Lecture 4: Production Functions and The Firm's Problem

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Last class

- We finished up looking at the household **individual** consumption-savings problem
- We covered what the implications of the permanent income hypothesis meant for Keynesian theories of consumption
- Keynesian consumption function descriptive, current consumption depends on current income
- Household consumption-savings problem, derived consumption as a choice and depends on permanent/lifetime income

The Representative Firm
And
The Production Function

The Firm

- Objective of the firm: maximize profits

- How? : A firm buys inputs (factors of productions) at some cost and converts them into output (consumption goods)

- A firm gets revenue from selling its output

- Profits are then given by:

$$\text{Profits} = \text{Revenue - Cost}$$

The Firm

Assumptions we will make:

- Firms are very smart (know how to maximize profits!)
- All firms have the same technology $\Rightarrow$ **focus on a representative firm**
- They use only two factors of productions: **capital and labor**
- Solve the same problem every period
- No financing issues

The Production Function

- Before we can talk about how the firm maximizes profits
- We need to know how it can convert inputs into output
- So we need to define our production function

The Production Function

- Production Function: specifies how much output (Y) can be produced given any number of inputs K and L
- Notation:
  - Labor: ℓ for an individual firm, L for aggregate
  - Capital: k for individual firm, K for aggregate
  - TFP (Solow Residual): $z \implies$ everything in Y not accounted by measurable inputs
  - Output: y for individual firm, Y for aggregate.
- Production Function: $Y = F(z, K, L)$

Marginal Product

Definition

- Marginal product of labour (capital) MPL (MPK) is the additional output produced by increasing labour (capital) by one unit, keeping fixed the other input.

- Mathematically, given a production function, the marginal product of labor is:

$$MPL = \frac{\partial F(z, K, L)}{\partial L}$$

- And the marginal product of capital as:

$$MPK = \frac{\partial F(z, K, L)}{\partial K}$$

Assumptions on Production functions

We will assume that production has the following properties

- **More input, more output** : Holding capital fixed, more labor produces more output
- **Constant returns to scale**
- **Diminishing marginal products**
- **Complementarity** : More capital, makes labor more productive

Cobb–Douglas Production Function

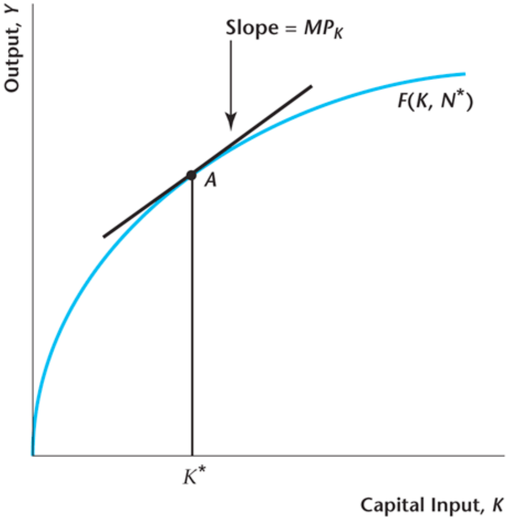
- A widely-used production function that fits these assumptions is the Cobb-Douglas (CD) production function:

  $$Y = zK^{\alpha}L^{1-\alpha},$$

  α is a parameter with $0 \leq \alpha \leq 1$

Properties of the Production Function

- **More Input, More Output** implies MPK and MPL are positive



- Under Cobb-Douglas production function:

$$MPK = \alpha z K^{\alpha-1} L^{1-\alpha} > 0$$

- Exercise: show that $MPL > 0$ for Cobb-Douglas on your own.

Properties of the Production Function

- **Constant returns to scale**: If all inputs increase by $x\%$, output increases by $x\%$
- The Cobb-Douglas production function features constant returns to scale (CRS):
  - To see this, we can show that if we double the inputs, it gives double the output

Properties of the Production Function

- **Diminishing marginal product**: implies F is concave
- Holding all else constant, output is increasing in its input at a diminishing rate
  - Cobb-Douglas production function features diminishing marginal products:

$$\frac{\partial^2 Y}{\partial K^2} = -\alpha(1-\alpha)zK^{\alpha-2}L^{1-\alpha} < 0$$

and

$$\frac{\partial^2 Y}{\partial L^2} = -\alpha(1-\alpha)zK^{\alpha}L^{-\alpha-1} < 0$$

Properties of Production Function

- **Complementarity**: K and L are complements
  - Cobb-Douglas features complementarity between K and L as seen from positive cross-partial:
$$\frac{\partial^2 Y}{\partial K \partial L} = \alpha(1-\alpha)zK^{\alpha-1}L^{-\alpha} > 0$$

From data to model

Measuring inputs: labour

- Total number of hours worked $(L) =$ total employed $(N) \times$ number of hours per worker (H)

Measuring Inputs: Capital

- A wide variety of capital goods used to produce output (e.g., equipment, buildings, software, etc.)

- Typically measure of capital K_t in terms of **value** of non-human inputs

- Common to measure K via the *Perpetual Inventory Method*:

  - Form an initial estimate of K_0 summing over all types capital goods in (real) constant dollar terms.

  - Update that estimate recursively using data on investment and depreciation.

  - Depreciation: capital wears out over time, capital stock decreases.

  - Investment: acquisition of new capital goods, capital stock increases

Capital accumulation

- The change in capital stock between two time periods:

$$\Delta K_t \equiv K_{t+1} - K_t = I_t - D_t$$

- Re-arrange:

$$K_{t+1} = K_t + (I_t - D_t)$$

  - K_t: capital stock at beginning of period t
  - I_t: new purchases of capital or **gross** investment in period t
  - D_t typically defined as fraction of K_t that wears out, $D_t = \delta K_t$, where $\delta =$ **the depreciation rate**

- So capital evolves according to:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

Total factor Productivity z_t

- Improvements in TFP make it possible to produce more output <u>without</u> additional inputs.
- Many factors can cause TFP to change:
  - (unmeasured) improvements (quality improvements) embodied in capital and labour inputs
  - Disembodied TFP changes that boost productivity in a more general way (changes to other productive factors not captured by K or L)
- TFP is hard to measure directly – it's often computed as a residual.

Growth Accounting

- **Growth accounting:** for a given production function, how much of output growth over a given period of time is due to growth in inputs, or changes in TFP?
  - growth in K weighted by capital's share α
  - growth in L weighted by labour's share $1 - \alpha$.
  - growth in TFP

Growth Accounting

- A useful note: the difference in (natural) logs approximates percentage growth
  - Suppose x_1 grew at rate g between period 1 and 2, i.e,
    $$x_2 = x_1(1+g)$$
  - Take natural logs and re-arrange:
    $$\ln(1+g) = \ln x_2 - \ln x_1$$
  - For small enough g, $\ln(1+g) \approx g$ (look up Taylor series expansion):
    $$g \approx \ln x_2 - \ln x_1$$
- A useful exercise (you will try this in your assignment): show that GDP growth rates can be approximated with the difference in logs.

Growth Accounting

- A useful note: the difference in (natural) logs approximates percentage growth
  - We have output $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$
  - Growth of Y is equal to the weighted sum of growth rates of its components:

$$
\begin{aligned}
g_{y,t} &= \ln Y_{t+1} - \ln Y_t \\
&= \overbrace{\ln z_{t+1} - \ln z_t}^{g_{z,t}} \\
&\quad + \alpha \underbrace{(\ln K_{t+1} - \ln K_t)}_{g_{K,t}} + (1-\alpha) \underbrace{(\ln L_{t+1} - \ln L_t)}_{g_{L,t}}
\end{aligned}
$$

  - Data on Y, K and L, and $1-\alpha$ can be estimated by the income share of labour $\Longrightarrow$ back out z and g_z as residual

Measures of productivity

- When you read the news, the word "productivity" is used to refer to many different objects
- Suppose the production function is $Y = zK^{\alpha}L^{1-\alpha}$
- Formally, we define:

$$\text{Total Factor Productivity, } TFP = z$$

$$\text{Avg. Product of Labor (Labor Productivity)} = Y/L$$

$$\text{Marginal Product of Labor, } MPL = \frac{\partial Y}{\partial L}$$

- Do the three measure move in the same direction?

Profit Maximization

Firm Profit Maximisation

- Goal of firms: maximize profits π
- Revenue: firms sell Y to consumers ($P = 1$, consumption $=$ numeraire good)
- Cost: in every period, firms rent capital and hire labor to produce output
- Markets are perfectly competitive and all firms are identical
  - $\implies$ can summarize the collective production of all firms by the production of **one representative firm**.

Firm Profit Maximisation

- Goal of firms: maximize profits $\implies \max \pi =$ Revenue $-$ Cost
- Revenue: $Y = F(z, K, L)$
- Cost: perfectly competitive markets means firms take prices – rental rate R and real wage rate w – as given
- Firm therefore solves:

$$\max_{K,L} \pi = F(z, K, L) - wL - RK$$

Firm Profit Maximisation

- Firm solves:

$$\max_{K,L} \pi = F(z, K, L) - wL - RK$$

  - Firm chooses how much capital and labour to use in production
  - No choice over w,R (perfect competition assumption)
  - No choice over z (TFP is exogenous)

Optimality

- Optimality entails marginal benefit $=$ marginal cost

$$MPL = w \quad \text{and} \quad MPK = R$$

- Why?
  - If $MPL > w$, each additional unit of labour brings the firm more additional revenue than it does to costs, firm should hire more labour!
  - If $MPL < w$, each additional unit of labour adds more to firm's costs than it does to revenue, firm should reduce labour hired
- Firm's profit maximized when marginal product $=$ marginal cost

An example of firm profit maximization with Cobb-Douglas Production Function

$$\max_{K,L} \pi = zK^{\alpha}L^{1-\alpha} - wL - RK$$

- Taking first order conditions (FOCs), we have:
  - Optimal labour demand:

$$MPL = (1-\alpha)z\left(\frac{K}{L}\right)^{\alpha} = w$$

  - Optimal capital demand:

$$MPK = \alpha z\left(\frac{K}{L}\right)^{\alpha-1} = R$$

An example of firm profit maximization with Cobb-Douglas

- Optimal labour demand:

$$MPL = (1-\alpha)z\left(\frac{K}{L}\right)^{\alpha} = w$$

  - Re-arrange:

$$L = \left[\frac{(1-\alpha)zK^{\alpha}}{w}\right]^{1/\alpha}$$

  - Given z and choice of K, firm tells you its demand schedule for L for any price w

Exercise: show that the firm's demand for K is declining in R

Implications of Cobb-Douglas production with perfect competition

- Profit maximization with perfect competition implies that **factors earn their marginal products**

- The share of income paid to capital is given by:

$$\frac{RK}{Y} = \alpha z \left(\frac{K}{L}\right)^{\alpha-1} \frac{K}{Y} = \alpha$$

- The share of income paid to labour is:

$$\frac{wL}{Y} = (1-\alpha) z \left(\frac{K}{L}\right)^{\alpha} \frac{L}{Y} = 1-\alpha$$

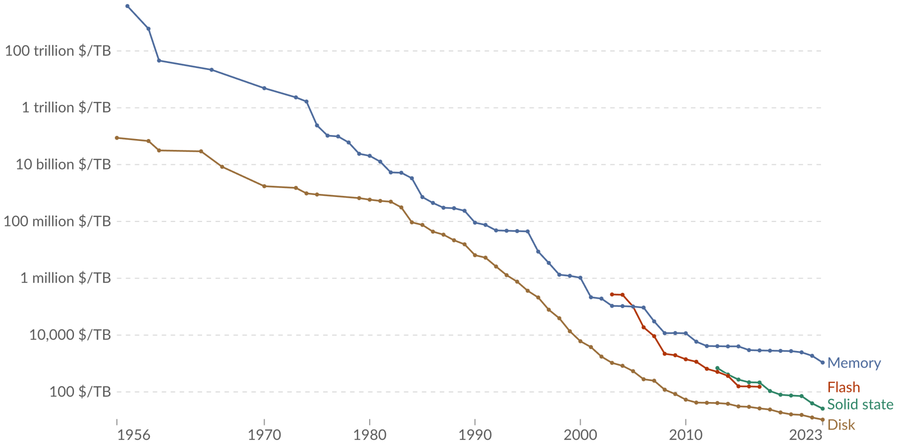
- This implies that firms earn zero profits: share of income paid to factors of production sum to 1.

$$\pi = Y - RK - wL = 0$$

A digression

Historical price of computer memory and storage

This data is expressed in US dollars per terabyte (TB), adjusted for inflation. "Memory" refers to random access memory (RAM), "disk" to magnetic storage, "flash" to special memory used for rapid data access and rewriting, and "solid state" to solid-state drives (SSDs).



Data source: John C. McCallum (2023); U.S. Bureau of Labor Statistics (2024)

Note: For each year, the time series shows the cheapest historical price recorded until that year. This data is expressed in constant 2020 US$.

Back to example of profit maximization with Cobb-Douglas

- Optimal labour demand:

$$L = \left[\frac{(1-\alpha)zK^{\alpha}}{w}\right]^{1/\alpha}$$

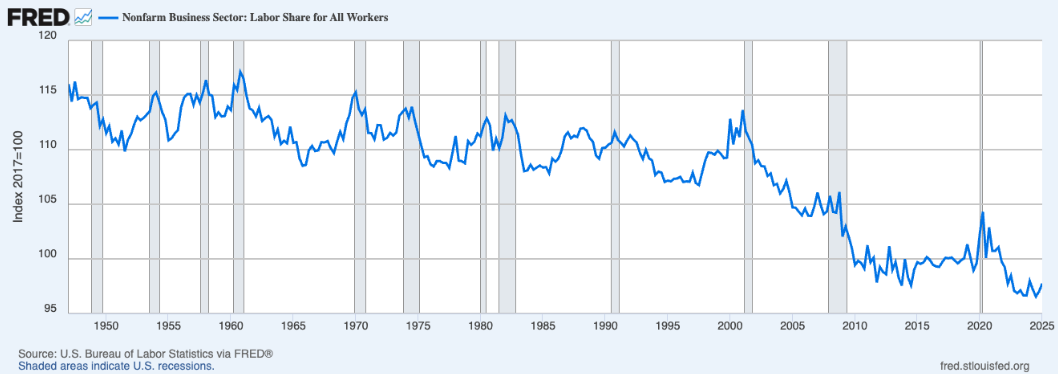
- Optimal capital demand:

$$K = \left[\frac{\alpha z L^{1-\alpha}}{R}\right]^{\frac{1}{1-\alpha}}$$

- If R falls over time, firms demand more capital, holding all else (z, w) constant.

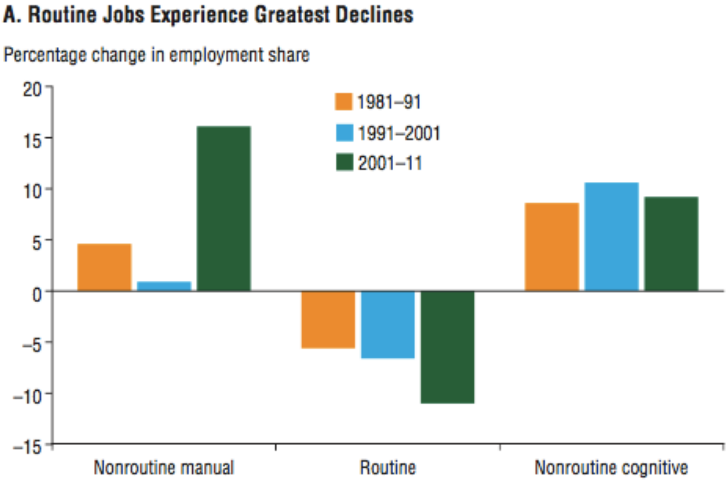
- But if firms demand more K, what happens to their labour demand? Why does L change in that direction?

Labour share declining in the US post 2000s



Production function that predicts constant factor shares may not be so realistic

Hollowing out of routine jobs



One simple alternative

- Suppose there are two types of labour: L_r and L_{nr}
- Production function is:

$$Y = z(K + L_r)^{\alpha} L_{nr}^{1-\alpha}$$

- K and L_r are complements to L_{nr}
- Are K and L_r complements to each other?

One simple alternative

- Firm's maximization problem is

$$\max_{K,L_r,L_{nr}} = z(K+L_r)^\alpha L_{nr}^{1-\alpha} - RK - w_r L_r - w_{nr} L_{nr}$$

- Firm can use K and/**or** L_r in production (perfect substitutes)
- Suppose R falls over time such that $R < w_r$, hold all else (w_r, w_{nr}, z) constant
- What do you think might happen to firm's demand for K, L_r and L_{nr} in this case?

A note

- We can see that the assumptions about production functions can affect the choices firms make regarding their inputs
- For the most part, we will assume that firms mainly make choices over K and L (and not over different types of labour inputs)
- In next week's tutorial: you will explore a CES production function which allows for different degrees of substitutability between K and L

Roadmap

- Today: production functions and firm profit maximization problem
- Next class: Introduction to an OLG model
- After that: General equilibrium in an OLG model