ECOM20002
Forecasting in Economics and Business
Tutorial 6 Solutions

1.) Consider the variables Y_t and X_t such that Y_t is described by an AR(1) model,

$$Y_t = \phi Y_{t-1} + \varepsilon_t$$

while X_t is described by the following restricted ARMA(4,1) model,

$$X_t = \beta X_{t-4} + u_t + \theta u_{t-1}$$

where both ε_t and u_t are white noise series and $|\phi| < 1$, $|\beta| < 1$ and $|\theta| < 1$ so that the stationarity and invertibility of Y_t and X_t are guaranteed.

Show that the variable $Z_t = Y_t + X_t$ can be described by an ARMA(5,4) model. (*Hint: The lag operator will be useful here*!)

Solution

Write $y_t = \phi_1 y_{t-1} + \varepsilon_t$ as

$$(1-\phi_1 L)y_t = \varepsilon_t \quad \text{or} \quad y_t = \frac{\varepsilon_t}{1-\phi_1 L}.$$

Similarly, write $x_t = \beta x_{t-4} + u_t + \theta u_{t-1}$ as

$$(1-\beta L^4)x_t = (1+\theta L)u_t \quad \text{or} \quad x_t = \frac{(1+\theta L)u_t}{1-\beta L^4}.$$

Then, $z_t = y_t + x_t$ can be written as

$$z_t = \frac{\varepsilon_t}{1-\phi_1 L} + \frac{(1+\theta L)u_t}{1-\beta L^4},$$

or

$$(1-\phi_1 L)(1-\beta L^4)z_t = (1-\beta L^4)\varepsilon_t + (1-\phi_1 L)(1+\theta L)u_t.$$

The model for z_t

$$(1-\phi_1 L)(1-\beta L^4)z_t = (1-\beta L^4)\varepsilon_t + (1-\phi_1 L)(1+\theta L)u_t$$

can be written as

$$\underbrace{(1-\phi_1 L-\beta L^4+\phi_1\beta L^5)}_{\text{AR(5) polynomial}} z_t = \underbrace{\varepsilon_t - \beta\varepsilon_{t-4} + u_t + (\theta-\phi_1)u_{t-1} - \phi_1\theta u_{t-2}}_{\text{no dependence beyond lag 4}}.$$

Hence, indeed z_t can be described by an ARMA(5,4) model.

2.) Consider the general MA(∞) representation for a stationary time series Y_t, that is,

$$Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots +$$

and suppose that the parameters $\theta_1, \theta_2, \ldots$ are known.

a.) What are the forecast errors for 3 and 4 steps ahead?

b.) What is the covariance between these errors

Solution

What is known at time T about the observation for time $T+h$?

$$y_{T+h} = \underbrace{\varepsilon_{T+h} + \theta_1 \varepsilon_{T+h-1} + \cdots + \theta_{h-1}\varepsilon_{T+1}}_{\text{unknown at time } T} + \underbrace{\theta_h \varepsilon_T + \theta_{h+1}\varepsilon_{T-1} + \ldots}_{\text{known at time } T}$$

The optimal h-step ahead point forecast is equal to

$$y_{T+h|T} = \mathsf{E}[y_{T+h}|\mathcal{Y}_T] = \theta_h \varepsilon_T + \theta_{h+1}\varepsilon_{T-1} + \theta_{h+2}\varepsilon_{T-2} + \ldots$$

with forecast error

$$e_{T+h|T} = \varepsilon_{T+h} + \theta_1 \varepsilon_{T+h-1} + \cdots + \theta_{h-1}\varepsilon_{T+1}\,.$$

For 3- and 4-steps ahead, this gives

$$\begin{aligned} e_{T+3|T} &= \varepsilon_{T+3} + \theta_1 \varepsilon_{T+2} + \theta_2 \varepsilon_{T+1}, \\ e_{T+4|T} &= \varepsilon_{T+4} + \theta_1 \varepsilon_{T+3} + \theta_2 \varepsilon_{T+2} + \theta_3 \varepsilon_{T+1} \end{aligned}$$

For the covariance between these errors it follows that

$$\begin{aligned} \mathsf{E}[e_{T+3|T} e_{T+4|T}] &= \mathsf{E}[\theta_1 \varepsilon_{T+3}^2 + \theta_1\theta_2 \varepsilon_{T+2}^2 + \theta_2\theta_3 \varepsilon_{T+1}^2] \\ &= (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3)\sigma^2 \end{aligned}$$

3.) Suppose that the time series Y_t is governed by the following process,

$$Y_t = Y_{t-1} + \varepsilon_t$$

Where ε_t is a white noise series with $E[\varepsilon_t] = 0$ and $E[\varepsilon_t^2] = \sigma^2$ for all t. Also suppose that that Y_t is observed every six months, but that it is aggregated to annually observed time series X_T by taking the sum of the two observations of Y in year T. Show that X_T can be described by

$$X_T = X_{T-1} + u_T$$

Where u_T is an MA(1) process with first order autocorrelation equal to $^1/_6$. (*Hint: Let periods t and* $t-1$ *be in year* T)

Solution

From the AR(1) specification for the observed series y, it follows that y_t can be expressed as

$$y_t = y_{t-2} + \varepsilon_t + \varepsilon_{t-1} = y_{t-2} + (1+L)\varepsilon_t.$$

Multiplying both sides with $1 + L$ results in

$$(1+L)y_t = (1+L)y_{t-2} + (1+L)(1+L)\varepsilon_t.$$

If the observations at times t and $t-1$ are in year T, this is equivalent to

$$x_T = x_{T-1} + u_T,$$

where u_T corresponds with $(1+L)(1+L)\varepsilon_t = \varepsilon_t + 2\varepsilon_{t-1} + \varepsilon_{t-2}$. It then follows that

$$\begin{aligned}
\mathsf{E}[u_T] &= \mathsf{E}[\varepsilon_t + 2\varepsilon_{t-1} + \varepsilon_{t-2}] = 0, && (26)\\
\mathsf{E}[u_T^2] &= \mathsf{E}[(\varepsilon_t + 2\varepsilon_{t-1} + \varepsilon_{t-2})^2] = 6\sigma^2, && (27)\\
\mathsf{E}[u_T u_{T-1}] &= \mathsf{E}[(\varepsilon_t + 2\varepsilon_{t-1} + \varepsilon_{t-2})(\varepsilon_{t-2} + 2\varepsilon_{t-3} + \varepsilon_{t-4})] = \sigma^2, && (28)\\
\mathsf{E}[u_T u_{T-k}] &= \mathsf{E}[(\varepsilon_t + 2\varepsilon_{t-1} + \varepsilon_{t-2})(\varepsilon_{t-2k} + 2\varepsilon_{t-2k-1} + \varepsilon_{t-2k-2})] = 0, \quad \text{for all } k > 1. && (29)
\end{aligned}$$

Thus, indeed u_T has an MA(1) structure (i.e. a non-zero first-order autocorrelation and all higher-order autocorrelations equal to zero), with first-order autocorrelation equal to $1/6$.

4.) For each of the following time series processes

a.) $Y_t = \mu + \beta Y_{t-1} + u_t$

b.) $Y_t = \mu + u_t + 0.6u_{t-1} + 0.2u_{t-2}$

Where u_t is a white noise process with $E[u_t] = 0$ and $E[u_t^2] = \sigma^2$

i.) Derive the unconditional mean $E[Y_t]$
ii.) Derive the unconditional variance $Var(Y_t)$
iii.) Derive the first-order autocovariance $Cov(Y_t, Y_{t-1})$

(a) (i) $Ey_t = \mu + \beta Ey_{t-1} + Eu_t \implies Ey_t = \mu + \beta Ey_t \implies (Ey_t)(1-\beta) = \mu$
$\implies Ey_t = \mu/(1-\beta)$

(ii) $\mathrm{Var}(y_t) = \beta^2\mathrm{Var}(y_{t-1}) + \mathrm{Var}(u_t) \implies \mathrm{Var}(y_t) = \beta^2\mathrm{Var}(y) + \sigma^2$
$\implies \mathrm{Var}(y) = \sigma^2/(1-\beta^2)$

(iii) $y_t - Ey_t = \mu + \beta y_{t-1} + u_t - E(\mu + \beta y_{t-1} + u_t) = \mu + \beta y_{t-1} + u_t - (\mu + \beta Ey_{t-1})$
$= \beta(y_{t-1} - Ey_{t-1}) + u_t$
So $E(y_t - Ey_t)(y_{t-1} - E(y_{t-1})) = E(\beta(y_{t-1} - Ey_{t-1}) + u_t)(y_{t-1} - Ey_{t-1})$
$= \beta E(y_{t-1} - Ey_{t-1})^2 = \beta\mathrm{Var}(y_t)$

(b) (i) $Ey_t = \mu$

(ii) $\mathrm{Var}(y) = E(y_t - \mu)^2 = (1 + .6^2 + .2^2)\sigma^2 = 1.4\sigma^2$

(iii) $E(y_t - Ey_t)(y_{t-1} - E(y_{t-1})) = E(u_t + .6u_{t-1} + .2u_{t-2})(u_{t-1} + .6u_{t-2} + .2u_{t-3})$
$= .6Eu_{t-1}^2 + .12Eu_{t-2}^2 = .72\sigma^2$