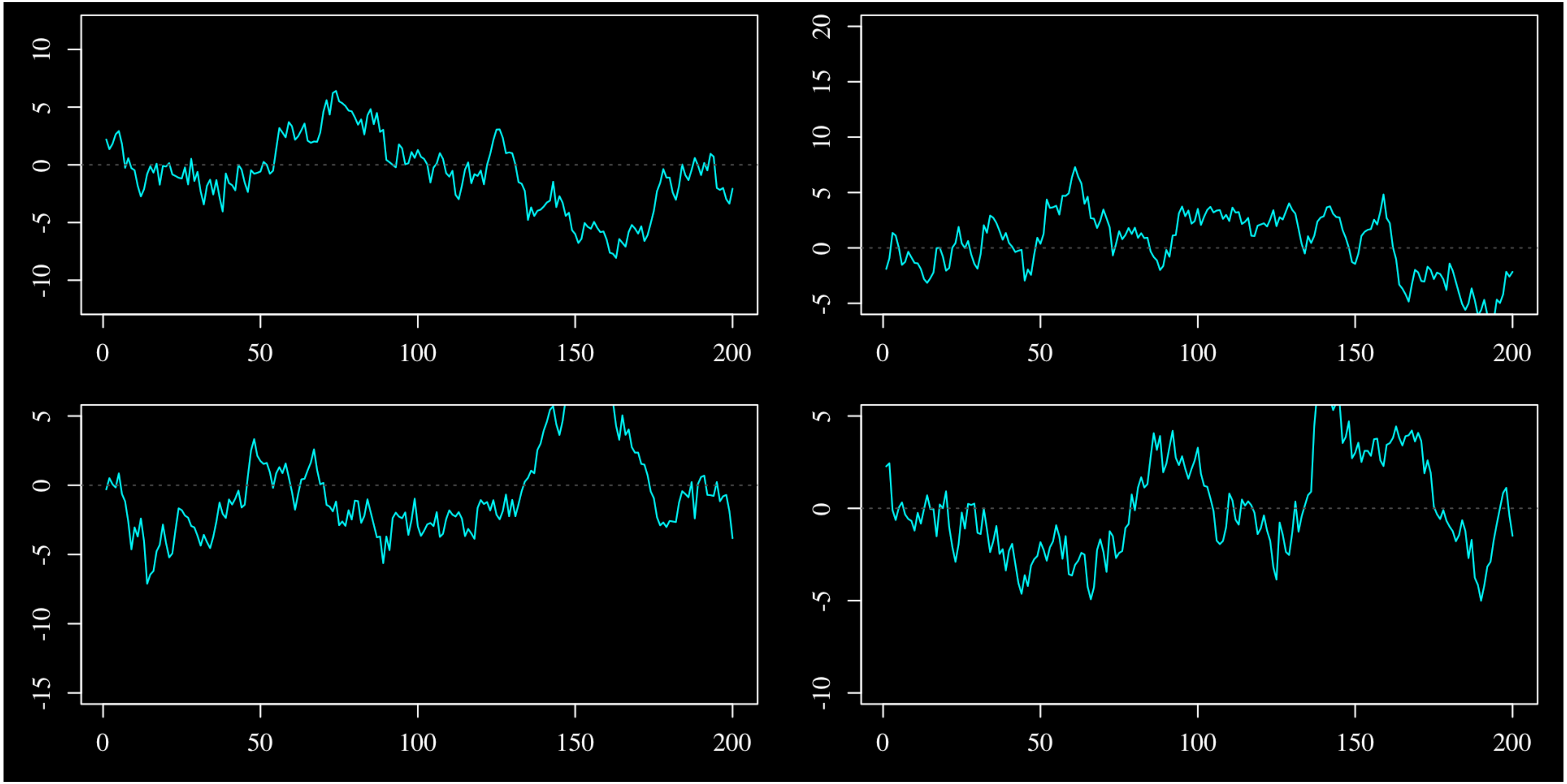
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.9$$



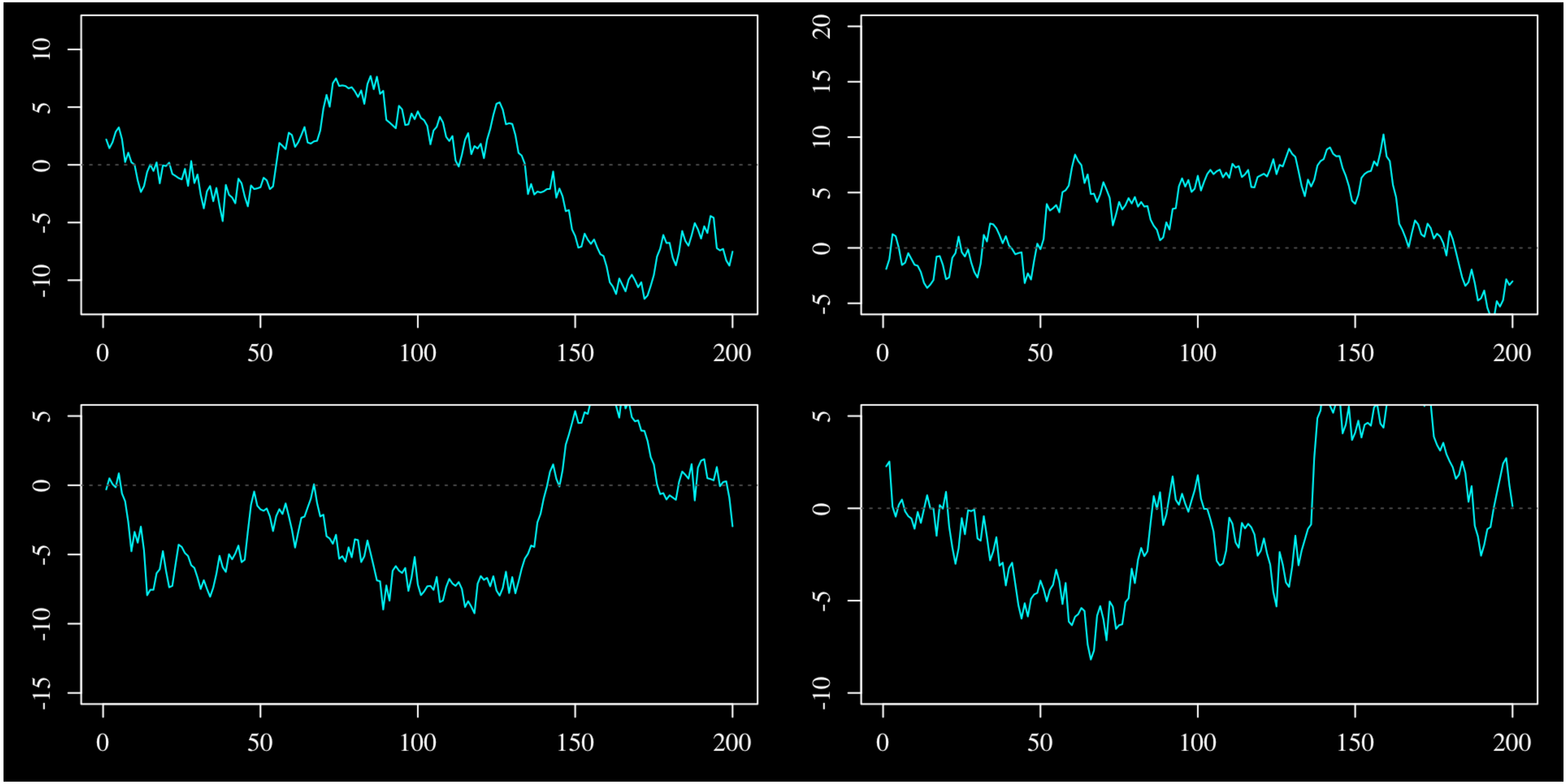
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.95$$



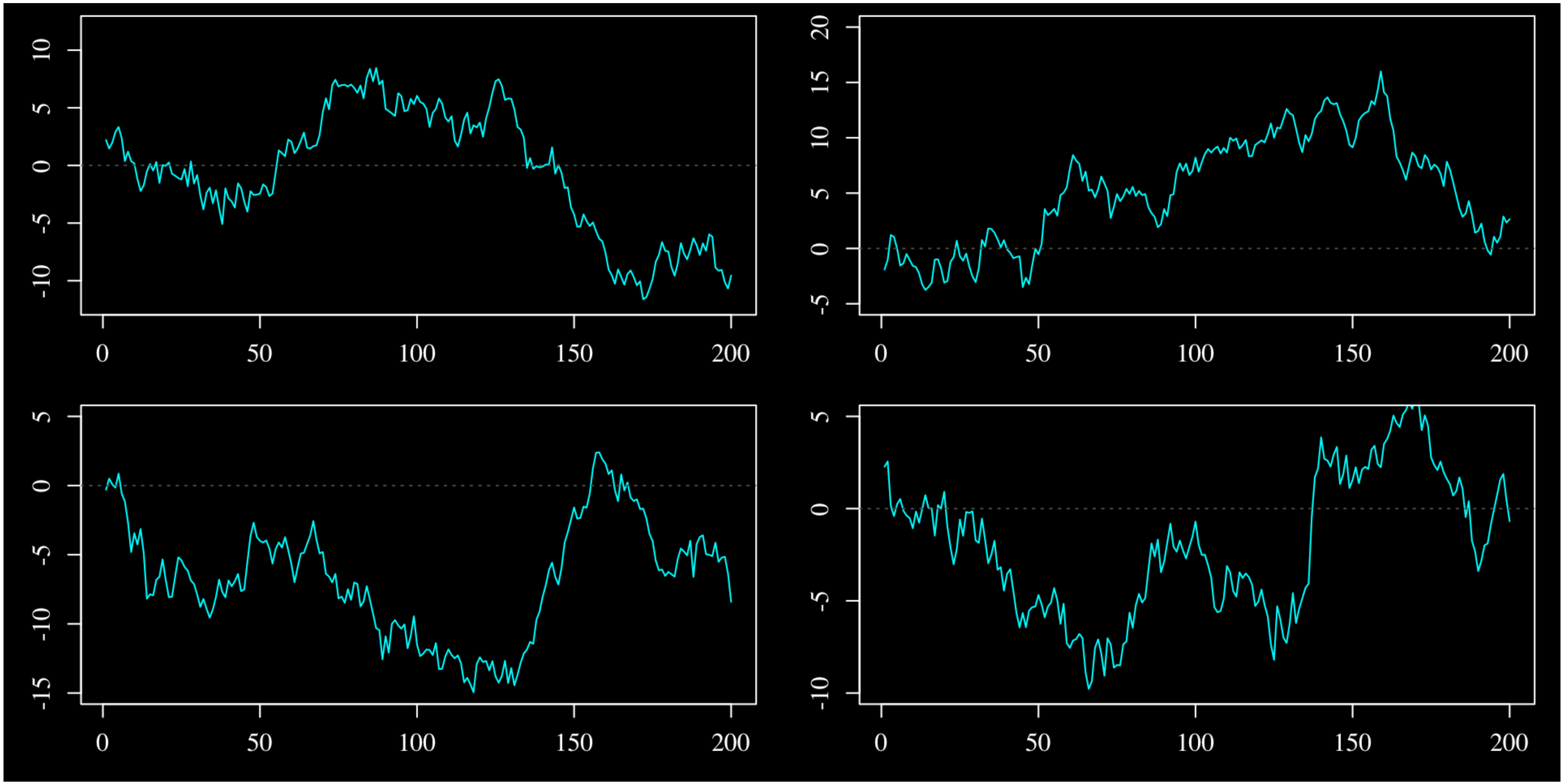
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 0.99$$



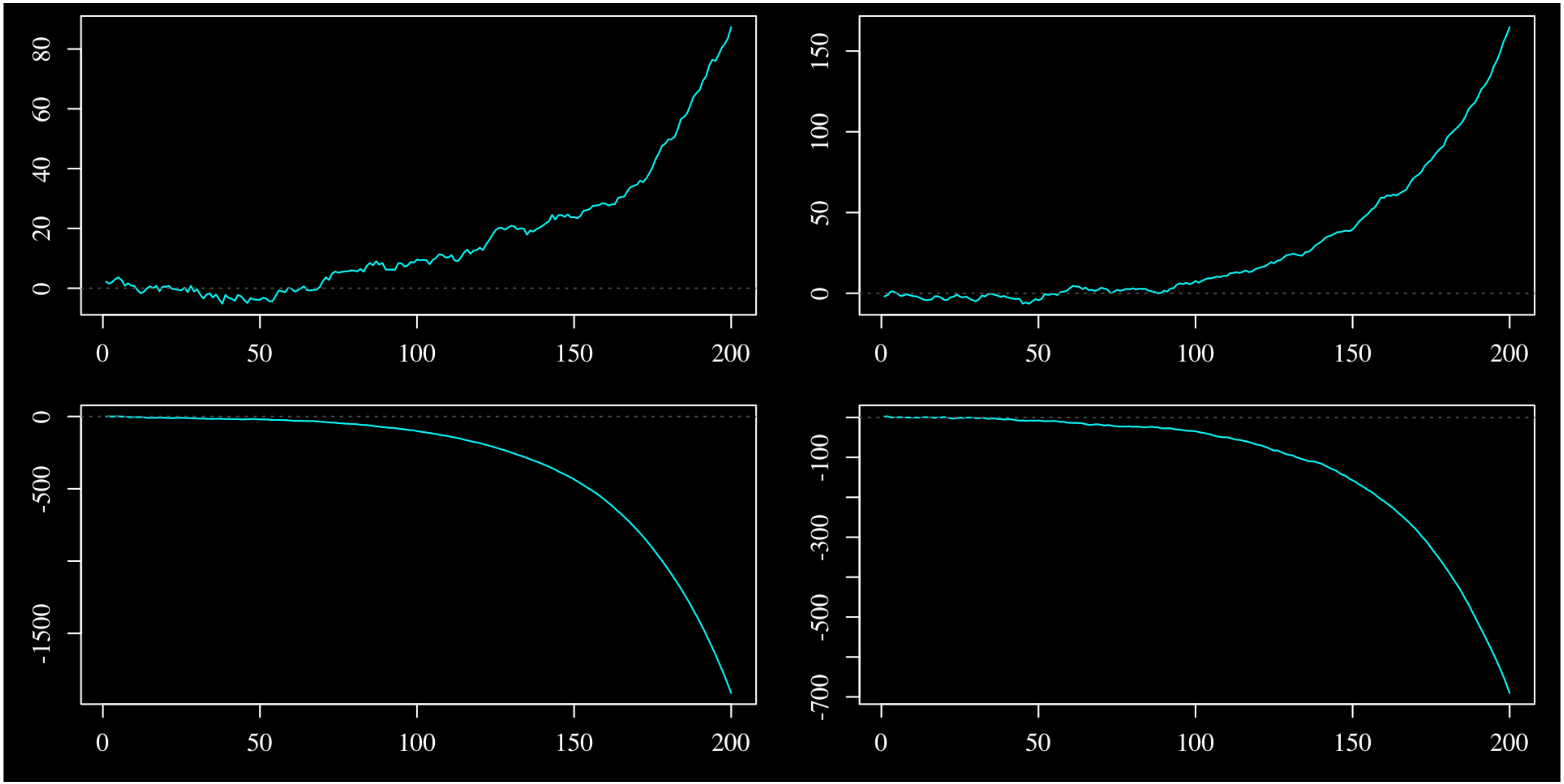
Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 1.0$$



Some simulated sample paths

$$Y_t = \phi_1 Y_{t-1} + U_t, \quad \phi_1 = 1.03$$



Differencing for Stationarity

Differencing for stationarity

The random walk is non-stationary:

$$Y_t = Y_{t-1} + U_t$$

The *first difference* is stationary:

$$\Delta Y_t = Y_t - Y_{t-1} = U_t$$

Differencing an explosive AR(1) does *not* produce stationarity. (Rare in practice.)

Differencing for stationarity

It is conventional to

- specify a deterministic regression
- apply differencing

to produce a stationary time series.

Differencing for stationarity

It is conventional to

- specify a deterministic regression
- apply differencing d times

to produce a stationary time series. Then

- specify an ARIMA(p, d, q) model.

Testing for differencing in an AR(1) model

$$
\begin{aligned}
Y_t &= X_t'\beta + Z_t \\
Z_t &= \phi_1 Z_{t-1} + U_t
\end{aligned}
$$

Testing for differencing in an AR(1) model

$$Y_t = X_t'\beta + Z_t$$
$$Z_t = \phi_1 Z_{t-1} + U_t$$

$\phi_1 = 1$: “unit root”, differencing required.

$\phi_1 < 1$: stationary, no differencing required.

Testing for differencing in an AR(1) model

$$Y_t = X_t'\beta + Z_t$$
$$Z_t = \phi_1 Z_{t-1} + U_t$$

$H_0 : \phi_1 = 1$: “unit root”, differencing required.

$H_1 : \phi_1 < 1$: stationary, no differencing required.

“Dickey-Fuller” test:

$$t_{\mathrm{DF}} = \frac{\widehat{\phi}_1 - 1}{\mathrm{s.e.}(\widehat{\phi}_1)} \quad \text{(nonstandard critical/}p\text{-value)}$$

The Augmented Dickey-Fuller test

AR(1):

$$Z_t = \phi_1 Z_{t-1} + U_t$$

can be written

$$- Z_{t-1} = \qquad - 1 \;\; Z_{t-1} + U_t$$

The Augmented Dickey-Fuller test

AR(1):

$$Z_t = \phi_1 Z_{t-1} + U_t$$

can be written

$$\begin{aligned} Z_t - Z_{t-1} &= (\phi_1 - 1)\, Z_{t-1} + U_t \\ \Delta Z_t &= \quad \varphi \quad Z_{t-1} + U_t \end{aligned}$$

$$\begin{aligned} H_0 &: \phi_1 = 1 \\ H_1 &: \phi_1 < 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} H_0 &: \varphi = 0 \\ H_1 &: \varphi < 0 \end{aligned}$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$-Z_{t-1} = \quad + \phi_2 - 1 \; Z_{t-1} - \phi_2 (Z_{t-1} - Z_{t-2}) + U_t$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$\begin{aligned} Z_t - Z_{t-1} &= (\phi_1 + \phi_2 - 1) Z_{t-1} - \phi_2 (Z_{t-1} - Z_{t-2}) + U_t \\ \Delta Z_t &= \qquad \varphi \qquad Z_{t-1} + \psi_1 \quad \Delta Z_{t-1} \qquad + U_t \end{aligned}$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$\Delta Z_t = \varphi Z_{t-1} + \psi_1 \Delta Z_{t-1} + U_t$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$\Delta Z_t = \qquad\qquad \psi_1 \Delta Z_{t-1} + U_t$$

$$\varphi = 0 \quad \Rightarrow \quad \text{AR}(1) \text{ model for } \Delta Z_t$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$\Delta Z_t = \qquad\qquad \psi_1 \Delta Z_{t-1} + U_t$$

$$\varphi = 0 \quad \Rightarrow \quad \text{AR}(1) \text{ model for } \Delta Z_t$$

The Augmented Dickey-Fuller test

AR(2):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + U_t$$

can be written

$$\Delta Z_t = \qquad\qquad \psi_1 \Delta Z_{t-1} + U_t$$

$$\varphi = 0 \quad \Rightarrow \quad \text{AR}(1) \text{ model for } \Delta Z_t$$

$$\Rightarrow \quad \text{ARIMA}(1, 1, 0) \text{ model for } Z_t$$

The Augmented Dickey-Fuller test

AR(p):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \ldots + \phi_p Z_{t-p} + U_t$$

can be written

$$\Delta Z_t = \varphi Z_{t-1} + \psi_1 \Delta Z_{t-1} + \ldots + \psi_{p-1} \Delta Z_{t-p+1} + U_t$$

The Augmented Dickey-Fuller test

AR(p):

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \ldots + \phi_p Z_{t-p} + U_t$$

can be written

$$\Delta Z_t = \qquad \psi_1 \Delta Z_{t-1} + \ldots + \psi_{p-1} \Delta Z_{t-p+1} + U_t$$

$$\begin{aligned} \varphi = 0 \quad & \Rightarrow \quad \text{AR}(p-1) \text{ model for } \Delta Z_t \\ & \Rightarrow \quad \text{ARIMA}(p-1, 1, 0) \text{ model for } Z_t \end{aligned}$$

The Augmented Dickey-Fuller test

$$Y_t = X_t'\beta + Z_t$$
$$\Delta Z_t = \varphi Z_{t-1} + \psi_1 \Delta Z_{t-1} + \ldots + \psi_{p-1} \Delta Z_{t-p+1} + U_t$$

1. Choose X_t based on time series plot.

2. Choose p by AIC (approximate model)

3. ADF test:

$H_0 : \varphi = 0$ unit root, difference Z_t

$H_1 : \varphi < 0$ Z_t stationary, no difference