ECOM40006/ECOM90013 Econometrics 3
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Week 7 Tutorial Exercise Solutions

Semester 1, 2025

Suppose that you estimate the following autoregressive model

$$y_t = \alpha + \rho y_{t-1} + u_t \tag{1}$$

by ordinary least squares when the true data generating process is given by

$$y_t = y_{t-1} + u_t. \tag{2}$$

where $y_0 = 0$ and the u_t are *iid* random variables with $\mathrm{E}[u_t] = 0$ and $\mathrm{E}[u_t^2] = \sigma^2$ for all $t = 1, \ldots, T$.

(i) Show that

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix} = \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^T u_{t-1} \\ \sum_{t=1}^T y_{t-1} u_t \end{bmatrix}.$$

Solution
This is nothing more that the usual treatment of OLS. Thus, in the model

$$y = X\beta + u$$

we know that

$$\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + u) = \beta + (X'X)^{-1}X'u,$$

so that

$$\hat{\beta} - \beta = (X'X)^{-1}X'u.$$

In this case $\hat{\beta} - \beta = [\hat{\alpha} - 0 \;\; \hat{\rho} - 1]' = [\hat{\alpha} \;\; \hat{\rho} - 1]'$,

$$X = \begin{bmatrix} 1 & y_0 \\ 1 & y_1 \\ 1 & y_2 \\ \vdots & \vdots \\ 1 & y_{T-1} \end{bmatrix}$$

and so

$$X'X = \begin{bmatrix} \sum_{t=1}^{T} 1 \times 1 & \sum_{t=1}^{T} 1 \times y_{t-1} \\ \sum_{t=1}^{T} y_{t-1} \times 1 & \sum_{t=1}^{T} y_{t-1} \times y_{t-1} \end{bmatrix} = \begin{bmatrix} T & \sum_{t=1}^{T} y_{t-1} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 \end{bmatrix},$$

and

$$X'u = \begin{bmatrix} \sum_{t=1}^{T} 1 \times u_t \\ \sum_{t=1}^{T} y_{t-1} \times u_t \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T} u_t \\ \sum_{t=1}^{T} y_{t-1} u_t \end{bmatrix}$$

as required.

(ii) Show that, in terms of orders in probability

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix} = \begin{bmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{bmatrix}^{-1} \begin{bmatrix} O_p(T^{1/2}) \\ O_p(T) \end{bmatrix}$$

and conclude that the quantity that might have a non-degenerate limiting distribution is

$$\begin{bmatrix} T^{1/2} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix}.$$

Solution

Noting that the order in probability of a random variable is just the reciprocal of the scaling required for it to have a non-degenerate limiting distribution, we have the following:

(a) $T = O_p(T)$ (Note: As T is non-stochastic we might equally write $T = O(T)$.)

(b) $\sum_{t=1}^{T} y_{t-1} = O_p(T^{3/2})$ (Follows from the result of Question 1.)

(c) $\sum_{t=1}^{T} y_{t-1}^2 = O_p(T^2)$
We essentially established this result in the lectures where we showed that (in the notation of the lecture)

$$T^{-2} \sum_{t=1}^{T} z_{t-1}^2 \xrightarrow{d} \sigma^2 \int_0^1 [W(r)]^2 \mathrm{d}r$$

(d) $\sum_{t=1}^{T} u_t = O_p(T^{1/2})$
In the lecture notes we showed that

$$X_T(r) = \left[\sigma\sqrt{T}\right]^{-1} \sum_{t=1}^{T} u_t \xrightarrow{d} W(r),$$

which establishes the order of the sum.

(e) $\sum_{t=1}^{T} y_{t-1} u_t = O_p(T)$
In lectures we showed that (in the notation of the lectures)

$$\frac{1}{\sigma^2 T} \sum_{t=1}^{T} z_{t-1} u_t \xrightarrow{d} \frac{1}{2}\left(\chi_1^2 - 1\right) = \frac{1}{2}\left([W(1)]^2 - 1\right)$$

which gives us the appropriate scaling.

Now, we need to choose functions $\omega_1(T)$ and $\omega_2(T)$ such that

$$\begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix}$$

has a non-degenerate limiting distribution. Noting that

$$\begin{aligned}
&\begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix} \\
&= \begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix} \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^T u_{t-1} \\ \sum_{t=1}^T y_{t-1} u_t \end{bmatrix} \\
&= \left(\begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix} \begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \right)^{-1} \\
&\qquad\qquad \times \begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^T u_{t-1} \\ \sum_{t=1}^T y_{t-1} u_t \end{bmatrix} \\
&= \left(\begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \begin{bmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{bmatrix} \begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \right)^{-1} \\
&\qquad\qquad \times \begin{bmatrix} \omega_1(T) & 0 \\ 0 & \omega_2(T) \end{bmatrix}^{-1} \begin{bmatrix} O_p(T^{1/2}) \\ O_p(T) \end{bmatrix} \\
&= \begin{bmatrix} \frac{O_p(T)}{\omega_1(T)^2} & \frac{O_p(T^{3/2})}{\omega_1(T)\omega_2(T)} \\ \frac{O_p(T^{3/2})}{\omega_1(T)\omega_2(T)} & \frac{O_p(T^2)}{\omega_2^2(T)} \end{bmatrix}^{-1} \begin{bmatrix} \frac{O_p(T^{1/2})}{\omega_1(T)} \\ \frac{O_p(T)}{\omega_2(T)} \end{bmatrix}
\end{aligned}$$

we want to choose $\omega_1(T)$ and $\omega_2(T)$ so that the resulting expression are $O_p(1)$. Clearly the choice $\omega_1(T) = T^{1/2}$ and $\omega_2(T) = T$ achieves this outcome.

(iii) Show that

$$\begin{aligned}
&\begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \\
&\qquad \xrightarrow{d} \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 1 & \int_0^1 W(r)\,\mathrm{d}r \\ \int_0^1 W(r)\,\mathrm{d}r & \int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \qquad (3)
\end{aligned}$$

Solution

We have all the results that we need; see the list of distributional results in the Solution to Part (ii) of this Question. Thus

$$\begin{aligned}
&\begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \\
&\qquad = \begin{bmatrix} 1 & T^{-3/2} \sum_{t=1}^T y_{t-1} \\ T^{-3/2} \sum_{t=1}^T y_{t-1} & T^{-2} \sum_{t=1}^T y_{t-1}^2 \end{bmatrix} \\
&\qquad \xrightarrow{d} \begin{bmatrix} 1 & \sigma \int_0^1 W(r)\,\mathrm{d}r \\ \sigma \int_0^1 W(r)\,\mathrm{d}r & \sigma^2 \int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix},
\end{aligned}$$

which yields the desired result.

(iv) Show that

$$\begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{T} u_t \\ \sum_{t=1}^{T} y_{t-1}u_t \end{bmatrix} \xrightarrow{d} \sigma \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} W(1) \\ \frac{1}{2}\left\{[W(1)]^2 - 1\right\} \end{bmatrix} \tag{4}$$

Solution
Again the Solution to Part (ii) of this Question lists the distributional results required. Thus,

$$\begin{aligned} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{T} u_t \\ \sum_{t=1}^{T} y_{t-1}u_t \end{bmatrix} &= \begin{bmatrix} T^{-1/2}\sum_{t=1}^{T} u_t \\ T^{-1}\sum_{t=1}^{T} y_{t-1}u_t \end{bmatrix} \\ &\xrightarrow{d} \begin{bmatrix} \sigma W(1) \\ \frac{\sigma^2}{2}\left([W(1)]^2 - 1\right) \end{bmatrix} \end{aligned}$$

as required.

(v) Combine the results of equations (3) and (4) to show that

$$\begin{bmatrix} T^{1/2}\hat{\alpha} \\ T(\hat{\rho}-1) \end{bmatrix} \xrightarrow{d} \Delta^{-1} \begin{bmatrix} \sigma W(1)\cdot \displaystyle\int_0^1 [W(r)]^2\,\mathrm{d}r - \frac{\sigma}{2}\left\{[W(1)]^2 - 1\right\}\cdot \int_0^1 W(r)\,\mathrm{d}r \\ \displaystyle\frac{1}{2}\left\{[W(1)]^2 - 1\right\} - W(1)\cdot \int_0^1 W(r)\,\mathrm{d}r \end{bmatrix}$$

where $\Delta = \int_0^1 [W(r)]^2\,\mathrm{d}r - \left[\int_0^1 W(r)\,\mathrm{d}r\right]^2$

Solution
Nearly there! Clearly we have established that

$$\begin{bmatrix} T^{1/2}\hat{\alpha} \\ T(\hat{\rho}-1) \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 & \sigma\int_0^1 W(r)\,\mathrm{d}r \\ \sigma\int_0^1 W(r)\,\mathrm{d}r & \sigma^2\int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix}^{-1} \begin{bmatrix} \sigma W(1) \\ \frac{\sigma^2}{2}\left([W(1)]^2 - 1\right) \end{bmatrix}.$$

Writing

$$\begin{bmatrix} 1 & \sigma\int_0^1 W(r)\,\mathrm{d}r \\ \sigma\int_0^1 W(r)\,\mathrm{d}r & \sigma^2\int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix}^{-1} = (\sigma^2\Delta)^{-1} \begin{bmatrix} \sigma^2\int_0^1 [W(r)]^2\,\mathrm{d}r & -\sigma\int_0^1 W(r)\,\mathrm{d}r \\ -\sigma\int_0^1 W(r)\,\mathrm{d}r & 1 \end{bmatrix}$$

our limiting distribution becomes

$$\begin{aligned} &\begin{bmatrix} 1 & \sigma\int_0^1 W(r)\,\mathrm{d}r \\ \sigma\int_0^1 W(r)\,\mathrm{d}r & \sigma^2\int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix}^{-1} \begin{bmatrix} \sigma W(1) \\ \frac{\sigma^2}{2}\left([W(1)]^2 - 1\right) \end{bmatrix} \\ &= (\sigma^2\Delta)^{-1} \begin{bmatrix} \sigma^2\int_0^1 [W(r)]^2\,\mathrm{d}r & -\sigma\int_0^1 W(r)\,\mathrm{d}r \\ -\sigma\int_0^1 W(r)\,\mathrm{d}r & 1 \end{bmatrix} \begin{bmatrix} \sigma W(1) \\ \frac{\sigma^2}{2}\left([W(1)]^2 - 1\right) \end{bmatrix} \\ &= (\sigma^2\Delta)^{-1} \begin{bmatrix} \sigma^3 W(1)\cdot\int_0^1 [W(r)]^2\,\mathrm{d}r - \frac{\sigma^3}{2}\left([W(1)]^2 - 1\right)\cdot\int_0^1 W(r)\,\mathrm{d}r \\ \frac{\sigma^2}{2}\left([W(1)]^2 - 1\right) - \sigma^2 W(1)\cdot\int_0^1 W(r)\,\mathrm{d}r \end{bmatrix} \end{aligned}$$

which yields the desired result.