ECOM40006/90013 Econometrics 3

Week 1 Extras

Question 1: Expectations and Variance

In this question, we'll revise some of the concepts of both univariate and multivariate probability. To begin with, consider two univariate random variables X and Y.

(a) Use the definition of variance to show that $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

(b) Then, show that $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

(c) Is it always the case that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$? Under what conditions is this statement true?

Now consider random *vectors* Z_1 and Z_2 and constant matrices A and B which are conformable with Z_1 and Z_2. Assume that Z is $n \times 1$ with $\text{cov}(Z_1, Z_2) = \Sigma$ where Σ is $n \times n$.

(d) Show that $\text{Var}(AZ_1) = A\text{Var}(Z_1)A'$.

(e) Show that $\text{cov}(AZ_1, BZ_2) = A\Sigma B'$.

Question 2: Indicator Random Variables

One of the most common applications in econometrics is to model a random variable that takes only two values: 0 and 1. Specifically, suppose there is an event A and an indicator variable $\mathbf{1}_A$ such that

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Suppose you also have an indicator variable $\mathbf{1}_B$ that returns 1 if B occurs and 0 otherwise. Furthermore, let the probability of A occurring be given by $\Pr(A) = p$.

(a) On a Venn diagram, show the areas where (i) $\mathbf{1}_A = 1$ and (ii) $\mathbf{1}_B = 1$.

(b) Provide a brief argument for each of the cases below as to why they are true, using the definition of an indicator random variable above:

(i.) $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ (note: A^c is the complement of A)

(ii.) $\mathbf{1}_A \mathbf{1}_B = \mathbf{1}_{A \cap B}$

(iii.) $\max\{\mathbf{1}_A, \mathbf{1}_B\} = \mathbf{1}_{A \cup B}$

For the last two cases, you will want to consider all possible combinations of events that can occur.

(c) Prove the following statement:

$$\mathbf{1}_A \textit{ and } \mathbf{1}_B \textit{ are independent if and only if } cov(\mathbf{1}_A, \mathbf{1}_B) = 0.$$

Several hints: (i) $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. (ii) You may find your answers above useful. (iii) Since this is a biconditional statement, you'll need to prove *two* separate statements. Can you figure out what those statements are? (iv) What is the definition for independent random variables?

(d) State the probability mass function of $\mathbf{1}_A$. For a bonus, give an explicit functional form for the probability mass function. (Hint: look up *Bernoulli* random variables.)

(e) Derive the mean and variance of $\mathbf{1}_A$.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. indicator random variables, each of them being equal with the same probability p. First, consider the expression $S_2 = X_1 + X_2$.

(f) Derive the mean and variance of S_2. Using this results, generalize this to find the mean and variance of $S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$.

Question 3: Moment Generating Functions

In this question, we're going to take a detour and look into a method that can be used to derive the properties of various distributions, specifically those which are continuous.[1] We'll first motivate the idea of a *moment generating function* (MGF), then use that to derive a known property of the normal distribution.

Let's first revisit the idea of transformations of random variables. Suppose that you took three draws from a random variable $X \sim N(\mu, \sigma^2)$, and that for the sake of illustration they happened to be

$$-1, \quad 2, \quad 4.$$

If someone told you they wanted to obtain draws from another distribution $Y = e^X$, you could get them by taking your original three draws and exponentiating them:

$$e^{-1}, \quad e^2, \quad e^4$$

These would then be said to be draws from $Y = e^X$.

(a) We're going to examine a very specific transformation of X, but before that let's revisit the idea of a *Taylor series*: a continuous function can be approximated as a polynomial

[1] For our discrete friends, the counterpart is the *probability generating function*, or PGF.

consisting of its derivatives. In particular, find the Taylor series centered around zero for the function

$$f(X) = e^{tX}, \qquad t \in \mathbb{R}.$$

If you have any trouble calculating this, remember that the Taylor series for a function centered around the point $x = a$ satisfies

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ is the n^{th} derivative of x evaluated at a.

(b) Now suppose that X is a random variable. Whenever we draw a value of X, let's plug it into the *Taylor series* for $f(X)$ above. Now take the expectation of $f(X)$. What would we need to know about X in order to calculate $\mathbb{E}(f(X))$?

(c) Based on what you found in (b) above, if you wanted to obtain the so-called 'raw moment' $\mathbb{E}(X)$, how would you do that? What if you wanted to get $\mathbb{E}(X^2)$? How about $\mathbb{E}(X^3)$? Assume that they all exist.

(d) For this question, take as given the following:

- The moment generating function (MGF) associated with a random variable uniquely determines its distribution.
- The MGF of a random variable $X \sim N(\mu, \sigma^2)$ is

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Note: As an optional exercise you can derive this MGF from its probability density function

$$f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}.$$

This is a standard derivation exercise for students in undergraduate probability and involves some manipulation, particularly one where you can eliminate half the expression by finding another PDF that integrates to 1 over its support. But we've got plenty of time to do that later if the need arises. For now, let's just focus on deriving properties.

Let $X \sim N(\mu, \sigma^2)$. Show that the *affine transformation* of X,

$$Y = aX + b, \qquad a, b \in \mathbb{R},$$

is **normally** distributed with mean $a\mu + b$ and variance $a^2\sigma^2$.