Lecture 8

Moving average invertibility and autoregressive approximation

Autoregressive approximation

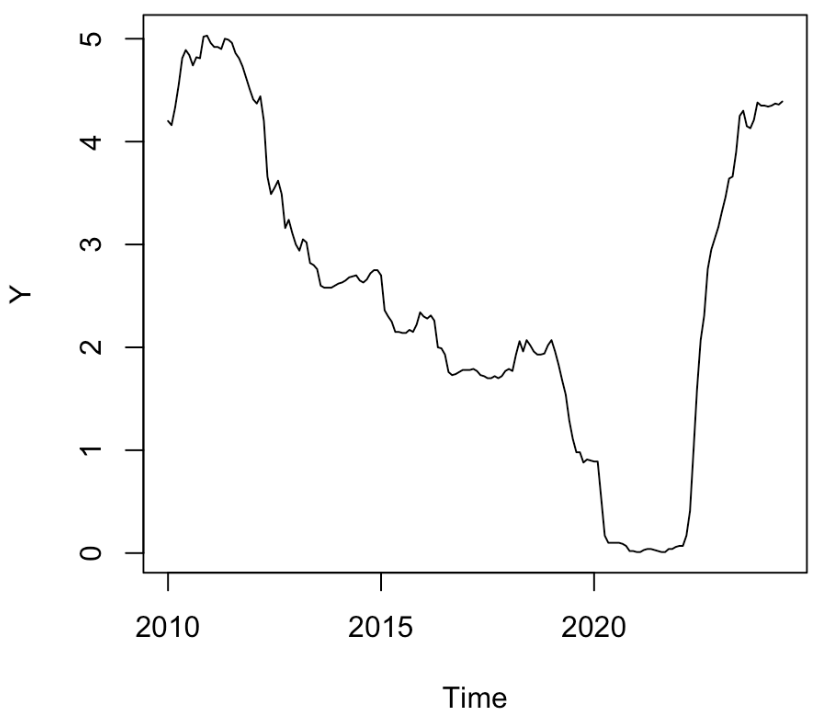
AR vs (AR)MA models

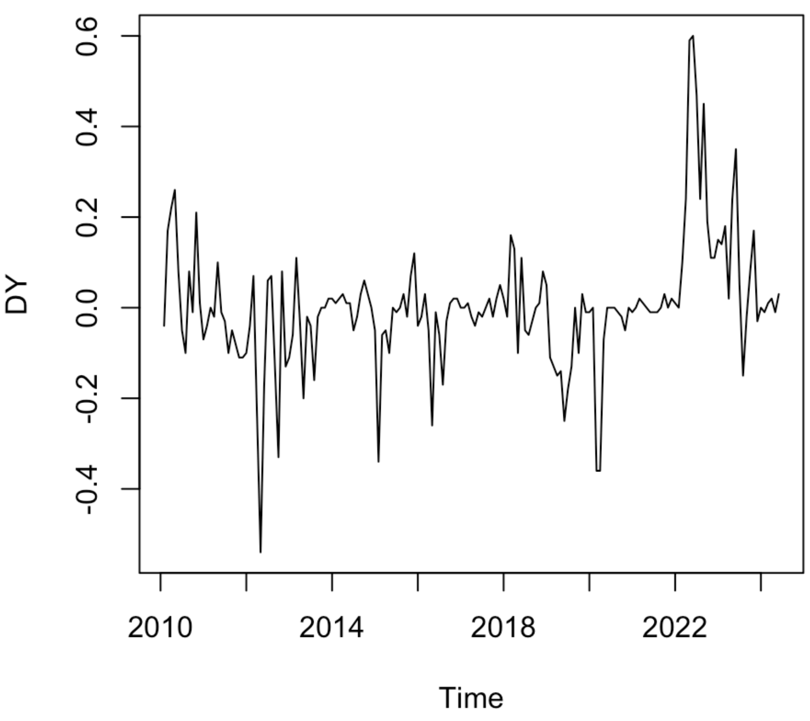
- Specification search is simpler in one (p) than two (p,q) dimensions.
- AR can be estimated by OLS
  MA requires numerical search
- AR models are easier to extend to models of multiple time series.

(There are not huge problems for MA.)

Interest rate forecasts

In Tutorial 8 we construct forecasting models for first differences of 3 month Bank Accepted Bills.





ARMA(*p*,*q*) search

AICc proportion of maximum:

	q=0	q=1	q=2	q=3	q=4
p=0	0.691	0.931	0.962	0.960	0.968
p=1	0.987	0.981	1.000	0.993	0.988
p=2	0.980	0.976	0.992	0.995	0.986
p=3	0.994	0.995	0.987	0.979	0.980
p=4	0.993	0.987	0.986	0.979	0.971
p=5	0.985	0.979	0.979	0.971	0.962
p=6	0.978	0.972	0.978	0.972	0.971

ARMA(1,2) model

ar1	ma1	ma2	intercept
0.904	-0.305	-0.310	0.004

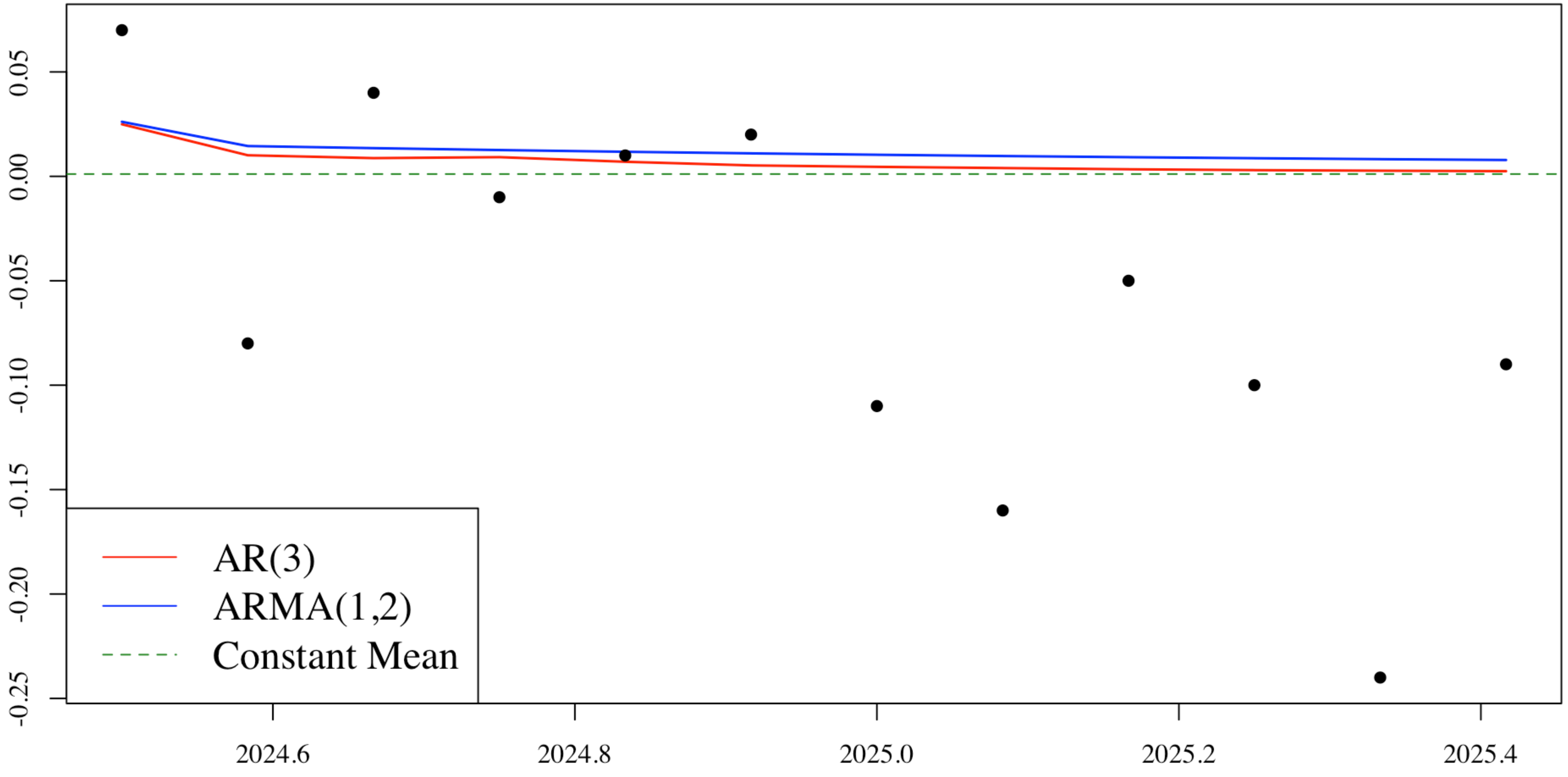
$$Y_t = 0.004 + Z_t$$
$$Z_t = 0.904 Z_{t-1} + U_t - 0.305 U_{t-1} - 0.310 U_{t-2}$$

AR(3) model

ar1	ar2	ar3	intercept
0.639	-0.153	0.183	0.002

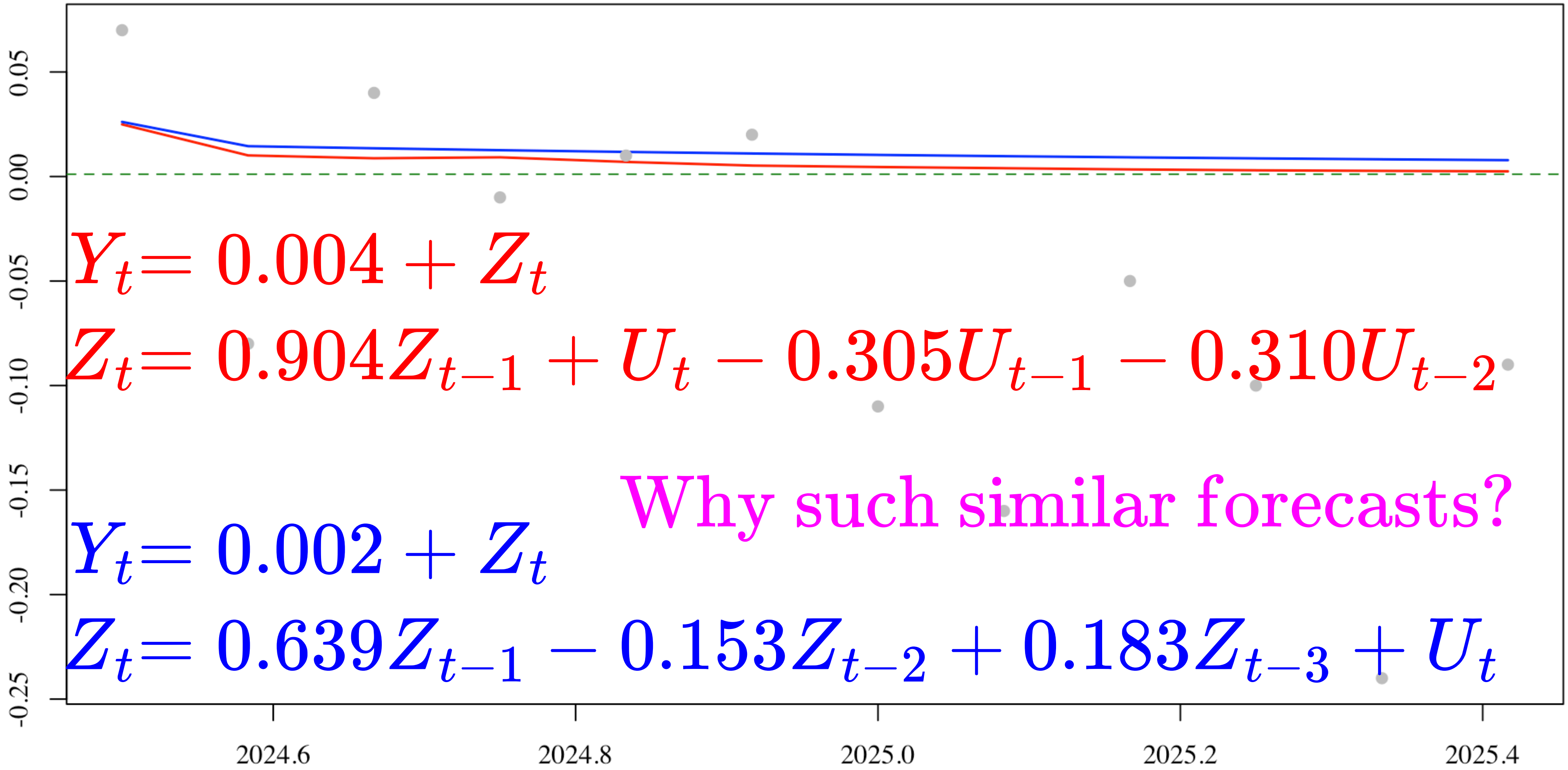
$$Y_t = 0.002 + Z_t$$
$$Z_t = 0.639 Z_{t-1} - 0.153 Z_{t-2} + 0.183 Z_{t-3} + U_t$$

Forecasts



0.05
0.00
-0.05
-0.10
-0.15
-0.20
-0.25
2024.6
2024.8
2025.0
2025.2
2025.4
AR(3)
ARMA(1,2)
Constant Mean

Forecasts



$$Y_t = 0.004 + Z_t$$

$$Z_t = 0.904 Z_{t-1} + U_t - 0.305 U_{t-1} - 0.310 U_{t-2}$$

Why such similar forecasts?

$$Y_t = 0.002 + Z_t$$

$$Z_t = 0.639 Z_{t-1} - 0.153 Z_{t-2} + 0.183 Z_{t-3} + U_t$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 U_{t-1}$$

as

$$U_t = -\theta_1 U_{t-1} + Z_t$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$U_t = -\theta_1 \, U_{t-1} + Z_t$$

$$U_{t-1} = -\theta_1 \, U_{t-2} + Z_{t-1}$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$U_t = -\theta_1 \left(-\theta_1 \, U_{t-2} + Z_{t-1} \right) + Z_t$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$\begin{aligned} U_t &= -\theta_1 \left(-\theta_1 \, U_{t-2} + Z_{t-1}\right) + Z_t \\ &= \theta_1^2 \, U_{t-2} - \theta_1 \, Z_{t-1} + Z_t \end{aligned}$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$U_t = \theta_1^2 \, U_{t-2} - \theta_1 \, Z_{t-1} + Z_t$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 U_{t-1}$$

as

$$U_t = \theta_1^2 U_{t-2} - \theta_1 Z_{t-1} + Z_t$$

$$U_{t-2} = -\theta_1 U_{t-3} + Z_{t-2}$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$U_t = \theta_1^2 \left(-\theta_1 \, U_{t-3} + Z_{t-2}\right) - \theta_1 \, Z_{t-1} + Z_t$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$\begin{aligned} U_t &= \theta_1^2 \left(-\theta_1 \, U_{t-3} + Z_{t-2}\right) - \theta_1 \, Z_{t-1} + Z_t \\ &= -\theta_1^3 \, U_{t-3} + \theta_1^2 \, Z_{t-2} - \theta_1 \, Z_{t-1} + Z_t \end{aligned}$$

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 \, U_{t-1}$$

as

$$U_t = -\theta_1^3 \, U_{t-3} + \theta_1^2 \, Z_{t-2} - \theta_1 \, Z_{t-1} + Z_t$$

Continuing this process…

Inverting an MA(1) Model

Write

$$Z_t = U_t + \theta_1 U_{t-1}$$

as

$$\begin{aligned} U_t &= -\theta_1^3 U_{t-3} + \theta_1^2 Z_{t-2} - \theta_1 Z_{t-1} + Z_t \\ &\vdots \\ U_t &= (-\theta_1)^t U_0 + (-\theta_1)^{t-1} Z_1 + (-\theta_1)^{t-2} Z_2 + \ldots \\ &\qquad + \theta_1^2 Z_{t-2} - \theta_1 Z_{t-1} + Z_t \end{aligned}$$

Inverting an MA(1) Model

$$U_t = (-\theta_1)^t\, U_0 + (-\theta_1)^{t-1} Z_1 + (-\theta_1)^{t-2} Z_2 + \ldots$$
$$-\theta_1^3 Z_{t-3} + \theta_1^2\, Z_{t-2} - \theta_1 Z_{t-1} + Z_t$$

Solve for Z_t :

$$Z_t = \theta_1 Z_{t-1} - \theta_1^2 Z_{t-2} + \theta_1^3 Z_{t-3} + \ldots$$
$$-(-\theta_1)^{t-2} Z_2 - (-\theta_1)^{t-1} Z_1$$
$$-(-\theta_1)^t U_0 + U_t$$

Inverting an MA(1) Model

$$Z_t = U_t + \theta_1 U_{t-1}$$

can be written

$$Z_t = -\sum_{j=1}^{t-1} (-\theta_1)^j \, Z_{t-j} + U_t - (-\theta_1)^t U_0$$

Inverting an MA(1) Model

$$Z_t = -\sum_{j=1}^{t-1} (-\theta_1)^j \, Z_{t-j} + U_t - (\theta_1)^t U_0$$

Inverting an MA(1) Model

$$Z_t = -\sum_{j=1}^{t-1} (-\theta_1)^j Z_{t-j} + U_t - (-\theta_1)^t U_0$$

If $|\theta_1| < 1$ then $(-\theta_1)^j \to 0$ as j increases.

Autoregressive approximation

$$Z_t = \sum_{j=1}^{t-1} -(-\theta_1)^j Z_{t-j} + U_t - (-\theta_1)^t U_0$$

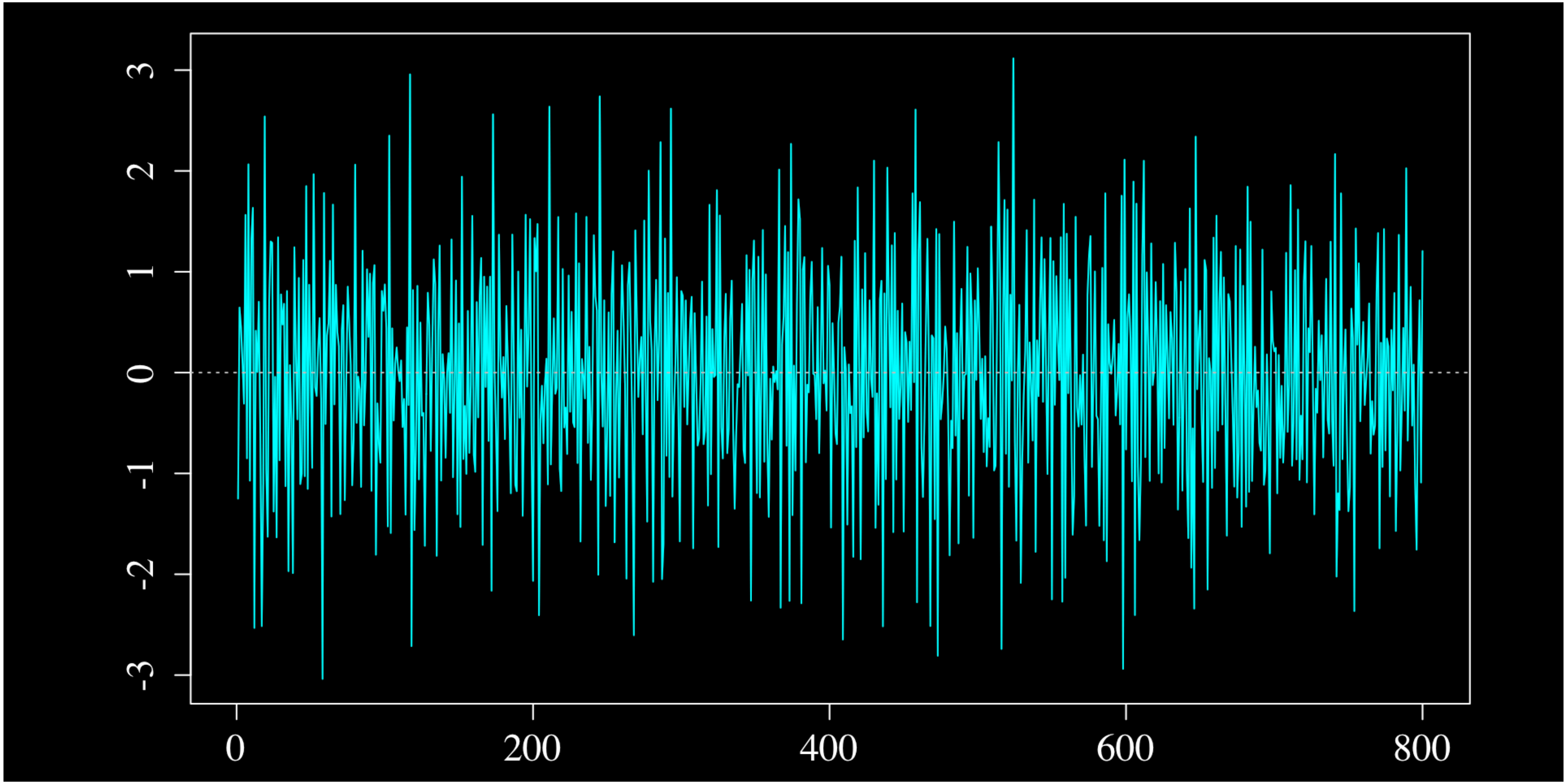
If $|\theta_1| < 1$ then $(-\theta_1)^j \to 0$ as j increases.

$$Z_t \approx \sum_{j=1}^{p} \phi_j Z_{t-j} + U_t$$

If p is “large enough” then the autoregressive approximation “works well”.

Autoregressive approximation

```r
Y <- arima.sim(n=800, list(ma=0.5))
```



Autoregressive approximation

```r
MA1 <- Arima(Y, order=c(0,0,1))
```

```
     ma1 intercept
 -0.5013   -0.0261
```

Autoregressive approximation

```
      ma1  intercept
  -0.5013    -0.0261
```

$$\hat{\theta}_1 = -0.5013$$

$$Z_t = \sum_{j=1}^{t-1} -(-\theta_1)^j Z_{t-j} + U_t - (-\theta_1)^t U_0$$

Autoregressive approximation

```
     ma1 intercept
 -0.5013   -0.0261
```

$\hat{\theta}_1 = -0.5013$

$$Z_t = \sum_{j=1}^{t-1} -(-\theta_1)^j Z_{t-j} + U_t - (-\theta_1)^t U_0$$

```
j -(-ma1)^j
1     -0.501
2     -0.251
3     -0.126
4     -0.063
5     -0.032
6     -0.016
```

Autoregressive approximation: coefficients

j	-(-ma1)^j	AR1	AR2	AR3	AR4
1	-0.501	-0.39	-0.464	-0.488	-0.497
2	-0.251		-0.191	-0.249	-0.266
3	-0.126			-0.126	-0.159
4	-0.063				-0.070
5	-0.032				
6	-0.016				
7	-0.008				

Autoregressive approximation: coefficients

j	-(-ma1)^j	AR5	AR6	AR7	AR8
1	−0.501	−0.492	−0.491	−0.492	−0.494
2	−0.251	−0.255	−0.255	−0.250	−0.251
3	−0.126	−0.141	−0.142	−0.145	−0.143
4	−0.063	−0.035	−0.037	−0.047	−0.049
5	−0.032	0.070	0.067	0.048	0.043
6	−0.016		−0.007	−0.044	−0.051
7	−0.008			−0.073	−0.088

Autoregressive approximation: forecasts

h	MA1	AR1	AR2	AR3	AR4
1	−0.397	−0.506	−0.394	−0.456	−0.471
2	−0.026	0.161	−0.090	0.011	−0.015
3	−0.026	−0.099	0.074	−0.092	−0.035
4	−0.026	0.002	−0.060	0.051	−0.039
5	−0.026	−0.037	−0.029	−0.052	0.012
6	−0.026	−0.022	−0.018	−0.024	−0.041
7	−0.026	−0.028	−0.029	−0.030	−0.026

Autoregressive approximation: forecasts

h	MA1	AR5	AR6	AR7	AR8
1	−0.397	−0.586	−0.571	−0.515	−0.485
2	−0.026	0.057	0.061	0.177	0.186
3	−0.026	−0.008	−0.016	0.032	0.079
4	−0.026	−0.095	−0.098	−0.173	−0.154
5	−0.026	0.097	0.104	0.076	0.046
6	−0.026	−0.114	−0.122	−0.057	−0.067
7	−0.026	0.000	0.007	−0.077	−0.055

Non-invertibility

Practically relevant is $\theta_1 = -1$:

$$Z_t = U_t - U_{t-1}$$

Why relevant? Suppose

$$Y_t = U_t$$

but we (mistakenly) take a first difference:

$$Z_t = \Delta Y_t = U_t - U_{t-1}.$$

Non-invertibility

Practically relevant is $\theta_1 = -1$:

$$Z_t = U_t - U_{t-1}$$

Autoregressive "approximation":

$$Z_t = \sum_{j=1}^{t-1} (-1)\, Z_{t-j} + U_t + (-1)\, U_0$$

$\uparrow$

These coefficients do not decline with j

AR(p) estimation is generally poor for any p.

Autoregressive approximation: coefficients

j	-(-ma1)^j	AR1	AR2	AR3	AR4
1	-1	-0.49	-0.649	-0.741	-0.813
2	-1		-0.327	-0.508	-0.637
3	-1			-0.280	-0.468
4	-1				-0.253
5	-1				
6	-1				
7	-1				

Autoregressive approximation: coefficients

j	-(-ma1)^j	AR5	AR6	AR7	AR8
1	-1	-0.835	-0.842	-0.856	-0.873
2	-1	-0.678	-0.707	-0.730	-0.756
3	-1	-0.523	-0.570	-0.627	-0.658
4	-1	-0.323	-0.384	-0.469	-0.525
5	-1	-0.086	-0.161	-0.266	-0.342
6	-1		-0.090	-0.216	-0.304
7	-1			-0.149	-0.252

Autoregressive approximation: forecasts

h	MA1	AR1	AR2	AR3	AR4
1	−0.81	−0.845	−0.745	−1.065	−1.566
2	0.00	0.413	−0.080	0.234	0.374
3	0.00	−0.203	0.294	−0.116	0.175
4	0.00	0.099	−0.166	0.263	−0.088
5	0.00	−0.049	0.011	−0.204	0.178
6	0.00	0.023	0.046	0.048	−0.269
7	0.00	−0.012	−0.035	−0.008	0.098

Autoregressive approximation: forecasts

h	MA1	AR5	AR6	AR7	AR8
1	−0.81	−1.601	−1.516	−1.428	−1.232
2	0.00	0.245	0.137	0.200	0.098
3	0.00	0.234	0.101	−0.079	−0.031
4	0.00	0.014	0.082	−0.137	−0.283
5	0.00	0.068	0.186	0.306	0.143
6	0.00	−0.134	−0.241	−0.032	0.079
7	0.00	−0.042	0.096	−0.081	0.064

Higher order models

Lag operator

Define the "lag operator" L :

$$L\,Y_t = Y_{t-1}$$

The lag operator can be applied multiple times:

$$L^2\,Y_t = L\,L\,Y_t = L\,Y_{t-1} = Y_{t-2}$$

Lag operator

Define the “lag operator” L :

$$L\,Y_t = Y_{t-1}$$

The lag operator can be applied multiple times:

$$\begin{aligned} L^2\,Y_t &= L\,L\,Y_t = L\,Y_{t-1} = Y_{t-2} \\ L^3\,Y_t &= Y_{t-3} \\ L^4\,Y_t &= Y_{t-4} \\ &\text{etc} \end{aligned}$$

Lag operator

Define the "lag operator" L :

$$L\,Y_t = Y_{t-1}$$

The lag operator can be applied multiple times:

$$L^j\,Y_t = Y_{t-j}$$

j could be negative, eg $L^{-1}Y_t = Y_{t+1}$.
Negative lag is a "lead".

Difference operator

$$\Delta = 1 - L$$

i.e. $\quad \Delta Y_t = (1 - L)Y_t = Y_t - Y_{t-1}.$

AR Models

AR(1):

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + U_t \\ \Rightarrow \quad (1 - \phi_1 L)\, Y_t &= U_t \end{aligned}$$

AR(p):

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + U_t \\ \Rightarrow \quad (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) Y_t &= U_t \end{aligned}$$

AR Models

AR(p):

$$
\begin{aligned}
& Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + U_t \\
\Rightarrow \quad & (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) Y_t = U_t
\end{aligned}
$$

$$\uparrow$$

Lag polynomial of order p

MA Models

MA(q):

$$
\begin{aligned}
Y_t &= U_t + \theta_1 U_{t-1} + \theta_2 U_{t-2} + \ldots + \theta_q U_{t-q} \\
\Rightarrow \quad Y_t &= (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) U_t
\end{aligned}
$$

MA Models

MA(q):

$$Y_t = U_t + \theta_1 U_{t-1} + \theta_2 U_{t-2} + \ldots + \theta_q U_{t-q}$$

$$\Rightarrow \quad Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) U_t$$

$$\uparrow$$

Lag polynomial of order q

Moving average invertibility

Invertibility of MA(1)

$$Y_t = U_t + \theta_1 U_{t-1}$$

is invertible if $|\theta_1| < 1$.

Invertibility of MA(1)

$$Y_t = (1 + \theta_1 L)U_t$$

is invertible if $|\theta_1| < 1$.

The root of the lag polynomial is the solution of

$$1 + \theta_1 z = 0$$

Invertibility of MA(1)

$$Y_t = (1 + \theta_1 L)U_t$$

is invertible if $|\theta_1| < 1$.

The root of the lag polynomial is the solution of

$$1 + \theta_1 z = 0 \quad \text{i.e.} \quad z = -\frac{1}{\theta_1}$$

The inverse root is $z^{-1} = -\theta_1$

The model is invertible if the inverse root z^{-1} of $(1 + \theta_1 L)$ satisfies $|z^{-1}| < 1$.

Invertibility of MA(q)

$$Y_t = U_t + \theta_1 U_{t-1} + \theta_2 U_{t-2} + \ldots + \theta_q U_{t-q}$$

$$\Rightarrow \quad Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) U_t$$

Invertibility of MA(q)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q)\, U_t$$

The roots of the q-order polynomial satisfy

$$1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_1 z^q = 0.$$

A q-order polynomial has q roots, $z_1, \ldots, z_q$.

Invertibility of MA(q)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) U_t$$

The roots of the q-order polynomial satisfy

$$1 + \theta_1 z + \theta_2 z^2 + \ldots + \theta_1 z^q = 0.$$

The model is invertible if the inverse roots satisfy $|z_j^{-1}| < 1$ *for all* $j = 1, \ldots, q$.

z_j may be a *complex number*.
$|z_j^{-1}| < 1 \;\Rightarrow\;$ "z_j^{-1} is inside the unit circle".

Invertibility of MA(2)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2)\, U_t$$

The roots satisfy

$$1 + \theta_1 z + \theta_2 z^2 = 0$$

i.e. quadratic with two solutions.

$$\zeta_1, \zeta_2 = \frac{-\theta_1 \pm \sqrt{\theta_1 - 4\theta_2}}{2\theta_2}$$

Invertibility of MA(2)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2)\, U_t$$

The roots satisfy

$$1 + \theta_1 z + \theta_2 z^2 = 0$$

i.e. quadratic with two solutions.

$$\zeta_1^{-1}, \zeta_2^{-1} = \frac{2\theta_2}{-\theta_1 \pm \sqrt{\theta_1 - 4\theta_2}}$$

Invertibility: $|\zeta_1^{-1}|$ and $|\zeta_2^{-1}| < 1$

Invertibility of MA(2)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2)\, U_t$$

The roots satisfy

$$1 + \theta_1 z + \theta_2 z^2 = 0$$
$$(1 - \zeta_1^{-1} z)(1 - \zeta_2^{-1} z) = 0$$

Example:

$$1 - 0.6L - 0.4L^2 = (1 - L)(1 + 0.4L)$$

Invertibility of MA(2)

$$Y_t = (1 + \theta_1 L + \theta_2 L^2)\, U_t$$

The roots satisfy

$$1 + \theta_1 z + \theta_2 z^2 = 0$$
$$(1 - \zeta_1^{-1} z)(1 - \zeta_2^{-1} z) = 0$$

Example:

$$1 - 0.6L - 0.4L^2 = (1 - 1L)(1 - (-0.4)L)$$
$$\zeta_1^{-1}, \zeta_2^{-1} = 1, -0.4$$

Not invertible

Invertibility of MA(2): Illustration

```r
set.seed(42)
Y <- arima.sim(n=400, list(ma=c(0.5,0.3)))
```

True model:

$$
\begin{aligned}
Y_t &= U_t + 0.5U_{t-1} + 0.3U_{t-2} \\
&= (1 + 0.5\,L + 0.3\,L^2)\,U_t
\end{aligned}
$$

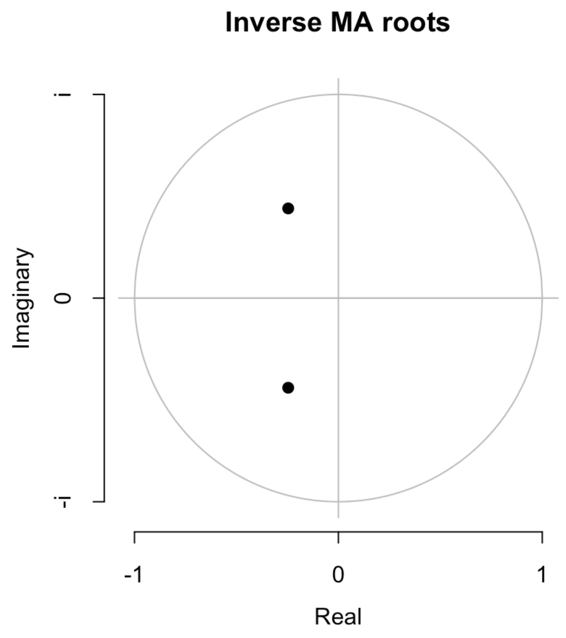
Invertibility of MA(2): Illustration

```r
set.seed(42)
Y <- arima.sim(n=400, list(ma=c(0.5,0.3)))
MA2 <- Arima(Y, order=c(0,0,2))
print(round(MA2$coef,4))
```

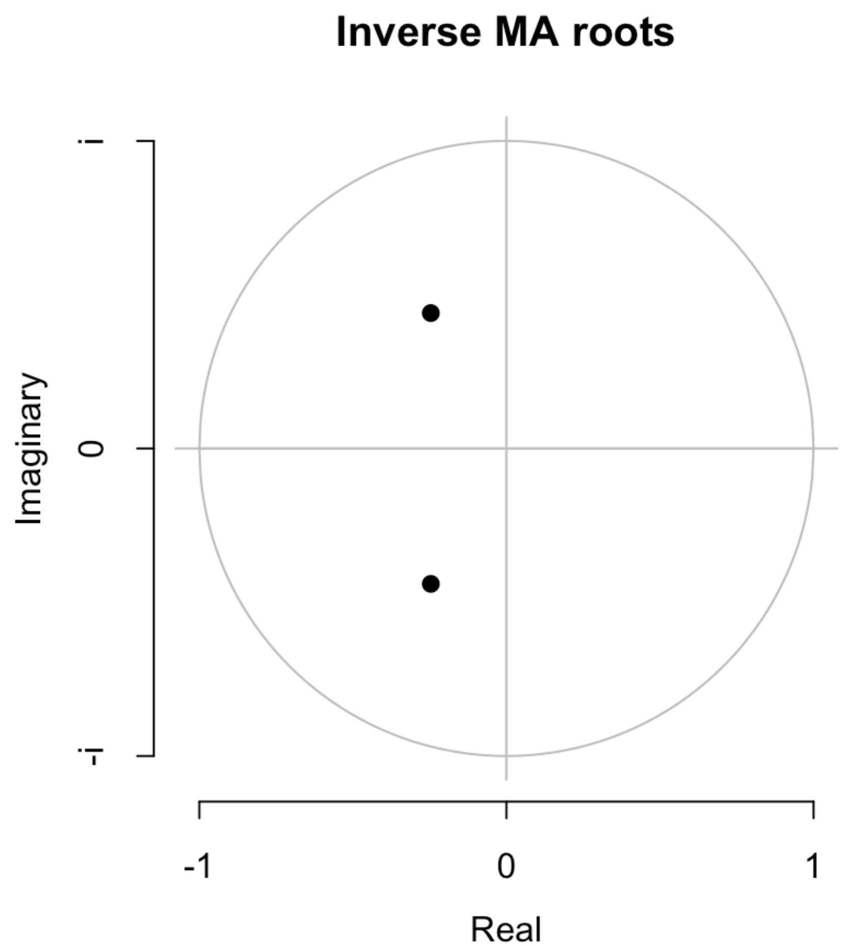
```
    ma1       ma2 intercept
 0.4924    0.2545   -0.0156
```

Invertibility of MA(2): Illustration

```r
set.seed(42)
Y <- arima.sim(n=400, list(ma=c(0.5,0.3)))
MA2 <- Arima(Y, order=c(0,0,2))
plot(MA2)
```



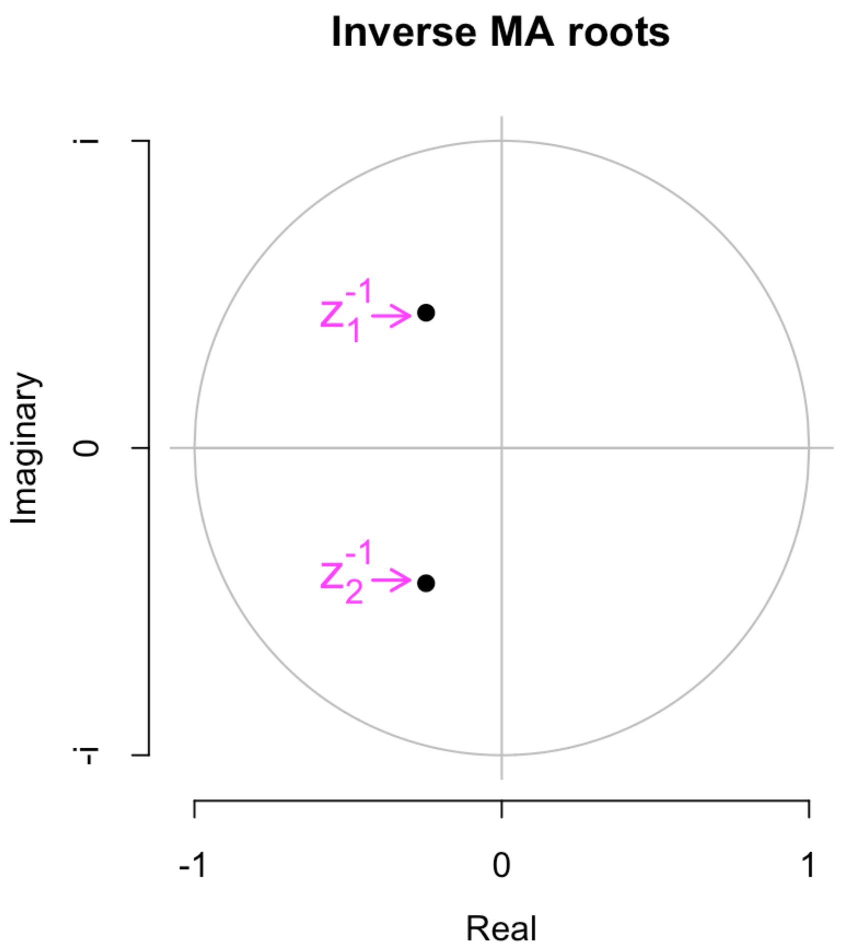
Invertibility of MA(2): Illustration



Invertibility of MA(2): Illustration

2 roots

Both are complex in this case.

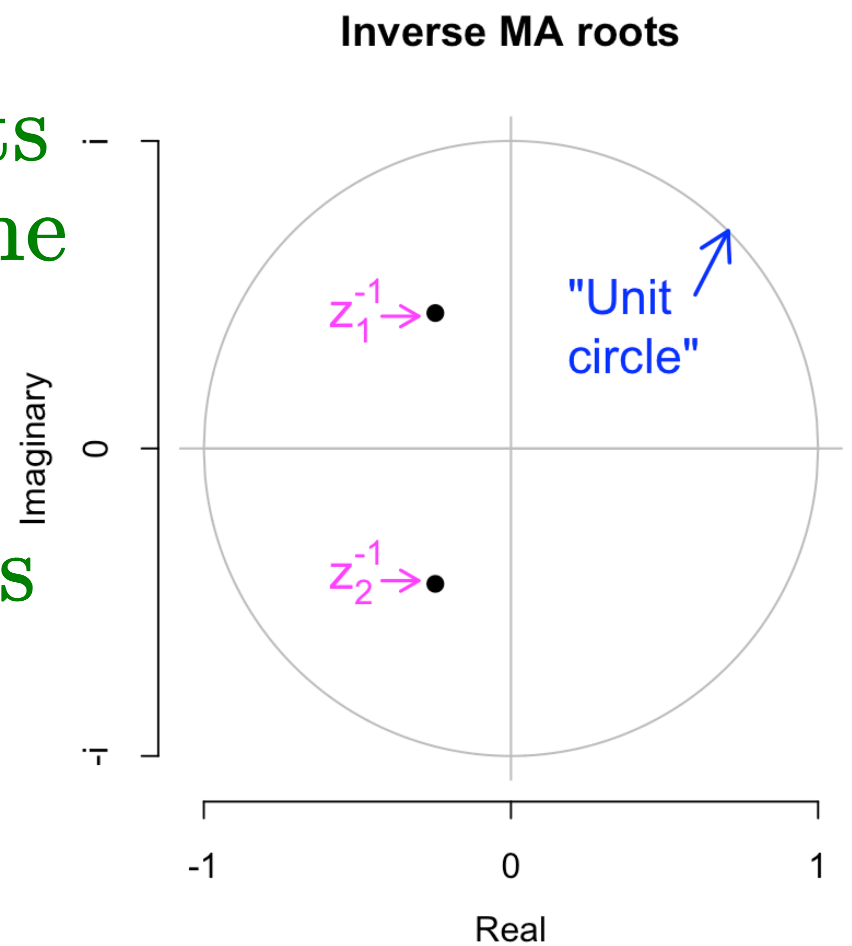


Invertibility of MA(2): Illustration

Inverse roots are inside the unit circle

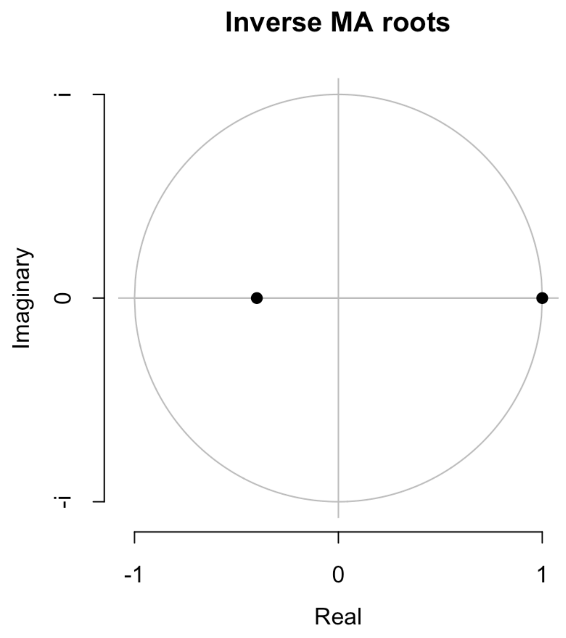
$\Downarrow$

this model is invertible.



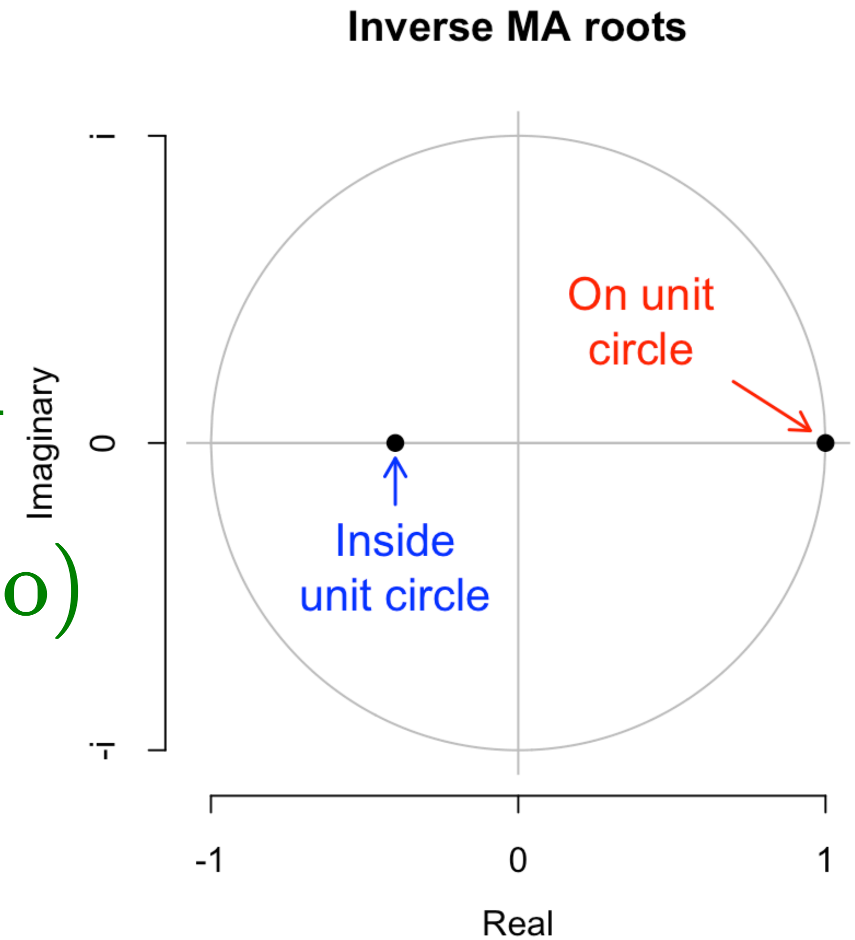
Non-Invertibility of MA(2): Illustration

```
set.seed(42)
Y <- arima.sim(n=400, list(ma=c(-0.6,-0.4)))
MA2 <- Arima(Y, order=c(0,0,2))
plot(MA2)
```

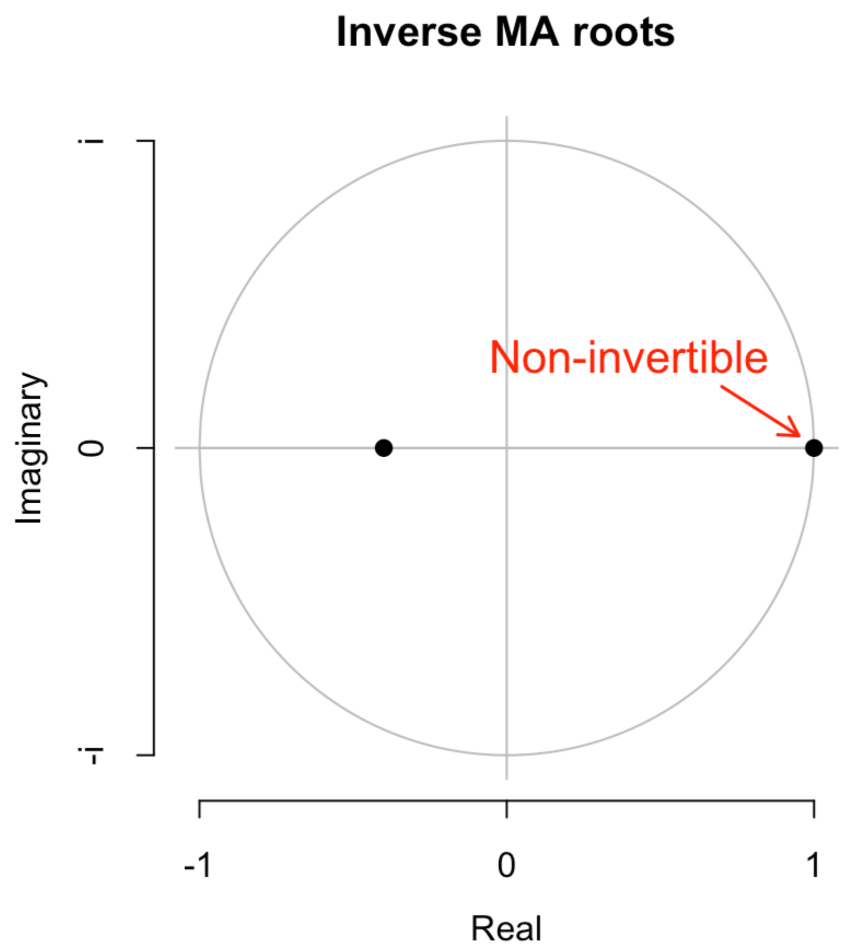


Non-Invertibility of MA(2): Illustration

Both inverse roots are real (imaginary parts are zero)



Non-Invertibility of MA(2): Illustration



Summary

- Invertible MA models can be approximated by AR models.
- AR models are more convenient, although more require more parameters.
- Invertibility is required for AR approximation.
- Invertibility is determined by the inverse roots of the MA lag polynomial.
- Be careful not to over-difference!