Lecture 5: Introduction to the OLG model

ECON30009/90080 Macroeconomics
Semester 2, 2025

Shu Lin Wee
Department of Economics
The University of Melbourne

Last class

- We finished up looking at the firm's **individual** profit maximization problem

- Everything we did so far, was in partial equilibrium, taking prices as given

- With the OLG model, we will be moving to general equilibrium analysis. Prices are determined via market clearing.

What is the OLG model?

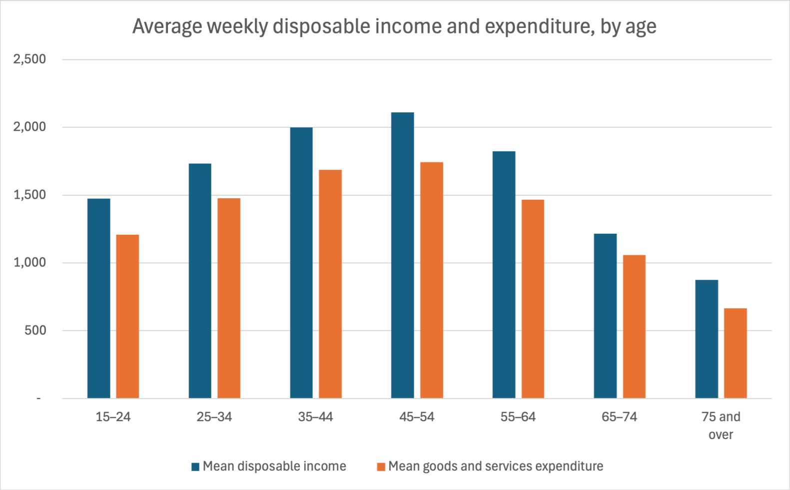
Introduction to the Life-cycle (OLG) Model

- OLG (Overlapping Generation) model is the second major workhorse of modern macroeconomics (other major model is neoclassical growth model).
- The OLG model links capital and labour supply to household saving and work decisions, and the demands for capital and labour to firm production decisions.
- Agents interact through markets. Given prices, agents make their choices and they tell you how much they would demand and/or supply given a price
- Prices adjust to make markets clear (demand equal to supply)

Introduction to the Life-cycle (OLG) Model

- Originally developed in the 1950s and 1960s to study economic growth.
- Similar to the Solow model: looks at how the economy grows through factor-input accumulation
- Unlike the Solow model – which is descriptive and takes savings rate as exogenous – how much to save and invest are choices and thus endogenous in the OLG model

Observed income and consumption dynamics over a lifetime



Source: ABS, Household expenditure survey 2015/16

Some things we will want to incorporate in our model: income declines over lifetime

Observed income dynamics over a lifetime

% of households with characteristic	15-24	25-34	35-44	45-54	55-64	65-75	75 and over
Zero or negative income	0.5	0.6	0.5	0.6	1.4	0.6	0.2
Employee income	69.2	83.8	83.4	80.2	68.6	19.2	4.5
Own unincorporated business income	4.0	3.4	3.6	3.7	4.4	2.0	0.5
Government pensions and allowances	12.2	7.8	9.9	10.5	14.3	47.2	67.4
Other income	15.3	4.2	2.8	4.6	11.4	31.3	27.6

Source: ABS, Survey of Income and Housing 2019–20

- Most of income when young stems from labour income
- Most of income when old stems from savings (public and private)

Set-up

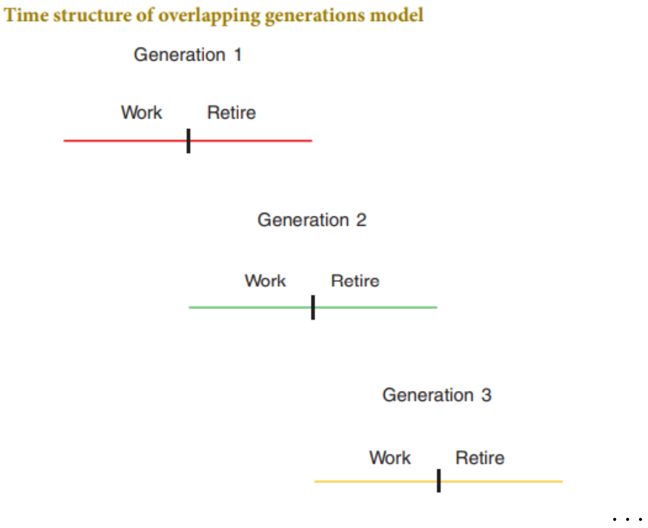
The Basic Structure of the Life-cycle (OLG) Model

- Model of a single economy.
- Agents of the economy are households and firms (for simplicity no government)
- Time is discrete, $t = 1, 2, 3, \ldots$ and goes on forever
- Economy goes on forever, but households in this economy live for two periods:
  - First period of life: young households work, choose how much to consume and save
  - Second period of life: old households retire and consume their savings (dis-save)

Life-cycle Model Structure

- Exists 2 generations each period
  - When the economy starts in period 1, there is an initial old generation, generation 0, who own an initial endowment of capital
    - This means that the capital supply at the start of each period is **pre-determined** and given by the savings of the current old
  - At the beginning of each period t, a new generation is born, and the previous young generation becomes old.
  - In every t, there is always one working (young) and one retired (old) generation.
  - There are N members of every generation. No population growth.
  - All individuals within each generation are identical.

The life-cycle model time line



Source: Chapter 8, Campante, Sturzenegger, Federico; and Velasco (2021).

Overview of Production

- Firms take TFP, z_t, as given (exogenous)

- Firms use labour and capital to produce output given rental rate, wage rate and z_t

- Assume capital fully depreciates after use in production ($\delta = 1$)

- All markets are **perfectly competitive**. (firms are price takers!)

- Because firms rent capital and hire labour each period, the firm's decision problem can be thought of as a series of static (1 period) problems

Timing

- Timing of events within a period t:
  - At the beginning of t, generation t households are born and enter the workforce. Generation $t-1$ households transition to retirement.
  - Firms demand labour from young and rent capital from old.
  - Young earn wages by inelastically supplying 1 unit of labour, and receive dividend income from firms (households own firms)
  - Old earn interest from renting out their capital
  - The young save by investing in physical capital
  - The old dis-save and consume their assets from the previous period

Households in the OLG model

Households

- Very similar to our consumption-savings problem from before!
- When young in period t, income = labour income + dividend income π_t
- As before, assume that consumption is the numeraire good, i.e., price of consumption = 1
- Denote the real wage rate per unit of labour in period t as w_t:

$$\text{real wage, } w_t = \frac{\overbrace{W_t}^{\text{nominal wage}}}{\underbrace{P_t}_{\text{assumed to be 1, consumption is numeraire good}}}$$

Households: budget constraints

- Each household inelastically supplies one unit of labour when young
- and receives dividend income π_t when young (household owns the firm)
- This implies that budget constraint when young in t is:

$$c_t^y + a_{t+1} = w_t + \pi_t$$

- Each household retires when old. Only income is interest income from savings
- This implies budget constraint of that generation when old in period $t+1$ is:

$$c_{t+1}^o = (1 + r_{t+1})a_{t+1}$$

Note: the household takes prices w_t, r_{t+1} and dividend income, π_t as given

Households: lifetime budget constraints

- Substitute out a_{t+1} to derive lifetime budget constraint (LBC):

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t$$

- LBC has PDV of consumption expenditure $=$ PDV of lifetime income

Households: utility maximization

- Household chooses consumption when young and old to maximize lifetime utility:

$$\max_{\{c_t^y, c_{t+1}^o\}} U(c_t^y, c_{t+1}^o)$$

s.t.

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t$$

- We can write this as an unconstrained problem using the Lagrangian:

$$\max_{c_t^y, c_{t+1}^o, \lambda_t} \mathcal{L}(c_t^y, c_{t+1}^o, \lambda_t) = U(c_t^y, c_{t+1}^o) + \lambda_t \left[w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} \right]$$

Household optimality conditions

$$\max_{c_t^y, c_{t+1}^o, \lambda_t} \mathcal{L}(c_t^y, c_{t+1}^o, \lambda_t) = U(c_t^y, c_{t+1}^o) + \lambda_t \left[w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} \right]$$

- Euler equation:

$$\frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_t^y} = (1 + r_{t+1}) \frac{\partial U(c_t^y, c_{t+1}^o)}{\partial c_{t+1}^o}$$

- Lifetime budget constraint :

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t$$

As before, these two equations characterize the solution to the household's problem taking prices as given

Firms in the OLG model

Firm's profit maximization problem

- Firm gets revenue from producing and selling output, incurs costs by hiring labour and renting capital

$$\max_{K_t, L_t} \pi_t = F(z_t, K_t, L_t) - w_t L_t - R_t K_t$$

- Assume production function is Cobb-Douglas: $Y_t = z_t K^\alpha L^{1-\alpha}$

- All firms are identical: focus on representative firm's problem (whose demand represents the demand of all firms)

Firm's optimality conditions

- Define $k_t \equiv \frac{K_t}{L_t}$

- Then optimal labor demand satisfies:

$$(1-\alpha)z_t \left(\frac{K_t}{L_t}\right)^{\alpha} = (1-\alpha)z_t k_t^{\alpha} = w_t$$

- And optimal capital demand satisfies:

$$\alpha z_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \alpha z_t k_t^{\alpha-1} = R_t$$

- Because of perfect competition and Cobb-Douglas production, $\pi_t = 0$ (which also implies that firms give zero dividend income to households)

Market clearing

Aggregation - Total Supply and Demand for Inputs

- Total supply and demand for inputs are obtained by aggregating all individual labour and capital supplies and total firm demand.
- There are N individuals in each generation (no population growth).
- Total labour supply in t (by generation t individuals): N
- Total labour demand in t (by firms): L_t
- Total capital supply in t (by generation $t-1$ individuals, each saves a_t): Na_t
- Total demand for capital in period t (by firms): K_t

Market clearing

- From the individual household and firm problem, each agent told us how much they would consume/demand given prices.

- Prices adjust to clear markets

- Households and firms interact in

  - a labour market

  - an asset market

  - a goods market

- Note: when looking at market clearing, we need to aggregate (sum up) the choices of all households.

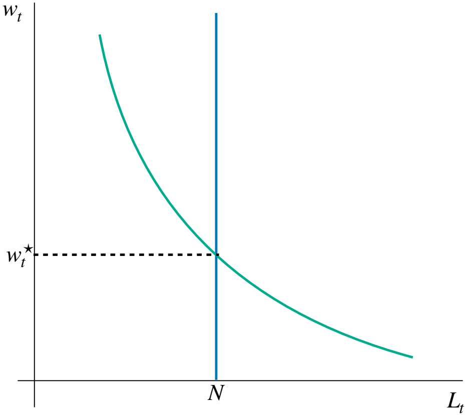
Labor market

- Young inelastically supply 1 unit of labour, $N_t = N \times 1 = N$

- Labor demand from firms:

$$L_t = \left[\frac{(1-\alpha) z_t K_t^{\alpha}}{w_t} \right]^{1/\alpha}$$

- In eqm, w_t adjust to make $L_t = N$



Asset market

- Capital supply **pre-determined**: Old own the capital at start of period: Na_t

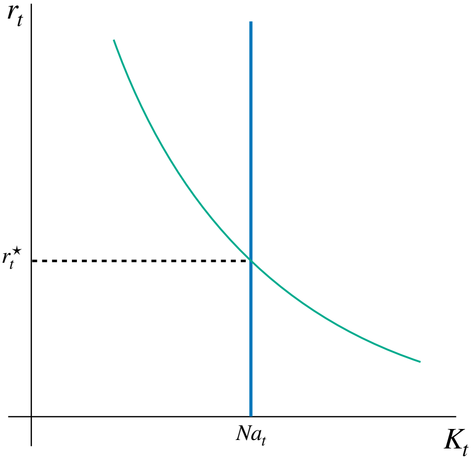
- Capital demand from firms:

$$K_t = \left[\frac{\alpha z_t L_t^{1-\alpha}}{R_t}\right]^{\frac{1}{1-\alpha}}$$

- In eqm, R_t adjust to make $K_t = Na_t$

- In eqm, rental rate of capital must equal gross return on savings:

$$R_t = 1 + r_t$$



Goods market

- Output supplied by firms: $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$

- Spending (output demanded):

  - young save by investing in capital stock $\implies Na_{t+1} = \overbrace{K_{t+1} = I_t}^{\text{because of full depreciation}}$

  - young household consumption: $Nc_t^y = C_t^y$

  - old household consumption: $Nc_t^o = C_t^o$

  $\implies$ total consumption in period $t = C_t = C_t^y + C_t^o$

Supply = Demand $\implies Y_t = C_t + K_{t+1} = C_t + I_t$

Equilibrium

- Equilibrium requires:
  - **Households** choose consumption and savings optimally
    - Euler Equation holds (MB of consuming today = MC of consuming today)
    - Lifetime Budget Constraint holds (must be afforable)
  - **Firms** choose capital and labour optimally (maximize profits)
  - All (labour, asset, goods) **markets clear**

An example

An example

- Suppose household preferences given by:

$$U(c_t^y, c_{t+1}^o) = \ln c_t^y + \beta \ln c_{t+1}^o$$

- Rest of set-up is same as what we just covered.
- Define $k_t = K_t/L_t$ (i.e., capital per worker).
- Solve for c_t^y, c_t^o, R_t, w_t, k_{t+1} in terms of z_t, k_t, α and β

An example

- From the firm's problem we have the following information:
  - Imposing labor market clearing $L_t = N$, optimal labour demand satisfies:

$$w_t = (1-\alpha)\, z_t k_t^{\alpha}$$

  - optimal capital demand satisfies:

$$R_t = \alpha z_t k_t^{-(1-\alpha)}$$

  - Perfect competition implies firms earn zero profits:

$$\pi_t = 0$$

From the firm's problem, and imposing market clearing, we have solved for prices in terms of **pre-determined** variable k_t, **exogenous** variable z_t and **parameter** α

An example

- From the household's problem we have the following information:
  - Euler Equation:
$$\frac{1}{c_t^y} = \beta \frac{1 + r_{t+1}}{c_{t+1}^o}$$
  - Lifetime budget constraint (LBC):
$$c_{t+1}^o = (1 + r_{t+1}) \left[w_t + \pi_t - c_y^y \right]$$
  - Budget constraint of young in t:
$$c_t^y + a_{t+1} = w_t + \pi_t$$
  - Budget constraint of old generation in t:
$$c_t^o = (1 + r_t) a_t$$

An example

- Plug in LBC into Euler

$$[w_t + \pi_t] = (1+\beta)\, c_t^y$$

An example

- Plug in LBC into Euler

$$[w_t + \pi_t] = (1+\beta)\, c_t^y$$

- From firm's optimality, we know π_t and w_t, plug in:

$$c_t^y = \frac{(1-\alpha)}{(1+\beta)} z_t k_t^\alpha$$

An example

- Plug in LBC into Euler

$$[w_t + \pi_t] = (1+\beta)\, c_t^y$$

- From firm's optimality, we know π_t and w_t, plug in:

$$c_t^y = \frac{(1-\alpha)}{(1+\beta)} z_t k_t^{\alpha}$$

- From young BC and capital market clearing every period, we know:

$$k_{t+1} = a_{t+1} = w_t + \pi_t - c_t^y$$

An example

- Plug in LBC into Euler

$$\left[w_t + \pi_t\right] = (1 + \beta)\, c_t^y$$

- From firm's optimality, we know π_t and w_t, plug in:

$$c_t^y = \frac{(1 - \alpha)}{(1 + \beta)} z_t k_t^\alpha$$

- From young BC and capital market clearing every period, we know:

$$k_{t+1} = \frac{\beta}{(1 + \beta)} (1 - \alpha)\, z_t k_t^\alpha$$

An example

- Plug in LBC into Euler

$$[w_t + \pi_t] = (1+\beta)\, c_t^y$$

- From firm's optimality, we know π_t and w_t, plug in:

$$c_t^y = \frac{(1-\alpha)}{(1+\beta)} z_t k_t^\alpha$$

- From young BC and capital market clearing every period, we know:

$$k_{t+1} = \frac{\beta}{(1+\beta)} (1-\alpha)\, z_t k_t^\alpha$$

- From old in t BC, we know:

$$c_t^o = (1+r_t)\, a_t = R_t k_t = \alpha z_t k_t^\alpha$$

An example

- Finally, let's verify that the total spending = total supply of resources:

$$\begin{aligned} k_{t+1} + c_t^y + c_t^o &= \frac{\beta}{(1+\beta)} (1-\alpha) z_t k_t^\alpha + \frac{(1-\alpha)}{(1+\beta)} z_t k_t^\alpha + \alpha z_t k_t^\alpha \\ &= z_t k_t^\alpha \\ &= y_t \end{aligned}$$

- Re-arrange and use fact $c_t^o + c_t^y = c_t$, and $k_{t+1} - (1-\delta)k_t = i_t$ where $\delta = 1$:

$$y_t = c_t + i_t$$

Congrats! You just solved a simple model of the aggregate economy!

An OLG model of the economy

- The OLG model is not a set of descriptive statistical relationships
- Rather, consumption and investment in the OLG model are outcomes from aggregating individual optimizing behaviour
- We assumed that these agents (firms and households) interacted by trading in markets
- And prices adjusted to make markets clear
- Now that we have developed a model, we can ask what are the predictions of our model.

Roadmap

- This class: Equilibrium in the OLG model and solution
- Next class: Growth in an OLG model