# ECOM40006/ECOM90013 Econometrics 3
# Department of Economics
# University of Melbourne

## Assignment 2 Solutions

Semester 1, 2025

1. Let $Y_1, Y_2, \ldots, Y_n$ denote a simple random sample of size $n$ from a Normal population with mean $\mu$ and variance 1. Consider the first observation $Y_1$ as an estimator for $\mu$.

   (a) Show that $Y_1$ is an unbiased estimator for $\mu$. (1 mark)

   *Solution*:
   Each sample element is an image of the population; that is, $Y_j \sim \mathrm{N}(\mu, 1)$, $j = 1, 2, \ldots, n$. Specifically, $Y_1 \sim \mathrm{N}(\mu, 1)$. It follows that $\mathrm{E}[Y_1] = \mu$, which makes $Y_1$ an unbiased estimator for $\mu$.

   (b) Find $\Pr(|Y_1 - \mu| \leq 1)$. (2 marks)

   *Solution*:
   $\Pr(|Y_1 - \mu| \leq 1) = \Pr\left(-1 \leq Z + \frac{Y_1 - \mu}{\sqrt{1}} \leq 1\right) \approx 0.6826.$

   (c) Based on your answer to 1(b), is $Y_1$ a consistent estimator for $\mu$? Explain your answer both theoretically and intuitively. (2 marks)

   *Solution*:
   Consistency of an estimator $\hat{\theta}$ for a parameter $\theta$ requires that

   $$\lim_{n \to \infty} \Pr\left(|\hat{\theta} - \theta| < \epsilon\right) = 1,$$

   for any $\epsilon > 0$. We see that $\Pr(|Y_1 - \mu| \leq 1) \approx 0.6826$ does not change as $n \to \infty$ (the probability is not a function of $n$). This is true for any other choice of $\epsilon$ too. Hence, $Y_1$ is not a consistent estimator for $\mu$ even though it is unbiased. Intuitively, the single draw from the population is no accumulating information as the sample size increases and so we do not get convergence to anything.

2. The Constant Elasticity of Substitution (CES) production function is of the form

   $$Q = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho},$$

   where $K$ and $L$ are the factor inputs, capital and labour say, and the parameters of the function are $A > 0$, $0 < \delta < 1$, and $-1 < \rho \neq 0$. In this model the elasticity of substitution can be shown to be $\varepsilon = 1/(1+\rho)$. Suppose that you have an estimator for the parameters of the CES production function with joint limiting distribution of the form

$$\sqrt{n}\left(\begin{bmatrix}\hat{A}\\ \hat{\delta}\\ \hat{\rho}\end{bmatrix}-\begin{bmatrix}A\\ \delta\\ \rho\end{bmatrix}\right)\xrightarrow{d}\mathrm{N}\left(0,\Sigma\right),\qquad \Sigma=\begin{bmatrix}\sigma_A^2 & \sigma_{A\delta} & \sigma_{A\rho}\\ \sigma_{A\delta} & \sigma_\delta^2 & \sigma_{\delta\rho}\\ \sigma_{A\rho} & \sigma_{\delta\rho} & \sigma_\rho^2\end{bmatrix}.$$

(a) What is the marginal limiting distribution of $\hat{\rho}$? (1 mark)

*Solution*:

$$\sqrt{n}(\hat{\rho}-\rho)\xrightarrow{d}\mathrm{N}(0,\sigma_\rho^2).$$

(b) If $\hat{\Sigma}$ denotes a consistent estimator for $\Sigma$, derive an operational 95% confidence interval for $\varepsilon$, where

$$\hat{\Sigma}=\begin{bmatrix}\hat{\sigma}_A^2 & \hat{\sigma}_{A\delta} & \hat{\sigma}_{A\rho}\\ \hat{\sigma}_{A\delta} & \hat{\sigma}_\delta^2 & \hat{\sigma}_{\delta\rho}\\ \hat{\sigma}_{A\rho} & \hat{\sigma}_{\delta\rho} & \hat{\sigma}_\rho^2\end{bmatrix}.$$

By 'operational' is meant that your answer cannot depend upon any unknown parameters. Be sure to include all steps of your derivation. (4 marks)

*Solution*:

By the delta method

$$n^{1/2}\left(g(\hat{\rho})-g(\rho)\right)\xrightarrow{d}\mathrm{N}\left(0,G^2(\rho)\sigma_\rho^2\right),\quad \text{where } G(\rho)=\frac{\mathrm{d}g(\rho)}{\mathrm{d}\rho}.$$

Here $g(\rho)=1/(1+\rho)$ and so $G(\rho)=-1/(1+\rho)^2$. Hence

$$n^{1/2}\left(\frac{1}{1+\hat{\rho}}-\varepsilon\right)\xrightarrow{d}\mathrm{N}\left(0,\frac{\sigma_\rho^2}{(1+\rho)^4}\right).$$

or

$$Z_\varepsilon=\frac{n^{1/2}\left(\frac{1}{1+\hat{\rho}}-\varepsilon\right)}{\frac{\sigma_\rho}{(1+\rho)^2}}\underset{a}{\sim}\mathrm{N}\left(0,1\right).$$

Using this approximation we have the following probability statements

$$\begin{aligned}
&\Pr\left(-z_{0.025}\leq Z_\varepsilon\leq z_{0.025}\right)=0.95\\
\Longrightarrow\ &\Pr\left(-\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\leq\frac{1}{1+\hat{\rho}}-\varepsilon\leq\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\right)=0.95\\
\Longrightarrow\ &\Pr\left(-\frac{1}{1+\hat{\rho}}-\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\leq-\varepsilon\leq-\frac{1}{1+\hat{\rho}}+\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\right)=0.95\\
\Longrightarrow\ &\Pr\left(\frac{1}{1+\hat{\rho}}+\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\geq\varepsilon\geq\frac{1}{1+\hat{\rho}}-\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\right)=0.95\\
\Longrightarrow\ &\Pr\left(\frac{1}{1+\hat{\rho}}-\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\leq\varepsilon\leq\frac{1}{1+\hat{\rho}}+\frac{z_{0.025}\sigma_\rho}{n^{1/2}(1+\rho)^2}\right)=0.95
\end{aligned}$$

where $z_{0.025}$ is that value that cuts off an upper tail area of 0.025 in a standard Normal distribution. Now these are not operational statements because they depend on the unknown values of $\rho$ and $\sigma_\rho$. We make them operational by replacing $\rho$ and $\sigma_\rho$ by consistent estimates of them. That is, the 95% confidence interval is

$$\left[\frac{1}{1+\hat{\rho}} - \frac{z_{0.025}\hat{\sigma}_\rho}{n^{1/2}(1+\hat{\rho})^2}, \frac{1}{1+\hat{\rho}} + \frac{z_{0.025}\hat{\sigma}_\rho}{n^{1/2}(1+\hat{\rho})^2}\right].$$

Your answers to the Assignment should be submitted via the LMS no later than 4:30pm, Thursday 17 April.

No late assignments will be accepted and an incomplete exercise is better than nothing.

Your mark for this assignment may contribute up to 10% towards your final mark in the subject.