Topic 8. Present Value Relationship and Price Variability

ECON30024 Economics of Finanical Markets

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Outline

1. Introduction
2. Net present value under certainty
3. Net present value under uncertainty
4. Asset price volatility
5. Bubbles and Ponzi schemes

Required reading: Bailey, Chapter 10

Further reading: Malkiel (2010), Diba and Grossman (1988), (Brunnermeier, 2008)

1. Introduction

- Both the CAPM and APT predict asset prices as present values of their expected unit payoffs:

$$p_j = \frac{1}{1 + \mu_j} E(v_j),$$

where μ_j is the expected rate of return on asset j as predicted by the CAPM or APT.

- In this topic, we discuss a general theory of asset price determination based on the net present value relationship.
  - It sets the equilibrium price (value) of an asset to the present value of a sequence of its expected payoffs.
  - The CAPM and APT can be viewed as special versions of this theory.

- The NPV pricing is widely applied, e.g., the NPV criteria constitute the basis for the selection of investment projects.
- We'll derive the NPV pricing equations in a **multiperiod** setting for two cases: payoffs of assets are **certain**, and **uncertain**.
  - In both cases, **no arbitrage** is assumed.
- We then discuss an application of the NPV theory in examining the volatility in asset prices.
  - Asset prices are found to be too volatile, compared with what the NPV predicts.
  - In history, there are episodes of extreme asset price fluctuations-**financial bubbles**.

2. Net Present Value under Certainty

- We first derive the NPV pricing equation assuming that
  - future dividends of assets, $\{d_{t+1}, d_{t+2}, \ldots\}$, are deterministic – known with **certainty** to all investors.
  - funds can be borrowed or lent in unlimited amounts at a constant interest rate r^f.
- First consider price determination of any finitely-lived asset j which lives for N periods from present date t, with maturity value p_{t+N}.
- Under certainty, **the arbitrage principle** implies that in equilibrium the rate of return on *any asset* equals r^f, i.e.,

$$r^j_{t+1} = r^f, \qquad \text{for all } j, \text{ all } t$$

We'll suppress the superscript j from now on.

- Recall from Topic 1 the definition for the rate of return:

$$r_{t+1} = \frac{(d_{t+1} + p_{t+1}) - p_t}{p_t} = \frac{d_{t+1} + p_{t+1}}{p_t} - 1$$

where d_{t+1} denotes the dividend or coupon received on 1 unit of the asset at date $t+1$.

- Then by the arbitrage principle we have

$$p_t = \frac{1}{1+r^f}(d_{t+1} + p_{t+1}) \tag{1}$$

This equation also holds for $t+1$, $t+2$, $\ldots$:

$$p_{t+1} = \frac{1}{1+r^f}(d_{t+2} + p_{t+2}), \quad p_{t+2} = \frac{1}{1+r^f}(d_{t+3} + p_{t+3}), \quad \cdots$$

i.e.,

$$p_{t+i} = \frac{1}{1+r^f}(d_{t+i+1} + p_{t+i+1}), \quad \text{for all } i \geq 0 \tag{2}$$

- Applying (2) successively to (1), we have:

$$
\begin{aligned}
p_t &= \frac{d_{t+1}}{1+r^f} + \frac{p_{t+1}}{1+r^f} = \frac{d_{t+1}}{1+r^f} + \frac{1}{1+r^f}\left(\frac{1}{1+r^f}(d_{t+2}+p_{t+2})\right) \\
&= \frac{d_{t+1}}{1+r^f} + \frac{d_{t+2}}{(1+r^f)^2} + \frac{p_{t+2}}{(1+r^f)^2} \\
&= \frac{d_{t+1}}{1+r^f} + \frac{d_{t+2}}{(1+r^f)^2} + \frac{1}{(1+r^f)^2}\left(\frac{1}{1+r^f}(d_{t+3}+p_{t+3})\right) \\
&= \frac{d_{t+1}}{1+r^f} + \frac{d_{t+2}}{(1+r^f)^2} + \frac{d_{t+3}}{(1+r^f)^3} + \frac{p_{t+3}}{(1+r^f)^3} \\
&= \cdots \\
&= \frac{d_{t+1}}{1+r^f} + \frac{d_{t+2}}{(1+r^f)^2} + \frac{d_{t+3}}{(1+r^f)^3} + \cdots + \frac{d_{t+N}}{(1+r^f)^N} + \frac{p_{t+N}}{(1+r^f)^N}, \\
&= \left(\sum_{i=1}^{N} \frac{d_{t+i}}{(1+r^f)^i}\right) + \frac{p_{t+N}}{(1+r^f)^N} \qquad (3)
\end{aligned}
$$

- Eqn. (3) is the NPV pricing equation for the **equilibrium** price of a finitely-lived asset under certainty and a constant risk-free rate:
  - Each dividend d_{t+i} is discounted back to the present by multiplying a **discount factor**, $\frac{1}{(1+r^f)^i}$.

- If we allow for a time-varying risk-free rate of return r_t^f, we can similarly show that (Q2 in Tutorial 9):

$$\begin{aligned} p_t &= \delta_{t+1}d_{t+1} + \delta_{t+2}d_{t+2} + \cdots + \delta_{t+N}d_{t+N} + \delta_{t+N}p_{t+N} \\ &= \left(\sum_{i=1}^{N} \delta_{t+i}d_{t+i}\right) + \delta_{t+N}p_{t+N}, \end{aligned} \tag{4}$$

where the discount factor δ_{t+i} is given by

$$\delta_{t+i} = \frac{1}{(1+r_{t+1}^f)(1+r_{t+2}^f)\cdots(1+r_{t+i}^f)}, \; i \geq 1.$$

- When N is finite, p_t is **uniquely determined** by the NPV equation (3) or (4).
- Now suppose the asset is **indefinitely lived**, i.e., $N \to \infty$.
  - First, by the arbitrage principle, p_t still satisfies

$$p_t = \frac{d_{t+1} + p_{t+1}}{1 + r^f_{t+1}}. \tag{5}$$

  - As before, successive substitution yields (analogous to (4)):

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} d_{t+i} + \lim_{N \to \infty} \delta_{t+N} p_{t+N}. \tag{6}$$

  - To ensure p_t is well defined, we need to impose:
    - $\sum_{i=1}^{\infty} \delta_{t+i} d_{t+i}$ is finite (the convergence condition).
    - $\lim_{N \to \infty} \delta_{t+N} p_{t+N}$ is finite.

– If we further impose

$$\lim_{N\to\infty} \delta_{t+N} p_{t+N} = 0 \quad \text{(the transversality condition)},$$

then we reach the NPV equation:

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} d_{t+i} \tag{7}$$

– However, equilibrium prices of the indefinitely-lived asset are not uniquely determined by (7) (see Section 5).

- A special case – **Gordon growth model**

– If the risk-free rate is r^f and dividend grows at a constant rate $g < r^f$: $d_{t+1+i} = (1+g)^i d_{t+1}$, $i \geq 1$, then (7) simplifies to (see exercise_topic8):

$$p_t = \frac{d_{t+1}}{r^f - g} \tag{8}$$

3. Net Present Value under Uncertainty

- Now consider the more general case: future payoffs of assets are uncertain, assets are indefinitely lived, the risk-free rate varies over time.

- Following (7), it is natural to write down the following NPV pricing equation:

$$p_t = \delta_{t+1} E_t d_{t+1} + \delta_{t+2} E_t d_{t+2} + \cdots = \sum_{i=1}^{\infty} \delta_{t+i} E_t d_{t+i}, \tag{9}$$

  where $E_t d_{t+i} \equiv E(d_{t+i}|\Omega_t)$ is the expected dividend in $t+i$, $i \geq 1$, conditional on information available at date t.

- However, a few assumptions are required to justify (9):

  (i) Investors are all **risk neutral**.

(ii) Investors have **homogeneous beliefs** about the probabilities of future dividends, or alternatively,

(ii') there is some true stochastic process governing the dividends and this process is known to all investors (i.e., investors have **rational expectations**).

- Under these assumptions, the arbitrage principle requires every asset yields an **expected** return equal to the risk-free rate.
  - Then we have an equation analogous to (5):

$$r^f_{t+1} = E_t(r_{t+1}) = \frac{E_t v_{t+1} - p_t}{p_t} = \frac{E_t(d_{t+1} + p_{t+1})}{p_t} - 1$$

$$\Rightarrow \quad p_t = \frac{E_t(d_{t+1} + p_{t+1})}{1 + r^f_{t+1}} \tag{10}$$

– Equation (10) holds for any date, $t+i$, in the future:

$$p_{t+i} = \frac{E_t(d_{t+i+1} + p_{t+i+1})}{1 + r^f_{t+i+1}}, \quad \text{for all } i \geq 0. \tag{11}$$

– Applying (11) successively to (10), following a similar procedure as before, we obtain an equation analogous to (6):

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} E_t d_{t+i} + \lim_{N \to \infty} \delta_{t+N} E_t p_{t+N}.$$

– Again, to make p_t well defined, we impose the convergence condition and the transversality condition

$$\sum_{i=1}^{\infty} \delta_{t+i} E_t d_{t+i} < \infty, \quad \lim_{N \to \infty} \delta_{t+N} E_t p_{t+N} = 0,$$

then we reach the NPV equation (9).

- The Gordon growth model under uncertainty (and all the assumptions above) is given by

$$p_t = \frac{E_t d_{t+1}}{r^f - g}, \quad \text{where } g < r^f \tag{12}$$

- What would be the NPV equation for asset prices if the assumption of risk neutrality is relaxed?
  - The discount factors δ_{t+i} need to be adjusted to account for risk-aversion, no longer closely tied to risk-free rates.
  - Example: in a single-period context, the CAPM prediction for asset prices:

$$p_j = \frac{E(v_j)}{1+\mu_j} = \frac{E(v_j)}{1+r_0+(\mu_M - r_0)\beta_j} \quad (\text{with } r_0 = r^f)$$

– δ_{t+i} can be subject to uncertainty as well. Rewrite (9) to accommodate this:

$$p_t = \sum_{i=1}^{\infty} E_t \left(\delta_{t+i} d_{t+i} \right). \tag{13}$$

– Different asset pricing models imply different discount factors δ_{t+i} (another example to be seen in Topic 9).

– The Gordon growth model can be modified as

$$p_t = \frac{E_t d_{t+1}}{\mu - g}, \quad \text{with } g < \mu \tag{14}$$

where μ denotes a constant rate of return per period required by **risk-averse** investors.

This model, combined with the CAPM or APT to determine μ, is widely applied.

4. Asset Price Volatility

- Our discussion so far focuses on the determination of expected returns on assets or the level of asset prices. How about the volatility?

- A claim, commonly heard among critics of financial markets, is that asset prices are 'too volatile'.

- This claim presumes a benchmark for asset prices.
  - Prices predicted by the NPV theory often serve as benchmark prices against which to judge the volatility of assets.

- A pioneering analysis of asset price volatility was done by Robert Shiller – Shiller's **variance bounds test**.

– To illustrate the basic idea, assume the discount factors are based on a constant interest rate r (r may not equal r^f).

– Assume the NPV relationship holds, hence the observed market price of an asset p_t should satisfy

$$p_t = \frac{E_t d_{t+1}}{1+r} + \frac{E_t d_{t+2}}{(1+r)^2} + \frac{E_t d_{t+3}}{(1+r)^3} + \cdots = \sum_{i=1}^{\infty} \frac{E_t d_{t+i}}{(1+r)^i} \quad (15)$$

– Shiller defines the **ex post rational asset price** p_t^* as

$$p_t^* = \frac{d_{t+1}}{1+r} + \frac{d_{t+2}}{(1+r)^2} + \frac{d_{t+3}}{(1+r)^3} + \cdots = \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \quad (16)$$

where d_{t+i}'s are **realised** dividends. So p_t^* is the price predicted by the NPV theory if investors have **perfect foresight** about future dividends.

– What is the relationship between p_t^* and p_t? First note that

$$d_{t+i} = E_t d_{t+i} + \varepsilon_{t+i},$$

where ε_{t+i} is an unobserved forecast error, $E(\varepsilon_{t+i}|\Omega_t) = 0$. Then (16) can be rewritten as

$$\begin{aligned} p_t^* &= \frac{E_t d_{t+1} + \varepsilon_{t+1}}{1+r} + \frac{E_t d_{t+2} + \varepsilon_{t+2}}{(1+r)^2} + \cdots \\ &= \left(\frac{E_t d_{t+1}}{1+r} + \frac{E_t d_{t+2}}{(1+r)^2} + \cdots \right) + \left(\frac{\varepsilon_{t+1}}{1+r} + \frac{\varepsilon_{t+2}}{(1+r)^2} + \cdots \right). \end{aligned}$$

Define u_t as the NPV of forecast errors, then

$$p_t^* = p_t + u_t, \tag{17}$$

That is, p_t plays the role of a forecast of p_t^*.

– (17) implies that

$$\mathrm{var}(p_t^*) = \mathrm{var}(p_t) + \mathrm{var}(u_t) + 2\ \mathrm{cov}(p_t, u_t) = \mathrm{var}(p_t) + \mathrm{var}(u_t),$$

Given that $\mathrm{var}(u_t) > 0$, we must have

$$\mathrm{var}(p_t) < \mathrm{var}(p_t^*) \tag{18}$$

– (18) provides an upper bound, $\mathrm{var}(p_t^*)$, on the volatility of observed market prices.

– Asset prices are regarded as too volatile if $\mathrm{var}(p_t) \geq \mathrm{var}(p_t^*)$.

- To make the variance bounds test in (18) operational, we need to estimate p_t^*, how?

– Shiller applies the arbitrage principle under perfect foresight (recall equation (1)):

$$p_t^* = \frac{d_{t+1} + p_{t+1}^*}{1+r}. \tag{19}$$

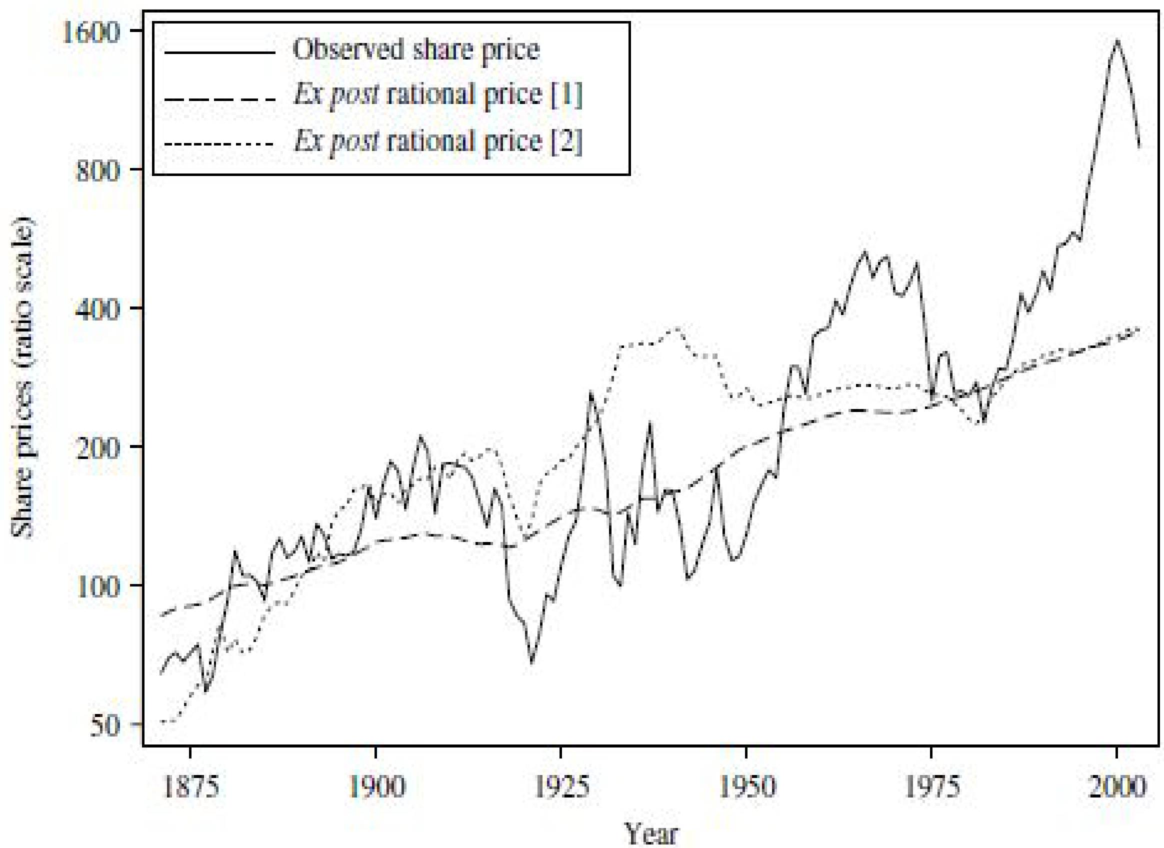
to construct estimates of ex post rational prices p_t^*.

– In Shiller (2003), the terminal year is 2003. A value of p_{2003}^* is estimated using the Gordon growth model.

– Then working backwards, using **observed** dividends, and successively applying (19) to obtain:

$$\begin{aligned} p_{2002}^* &= \frac{1}{1+r}\left(d_{2003} + p_{2003}^*\right) \\ p_{2001}^* &= \frac{1}{1+r}\left(d_{2002} + p_{2002}^*\right) \\ &\cdots \end{aligned}$$

- The following figure from Shiller (2003) plots the observed US stock prices and the constructed ex post rational prices for two measures of interest rates r_t.



- Contrary to (18), the observed stock prices are much more volatile than both ex post rational price series.
  - Shiller shows that this result is robust to alternative rules for estimating p_t^*, the choice of sample period and data set.
  - Shiller draws a strong conclusion: stock prices are too volatile for compatibility with asset market efficiency.
- Shiller's claim has been extensively re-examined and critiqued in the literature (see Bailey for more details).
  - Many empirical studies also find asset prices are too volatile than predicted by the standard NPV pricing equations.

- Some criticism questions the rule for constructing p_t^*.
- Some challenges the assumptions underlying the NPV equations with uncertainty.
- Some interprets the finding as empirical evidence against the fairly simple but widely accepted NPV benchmark.

- All these have motivated efforts to account for asset price volatility. A variety of routes have been pursued.
  - Dynamic asset pricing model which considers individual investor's dynamic portfolio selection decision under uncertainty and risk aversion (Topic 9)
  - Behavioural finance (Topic 11)

5. Bubbles and Ponzi Schemes

- Asset price volatility sometimes takes the form of spectacular increases in prices followed by equally spectacular collapses.
- Many such historical episodes have been documented, with some typical characteristics:
  (a) Often start with some development that can be interpreted rationally as presenting large future prospects for profit.
  (b) Then followed by a period of rapid price increases fueled by manic optimism or frenzy.
  (c) Then a crisis of confidence is triggered by some adverse news or events, causing prices to decline.
  (d) Earlier optimism is replaced by intense pessimism, leading to collapse of prices and economic-wide distress.

- Some most notorious historical examples
  - Tulipmania, 1636-37 (video)
  - The Mississippi and South Sea Bubbles, 1719-20 (video)
  - The Wall Street Crash of 1929
  - The Stock Market Crash of 1987
  - The dot.com bubble of 1999-2000
  - The global financial crisis in 2008
- These phenomena have commonly been called **'bubbles'**.
- Can bubbles exist in equilibrium prices of assets?
  - Yes.

- To illustrate this point, recall the equilibrium price of an indefinitely-lived asset under certainty given by

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} d_{t+i} \tag{7}$$

However, prices determined by (7) are not the only equilibrium prices that satisfy the arbitrage principle.

- Consider any sequence of numbers $\{b_t, b_{t+1}, \ldots, b_{t+i}, \ldots\}$, denoted as $\{b_t\}$, that satifies

$$b_{t+1} = (1 + r^f) b_t,$$

and let p_t be given by the following equation

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} d_{t+i} + b_t \tag{20}$$

where $\delta_{t+i} = \frac{1}{(1+r^f)^i}$.

– We can show that the prices defined by (20) also satisfies the no arbitrage condition (Q3 in Tutorial 9):

$$p_t = \frac{d_{t+1} + p_{t+1}}{1 + r^f} \tag{5}$$

– Because $\{b_t\}$ is not unique, (20) represents a lot of solutions for equilibrium asset prices – indeterminacy of asset prices under infinite horizon.

- Difference between equilibrium prices given by (7) and by (20)? Asset prices given by (20) violate the **transversality condition** (TVC) (see exercise_topic 8):

$$\lim_{N \to \infty} \frac{p_{t+N}}{(1 + r^f)^N} = 0,$$

which requires that asset prices cannot grow too fast.

- It is possible to allow for uncertainty and obtain:

$$p_t = \sum_{i=1}^{\infty} \delta_{t+i} E_t d_{t+i} + b_t, \quad \text{(recall equation (9))} \qquad (21)$$

where $\{b_t\}$ satisfies $E_t b_{t+1} = (1 + r^f) b_t$.

  – Again can show that (21) satisfies the arbitrage principle under uncertainty:

$$p_t = \frac{E_t(d_{t+1} + p_{t+1})}{1 + r^f}$$

- The term b_t in (20) and (21) is called a **'bubble'**, while the discounted value of the dividend stream is called the **'fundamental value'** of the asset.

- As shown in (20) and (21), the presence of bubbles causes the prices of assets to deviate from their fundamental values.
- Bubbles could be driven by speculation, self-fufilling expectations, herding behaviour, etc.
- In economic models with rational investors and homogeneous beliefs, bubbles can be ruled out by imposing the TVC.
- However, bubbles can emerge and persist in models with irrational investors, asymmetric information and heterogeneous beliefs (Brunnermeier (2008)).

* Much research has been devoted to theoretical and empirical aspects of bubbles.

- Bubbles cause asset prices to deviate from fundamental values which can distort the real allocation of an economy.
- Bubbles can lead to financial instability and economy-wide distress.
- There is hot debate on whether central banks should include asset price movements in their policy reaction function (see discussion in Malkiel(2010)).
- Theoretical research seeks to explore what factors may induce a bubble to occur and what conditions are needed to ensure the absence of bubbles.
- Empirical research seeks to identify whether bubbles exist in a particular asset market (Diba and Grossman, 1988).

- **Ponzi schemes**: Ponzi schemes are named after Charles Ponzi who designed a fraudulent investment scheme to earn arbitrage profits in the 1920s (video).
  - A Ponzi scheme is a fraudulent investing scam which generates returns for earlier investors with money taken from later investors.
  - The scheme usually promises a rate of return much higher than could be obtained from any genuine investment.
  - Ponzi schemes are much like bubbles. The difference is that bubbles may not involve a **fraud** while a Ponzi scheme does.
  - Despite the inherent fragility of Ponzi schemes, they are commonplace in financial markets.
  - Private Ponzi schemes inevitably end in collapse.

Review questions

1. Derive the NPV equation under certainty and a constant interest rate, i.e., equation (3), and understand its intuition.
2. Derive the NPV equation under certainty and a variable interest rate, i.e., equation (4). Understand why δ_{t+i} is expressed that way (see exercise_topic 8).
3. Understand why p_t is uniquely determined when N is finite.
4. Derive the NPV equation under certainty for infinite horizon, i.e. equation (7).
5. Derive the Gordon growth model under certainty, and understand how we have reached this equation.
6. Write down the equation that ensures the absence of arbitrage opportunities under uncertainty and risk neutrality, i.e., equation (10).
7. Write down the NPV equation and the Gordon growth model under uncertainty for infinite horizon, i.e., equation (9) and (12), and understand these equations.
8. What is the discount factor in the CAPM model? Understand how

it reflects the risk aversion of investors.

9. Be able to combine the Gordon growth model and the CAPM or APT in practical applications (Quiz 9 contains one example).

10. Understand the basic idea of Shiller's volatility test. In particular, what is the benchmark price used to assess the volatility of observed asset prices?

11. Roughly understand the typical characteristics of financial bubbles. Name a few bubble periods in the history.

12. Understand the formal definition of bubble, as given in equation (20) and (21). In particular, understand why asset prices given in (20) and (21) can be equilibrium asset prices, i.e., compatible with the arbitrage principle; why such prices can grow rapidly over a period of time; how different such asset prices compare with the asset prices fully determined by fundamentals, as given in (7) and (9).

13. What is the crucial feature of a Ponzi scheme? Understand the inherent fragility of a Ponzi Scheme.