ECON30009/90080 – Tutorial 2 Solutions

This Version: Semester 2, 2025

Question 1: A 2 period consumption savings model

Suppose the household has utility given by $U(c^y, c^o) = (c^y)^\alpha (c^o)^{1-\alpha}$ where $0 < \alpha < 1$. The household also receives exogenous income y^y and y^o when young and old, respectively. There exists an asset that the household can choose to save in which has a gross rate of return $R = (1 + r)$.

Answers in Red

a) Write down the household's problem

$$\max_{c^y, c^o} (c^y)^\alpha (c^o)^{1-\alpha}$$

s.t.

$$c^y + \frac{c^o}{R} = y^y + \frac{y^o}{R}$$

where $R = (1 + r)$

b) Derive the household's optimality conditions

Set up Lagrangian to make this an unconstrained problem:

$$\mathcal{L} = (c^y)^\alpha (c^o)^{1-\alpha} + \lambda \left[y^y + \frac{y^o}{R} - c^y - \frac{c^o}{R} \right]$$

Take FOCs:

$$(c^y) : \alpha \left(\frac{c^o}{c^y} \right)^{1-\alpha} = \lambda$$

$$(c^o) : (1 - \alpha) \left(\frac{c^o}{c^y} \right)^{-\alpha} = \frac{\lambda}{R}$$

$$(\lambda) : y^y + \frac{y^o}{R} - c^y - \frac{c^o}{R} = 0$$

Combine FOC wrt c^y and c^o to get **Euler** equation:

$$c^o = \frac{1 - \alpha}{\alpha} R c^y$$

The Euler equation together with our lifetime budget constraint are the two optimality conditions from the household's problem.

c) Solve for optimal c^y and c^o in terms of variables the household takes as given. Compute also the ratio of consumption when old to consumption when young.

Plug Euler into LBC:

$$c^y = \alpha \left(y^y + \frac{y^o}{R} \right)$$

And use Euler to get c^o:

$$c^o = (1-\alpha)R \left(y^y + \frac{y^o}{R} \right)$$

Finally take c^o and divide by c^y to get ratio:

$$\frac{c^o}{c^y} = \frac{1-\alpha}{\alpha} R$$

d) Now suppose there is a tax on interest income. That is, τ proportion of your returns to savings Ra is taxed. How does c^o/c^y vary in response to the introduction of a tax on interest income? Provide an explanation for why the ratio changes in that direction

The introduction of $0 < \tau < 1$ reduces the return to savings and thus the relative price of consuming today when young. Following the same steps as before, we can show that the ratio of consumption when old to consumption when young is now given by

$$\frac{c^o}{c^y} = \frac{1-\alpha}{\alpha}(1-\tau)R$$

As τ gets larger (grows towards 1), c^o falls relative to c^y. This is because households observe a reduced incentive to save when the opportunity cost of consuming today when young is lower.

Question 2: Habit Persistence in the Household Problem

Consider the two period household consumption-savings problem. The household receives exogenous income of y^y and y^o when young and old, respectively. There exists an asset a that if you save in, gives a gross return of $R = 1+r$. Suppose the household's preferences feature **habit persistence**. Habit persistence (sometimes also known as habit formation) is the feature where household's utility from consumption not just on today's consumption but also his/her history of past consumption. That is, the household is only happier if he/she is able to consume more today than she did yesterday. Specifically, we will assume that the household's preferences are given by:

$$U(c^y, c^o) = \ln c^y + \beta \ln (c^o - \eta c^y)$$

where $0 \leq \eta \leq 1$.

a State what are the endogenous choice variables of the household.
The endogenous variables are c^y, c^o, a

b Set up the household's problem
The household's problem is to maximize her lifetime utility subject to her lifetime budget constraint:

$$\max U(c^y, c^o) = \ln c^y + \beta \ln(c^o - \eta c^y)$$

s.t.

$$c^y + \frac{c^y}{R} = y^y + \frac{y^o}{R}$$

We can also write this as a Lagrangian:

$$\mathcal{L} = \ln c^y + \beta \ln(c^o - \eta c^y) + \lambda \left[y^y + \frac{y^o}{R} - c^y - \frac{c^o}{R} \right]$$

c Derive the household's optimality conditions
Taking FOC wrt c^y, we have:

$$\frac{1}{c^y} - \frac{\eta\beta}{c^o - \eta c^y} = \lambda$$

FOC wrt c^o, we have:

$$\frac{\beta}{c^o - \eta c^y} = \frac{\lambda}{R}$$

and FOC wrt λ, we get back lifetime budget constraint (LBC):

$$\boxed{y^y + \frac{y^o}{R} - c^y - \frac{c^o}{R} = 0}$$

The LBC is one of the optimality conditions of the household.

We can further combine FOC wrt c^y with FOC wrt c^o to get the other household optimality condition, the household's Euler equation:

$$\boxed{\frac{1}{c^y} - \frac{\eta\beta}{c^o - \eta c^y} = \frac{R\beta}{c^o - \eta c^y}}$$

which (re-arranging) is equivalent to:

$$c^o = \left[\eta\left(1 + \beta\right) + R\beta\right] c^y$$

d Solve for c^y in terms of y^y, y^o, R, β and η. Explain how c^y varies with η, which is the weight households put on past consumption when old (the

habit). Give a brief intuition as to why c^y varies with η in that direction.
Plug the Euler equation into the LBC to get:

$$y^y + \frac{y^o}{R} = c^y + \frac{c^o}{R} = c^y\,(1+\beta)\,\frac{\eta + R}{R}$$

Make c^y the subject of the equation:

$$c^y = \frac{R}{(1+\beta)\,(\eta+R)}y^y + \frac{1}{(1+\beta)\,(\eta+R)}y^o$$

Once we know c^y, we effectively also know c^o from the Euler equation.

Notice that c^y depends negatively on η (η appears in the denominator). Intuitively, a higher η makes c^y lower as the household recognizes that consuming alot today would require them to consume even more tomorrow since the household only gets utility from c^o if their level of consumption is higher than what they consumed when they were young.

Question 3: A CES production function

Assume $Y = (K^\gamma + L^\gamma)^{1/\gamma}$ where Y is output, K is capital and L is labor. γ is a parameter that determines the elasticity of substitution. Does this production function satisfy all the properties we would like in a production function?

a) Show that output is increasing in its inputs

$$\frac{\partial Y}{\partial K} = (K^\gamma + L^\gamma)^{\frac{1-\gamma}{\gamma}}\,K^{\gamma-1} > 0$$
$$\frac{\partial Y}{\partial L} = (K^\gamma + L^\gamma)^{\frac{1-\gamma}{\gamma}}\,L^{\gamma-1} > 0$$

b) Show that the production function features diminishing marginal returns

$$\frac{\partial^2 Y}{\partial K^2} = -(1-\gamma)\,(K^\gamma + L^\gamma)^{\frac{1-2\gamma}{\gamma}}\,K^{\gamma-2}L^\gamma < 0$$
$$\frac{\partial^2 Y}{\partial L^2} = -(1-\gamma)\,(K^\gamma + L^\gamma)^{\frac{1-2\gamma}{\gamma}}\,L^{\gamma-2}K^\gamma < 0$$

c) Show that the function satisfies constant returns to scale
Here we show that if we scale the inputs by x, this is equivalent to scaling the output by x.

$$[(xK)^\gamma + (xL)^\gamma]^{1/\gamma} = [x^\gamma(K^\gamma + L^\gamma)]^{1/\gamma} = x\,[K^\gamma + L^\gamma] = xY$$

d) Show that K and L are complements for $0 < \gamma < 1$

$$\frac{\partial^2 Y}{\partial K \partial L} = (1-\gamma)\left(K^\gamma + L^\gamma\right)^{\frac{1-2\gamma}{\gamma}} K^{\gamma-1} L^{\gamma-1} > 0$$

e) Write down the firm's profit maximization problem taking the wage rate w and rental rate of capital R as given.

$$\max_{K,L} (K^\gamma + L^\gamma)^{1/\gamma} - RK - wL$$

f) Suppose $0 < \gamma < 1$. Derive the firm's optimality conditions

$$\left(K^\gamma + L^\gamma\right)^{\frac{1-\gamma}{\gamma}} K^{\gamma-1} = R$$
$$\left(K^\gamma + L^\gamma\right)^{\frac{1-\gamma}{\gamma}} L^{\gamma-1} = w$$

g) Suppose $R < w$ and $\gamma = 1$. Would the firm like to use labour in production? Provide some brief intuition to your answer.
In this case, when $\gamma = 1$, capital and labour are perfect substitutes. The firm would prefer to use the cheaper input in production and would not hire any labour. Note that this is a **corner** solution, and we cannot apply our usual first order conditions to solve for K and L.