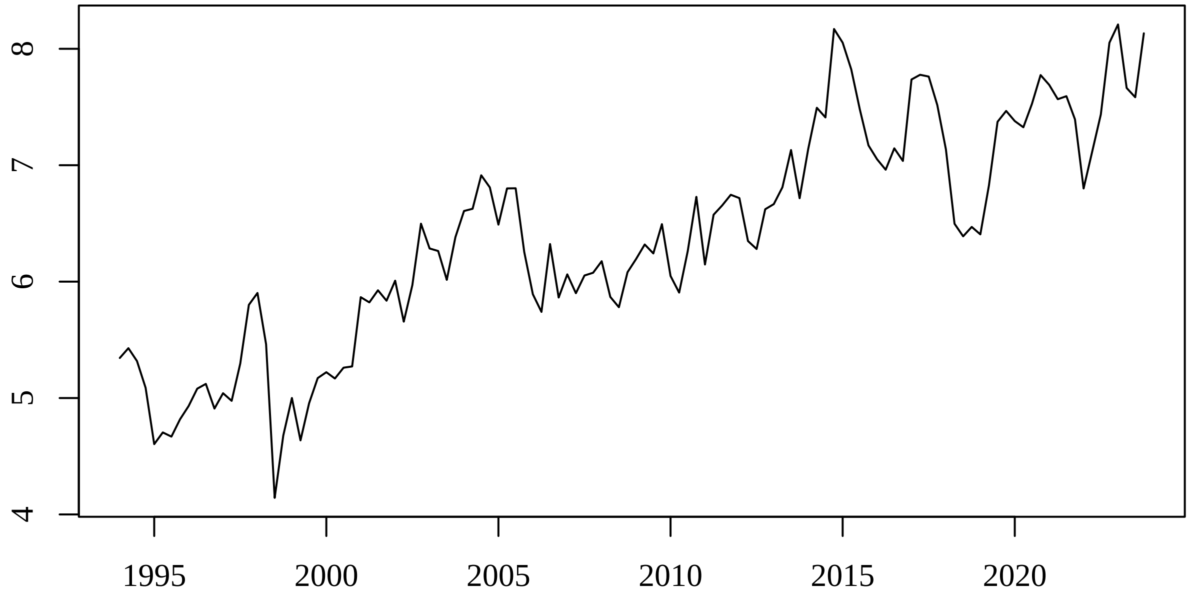
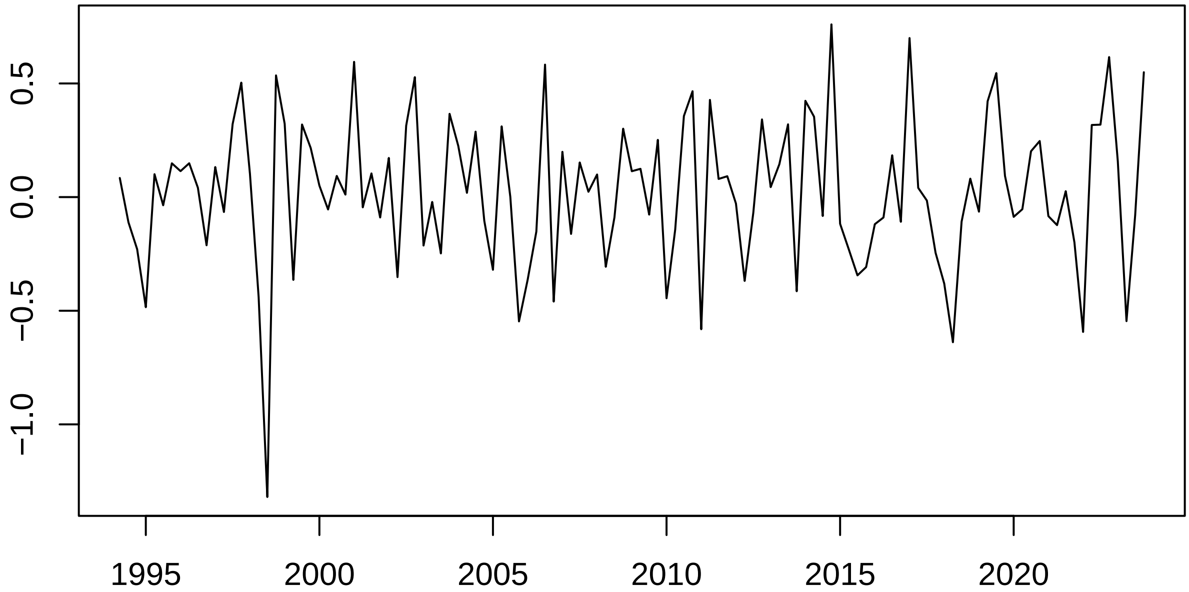
Question 2.

Following is a plot of a quarterly time series with 120 observations, from 1994q1 to 2023q4.



The first difference of the time series shown here:



(a) Purely from visual inspection, do you think the original time series is stationary? What about the first difference time series? Explain.

The original time series appears non-stationary because of the positive trend. (This conclusion can be drawn without speculating on whether it also has a unit root.)

The first difference time series appears to be stationarity. There are no clear signs of trends, variance changes, or the "random wandering" that is characteristic of a unit root.

A Dickey-Fuller test was computed for the original time series, including a trend and two lagged differenced in the test equation. For reference, the command was:

```
adf <- ur.df(Y, type=c("trend"), lags=2, selectlags="Fixed")
```

The following summary results were obtained:

```
###############################################
# Augmented Dickey-Fuller Test Unit Root Test #
###############################################
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.453860   0.360766   4.030 0.000102 ***
z.lag.1     -0.296664   0.073029  -4.062 9.05e-05 ***
tt           0.007527   0.001964   3.833 0.000210 ***
z.diff.lag1  0.141602   0.095048   1.490 0.139089
z.diff.lag2 -0.014406   0.095089  -0.152 0.879851

Critical values for test statistics:
      1pct  5pct 10pct
tau3 -3.99 -3.43 -3.13
```

(b) Write out the estimated ADF test regression in equation form.

$$\Delta\widehat{Y}_t = 1.454 - 0.297Y_{t-1} + 0.008t + 0.142\Delta Y_{t-1} - 0.014\Delta Y_{t-2}$$

(c) Carry out the unit root test at the 5% level of significance, including specifying the hypotheses, test statistic, decision rule and conclusion.

H_0 : unit root vs H_1 : no unit root, or specifically $H_0 : \varphi = 0$ vs $H_1 : \varphi < 0$, where $\Delta Y_t = \beta_0 + \beta_1 t + \varphi Y_{t-1} + \psi_1 \Delta Y_{t-1} + \psi_2 \Delta Y_{t-2} + U_t$

The test statistic is $t = -4.062$.

The 5% critical value is -3.43.

The decision rule is to reject H_0 if $t <$ cv, so the conclusion is that H_0 is rejected and the time series does not have a unit root. It is stationary around the trend, or equivalently the deviations of the time series from the linear are stationary.

(d) Suppose we carried out a unit root test for the differenced time series. What do you think the conclusion would be, and why? What other feature we have discussed in the subject would be relevant for the differenced time series?

The original levels time series was found not to have a unit root, so after taking a first difference the time series would still not have a unit root.

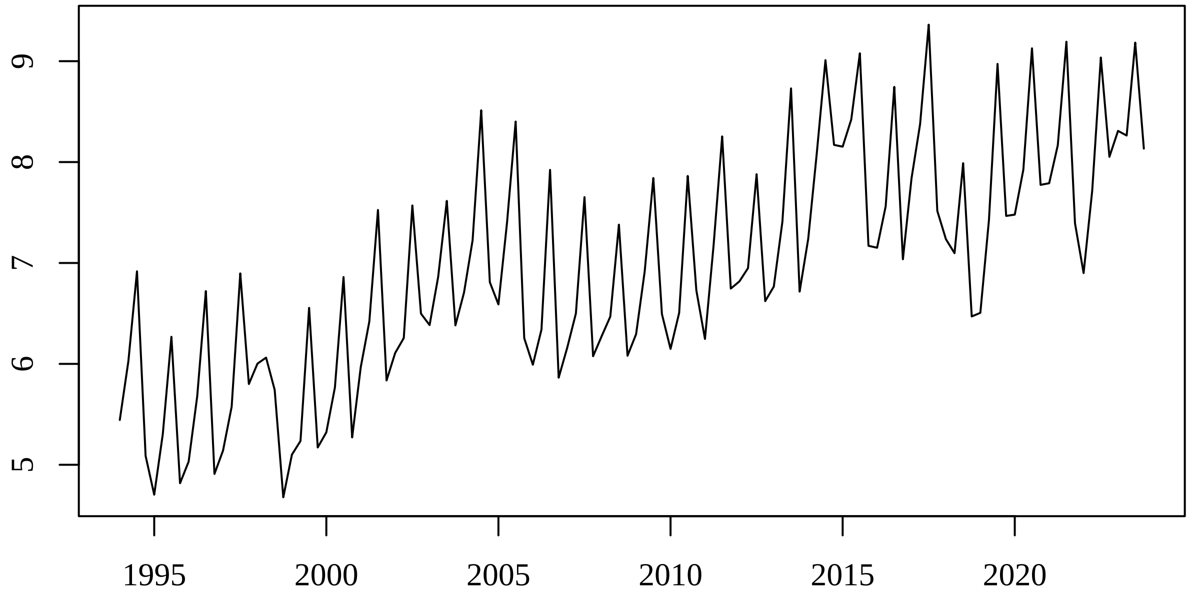
This can be understood as "over-differencing" the time series, which would produce a non-invertible representation.

(e) Suppose the unit root regression above has produced an ADF t statistic of -3.13. What would the p-value be for this statistic? Explain.

This is the 10% critical value, so the p-value would be 0.10.

i.e. the 10% critical value for a lower tailed test is the number below which the probability is 0.10.

Here is another closely quarterly time series over the same time period:



(f) What additional feature is evident in this time series that is not in the original time series?

This time series has a clear seasonal pattern.

(g) Write out the form of the ADF test equation you would want to estimate to test for a unit root in this time series.

$$\Delta Y_t = \beta_0 + \beta_1 t + \beta_2 Q_{2,t} + \beta_3 Q_{3,t} + \beta_4 Q_{4,t}$$
$$+ \varphi Y_{t-1} + \psi_1 \Delta Y_{t-1} + \psi_{p-1} \Delta Y_{t-p+1} + U_t$$

where $Q_{2,t}$, $Q_{3,t}$, $Q_{4,t}$ are seasonal dummy variables.

(h) Give an outline of the steps you would implement to obtain the appropriate critical value for the unit root in the previous part. (Do not give R code, just simple descriptions of the computational steps required.)

Standard ADF critical values are not applicable because of the quarterly dummies in the test equation.

Therefore implement a simulation of (say) 10,000 replications of the steps:

1. Simulate $U_1, \ldots, U_n$ from a standard normal distribution with $n = 120$.
2. Generate the random walk $Y_t = Y_{t-1} + U_t$ with $Y_0 = 0$.
   Equivalently $Y_t = \sum_{j=1}^{t} U_j$
3. Estimate the test regression in (g) with no lagged differences.
4. Obtain the t statistic on Y_{t-1}

From these 10,000 simulated t statistics, obtain the 0.05 quantile as the 5% critical value for the lower tailed test.