Topic 3. Introduction to Portfolio Selection under Uncertainty

ECON30024 Economics of Finanical Markets

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Semester 1, 2025

Outline

1. Risk and uncertainty.

2. The expected utility hypothesis (EUH)

    - Expected utility

    - Risk aversion

    - Portfolio selection with expected utility

3. The Mean-Variance approach

Required reading: Chap. 4 of Bailey

1. Risk and Uncertainty

- As discussed in Topic 1, a theory of asset price determination requires modeling portfolio selection under uncertainty.
- Such decision also involves specifying a utility function and a budget constraint for investors.
- How people incorporate **uncertainty** in their preference or utility function?
- The **expected utility hypothesis** (EUH) remains the most popular approach to modeling uncertainty in economics.
- The **mean-variance** objective is a special case of expected utility.

- In economics, **risk** refers to those unknown events for which '**objective** probabilities' can be assigned.

- **Uncertainty** applies to events for which probabilities cannot be assigned without being **subjective**.

- Unknown events in financial markets are typically located somewhere between the two extremes.

- We'll assume probabilities for unknown events can be assigned (objectively or subjectively), and use the two terms interchangeably.

2. The Expected Utility Hypothesis (EUH)

- Basic ingredients in portfolio selection problem under uncertainty

  1) **States**: uncertainty is represented by states of the world:

  $$S = \{s_1, s_2, ..., s_K\},$$

  where each s_k denotes a state that could occur.

  2) **Actions**: a portfolio choice made **prior to** the state of the world being revealed:

  $$(x_1, x_2, .., x_n)$$

  where each x_j denotes units of asset j chosen to hold.

3) **Consequences**: terminal wealth from the portfolio in each state of the world

$$W_k = v_{k1}x_1 + \cdots + v_{kn}x_n = \sum_{j=1}^{n} v_{kj}x_j, \quad k = 1, \ldots, K \tag{1}$$

where v_{kj} is the unit payoff of asset j in state k.

4) **Preferences**: consequences are valued according to a utility function:

$$U(W_1, W_2, .., W_K)$$

5) **Constraints**: we focus on budget constraint

$$\sum_{j=1}^{n} p_j x_j = A \tag{2}$$

where A is initial wealth, p_j is unit price of asset j.

- An investor's portfolio selection problem: choose $(x_1, \ldots, x_n)$ to maximise U, subject to the budget constraint.

2.1 Expected utility

- Under the expected utility hypothesis, preference U is specified as an expected utility:

$$E(u(W)) = \pi_1 u(W_1) + \pi_2 u(W_2) + \ldots + \pi_K u(W_K) \tag{3}$$

  - π_k is the probability the investor assigns to state s_k, reflecting her **belief** about which state will occur.
  - $u(\cdot)$ is a function of a **single** possible consequence that is the same for all states, reflecting the investor's **preferences** over the consequences.
  - u is known as the **von Neumann-Morgenstern** (vNM) utility function.

- The expected utility form seems quite natural, however, it was mathematically derived under some strong assumptions on preferences.
  - This is established in the Expected Utility Theorem.
  - For instance, one assumption is that preferences are independent of beliefs.
  - Further discussion on the underlying assumptions are in Tutorial 3.
  - Despite evidence on individual behaviour that contradicts its assumptions (Topic 11), EUH remains the mainstream approach in modeling decision making under uncertainty.

2.2 Risk aversion

- At the center of the expected utility form in (3) is the vNM utility function u.
- First, u reflects investors' attitude toward wealth.
  - It's natural to assume more wealth is preferred to less.
  - We assume $u(W)$ is **strictly increasing** in W, or $u'(W) > 0$.
- Second, u reflects investors' risk preference.
  - Do you like **variability of outcomes across states**?
    - No: **Risk averse**
    - Indifferent: **Risk neutral**
    - Yes: **Risk loving**

– An example: Consider a bet that pays \$0 with probability 0.5 and \$100 with probability 0.5.

Would you choose the bet or \$50 for sure?

What can you infer about your utility function u?

– Formalise risk preference by the **curvature** of u

$$\begin{aligned} \text{Risk averse :} &\quad E[u(W)] < u(E[W]) \\ \text{Risk neutral :} &\quad E[u(W)] = u(E[W]) \\ \text{Risk loving :} &\quad E[u(W)] > u(E[W]) \end{aligned}$$

Equivalently,

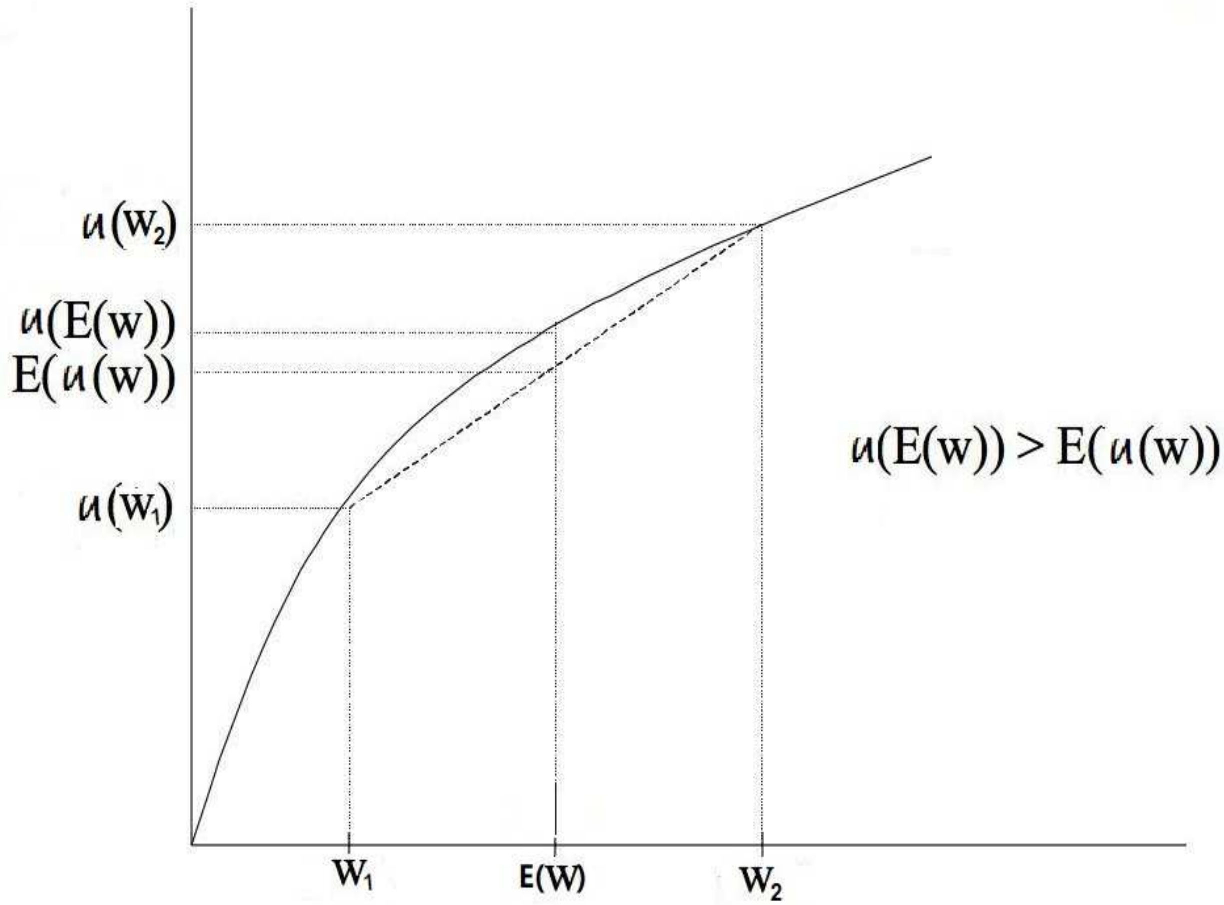
$$\text{Risk averse : } u \text{ is strictly concave, or } u''(W) < 0$$

$$\text{Risk neutral : } u \text{ is linear, or } u''(W) = 0$$

$$\text{Risk loving : } u \text{ is strictly convex, or } u''(W) > 0$$

– There is considerable empirical evidence that individuals are **risk averse**, so u is assumed to be **strictly concave**.

Figure 3.1: A concave vNM utility function $u(W)$



- Third, further assumptions on u to reflect the degree of risk aversion
  - **Absolute risk aversion (ARA)**: would you increase or decrease the *absolute amount* of wealth invested in risky assets if your wealth increases?
    - increase: decreasing ARA
    - no change: constant ARA
    - decrease: increasing ARA
  - **Relative risk aversion (RRA)**: would you increase or decrease the *percentage* of wealth invested in risky assets if your wealth increases?
    - increase: decreasing RRA
    - no change: constant RRA
    - decrease: increasing RRA

– Formalise the degree of absolute risk aversion:

- Measure of ARA:

$$R_A(W) = -\frac{u''(W)}{u'(W)} > 0$$

- Examples of $u(W)$ that exhibit increasing, constant, or decreasing ARA:

IARA	$R'_A(W) > 0$	$u(W) = W - bW^2$, $b > 0$, $0 \leq W < \frac{1}{2b}$
CARA	$R'_A(W) = 0$	$u(W) = -\exp(-cW)$, $c > 0$
DARA	$R'_A(W) < 0$	$u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$

– Formalise the degree of relative risk aversion:

· Measure of RRA:

$$R_R(W) = W R_A(W) = -\frac{W u''(W)}{u'(W)} > 0$$

· Examples of $u(W)$ that exhibit increasing, constant, or decreasing RRA:

IRRA	$R_R'(W) > 0$	$u(W) = -\exp(-cW)$, $c > 0$
CRRA	$R_R'(W) = 0$	$u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, $\gamma > 0, \gamma \neq 1$
DRRA	$R_R'(W) < 0$	$u(W) = -\frac{1}{W-1}$

– Negative exponential utility ($-\exp(-cW)$) and power utility ($\frac{W^{1-\gamma}}{1-\gamma}$) are often used in the literature.

· $u(w) = -\exp(-cW)$ exhibits CARA:

$$R_A(W) \equiv -\frac{u''(W)}{u'(W)} = c$$

· $u(w) = \frac{W^{1-\gamma}}{1-\gamma}$ exhibits CRRA:

$$R_R(W) \equiv -\frac{W u''(W)}{u'(W)} = \gamma$$

Verify these results and check your work with the solution in Exercise_Topic3.

- Relationship between ARA and RRA (see Exercise_Topic3)
  - If an investor has CRRA, then she has increasing, decreasing or constant ARA?
- There is empirical evidence of CRRA and DARA (Blume and Friend, 1975, AER).

- In summary, typical assumptions imposed on the vNM utility function $u(W)$
  - u is strictly increasing
  - u is strictly concave
  - u exhibits CRRA (often assumed, not always)

2.3 Portfolio selection with expected utility

- The static portfolio selection problem for an investor:

$$\max_{(x_1,\ldots,x_n)} E(u(W))$$

$$\text{s.t.} \quad \sum_{j=1}^{n} p_j x_j = A$$

  – Recall that $E(u(W)) \equiv \sum_{k=1}^{K} \pi_k u(W_k)$, where

$$W_k = v_{k1}x_1 + \cdots + v_{kn}x_n = \sum_{j=1}^{n} v_{kj}x_j, \quad k = 1, \ldots, K \tag{1}$$

  – Since (1) holds for all k, we have

$$W = \sum_{j=1}^{n} v_j x_j \tag{4}$$

  where v_j is asset j's unit payoff (a random variable).

- Alternative formulation of the portfolio selection problem
  - Let $a_j \equiv \frac{p_j x_j}{A}$ be the proportion of initial wealth invested in asset j, then the budget constraint is re-written as

$$\sum_{j=1}^{n} a_j = 1$$

  - Let r_j denote the rate of return on asset j, $j = 1, \ldots, n$, then

$$r_j = \frac{v_j - p_j}{p_j} \quad \Rightarrow \; v_j = (1 + r_j)p_j$$

  - Then (4) becomes

$$\begin{aligned} W &= \sum_{j=1}^{n} v_j x_j = \sum_{j=1}^{n} (1 + r_j) p_j x_j = \sum_{j=1}^{n} (1 + r_j) a_j A \\ &= \left(\sum_{j=1}^{n} a_j + \sum_{j=1}^{n} a_j r_j \right) A = \left(1 + \sum_{j=1}^{n} a_j r_j \right) A \end{aligned}$$

– Define

$$r_P \equiv \sum_{j=1}^{n} a_j r_j \tag{5}$$

then

$$W = (1 + r_P)A$$

Note that r_P is the rate of return on the portfolio.

– The portfolio selection problem can be re-written as:

$$\max_{(a_1,\ldots,a_n)} E\Big[u\left[(1 + r_P)A\right]\Big], \qquad \text{s.t.} \quad \sum_{j=1}^{n} a_j = 1$$

– Given u and the distributions of r_j's, the constrained maximisation problem can be solved (though not easy).

3. The Mean-Variance Approach (MVA)

- In the MVA, the preference or objective of the investor is assumed to be

$$G(\mu_P, \sigma_P^2)$$

  - μ_P: the expected rate of return on the portfolio
  - σ_P^2: the variance of the rate of return on the portfolio, representing the 'risk' of the portfolio.
  - G is increasing in μ_P and decreasing in σ_P, i.e., μ_P is a 'good', and σ_P is a 'bad'.

- Example: $G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$, what is the investor's risk preference if $\alpha > 0$, or $\alpha = 0$?

- The mean-variance objective is a special case of expected utility.
  - If u is a quadratic function, the expected utility $E(u(W))$ can be written as a mean-variance objective.
    (see Exercise_Topic3)
- Mean-variance objective is a reasonable approximation to investors' preference.
- However, it could ignore other things about the portfolio which investors may view as important.
  (Tutorial 3 discussion)

- Indifference curves of $G(\mu_P, \sigma_P^2)$
  - Indifference curves are drawn in (σ_P, μ_P) space.
  - Each indifference curve represents the combinations of σ_P and μ_P that give the same level of utility.

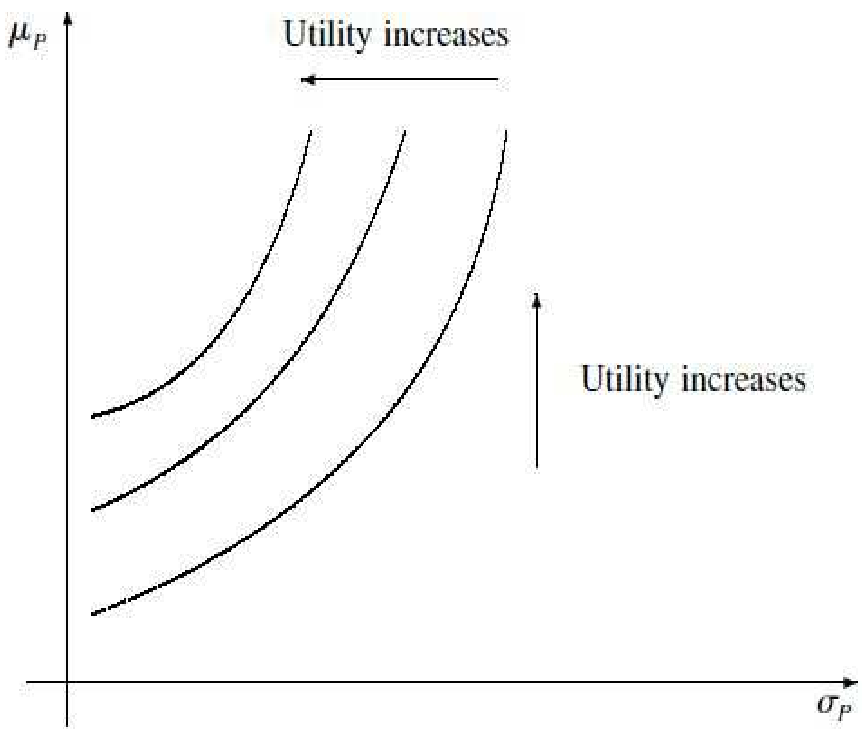
  An example: Let $G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$, where $\alpha > 0$, plot an indifference curve of G.

$$G(\mu_P, \sigma_P^2) = g_0, \text{ i.e., } \mu_P - \alpha\sigma_P^2 = g_0$$

$$\Rightarrow \mu_P = \alpha\sigma_P^2 + g_0.$$

  That is, μ_P is an **increasing convex** function of σ_P.

Figure 3.2: Indifference curves of a mean-variance objective



Why are indifference curves of a mean-variance objective upward slopping and convex? (Tutorial 3)

Review Questions

1. What is the difference between 'risk' and 'uncertainty'?
2. Understand the definitions of states, actions, consequences and preferences.
3. Under the EUH, how can you express the preferences of investors?
4. Understand the concepts of risk averse, risk neutral, and risk loving.
5. What are the mathematical properties of the vNM utility function for it to exhibit risk aversion, risk neutrality, and risk loving?
6. Why do we use utility functions that exhibit risk aversion in our analysis?
7. Understand the concept of absolute risk aversion and relative risk aversion.
8. What are the relationships between ARA and RRA?
9. How to measure ARA and RRA?
10. Give a utility function that exhibits constant RRA.
11. How to formulate the portfolio selection problem with expected

utility in two alternative ways? One way is to choose the quantity of each asset to hold, another is to choose the proportion of initial wealth invested in each asset.

12. How is the rate of return on a portfolio defined? Why it is a weighted sum of the rate of returns on the individual assets? Derive this expression mathematically.
13. What is the mean-variance objective function? Why is it increasing in μ_P and decreasing in σ_P?
14. Understand why the indifference curves of a mean-variance objective are upward sloping and convex.
15. For $G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$, be able to draw the indifference curves.
16. Roughly understand why the mean-variance objective is a special case of expected utility.
17. Think about other things that you care about other than expected return and variance of return on the portfolio when choosing a portfolio of assets to invest in.