ECON30024
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Additional exercise questions on Topic 4

1. Solve the portfolio selection problem with one risky asset and a risk free asset (Topic 4, slide 9)

$$\max_{(\sigma_P,\mu_P)} \left\{\mu_P - \alpha\sigma_P^2\right\}$$
$$\text{s.t.} \quad \mu_P = r_0 + \left(\frac{\mu_1 - r_0}{\sigma_1}\right)\sigma_P \tag{1}$$

2. You are given the following information about three assets:

Asset	μ_j	σ_j
1	8%	8%
2	16%	50%
3	11%	10%

The risk-free rate is $r_0 = 6\%$.

(a) Calculate the Sharpe ratio for each of the three assets. Comment on the comparative performance of the three assets.

(b) You are now told that a mean-variance efficient portfolio, Z, is available with $\mu_Z = 13.5\%$ and $\sigma_Z = 15\%$. Are the three assets mean-variance efficient as well?

3. Find the minimum risk portfolio of two risky assets when $\rho_{12} = 0$ (Topic 4, slide 26).

4. Suppose there are two risky assets, Asset 1 and Asset 2. The expected rates of return on Asset 1 and Asset 2 are 0.12 and 0.08, respectively. And the standard deviations of the rates of return on Asset 1 and 2 are 0.3 and 0.2, respectively. The correlation coefficient between the two rates of return is 0.5.[1]

(a) Sketch the portfolio frontier.

(b) Find the minimum risk portfolio.

(c) Suppose an investor's objective function is given by $G(\mu_P, \sigma_P) = \mu_P - 0.5\sigma_P^2$. Find her optimum portfolio of Asset 1 and Asset 2?

[1] Working through this question can deepen your understanding of portfolio selection theory and prepare you for assignment 2 which requires you to choose stocks in the ASX to construct your own portfolio frontier and optimum portfolio.

1. Solution: Using equation (1) to substitute out μ_P in the objective function, we can rewrite the problem into an unconstrained maximisation problem:

$$\max_{\sigma_P} \left\{ r_0 + \left(\frac{\mu_1 - r_0}{\sigma_1}\right)\sigma_P - \alpha\sigma_P^2 \right\}$$

Note that σ_P is the only choice variable. So the first-order condition is given by differentiating the objective function with respect to σ_P and setting it to zero:

$$\frac{\mu_1 - r_0}{\sigma_1} - 2\alpha\sigma_P = 0,$$

This gives the optimal σ_P as

$$\sigma_P = \frac{\mu_1 - r_0}{2\alpha\sigma_1},$$

and the corresponding μ_P is given by

$$\mu_P = r_0 + \left(\frac{\mu_1 - r_0}{\sigma_1}\right)\frac{\mu_1 - r_0}{2\alpha\sigma_1} = r_0 + \frac{(\mu_1 - r_0)^2}{2\alpha\sigma_1^2}.$$

Note that these values are exactly the same as the expected return and standard deviation of return of the optimum portfolio we derived earlier using the other approach (choosing a_1), (μ_E, σ_E) on slide 7.

2. Solution:

   (a) The Sharpe ratio for asset j is defined by $s_j = \frac{\mu_j - r_0}{\sigma_j}$. For the three assets in the table, their Sharpe ratios are given by
   Asset 1: $s_1 = \frac{0.08-0.06}{0.08} = 0.25$
   Asset 2: $s_2 = \frac{0.16-0.06}{0.5} = 0.2$
   Asset 3: $s_3 = \frac{0.11-0.06}{0.1} = 0.5$
   The Sharpe ratio can be interpreted for each asset as its risk-adjusted return, that is, the expected return in excess of the risk-free rate per unit of return volatility or risk. The Sharpe ratio provides a measurement of an asset's outperformance relative to its risk. In the question, Asset 3 has the highest Sharpe ratio, so Asset 3 outperforms Asset 1 and 2.

   (b) The Sharpe ratio for portfolio Z is given by

   $$s_Z = \frac{0.135 - 0.06}{0.15} = 0.5.$$

   Since only asset 3 has the same Sharpe ratio as portfolio Z, only asset 3 is mean-variance efficient, i.e., it locates on the the portfolio frontier. Asset 1 and 2 lie below the portfolio frontier since they have lower Sharpe ratios, and they are not mean-variance efficient.

3. We need to find the portfolio that has the minimum σ_P^2. With $\rho_{12} = 0$, equation (4) on Topic 4 slides becomes

$$\sigma_P^2(\mu_P) = (\sigma_1^2 + \sigma_2^2)\left(\frac{\mu_P - \mu_2}{\mu_1 - \mu_2} - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2 + \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Note that when

$$\frac{\mu_P - \mu_2}{\mu_1 - \mu_2} - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = 0,$$

i.e., when

$$\mu_P = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\mu_2,$$

the first term (which is nonnegative) in the expression of σ_P^2 disappears such that σ_P^2 is minimised and the minimum value is

$$\sigma_P^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

So the minimum risk portfolio is given by the point $(\sigma_{mrp}, \mu_{mrp})$ in the (σ_P, μ_P) space, where

$$\mu_{mrp} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\mu_2, \quad \sigma_{mrp}^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

Alternatively, we can express the minimum risk portfolio as proportions of assets, $(a_1, 1 - a_1)$, where

$$a_1 = \frac{\mu_P - \mu_2}{\mu_1 - \mu_2} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

(Note that $a_1 = \frac{\mu_P - \mu_2}{\mu_1 - \mu_2}$ is from slide 19 of Topic 4.)

4. Solution: The following values are given

$$\mu_1 = 0.12,\ \mu_2 = 0.08,\ \sigma_1 = 0.3,\ \sigma_2 = 0.2,\ \rho_{12} = 0.5.$$

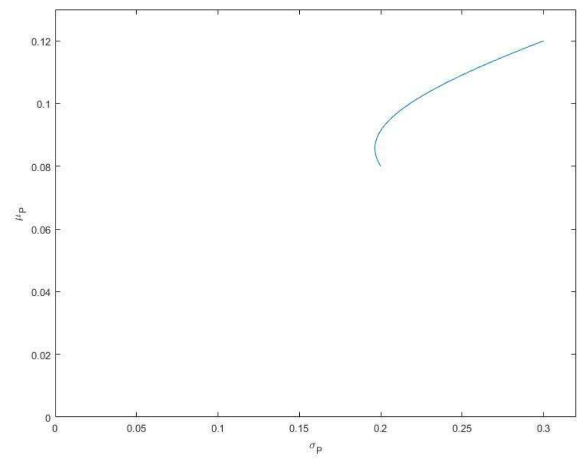
   (a) Let a_1 denote the proportion of wealth invested in Asset 1, then the expected value and the variance of the rate of return on the portfolio are given by

$$\begin{aligned}
\mu_P &= a_1\mu_1 + (1 - a_1)\mu_2 = a_1(0.12) + (1 - a_1)(0.08) = 0.08 + 0.04a_1, \qquad (1)\\
\sigma_P^2 &= a_1^2\sigma_1^2 + 2a_1(1 - a_1)\rho_{12}\sigma_1\sigma_2 + (1 - a_1)^2\sigma_2^2\\
&= a_1^2(0.09) + 2a_1(1 - a_1)(0.5)(0.3)(0.2) + (1 - a_1)^2(0.04)\\
&= (0.09)a_1^2 + (0.06)a_1 - (0.06)a_1^2 + (0.04)(1 - 2a_1 + a_1^2)\\
&= (0.07)a_1^2 - (0.02)a_1 + 0.04 \qquad (2)
\end{aligned}$$

Let a_1 take on a few values, and use (1) and (2) to find the corresponding values for μ_P and σ_P:

$$
\begin{aligned}
a_1 &= 0 \Rightarrow \mu_P = \mu_2 = 0.08,\ \sigma_P = \sigma_2 = 0.2 \\
a_1 &= 1 \ \Rightarrow \mu_P = \mu_1 = 0.12,\ \sigma_P = \sigma_1 = 0.3 \\
a_1 &= 0.5 \ \Rightarrow \mu_P = 0.08 + (0.04)(0.5) = 0.1, \\
&\qquad\qquad \sigma_P = \sqrt{(0.07)(0.5)^2 - (0.02)(0.5) + 0.04} \approx 0.2179 \\
&\quad \cdots
\end{aligned}
$$

In the (σ_P, μ_P) space, connect these points, you'll get a sketch of the portfolio frontier. To obtain a more accurate portfolio frontier, I used Matlab and let a_1 take on values $0, 0.01, 0.02, \ldots, 0.99, 1$, then I got the following figure.



(b) To find the minimum risk portfolio (MRP), we need to find the value of a_1 that minimises the variance given in (2):

$$\min_{a_1 \in [0,1]} \sigma_P^2 = (0.07)a_1^2 - (0.02)a_1 + 0.04$$

The first-order condition is given by

$$(0.07)(2a_1) - 0.02 = 0 \quad \Rightarrow a_1 = \frac{0.02}{2(0.07)} = \frac{1}{7}.$$

When $a_1 = \frac{1}{7}$, the corresponding values for μ_P and σ_P are given by

$$
\begin{aligned}
\mu_{mrp} &= 0.08 + 0.04/7 \approx 0.0857 \\
\sigma_{mrp} &= \sqrt{(0.07)(1/7)^2 - (0.02)(1/7) + 0.04} \approx 0.1964.
\end{aligned}
$$

So the minimum risk portfolio is represented by $(\sigma_{mrp}, \mu_{mrp})$ in the (σ_P, μ_P) space.

(c) From (1),

$$a_1 = \frac{\mu_P - 0.08}{0.04}.$$

Plugging this expression into (2),

$$\begin{aligned}\sigma_P^2 &= (0.07)\left(\frac{\mu_P - 0.08}{0.04}\right)^2 - (0.02)\left(\frac{\mu_P - 0.08}{0.04}\right) + 0.04 \\ &= (43.75)\,(\mu_P - 0.08)^2 - (0.5)\,(\mu_P - 0.08) + 0.04 \qquad (3)\end{aligned}$$

Note that you can use equation (3) to draw the portfolio frontier as well. That is, let μ_P take on a range of different values, and find the corresponding σ_P using (3), then plot these points in the (σ_P, μ_P) space to obtain the portfolio frontier. Do this in Excel for yourself and compare with what we get in (a). Do you see any difference, and why? You can discuss your answer with me.

The the **portfolio selection problem** is formulated as

$$\max_{\mu_P} \left\{\mu_P - 0.5\sigma_P^2\right\},$$

where σ_P^2 is given by (3).

The first-order condition with respect to μ_P is given by:

$$\begin{aligned}1 - 0.5\,[(43.75)(2)(\mu_P - 0.08) - 0.5] &= 0 \\ 1 - 0.5\,[87.5\mu_P - 7.5] &= 0 \\ 1 - (43.75)\mu_P + (0.5)(7.5) &= 0 \\ \mu_P^* = \frac{4.75}{43.75} &\approx 0.1086\end{aligned}$$

The corresponding σ_P^* is given by

$$\sigma_P^* = \sqrt{(43.75)\,(\mu_P^* - 0.08)^2 - (0.5)\,(\mu_P^* - 0.08) + 0.04} \approx 0.2478$$

So the optimum portfolio is represented by (σ_P^*, μ_P^*) in the (σ_P, μ_P) space. Note that it has a higher return and a higher risk than the minimum risk portfolio.

The optimal proportion of wealth invested in Asset 1 is given by

$$a_1^* = \frac{\mu_P^* - 0.08}{0.04} = 0.7150.$$

Because the investor is not very risk averse ($\alpha = 0.5$), investing the majority of wealth in the asset with higher risk and return turns out to be optimal for her.

Recall that the minimum risk portfolio involves $a_1 = \frac{1}{7} < a_1^*$. This question also illustrates that the minimum risk portfolio is not necessarily the optimal portfolio that would be chosen by an investor because the latter maximises the investor's mean-variance objective while the former just minimises her risk.