

Lecture 10

VARIANCE FORECASTING

Recall ARMA 1-step-ahead recursive forecasts

$$Y_t \mid \mathcal{Y}_{t-1} \sim N \left(\sum_{j=1}^{t-1} \lambda_j^{(1)} Y_{t-j}, \sigma^2 \right)$$

where $\lambda_j^{(1)}$ are functions of the ARMA coefficients.

Recall ARMA 1-step-ahead recursive forecasts

$$Y_t | \mathcal{Y}_{t-1} \sim N \left(\sum_{j=1}^{t-1} \lambda_j^{(1)} Y_{t-j}, \sigma^2 \right)$$

where $\lambda_j^{(1)}$ are functions of the ARMA coefficients.

Prediction interval methods so far assume that
 σ^2 is constant over time.

(Both normal and bootstrap intervals)

Recall ARMA 1-step-ahead recursive forecasts

$$Y_t | \mathcal{Y}_{t-1} \sim N \left(\sum_{j=1}^{t-1} \lambda_j^{(1)} Y_{t-j}, ??? \right)$$

where $\lambda_j^{(1)}$ are functions of the ARMA coefficients.

Just as we model the *mean* $E(Y_t | \mathcal{Y}_{t-1})$
we can model the *variance* $\text{var}(Y_t | \mathcal{Y}_{t-1})$.

This will permit

- variance forecasts
- corresponding prediction intervals

Autoregressive Conditional Heteroskedasticity (ARCH)

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Includes a deterministic trend equation.

In general Z_t are the deviations of Y_t from the deterministic trend.

We can estimate/plot the trend $X'_t \hat{\beta}$ and/or the deviations \hat{Z}_t .

AR(1)-ARCH(1) Model

$$Y_t = \mathbf{X}'_t \boldsymbol{\beta} + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Includes a deterministic trend equation.

The *unconditional* mean of Y_t is $E(Y_t) = \mathbf{X}'_t \boldsymbol{\beta}$.

AR(1)-ARCH(1) Model

$$Y_t = \mathbf{X}'_t \boldsymbol{\beta} + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Includes a deterministic trend equation.

For simplicity of explanation we sometimes

- omit this, so $Y_t = Z_t$, or
- use $X_t = 1$, $\beta = \beta_0$ is a *constant* mean.

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1) model for the *conditional* mean.

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1) model for the *conditional* mean.

$$E(U_t | \mathcal{Z}_{t-1}) = 0 \quad (\text{since } U_t \text{ is a prediction error.})$$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1) model for the *conditional* mean.

$$E(U_t | \mathcal{Z}_{t-1}) = 0$$

$$\Rightarrow E(Z_t | \mathcal{Z}_{t-1}) = \phi_1 Z_{t-1}$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1) model for the *conditional* mean.

$$E(U_t | \mathcal{Z}_{t-1}) = 0$$

$$\Rightarrow E(Z_t | \mathcal{Z}_{t-1}) = \phi_1 Z_{t-1}$$

$$\Rightarrow E(Y_t | \mathcal{Y}_{t-1}) = X'_t \beta + \phi_1 (Y_{t-1} - X'_{t-1} \beta)$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(1) model for the *conditional variance*.

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[(U_t - E(U_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}]$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned} \text{var}(U_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[(U_t - E(U_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}] \\ &= 0 \end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[U_t^2 | \mathcal{Y}_{t-1}]$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned}\text{var}(U_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[U_t^2 | \mathcal{Y}_{t-1}] \\ &= \omega + \alpha_1 U_{t-1}^2\end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

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AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned}\text{var}(U_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[U_t^2 | \mathcal{Y}_{t-1}] \\ &= \omega + \alpha_1 U_{t-1}^2\end{aligned}$$

$$\Rightarrow U_t^2 = \omega + \alpha_1 U_{t-1}^2 + V_t, \quad E[V_t | \mathcal{Y}_{t-1}] = 0$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned}\text{var}(U_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[U_t^2 | \mathcal{Y}_{t-1}] \\ &= \omega + \alpha_1 U_{t-1}^2\end{aligned}$$

$$\Rightarrow U_t^2 = \omega + \alpha_1 U_{t-1}^2 + V_t, \quad E[V_t | \mathcal{Y}_{t-1}] = 0$$

AR(1) model for U_t^2

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

The model for $\text{var}(U_t | \mathcal{Y}_{t-1})$ implies a model for $\text{var}(Z_t | \mathcal{Y}_{t-1})$ and $\text{var}(Y_t | \mathcal{Y}_{t-1})$.

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[(Z_t - E(Z_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}]$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[(Z_t - E(Z_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}]$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[(Z_t - \phi_1 Z_{t-1})^2 | \mathcal{Y}_{t-1}]$$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned}\text{var}(Z_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[(Z_t - \phi_1 Z_{t-1})^2 | \mathcal{Y}_{t-1}] \\ &= E[U_t^2 | \mathcal{Y}_{t-1}]\end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

$$\begin{aligned}\text{var}(Z_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[(Z_t - \phi_1 Z_{t-1})^2 | \mathcal{Y}_{t-1}] \\ &= E[U_t^2 | \mathcal{Y}_{t-1}] \\ &= \omega + \alpha_1 (Z_{t-1} - \phi_1 Z_{t-2})^2\end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Similarly:

$$E(Y_t | \mathcal{Y}_{t-1}) = X'_t \beta + \phi_1 (Y_{t-1} - X'_{t-1} \beta)$$

$$\begin{aligned} \text{var}(Y_t | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 ((Y_{t-1} - X'_{t-1} \beta) \\ &\quad - \phi_1 (Y_{t-2} - X'_{t-2} \beta))^2 \end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

- Variance must be positive.
- Hence $\omega > 0$ and $\alpha_1 \geq 0$ are imposed.

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Recall $E(U_t | \mathcal{Y}_{t-1}) = 0 \stackrel{\text{LIE}}{\Rightarrow} E(U_t) = 0$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Recall $E(U_t | \mathcal{Y}_{t-1}) = 0 \stackrel{\text{LIE}}{\Rightarrow} E(U_t) = 0$

and $\text{var}(U_t | \mathcal{Y}_{t-1}) = E(U_t^2 | \mathcal{Y}_{t-1})$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Recall $E(U_t | \mathcal{Y}_{t-1}) = 0 \stackrel{\text{LIE}}{\Rightarrow} E(U_t) = 0$

and $\text{var}(U_t | \mathcal{Y}_{t-1}) = E(U_t^2 | \mathcal{Y}_{t-1})$

and $\text{var}(U_t) = E(U_t^2).$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Variance of U_t :

$$\text{var}(U_t) = E(U_t^2) \stackrel{\text{LIE}}{=} E[E(U_t^2 | \mathcal{Y}_{t-1})]$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Variance of U_t :

$$\begin{aligned} \text{var}(U_t) &= E(U_t^2) \stackrel{\text{LIE}}{=} E[\textcolor{red}{E}(U_t^2 | \mathcal{Y}_{t-1})] \\ &= E[\omega + \alpha_1 U_{t-1}^2] \end{aligned}$$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Variance of U_t :

$$\text{var}(U_t) = E(U_t^2) = \omega + \alpha_1 E(U_{t-1}^2)$$

AR(1)-ARCH(1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Variance of U_t , under stationarity:

$$\text{var}(U_t) = E(U_t^2) = \omega + \alpha_1 E(U_{t-1}^2)$$

AR(1)-ARCH(1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Variance of U_t , under stationarity:

$$\text{var}(U_t) = E(U_t^2) = \omega + \alpha_1 E(U_t^2)$$

$$\Rightarrow E(U_t^2) = \frac{\omega}{1 - \alpha_1} \quad \text{if } \alpha_1 < 1$$

AR(p)-ARCH(r) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + \dots + \phi_p Z_{t-p} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2 + \dots + \alpha_r U_{t-r}^2$$

This is an obvious generalisation.

But in practice ARCH(r) is not often used, even though AR(p) is very common.

Generalised
Autoregressive Conditional
Heteroskedasticity
(GARCH)

AR(1)-GARCH(1,1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \text{var}(U_{t-1} | \mathcal{Y}_{t-2})$$

AR(1)-GARCH(1,1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\text{var}(U_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \text{var}(U_{t-1} | \mathcal{Y}_{t-2})$$

It is convenient to write

$$\sigma_t^2 = \text{var}(U_t | \mathcal{Y}_{t-1}), \quad \sigma_{t-1}^2 = \text{var}(U_{t-1} | \mathcal{Y}_{t-2})$$

AR(1)-GARCH(1,1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

AR(1)-GARCH(1,1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

Recall $E(U_t | \mathcal{Y}_{t-1}) = 0 \Rightarrow \sigma_t^2 = E(U_t^2 | \mathcal{Y}_{t-1}).$

AR(1)-GARCH(1,1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$$\Rightarrow E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 E(U_{t-1}^2 | \mathcal{Y}_{t-2})$$

AR(1)-GARCH(1,1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$$\begin{aligned}\Rightarrow E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \color{blue}{\alpha_1 U_{t-1}^2} + \color{red}{\gamma_1 E(U_{t-1}^2 | \mathcal{Y}_{t-2})} \\ &= \omega + (\color{blue}{\alpha_1} + \color{green}{\gamma_1}) \color{blue}{U_{t-1}^2} \\ &\quad - \color{green}{\gamma_1} (\color{green}{U_{t-1}^2} - \color{red}{E(U_{t-1}^2 | \mathcal{Y}_{t-2})})\end{aligned}$$

AR(1)-GARCH(1,1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$$\begin{aligned} \Rightarrow E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 + \gamma_1 E(U_{t-1}^2 | \mathcal{Y}_{t-2}) \\ U_t^2 &= \omega + (\alpha_1 + \gamma_1) U_{t-1}^2 \\ &\quad + V_t - \gamma_1 V_{t-1} \end{aligned}$$

where $V_t = U_t^2 - E(U_t^2 | \mathcal{Y}_{t-1})$.

AR(1)-GARCH(1,1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$$\begin{aligned} \Rightarrow E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 + \gamma_1 E(U_{t-1}^2 | \mathcal{Y}_{t-2}) \\ U_t^2 &= \omega + (\color{blue}{\alpha_1 + \gamma_1}) U_{t-1}^2 \\ &\quad + V_t - \color{green}{\gamma_1 V_{t-1}} \end{aligned}$$

ARMA(1,1) in U_t^2 with AR coefficient $\color{blue}{\alpha_1 + \gamma_1}$ and
MA coefficient $\color{green}{-\gamma_1}$.

AR(1)-GARCH(1,1) Model

$$Y_t = X_t' \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$$\begin{aligned} \Rightarrow E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 + \gamma_1 E(U_{t-1}^2 | \mathcal{Y}_{t-2}) \\ U_t^2 &= \omega + (\textcolor{blue}{\alpha_1 + \gamma_1}) U_{t-1}^2 \\ &\quad + V_t - \textcolor{green}{\gamma_1 V_{t-1}} \end{aligned}$$

Stationarity requires $|\textcolor{blue}{\alpha_1 + \gamma_1}| < 1$.

AR(1)-GARCH(1,1) Model

$$Y_t = X'_t \beta + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + U_t$$

$$\sigma_t^2 = \omega + \alpha_1 U_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$$

$\sigma_t^2 > 0$ requires $\omega > 0, \alpha_1 \geq 0, \gamma_1 \geq 0$.

GARCH modelling in practice

Model Specification

- Specify the mean model as usual
i.e. choose X_t and AR(MA) order(s).
- ARCH LM test on model residuals.
Analogous to Ljung-Box autocorrelation test.
- AIC can be used for GARCH orders.
But ARCH(1) or (usually) GARCH(1,1) are appropriate.

LM test for ARCH

Recall models of $E(Y_t | \mathcal{Y}_{t-1})$ must produce prediction errors U_t that satisfy $E(U_t | \mathcal{Y}_{t-1}) = 0$.

\Rightarrow no autocorrelation in U_t .

GARCH models $E(U_t^2 | \mathcal{Y}_{t-1})$.

The residuals from a properly specified GARCH model would exhibit no autocorrelation *in their squares*.

LM test for ARCH

The residuals from a properly specified GARCH model would exhibit no autocorrelation *in their squares.*

The LM test for ARCH is nR^2 , with R^2 from regression of \hat{U}_t^2 on a constant and $\hat{U}_{t-1}^2, \dots, \hat{U}_{t-l}^2$.

$H_0 : U_t$ has no conditional heteroskedasticity

Under H_0 , $nR^2 \stackrel{a}{\sim} \chi_l^2$.

Forecasting: AR(1)-ARCH(1)

$$Y_t = X'_t \beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$Z_t = \phi_1 Z_{t-1} + U_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Forecasting: AR(1)-ARCH(1)

$$\begin{aligned} Y_t &= X'_t \beta + Z_t & E(U_t | \mathcal{Y}_{t-1}) &= 0 \\ Z_t &= \phi_1 Z_{t-1} + U_t & E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 \end{aligned}$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) \stackrel{\text{defn}}{=} E[(Z_t - E(Z_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}]$$

Forecasting: AR(1)-ARCH(1)

$$\begin{aligned} Y_t &= X'_t \beta + Z_t & E(U_t | \mathcal{Y}_{t-1}) &= 0 \\ Z_t &= \phi_1 Z_{t-1} + U_t & E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 \end{aligned}$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\begin{aligned} \text{var}(Z_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[(Z_t - E(Z_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}] \\ &= E[(Z_t - \phi_1 Z_{t-1})^2 | \mathcal{Y}_{t-1}] \end{aligned}$$

Forecasting: AR(1)-ARCH(1)

$$Y_t = X'_t \beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$\textcolor{red}{Z_t} = \phi_1 Z_{t-1} + \textcolor{violet}{U}_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\begin{aligned} \text{var}(Z_t | \mathcal{Y}_{t-1}) &\stackrel{\text{defn}}{=} E[(Z_t - E(Z_t | \mathcal{Y}_{t-1}))^2 | \mathcal{Y}_{t-1}] \\ &= E[(\textcolor{red}{Z_t} - \phi_1 Z_{t-1})^2 | \mathcal{Y}_{t-1}] \\ &= E[\textcolor{violet}{U}_t^2 | \mathcal{Y}_{t-1}] \end{aligned}$$

Forecasting: AR(1)-ARCH(1)

$$Y_t = X'_t \beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$Z_t = \phi_1 Z_{t-1} + U_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) = E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Forecasting: AR(1)-ARCH(1)

$$Y_t = X'_t \beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$\textcolor{orange}{Z_t} = \phi_1 Z_{t-1} + \textcolor{red}{U_t} \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\begin{aligned} \text{var}(Z_t | \mathcal{Y}_{t-1}) &= E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 \textcolor{red}{U}_{t-1}^2 \\ &= \omega + \alpha_1 (\textcolor{orange}{Z}_{t-1} - \phi_1 Z_{t-2})^2 \end{aligned}$$

Forecasting: AR(1)-ARCH(1)

$$Y_t = X'_t \beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$Z_t = \phi_1 Z_{t-1} + U_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(Z_t | \mathcal{Y}_{t-1}) = \phi_1 Z_{t-1}$$

Conditional variance of Z_t :

$$\text{var}(Z_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 (Z_{t-1} - \phi_1 Z_{t-2})^2$$

Forecasting: AR(1)-ARCH(1)

$$\textcolor{blue}{Y_t} = \textcolor{blue}{X}'_t \beta + \textcolor{red}{Z}_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$Z_t = \phi_1 Z_{t-1} + U_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(\textcolor{red}{Z}_t | \mathcal{Y}_{t-1}) = \phi_1 \textcolor{red}{Z}_{t-1}$$

Conditional variance of Z_t :

$$\text{var}(\textcolor{red}{Z}_t | \mathcal{Y}_{t-1}) = \omega + \alpha_1 (\textcolor{red}{Z}_{t-1} - \phi_1 \textcolor{red}{Z}_{t-2})^2$$

Forecasting: AR(1)-ARCH(1)

$$\textcolor{blue}{Y_t} = \textcolor{blue}{X'_t}\beta + Z_t \quad E(U_t | \mathcal{Y}_{t-1}) = 0$$

$$Z_t = \phi_1 Z_{t-1} + U_t \quad E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

Conditional mean of Z_t :

$$E(\textcolor{blue}{Y}_t - \textcolor{blue}{X}'_t\beta | \mathcal{Y}_{t-1}) = \phi_1(\textcolor{blue}{Y}_{t-1} - \textcolor{blue}{X}'_{t-1}\beta)$$

Conditional variance of Z_t :

$$\begin{aligned} \text{var}(\textcolor{blue}{Y}_t - \textcolor{blue}{X}'_t\beta | \mathcal{Y}_{t-1}) &= \omega + \alpha_1((\textcolor{blue}{Y}_{t-1} - \textcolor{blue}{X}'_{t-1}\beta) \\ &\quad - \phi_1(\textcolor{blue}{Y}_{t-2} - \textcolor{blue}{X}'_{t-2}\beta))^2 \end{aligned}$$

Forecasting: AR(1)-ARCH(1)

$$\begin{aligned} Y_t &= X_t' \beta + Z_t & E(U_t | \mathcal{Y}_{t-1}) &= 0 \\ Z_t &= \phi_1 Z_{t-1} + U_t & E(U_t^2 | \mathcal{Y}_{t-1}) &= \omega + \alpha_1 U_{t-1}^2 \end{aligned}$$

Conditional mean of $\textcolor{magenta}{Y}_t$:

$$E(Y_t | \mathcal{Y}_{t-1}) = X_t' \beta + \phi_1(Y_{t-1} - X_{t-1}' \beta)$$

Conditional variance of $\textcolor{magenta}{Y}_t$:

$$\begin{aligned} \text{var}(Y_t | \mathcal{Y}_{t-1}) &= \omega + \alpha_1((Y_{t-1} - X_{t-1}' \beta) \\ &\quad - \phi_1(Y_{t-2} - X_{t-2}' \beta))^2 \end{aligned}$$

(since X_t is deterministic)

Forecasting: AR(1)-ARCH(1)

Conditional mean of Y_t :

$$E(Y_t | \mathcal{Y}_{t-1}) = X'_t \beta + \phi_1(Y_{t-1} - X'_{t-1} \beta)$$

Conditional variance of Y_t :

$$\begin{aligned} \text{var}(Y_t | \mathcal{Y}_{t-1}) = & \omega + \alpha_1((Y_{t-1} - X'_{t-1} \beta) \\ & - \phi_1(Y_{t-2} - X'_{t-2} \beta))^2 \end{aligned}$$

These *recursively* imply formulae for

$$\begin{aligned} E(Y_{n+h} | \mathcal{Y}_n) & \quad \text{for } h = 1, 2, 3 \dots \\ \text{var}(Y_{n+h} | \mathcal{Y}_n) \end{aligned}$$

Forecasting: ARMA-GARCH

ARMA-GARCH provides forecasts

$$\begin{aligned} E(Y_{n+h} | \mathcal{Y}_n) & \quad \text{for } h = 1, 2, 3 \dots \\ \text{var}(Y_{n+h} | \mathcal{Y}_n) \end{aligned}$$

using *both* mean *and* variance.

- $E(Y_{n+h} | \mathcal{Y}_n)$ provides a *point* forecast for Y_{n+h} .
- $\text{var}(Y_{n+h} | \mathcal{Y}_n)$ forecasts future variance
 - used for prediction intervals
 - directly useful in financial applications.

Application:
Interest rates on
3 month Bank Accepted Bills

AR(3)-ARCH(1) model

$$\Delta \text{BAB3}_t = \beta_0 + Z_t$$

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \phi_3 Z_{t-3} + U_t$$

$$E(U_t^2 | \mathcal{Y}_{t-1}) = \omega + \alpha_1 U_{t-1}^2$$

AR(3)-ARCH(1) model

```
1 print(round(coef(AR3ARCH1), 3))
```

mu	ar1	ar2	ar3	omega	alpha1
0.026	0.604	-0.084	0.166	0.005	0.828

$$\Delta \text{BAB3}_t = 0.026 + \hat{Z}_t$$

$$Z_t = 0.604Z_{t-1} - 0.084Z_{t-2} + 0.166Z_{t-3} + \hat{U}_t$$

$$\hat{E}(U_t^2 | \mathcal{Y}_{t-1}) = 0.005 + 0.828 U_{t-1}^2$$

AR(3)-ARCH(1) model

Weighted ARCH LM Tests

	Statistic	Shape	Scale	P-Value
ARCH Lag[2]	0.1092	0.500	2.000	0.7411
ARCH Lag[4]	0.4192	1.397	1.611	0.8940
ARCH Lag[6]	0.5775	2.222	1.500	0.9628

There is no evidence of unmodelled conditional heteroskedasticity in the residuals of this model.

AR(3)-ARCH(1) model

```
1 # h step ahead forecasts AR3-ARCH1
2 AR3ARCH1f <- ugarchforecast(AR3ARCH1,
3                               n.ahead=12)
4
5 # Point forecasts
6 Point <- ts(c(fitted(AR3ARCH1f)),
7               start=c(2024,7),
8               end=c(2025,6),
9               frequency=12)
```

AR(3)-ARCH(1) model

	Time	Lower	Point	Upper	Actual	Sigma
1	2024.500	-0.111	0.030	0.172	0.07	0.072
2	2024.583	-0.162	0.022	0.207	-0.08	0.094
3	2024.667	-0.189	0.024	0.238	0.04	0.109
4	2024.750	-0.209	0.026	0.261	-0.01	0.120
5	2024.833	-0.226	0.026	0.277	0.01	0.128
6	2024.917	-0.239	0.025	0.290	0.02	0.135
7	2025.000	-0.249	0.026	0.300	-0.11	0.140

$$\hat{E}(Y_{n+h} | \mathcal{Y}_n) \quad \sqrt{\widehat{\text{var}}(Y_{n+h} | \mathcal{Y}_n)}$$

AR(3)-ARCH(1) model

