ECOM40006/ECOM90013 Econometrics 3
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Week 4 Tutorial Exercise Solutions

Semester 1, 2025

1. Take the opportunity to ask any questions that you may have about the lecture material.

2. Consider the simple linear regression model

$$y_i = x_i + u_i, \qquad , i = 1, \dots, n, \tag{1}$$

so that $\beta = 1$ is the true parameter value. Also assume that the classical assumptions about the disturbance term are satisfied to that $\mathrm{E}[u_i \mid x_i] = 0$ and that the u_i given x_i are independent and identically distributed, with $0 < \mathrm{Var}[u_i \mid x_i] = \sigma^2 < \infty$. The OLS estimator for β is

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}. \tag{2}$$

Given these assumptions show that:

(a) If $x_i = i$ then $\operatorname{plim}_{n\to\infty} \hat{\beta} = \beta = 1$.

Hint: $\displaystyle\sum_{i=1}^n i^2 = \frac{1}{6}n(n+1)(2n+1)$.

Solution:
First, substitute for y_i in (2), so that

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i y_1}{\sum_{i=1}^n x_i^2} = \frac{\sum_{i=1}^n x_i(x_i + u_i)}{\sum_{i=1}^n x_i^2} = 1 + \frac{\sum_{i=1}^n x_i u_i}{\sum_{i=1}^n x_i^2}. \tag{3}$$

Consistency requires that

$$\hat{\beta} - 1 = \frac{\sum_{i=1}^n x_i u_i}{\sum_{i=1}^n x_i^2} = \frac{n^{-p}\sum_{i=1}^n x_i u_i}{n^{-p}\sum_{i=1}^n x_i^2} \xrightarrow{p} 0.$$

Note that we have scaled the sums as usual, but by a factor of n^{-p} rather than our usual n^{-1}. The reasons will soon become obvious.

$$\operatorname{plim}(\hat{\beta} - 1) = \operatorname{plim} \frac{n^{-p}\sum_{i=1}^n x_i u_i}{n^{-p}\sum_{i=1}^n x_i^2} = \frac{\operatorname{plim} n^{-p}\sum_{i=1}^n x_i u_i}{\operatorname{plim} n^{-p}\sum_{i=1}^n x_i^2} = \frac{\operatorname{plim} n^{-p}\sum_{i=1}^n x_i u_i}{\lim n^{-p}\sum_{i=1}^n x_i^2}.$$

Because the x_i are not random and so the probability limit in the denominator is identically equal to a limit, in this case. Looking first at this denominator,

$$\begin{aligned}\lim n^{-p} \sum_{i=1}^{n} x_i^2 &= \lim n^{-p} \sum_{i=1}^{n} i^2 = \lim \frac{1}{6n^p} n(n+1)(2n+1) \\ &= \frac{1}{6} \lim \frac{1}{n^p} n(n+1)(2n+1).\end{aligned}$$

If we choose $p = 3$ then we can write

$$\begin{aligned}\lim n^{-p} \sum_{i=1}^{n} x_i^2 &= \frac{1}{6} \times \lim \frac{n}{n} \times \lim \left(\frac{n+1}{n}\right) \times \lim \left(\frac{2n+1}{n}\right) \\ &= \frac{1}{6} \times 1 \times \lim \left(1 + \frac{1}{n}\right) \times \lim \left(2 + \frac{1}{n}\right) = \frac{2}{6} = \frac{1}{3} = O(1).\end{aligned}$$

Clearly, $p = 3$ is the right scaling to control the denominator and so the consistency of $\hat{\beta}$ will depend on the behaviour of the numerator given this choice of p. To establish the probability limit of the numerator we will try to establish mean square convergence. Thus, remembering that we have set p to 3,

$$\mathrm{E}\left[n^{-3} \sum_{i=1}^{n} x_i u_i\right] = n^{-3} \sum_{i=1}^{n} i\, \mathrm{E}\left[u_i\right] = n^{-3} \sum_{i=1}^{n} i \times 0 = 0, \quad \text{for all } n$$

where the first equality follows because $x_i = i$ is not random. Hence,

$$\lim_{n\to\infty} \mathrm{E}\left[n^{-3} \sum_{i=1}^{n} x_i u_i\right] = \lim_{n\to\infty} 0 = 0.$$

Next,

$$\mathrm{Var}\left[n^{-3} \sum_{i=1}^{n} x_i u_i\right] = n^{-6} \sum_{i=1}^{n} i^2\, \mathrm{E}\left[u_i^2\right] = \frac{\sigma^2}{n^6} \sum_{i=1}^{n} i^2 = \frac{\sigma^2}{n^3} \times \frac{1}{n^3} \sum_{i=1}^{n} i^2,$$

where the independence of the u_i has meant that all expectations of the form $\mathrm{E}\left[u_i u_j\right] = 0$ whenever $i \neq j$. Consequently,

$$\lim_{n\to\infty} \mathrm{Var}\left[n^{-3} \sum_{i=1}^{n} x_i u_i\right] = \lim \frac{\sigma^2}{n^3} \times \lim \frac{1}{n^3} \sum_{i=1}^{n} i^2 = 0 \times \frac{1}{3} = 0.$$

The combination of these two limiting results establishes convergence in mean square to zero, hence

$$n^{-3} \sum_{i=1}^{n} x_i u_i \xrightarrow{m} 0 \implies n^{-3} \sum_{i=1}^{n} x_i u_i \xrightarrow{p} 0.$$

Finally, we have established that

$$\mathrm{plim}(\hat{\beta} - 1) = \frac{\mathrm{plim}\, n^{-3} \sum_{i=1}^{n} x_i u_i}{\lim n^{-3} \sum_{i=1}^{n} x_i^2} = \frac{0}{1/3} = 0 \implies \mathrm{plim}\, \hat{\beta} = 1 = \beta.$$

That is, even though the upward trending regressor required a different scaling than usual to work our magic, OLS remains consistent for the regression coefficient in this model. Given the stronger scaling required what we observe in this case is faster convergence to the limit that the standard case. OLS is sometimes described as being super-consistent in this model. Intuitively what it means is that, with a trending regressor, the slope of the line becomes revealed much more quickly that when the data are all bobbing around some fixed mean and so OLS is able to discern that slope both quickly and easily.

(b) If $x_i = i^{-1}$ then $\operatorname{plim}_{n\to\infty} \hat{\beta} \neq \beta = 1$.

Hint: $\displaystyle\sum_{i=1}^{n} i^{-2} = \frac{\pi^2}{6}$.

Solution:

Everything works much the same as laid out in tedious detail in solution to the previous part and so this solution will skip a few steps. We see immediately that consistency of $\hat{\beta}$ requires that

$$\hat{\beta} - 1 = \frac{\sum_{i=1}^{n} x_i u_i}{\sum_{i=1}^{n} x_i^2} = \frac{n^{-p}\sum_{i=1}^{n} x_i u_i}{n^{-p}\sum_{i=1}^{n} x_i^2} \xrightarrow{p} 0.$$

Looking at the denominator

$$\lim n^{-p} \sum_{i=1}^{n} x_i^2 = \lim n^{-p} \sum_{i=1}^{n} i^{-2} = \lim \frac{\pi^2}{6n^p} = \frac{\pi^2}{6} \lim \frac{1}{n^p} = 0,$$

for all $p > 0$. Setting $p = 0$ and evaluating the numerator we see that we need to establish the limiting behaviour of $\sum_{i=1}^{n} x_i u_i$. Clearly

$$\mathrm{E}\left[\sum_{i=1}^{n} x_i u_i\right] = \sum_{i=1}^{n} x_i \,\mathrm{E}\left[u_i\right] = 0 \implies \lim \mathrm{E}\left[\sum_{i=1}^{n} x_i u_i\right] = 0,$$

as before, but

$$\begin{aligned} \mathrm{Var}\left[\sum_{i=1}^{n} x_i u_i\right] &= \sum_{i=1}^{n} \mathrm{Var}\left[u_i\right]/i^2 = \sigma^2 \sum_{i=1}^{n} i^{-2} = \frac{\sigma^2\pi^2}{6} \\ \implies \lim \mathrm{Var}\left[\sum_{i=1}^{n} x_i u_i\right] &= \frac{\sigma^2\pi^2}{6} \neq 0 \end{aligned}$$

unless $\sigma^2 = 0$, which it doesn't by assumption. Hence, we have failed to establish mean square convergence to zero. Does this necessarily preclude convergence in probability to zero. Strictly no, because convergence in mean square is a stronger result than convergence in probability, and so the latter may hold even if the former doesn't. However, this is most likely to occur because moments don't exist in the limit. What we have seen here is that the moments of interest are well-behaved in the limit. Indeed, they are so well-behaved that the variance of the random variable has a nice positive value in the limit. What this is telling us is that, in the limit, $\hat{\beta} - 1$ remains a random variable and so $\hat{\beta}$ is not consistent for $\beta = 1$, although its limiting distribution is centred at this value.

What do these two results suggest about consistency and the accumulation of knowledge?

Solution:

The intuition behind asymptotics is that, as the sample size grows, you accumulate information about the underlying data generating process and it is this accumulation of information that allows us to learn about the population. In the case of a trending regressor, we are accumulating information faster than usual but that causes us no problems at all. However, when we have a variable trending to zero then, as the sample size increases, this variable gets closer and closer to zero and so, after a point, additional observations are no longer contributing to our store of knowledge about the population. Consequently, there is no benefit to the accumulation of data and so large sample properties like consistency do not hold.