ECOM40006/ECOM90013 Econometrics 3
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Week 5 Tutorial Exercise Solutions

Semester 1, 2025

2. Suppose that for some estimator $\hat{\theta}$ of the scalar parameter θ, the limiting distribution is known to be
$$n^{1/2}(\hat{\theta} - \theta) \xrightarrow{d} \mathrm{N}(0, 1).$$
Construct a 95% confidence interval for e^{θ}.

Solution:
The delta method tells us that, if $n^{1/2}(\hat{\theta} - \theta) \xrightarrow{d} \mathrm{N}(0, 1)$ then
$$n^{1/2}(g(\hat{\theta}) - g(\theta)) \xrightarrow{d} \mathrm{N}\left(0, [g'(\theta)]^2\right).$$
In this case, $g(\theta) = e^{\theta}$. We note that $g'(\theta) = \mathrm{d}g(\theta)/\,\mathrm{d}\theta = e^{\theta}$. Thus, the relevant limiting distribution is
$$n^{1/2}(e^{\hat{\theta}} - e^{\theta}) \xrightarrow{d} \mathrm{N}\left(0, e^{2\theta}\right),$$
from which we may deduce the asymptotic distribution
$$e^{\hat{\theta}} \underset{a}{\sim} \mathrm{N}\left(e^{\theta}, n^{-1}e^{2\theta}\right).$$
Consequently, a 95% confidence interval is may be calculated from the probability statement
$$\Pr\left(-z_{0.025} \leq \frac{e^{\hat{\theta}} - e^{\theta}}{n^{-1/2}e^{\theta}} \leq z_{0.025}\right) = 0.95,$$
where $z_{0.025} = 1.96$ is that value which cuts off a 0.025 upper tail probability in a standard Normal distribution. This suggests
$$\Pr\left(-z_{0.025}n^{-1/2} \leq \frac{e^{\hat{\theta}}}{e^{\theta}} - 1 \leq z_{0.025}n^{-1/2}\right) = 0.95$$
$$\implies \Pr\left(1 - z_{0.025}n^{-1/2} \leq \frac{e^{\hat{\theta}}}{e^{\theta}} \leq 1 + z_{0.025}n^{-1/2}\right) = 0.95$$
$$\implies \Pr\left(\frac{1}{1 - z_{0.025}n^{-1/2}} \geq \frac{e^{\theta}}{e^{\hat{\theta}}} \geq \frac{1}{1 + z_{0.025}n^{-1/2}}\right) = 0.95$$
$$\implies \Pr\left(\frac{e^{\hat{\theta}}}{1 + z_{0.025}n^{-1/2}} \leq e^{\theta} \leq \frac{e^{\hat{\theta}}}{1 - z_{0.025}n^{-1/2}}\right) = 0.95$$

So, for a sample of size n and given our variance assumption, a 95% confidence interval for e^{θ} is given by

$$\left[\frac{e^{\hat{\theta}}}{1+z_{0.025}n^{-1/2}}, \frac{e^{\hat{\theta}}}{1-z_{0.025}n^{-1/2}}\right].$$

3. It is common for wage equations to be estimated with some measure of the level of education as one of the expanatory variables. To allow for a non-linear response, this variable often enters the equation in both level and squared forms. So, for the i-th individual the equation may look something like

$$wages_i = \beta educ_i + \delta educ_i^2 + x_i'\theta + u_i, i = 1, \ldots, n,$$

where w_i denotes the wage of the i-th individual, $educ_i$ their level of educational attainment, and x_i is a vector of observations on all the other expanators in the equation (including the intercept). Note that we have said nothing about how any of these variables are measured. In any event, the postulated model allows for a quadratic relationshp between wages and education and an obvious question to ask is where is the turning point in the relationship between wages and education given the x's. Elementary mathematics tells us that this occurs where $educ_i = -\beta/(2\delta)$. Suppose that the joint asymptotic distribution of $[\hat{\beta}, \hat{\delta}, \hat{\theta}']'$ is of the form

$$\begin{bmatrix}\hat{\beta}\\ \hat{\delta}\\ \hat{\theta}\end{bmatrix} \underset{a}{\sim} \mathrm{N}\left(\begin{bmatrix}\beta\\ \delta\\ \theta\end{bmatrix}, n^{-1}\begin{bmatrix}\sigma_\beta^2 & \sigma_{\beta\delta}^2 & \Sigma_{\beta\theta}'\\ \sigma_{\beta\delta}^2 & \sigma_\delta^2 & \Sigma_{\delta\theta}'\\ \Sigma_{\beta\theta} & \Sigma_{\delta\theta} & \Sigma_\theta\end{bmatrix}\right).$$

(a) What is the marginal asymptotic distribution of $[\hat{\beta}, \hat{\delta}]$?

Solution:

We can write down directly that

$$\begin{bmatrix}\hat{\beta}\\ \hat{\delta}\end{bmatrix} \underset{a}{\sim} \mathrm{N}\left(\begin{bmatrix}\beta\\ \delta\end{bmatrix}, n^{-1}\begin{bmatrix}\sigma_\beta^2 & \sigma_{\beta\delta}^2\\ \sigma_{\beta\delta}^2 & \sigma_\delta^2\end{bmatrix}\right).$$

(b) Using your answer to Question 3(a) of the Exercise, find the asymptotic distribution for the turning point of the wage equation as a function of education.

Solution:

Let $g(\beta, \delta) = \beta/(2\delta)$. Therefore

$$g^{(1)}(\hat{\beta}, \hat{\delta}) = \frac{\partial g(\beta,\delta)}{\partial[\beta,\delta]} = \begin{bmatrix}\dfrac{\partial g(\beta,\delta)}{\partial\beta}\\ \dfrac{\partial g(\beta,\delta)}{\partial\delta}\end{bmatrix} = \begin{bmatrix}(2\delta)^{-1}\\ -\beta/(2\delta^2)\end{bmatrix}.$$

The delta method tells us that

$$\begin{aligned} g(\hat{\beta}, \hat{\delta}) &\underset{a}{\sim} \mathrm{N}\left(g(\beta,\delta), n^{-1}g^{(1)}(\beta,\delta)'\begin{bmatrix}\sigma_\beta^2 & \sigma_{\beta\delta}^2\\ \sigma_{\beta\delta}^2 & \sigma_\delta^2\end{bmatrix}g^{(1)}(\beta,\delta)\right)\\ &= \mathrm{N}\left(\frac{\beta}{2\delta}, \frac{\delta^2\sigma_\beta^2 - 2\beta\delta\sigma_{\beta\delta}^2 + \beta^2\sigma_\delta^2}{4n\delta^4}\right). \end{aligned}$$