# Topic 9. Dynamic Asset Pricing and the Equity Premium Puzzle

**ECON30024 Economics of Finanical Markets**

**Shuyun May Li**

**Department of Economics**

**The University of Melbourne**

**Semester 1, 2025**

# Outline

1. Introduction
2. A simple two-period model with certainty
3. Uncertainty and the stochastic discount factor
4. The consumption CAPM
5. An infinite-horizon asset pricing model
6. The equity premium puzzle

Required reading: Bailey, Chapter 11

Further reading: Mehra and Prescott (1985), and Kocherlakota (1996)

# 1. Introduction

- The CAPM is a static asset pricing model, in which individuals' portfolio decision making is limited to a single date.
- The NPV theory admits multiple periods and infinite horizon, but individual portfolio decisions are neglected.
- In this topic, the optimising choice of investors is extended to a multiperiod setting.
- Investors are treated as consumers, who aim to maximise expected lifetime utility by optimally allocating consumption across time–**intertemporal optimisation**.

- Consumption allocation across time is achieved through accumulating and transferring wealth across time.
- Wealth transfer and accumulation over time is through saving, and saving is made by holding a portfolio of assets

- The optimal intertemporal choice is characterised by the **intertemporal consumption Euler equation**.
  - It implies an NPV equation with a **stochastic discount factor**.

- We'll begin with a simple two-period model with certainty, then extend it to uncertainty, many assets and multiple periods.
  - The Euler equations in each case take a similar form.

- We then derive a prediction from the Euler equation – the **consumption CAPM**.

- Finally, we briefly introduce an infinite-horizon asset pricing model and discuss a **quantitative** implication of the model – the **equity premium puzzle**.

# 2. A Simple Two-period Model with Certainty

## 2.1 The model

- Consider a single investor's utility maximisation problem.
- Suppose the investor can live for two periods: born in $t$, and passes away in $t+1$.
  - Born with an exogenous endowment of wealth, $W_t$, which can be allocated between consumption and savings.
  - Consumption in $t$ and $t+1$ are denoted by $C_t$ and $C_{t+1}$, and savings in $t$ is denoted by $S_t$.
- She can save through holding assets.
  - Under certainty, no arbitrage implies that all assets have the same rate rate of return from $t$ to $t+1$.

- Without loss of generality, assume that there is one risk free asset with interest rate $r_{t+1}$.

- The investor's preference is captured by a lifetime utility function:

$$U(C_t, C_{t+1}) = u(C_t) + \delta u(C_{t+1}). \tag{1}$$

  - The parameter $0 < \delta \leq 1$ denotes a **subjective discount factor**, which reflects the time preference of the investor.
  - The function $u(\cdot)$ is called the flow/periodic utility function.
  - Typical assumptions on $u$ (recall Topic 3):
    $u'(\cdot) > 0$, $u''(\cdot) < 0$,
    and $\lim_{C \to 0} u'(C) = \infty$ (the Inada condition).

## 2.2 The optimal consumption plan

- How to formulate the individual's utility maximisation problem?
- First, we need to write down the individual's budget constraint in each period.

Period $t$ :

Period $t+1$ :

Combining gives

or equivalently,

$$C_t + \frac{C_{t+1}}{1+r_{t+1}} = W_t \tag{2}$$

This is the **lifetime budget constraint** (economic intuition?)

- The investor's utility maximisation problem is to maximise (1), subject to (2), taking $r_{t+1}$ as given:

$$\max_{C_t, C_{t+1} \geq 0} \; U(C_t, C_{t+1}) = u(C_t) + \delta u(C_{t+1})$$

$$\text{s.t.} \qquad C_t + \frac{C_{t+1}}{1 + r_{t+1}} = W_t \tag{2}$$

  – The constrained maximisation problem above can be solved using the Lagrangian method.

  – Or, we can substitute out $C_{t+1}$ to transform it into an **unconstrained** maximisation problem:

$$\max_{C_t \geq 0} \; u(C_t) + \delta u\Big((1 + r_{t+1})(W_t - C_t)\Big)$$

- The first-order condition for this problem (the Inada condition implies optimal $C_t > 0$):

  Simplifying gives

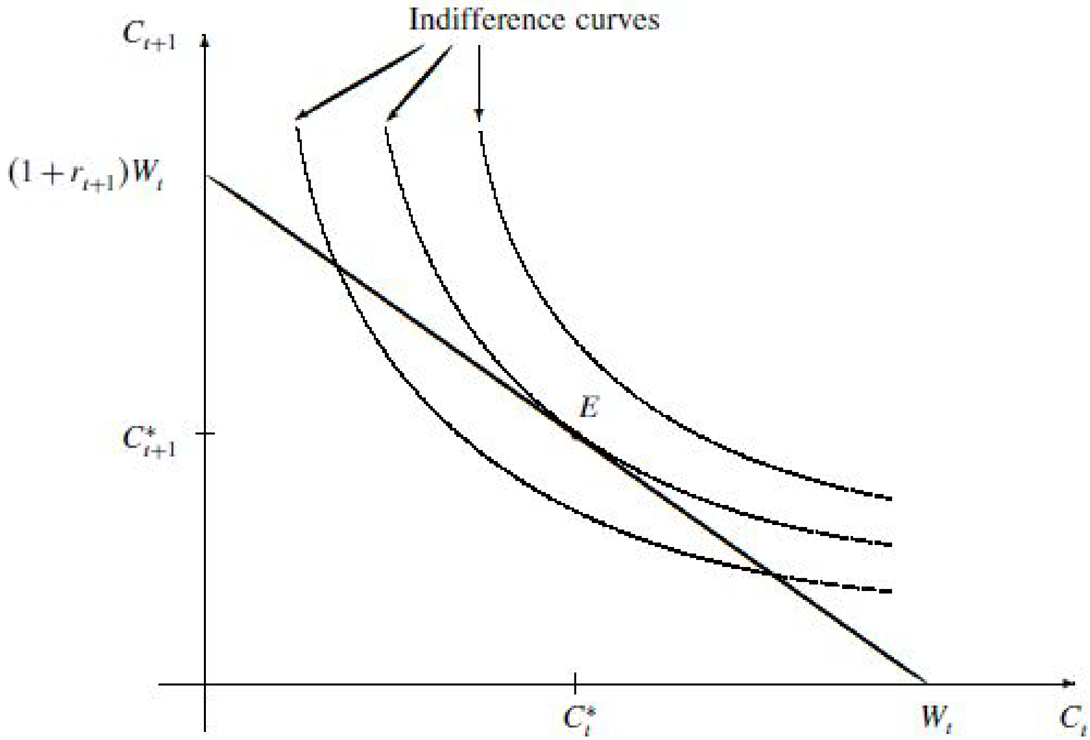
$$u'(C_t) = \delta u'(C_{t+1})(1 + r_{t+1}) \tag{3}$$

  Equation (3) is known as the **intertemporal consumption Euler equation**.

  – What is the economic intuition here?

- The optimal consumption plan $(C_t^*, C_{t+1}^*)$ is determined by the Euler equation (3) and the lifetime budget constraint (2).

- A graphical illustration is given in Figure 1




The slope of the budget line: $-(1+r_{t+1})$

The slope of an indifference curve at $(C_t, C_{t+1})$: $-\frac{u'(C_t)}{\delta u'(C_{t+1})}$

- An example: Let $u(C) = \log(C)$, then the utility maximisation problem is given by

$$\max_{C_t, C_{t+1} \geq 0} U(C_t, C_{t+1}) = \log(C_t) + \delta \log(C_{t+1})$$

$$\text{s.t.} \qquad C_t + \frac{C_{t+1}}{1 + r_{t+1}} = W_t$$

This problem can be solved in closed form (see Appendix A):

$$C_t^* = \frac{1}{1+\delta} W_t$$

$$S_t^* = \frac{\delta}{1+\delta} W_t$$

$$C_{t+1}^* = \frac{\delta}{1+\delta} (1 + r_{t+1}) W_t$$

**Question**: How do $C_t^*$, $S_t^*$ and $C_{t+1}^*$ depend on $\delta$ and $r_{t+1}$?

- The Euler equation (3) can be equivalently written as

$$1 = \left[\delta \frac{u'(C_{t+1})}{u'(C_t)}\right](1 + r_{t+1}). \tag{4}$$

Using $1 + r_{t+1} = \frac{p_{t+1}+d_{t+1}}{p_t}$, (4) implies that

$$p_t = \left[\delta \frac{u'(C_{t+1})}{u'(C_t)}\right](d_{t+1} + p_{t+1}). \tag{4'}$$

  – Note that (4') is an NPV equation for the price of the asset, where the **discount factor** is given by

$$\delta \frac{u'(C_{t+1})}{u'(C_t)}$$

- Equation (4) implies that there is a relationship between the rate of return on assets and **consumption dynamics**.

# 3. Uncertainty and the Stochastic Discount Factor

## 3.1 Uncertainty

- Now suppose the investor can save through holding **one risky asset** with stochastic rate of return $r_{t+1}$.
  - Then period $t+1$ consumption $C_{t+1}$ becomes uncertain:

$$C_{t+1} = (1 + r_{t+1})(W_t - C_t).$$

- Assume that the investor acts to maximise her **expected** lifetime utility (recall Topic 3):

$$E_t U(C_t, C_{t+1}) = u(C_t) + \delta E_t u(C_{t+1}). \tag{5}$$

- The first-order condition for the utility maximisation problem is given by

$$u'(C_t) + \delta E_t \Big[ u'(C_{t+1})(-(1 + r_{t+1})) \Big] = 0$$

$$\Rightarrow \quad u'(C_t) = \delta E_t \Big[ u'(C_{t+1})(1 + r_{t+1}) \Big]$$

$$\Rightarrow \quad 1 = E_t \left[ \delta \frac{u'(C_{t+1})}{u'(C_t)} (1 + r_{t+1}) \right]. \tag{6}$$

This is the intertemporal consumption Euler equation under uncertainty (uncertainty version of (4)), where

$$\delta \frac{u'(C_{t+1})}{u'(C_t)}$$

is known as the **stochastic discount factor**.

- Again the optimal consumption plan is determined by the Euler equation (6) and the lifetime budget constraint (2).

## 3.2 Multiple assets

- Now we allow for saving through holding a portfolio of $n$ assets with returns given by $r_{j,t+1}$, $j = 1, 2, \ldots, n$.

- Denote the portfolio as $(a_1, a_2, \ldots, a_n)$, $\sum_{j=1}^{n} a_j = 1$, where $a_j$ is the proportion of total savings invested in asset $j$.

  – The investment expenditure on asset $j$ is:

- So the period $t+1$ consumption is given by

$$ \tag{7}$$

- The investor's problem is to maximise her expected lifetime utility given in (5), by choosing an **optimal consumption and portfolio** plan that satisfy (7):

$$\max_{C_t, a_1, \ldots, a_n} \quad E_t U(C_t, C_{t+1}) = u(C_t) + \delta E_t u(C_{t+1})$$

$$\text{s.t.} \qquad C_{t+1} = \left( \sum_{j=1}^{n} (1 + r_{j,t+1}) a_j \right) (W_t - C_t) \tag{7}$$

$$\sum_{j=1}^{n} a_j = 1$$

Eq. (7) can be used to substitute out $C_{t+1}$ from the objective function, so there is only one constraint for the problem.

- Using the Lagrangian method, we can derive the intertemporal consumption Euler equation (Appendix B):

$$u'(C_t) = \delta E_t \Big[ u'(C_{t+1})(1 + r_{j,t+1}) \Big], \quad \text{for all } j = 1, 2, \ldots, n$$

– $\delta E_t \Big[ u'(C_{t+1})(1 + r_{j,t+1}) \Big]$ represents the increase in expected utility resulting from a marginal increase in the investment in asset $j$.

– This term is equalised across assets. Otherwise, the investor could achieve a higher expected utility by reallocating her portfolio.

- We can rewrite the Euler equation to obtain an equation analogous to (6):

$$1 = E_t \left[ \delta \frac{u'(C_{t+1})}{u'(C_t)} \left(1 + r_{j,t+1}\right) \right]. \tag{8}$$

Again, $\delta \frac{u'(C_{t+1})}{u'(C_t)}$ is the stochastic discount factor.

## 3.3 Long time horizon

- Now suppose at $t$ the investor can live for $T+1 \geq 2$ periods.
- Her expected lifetime utility is given by

$$E_t U(C_t, C_{t+1}, \ldots, C_{t+T}) = u(C_t) + \delta E_t u(C_{t+1}) + \cdots + \delta^T E_t u(C_{t+T})$$

- At $t$, the investor chooses a consumption and portfolio plan to maximise $E_t U(C_t, C_{t+1}, \ldots, C_{t+T})$.
  - The consumption plan: $(C_t, C_{t+1}, \ldots, C_{t+T})$
  - The portfolio plan (choose a portfolio of assets in every period except in last period):

$$(a_{1,t}, a_{2,t}, \ldots, a_{n,t}),\ (a_{1,t+1}, a_{2,t+1}, \ldots, a_{n,t+1}),\ \ldots,$$

$$(a_{1,t+T-1}, a_{2,t+T-1}, \ldots, a_{n,t+T-1})$$

- Despite the complexity, the intertemporal consumption Euler equations take the same form as (8).

$$1 = E_t \left[ \delta \frac{u'(C_{t+1})}{u'(C_t)} \left(1 + r_{j,t+1}\right) \right], \quad j = 1, 2, \ldots, n$$

  This equation holds for any two adjacent dates over the investor's life.

- How to solve the multiperiod utility maximisation problem?
  - If $T$ is large, it's hard to solve the problem directly, even numerically.
  - The strategy is to use a *recursive approach*: solving a sequence of two-period problems, starting from the second last period (to be covered in graduate school).

# 4. The Consumption CAPM

- The Euler equation (8) suggests that there is a relationship between the expected returns on assets and consumption.
- A straightforward application of this theoretical result is to use it to derive the **consumption CAPM** prediction.
- A few assumptions are needed to ensure the stochastic discount factor is the same for all investors and it is uniquely determined.
  - Investors have homogeneous beliefs and preferences.
  - Frictionless markets

- Deriving the consumption CAPM
  - First, we define the stochastic discount factor as $H_{t+1} \equiv \delta \frac{u'(C_{t+1})}{u'(C_t)}$, then (8) becomes

$$1 = E_t \left[ H_{t+1}(1 + r_{j,t+1}) \right].$$

Omitting the time subscripts for simplicity:

$$1 = E \left[ H(1 + r_j) \right]. \tag{9}$$

  - As (9) holds for all assets, it holds for an asset whose return has zero correlation with $H$, call it asset 0:

$$1 = E[H(1 + r_0)],$$

where $r_0$ satisfies $cov(r_0, H) = 0$.

– Combining the two equations gives

$$E[H(r_j - r_0)] = 0.$$

– Using the formula $E(XY) = cov(X, Y) + E(X)E(Y)$,

$$E[H(r_j - r_0)] = cov(r_j - r_0, H) + E(r_j - r_0)\,E(H)$$

$$= cov(r_j, H) + [E(r_j) - E(r_0)]\,E(H) = 0$$

$$\begin{aligned} \Rightarrow E[r_j] - E[r_0] &= -\frac{cov(r_j, H)}{E(H)} \\ &= \left(\frac{cov(r_j, H)}{var(H)}\right)\left(-\frac{var(H)}{E(H)}\right) \\ &\equiv \beta_{jH}\;\theta_H, \end{aligned}$$

where $\beta_{jH} \equiv \frac{cov(r_j, H)}{var(H)}, \quad \theta_H \equiv -\frac{var(H)}{E(H)}$

- Then we reach the following consumption CAPM prediction:

$$\mu_j - \mu_0 = \beta_{jH}\,\theta_H, \quad \beta_{jH} \equiv \frac{cov(r_j, H)}{var(H)} \tag{10}$$

- A comparison with the Black CAPM prediction:

$$\mu_j - \omega = \beta_j\,(\mu_M - \omega), \;\; \beta_j = \frac{cov(r_j, r_M)}{var(r_M)}$$

  - The stochastic discount factor $H$ is in place of the market return $r_M$, as the systematic factor.
  - $\beta_{jH}$ is analogous to $\beta_j$, also measuring the risk of asset $j$ caused by variations in the systematic factor.
  - $\mu_0$ is analogous to $\omega$, the expected return on an asset with zero correlation with the systematic factor.
  - $\theta_H$ is in place of $\mu_M - \omega$, also the same for all assets.

- In empirical applications, $u$ often takes the CRRA form $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$, then the consumption CAPM prediction becomes

$$\mu_j - \mu_0 = \beta_{jc}\theta_c, \tag{11}$$

$$c \equiv \log\left(\frac{C_{t+1}}{C_t}\right),\ \beta_{jc} = \frac{cov(r_j, c)}{var(c)},\ \theta_c \equiv \frac{\gamma var(c)}{1 - \gamma E(c)},$$

and $\mu_0$ is the expected return on an asset whose return has zero correlation with $c$ (Q3 in week 11 tutorial).

- Empirical tests of the consumption CAPM can be formulated in the same way as for the CAPM, with the **growth rate of consumption** $c$ replacing $r_M$.

# Summary

- We focused on an individual investor's intertemporal choice of consumption and portfolio plan that aims to maximise her expected lifetime utility.
- The decision problem is formulated as a constrained maximisation problem, subject to a lifetime budget constraint.
- The optimal consumption and portfolio plan is characterised by the intertemporal consumption Euler equation, which implies an NPV equation with a stochastic discount factor.
- An application of this equation is the consumption CAPM, which is comparable to the Black CAPM, with the stochastic discount factor replacing the market return.

## 5. An Infinite-horizon Asset Pricing Model

- Now extend the setting to infinite horizon, and assume investors are identical and there are only two assets: a bond and an equity (stock).
- The model is essentially the asset pricing model proposed by Mehra and Prescott (1985) which builds a basic framework for modern asset pricing theory.
- We'll briefly describe the model and its quantitative implication for the risk premium on equity – the famous **equity premium puzzle**.
- The model setup
  - All consumers/investors are identical, so can be summarised by a representative consumer.

– The representative investor is infinitely lived, with expected lifetime utility given by

$$U(C_t, C_{t+1}, C_{t+2}, \cdots) = u(C_t) + \delta E_t u(C_{t+1}) + \delta^2 E_t u(C_{t+2}) + \cdots$$
$$= E_t \left\{ \sum_{s=0}^{\infty} \delta^s u(C_{t+s}) \right\}, \quad 0 < \delta < 1 \qquad (12)$$

– In every period $t$, she is endowed with an exogenous wealth, $W_t$.

– In every period $t$, she can invest in a **risk free bond**, $B_t$, and a **risky stock or equity**, $S_t$.

– The rates of return on the assets from period $t$ to $t+1$:

$$r^f_{t+1} : \text{ for the bond, known at time } t$$
$$r_{t+1} : \text{ for the stock, realised at time } t+1.$$

- The representative investor's **utility maximisation problem**
  - The investor's problem is to choose an optimal consumption and portfolio plan to maximise her expected lifetime utility.
  - The optimal plan must satisfy her budget constraints in all periods.
  - Her budget constraint in a typical period $t$ is given by

$$C_t + B_t + S_t = (1 + r_t^f)B_{t-1} + (1 + r_t)S_{t-1} + W_t \qquad (13)$$

    for all $t$.
  - So her utility maximisation problem is to maximise (12), subject to a sequence of budget constraints (13).

- The **intertemporal consumption Euler equations** take the same form as in finite-horizon models:

$$1 = E_t\left[\frac{\delta u'(C_{t+1})}{u'(C_t)}\right](1 + r^f_{t+1}) \tag{14}$$

$$1 = E_t\left[\frac{\delta u'(C_{t+1})}{u'(C_t)}(1 + r_{t+1})\right] \tag{15}$$

  - The stochastic discount factor is again given by $\delta \frac{u'(C_{t+1})}{u'(C_t)}$.
  - If the investor is risk neutral, i.e., $u(C)$ is a linear function of $C$, then $u'(C)$ is a constant and the Euler equations become

$$1 = \delta(1 + r^f_{t+1}) \quad \Rightarrow r^f_{t+1} = \frac{1}{\delta} - 1$$

$$1 = E_t\left[\delta(1 + r_{t+1})\right] \quad \Rightarrow E_t r_{t+1} = \frac{1}{\delta} - 1$$

$$\Rightarrow r^f_{t+1} = E_t r_{t+1}$$

(no arbitrage condition under risk neutrality)

- What is the model's prediction for the equity premium?
  - Again, assuming the CRRA utility function, $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$, the Euler equations can be rewritten as

$$1 = E_t \left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \right] (1 + r^f_{t+1})$$

$$1 = E_t \left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} (1 + r_{t+1}) \right]$$

  - Assume that $c_{t+1} \equiv \log\left(\frac{C_{t+1}}{C_t}\right)$, $\log(1 + r^f_{t+1})$, and $\log(1 + r_{t+1})$ are jointly Normally distributed, the following equation can be derived from the Euler equations:

$$\bar{r} - \bar{r}^f \approx \gamma\, cov(c, r) = \gamma\, \rho_{rc}\, \sigma_c \sigma_r, \tag{16}$$

where

- $c$: consumption growth rate, $r$: return on equity, $r^f$: return on bond.
- $\bar{r} - \bar{r}^f$: average **equity premium**.
- $\gamma$: degree of relative risk aversion of the representative consumer.

– Equation (16) is the model's prediction on the equity premium.

- Bring the model's prediction (16) to the data.
  - In the US data, $\sigma_r = 0.167$, $\sigma_c = 0.036$, and $\rho_{rc} = 0.4$, then (16) predicts that
    - if $\gamma = 1$, $\bar{r} - \bar{r}^f = 0.24\%$.
    - if $\gamma = 10$, $\bar{r} - \bar{r}^f = 2.4\%$.
    - if $\gamma = 25$, $\bar{r} - \bar{r}^f = 6\%$.
  - The average equity premium in the US data is roughly 7%.
  - For reasonable values of $\gamma$ (less than 10) estimated in empirical studies, the model implies an equity premium that is much smaller than in the data.
  - This is the **equity premium puzzle**.

- More studies confirmed the robustness of the equity premium puzzle, and there have been many extensions to the basic asset pricing model in the literature that aim to resolve the puzzle (refer to the readings).

- To summarise, the equity premium puzzle is a **quantitative** puzzle regarding the gap between:

  (a) the observed average equity premium in the US data

  (b) the average equity premium implied by a representative agent model with CRRA utility.

  The model can explain the existence of an equity premium, but not its magnitude.

# Review questions

1. Be able to formulate the utility maximisation problem of an individual investor who lives for two periods, under certainty or uncertainty, with multiple assets.

2. Be able to solve the utility maximisation problem of an individual investor who lives for two periods under certainty, for a given utility function, and graphically illustrate the solution, and understand the intuition behind this solution.

3. Be able to write down the intertemporal consumption Euler equation under uncertainty, multiple assets, and multiple periods, and understand why it is actually an NPV equation with a stochastic discount factor.

4. What is the stochastic discount factor in these models? How is risk aversion captured in the stochastic discount factor?

5. Understand how the consumption CAPM is derived, and understand how it compares with the Black CAPM.

6. Be able to derive the specific form of the consumption CAPM for the CRRA utility function, i.e., equation (11).

7. Roughly understand the setup of the basic asset pricing model that is used to derive the equity premium puzzle.

8. Be able to briefly explain what the equity premium puzzle is, without using any equations (your answer should cover a brief description about the data fact on the equity premium, the model, and the model's quantitative prediction, and why it is a puzzle).

# Appendix A

The utility maximisation problem can be rewritten as

$$\max_{C_t \geq 0} \log(C_t) + \delta \log((1 + r_{t+1})(W_t - C_t))$$

The first-order condition is given by

$$\frac{1}{C_t} + \delta \frac{-(1 + r_{t+1})}{(1 + r_{t+1})(W_t - C_t)} = 0,$$

$$\Leftrightarrow \quad \frac{1}{C_t} = \frac{\delta}{W_t - C_t}.$$

$$\Leftrightarrow \quad \delta C_t = W_t - C_t$$

So $\quad C_t^* = \frac{1}{1+\delta} W_t,$

$$S_t^* = W_t - C_t = \left(1 - \frac{1}{1+\delta}\right) W_t = \frac{\delta}{1+\delta} W_t$$

$$C_{t+1}^* = (1 + r_{t+1}) S_t^* = \frac{\delta}{1+\delta}(1 + r_{t+1}) W_t$$

# Appendix B

Form the Lagrangian:

$$\mathcal{L} = u(C_t) + \delta E_t u(C_{t+1}) + \lambda \left(1 - \sum_{j=1}^{n} a_j\right), \ C_{t+1} = \left(\sum_{j=1}^{n}(1 + r_{j,t+1})a_j\right)(W_t - C_t)$$

The first order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial a_j} = \delta E_t \Big[u'(C_{t+1})(1 + r_{j,t+1})(W_t - C_t)\Big] - \lambda = 0$$

$$\Rightarrow \ \delta E_t \Big[u'(C_{t+1})(1 + r_{j,t+1})\Big] = \frac{\lambda}{W_t - C_t}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = u'(C_t) + \delta E_t \left[u'(C_{t+1}) \left(-\sum_{j=1}^{n}(1 + r_{j,t+1})a_j\right)\right] = 0$$

$$\Rightarrow \ u'(C_t) = \delta E_t \left[u'(C_{t+1}) \left(\sum_{j=1}^{n}(1 + r_{j,t+1})a_j\right)\right]$$

$$\Rightarrow \ u'(C_t) = \sum_{j=1}^{n} \Big\{\delta E_t \left[u'(C_{t+1})(1 + r_{j,t+1})\right] a_j\Big\} = \frac{\lambda}{W_t - C_t}$$

$$\Rightarrow \ u'(C_t) = \delta E_t \Big[u'(C_{t+1})(1 + r_{j,t+1})\Big], \quad \text{for all } j = 1, 2, \ldots, n$$