Quantitative Analysis of Finance I

ECON90033

HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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January 2023

LINEAR DIFFERENCE EQUATIONS

- A difference equation relates the current value of a variable to its own history (in the form of lagged values), time, and other variables.

Many economic theories can be represented as stochastic difference equations, like e.g.,

i. The random walk hypothesis about the day-to-day changes in the price of a share of stock (y_t) suggests that

$$\boxed{y_t = y_{t-1} + \varepsilon_t}$$

where ε_t is an uncorrelated random disturbance term with zero expected value.

$$\longrightarrow \quad \boxed{\Delta y_t = y_t - y_{t-1} = (1-L)y_t = \varepsilon_t}$$

ii. The error-correction representation of the unbiased forward rate hypothesis.

Let s_t denote the AUS\$ price in period t of one US\$ on the Melbourne spot market, and f_t the AUS\$ price in period t of one US\$ on the Melbourne forward market to be delivered in period $t+1$.

If in period t somebody buys one US$ to be delivered one period later for f_t, and then sells it in period $t+1$ for s_{t+1}, then the profit (loss) on this transaction is

$$\boxed{s_{t+1} - f_t}$$

According to the unbiased forward rate hypothesis, given risk neutrality and rational expectations, the forward rate (f_t) is an unbiased predictor of the future spot rate (s_{t+1}), and hence the expected profit is zero, i.e.

$$\boxed{s_{t+1} - f_t = \varepsilon_{t+1}} \quad \text{and} \quad \boxed{E_t\left(\varepsilon_{t+1}\right) = 0}$$

However, the spot and forward markets are in equilibrium if not only the expected profit, but the actual profit itself, i.e. ε_{t+1}, is also zero.

Otherwise, some sort of adjustment is necessary to restore equilibrium.

For example,

$$\boxed{s_{t+2} = s_{t+1} - \alpha\varepsilon_{t+1} + \delta_{t+2} \quad , \quad \alpha > 0}$$

$$\boxed{f_{t+1} = f_t + \beta\varepsilon_{t+1} + \varphi_{t+1} \quad , \quad \beta > 0}$$

where $\alpha\varepsilon_{t+1}$ and $\beta\varepsilon_{t+1}$ represent the adjustments, and δ_{t+2}, φ_{t+1} have zero conditional expected values.

The previous equations are equivalent to,

$$\boxed{\Delta s_{t+2} = -\alpha\varepsilon_{t+1} + \delta_{t+2}}$$

$$\boxed{\Delta f_{t+1} = \beta\varepsilon_{t+1} + \varphi_{t+1}}$$

This dynamic system, is an error-correction model that relates the short-run movement of the variables (Δs_{t+2}, Δf_{t+1}) to the previous period's deviation from equilibrium (ε_{t+1}).

Namely, if the markets are in equilibrium in period $t+1$, then $\varepsilon_{t+1} = 0$ and the spot and forward rates are expected to remain unchanged.

On the other hand, if $\varepsilon_{t+1} > 0$, then the spot rate is likely to decrease ($s_{t+2} < s_{t+1}$) and the forward rate is likely to increase ($f_{t+1} > f_t$).

Similarly, if $\varepsilon_{t+1} < 0$, then the spot rate is expected to rise ($s_{t+2} > s_{t+1}$) and the forward rate is expected to fall ($f_{t+1} < f_t$).

- All difference equations considered in this course are special cases of the following linear difference equation:

$$
\begin{aligned}
y_t &= a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \ldots + a_n y_{t-n} + x_t \\
&= a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t
\end{aligned}
$$

where a_0, a_1, …, a_n are constant parameters and x_t, the so-called forcing process, can be some function of the t time variable, current and lagged values of other variables and stochastic disturbances.

This difference equation is

i. *linear* because y_t is a linear function of the right-hand side variables;

ii. of *order n* because the largest time lag of y is n;

$$\longrightarrow \quad \boxed{y_t = a_0 + a_1 y_{t-1} + x_t}$$

is a linear first-order difference equation.

iii. *homogeneous* if $a_0 + x_t = 0$ and *non-homogeneous* otherwise.

In general, difference equations have three further important properties.

1) If $\Delta y_t = y_t - y_{t-1}$ is positive or negative for all t, i.e. the differences between adjacent terms are of the same sign, then the dynamics generated by a difference equation is said to be monotonic.

   On the other hand, if the differences between adjacent terms alternate in sign then the dynamics is oscillatory.

2) The equilibrium value (also called stationary or steady state value) of a difference equation is y^* if it satisfies the difference equation.

   $\longrightarrow$ If y^* exists and y_t is equal to it, then the $\{y_t\}$ series remains at this constant level thereafter; i.e. $y^* = y_t = y_{t+1} = \ldots = y_{t+i} = \ldots$

   For example, in case of a linear first-order difference equation

$$\boxed{y^* = a_0 + a_1 y^* + x_t} \longrightarrow \boxed{y^* = \frac{a_0 + x_t}{1 - a_1} \quad if \quad a_1 \neq 1}$$

$$\boxed{y^* = y_0 \quad if \quad a_1 = 1, a_0 = x_t = 0}$$

   However, y^* does not exist when $a_1 = 1$ and $a_0 \neq 0$ and/or $x_t \neq 0$.

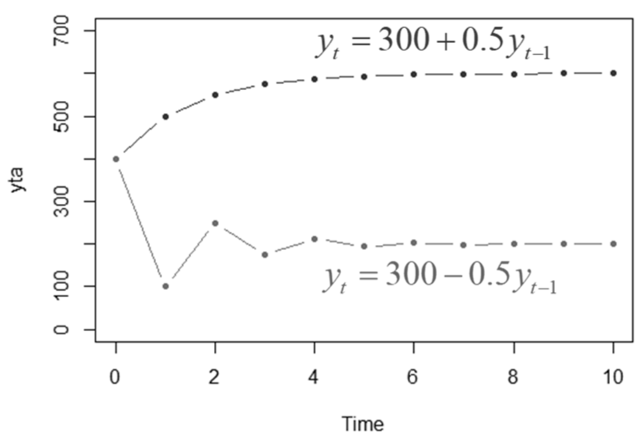
3) A difference equation is said to be stable if it has an equilibrium value y^* and, regardless of the initial value(s), $\{y_t\}$ converges to y^*.

   Otherwise it is called unstable.

<u>Ex 1</u>:

Consider the following three pairs of non-homogeneous first-order linear difference equations. In each case the initial value is $y_0 = 400$, $a_0 = 300$, the forcing process is $x_t = 0$, but the a_1 parameters are different. We simulate $\{y_t\}$ for $t = 1, 2, \ldots, 10$.

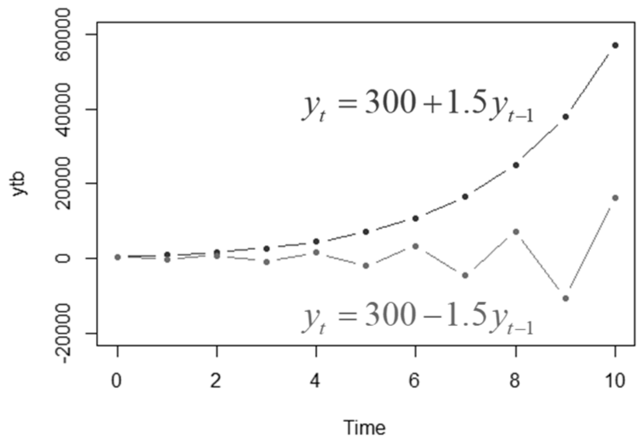
a) $a_1 = \pm 0.5 \;\; \rightarrow \;\; |a_1| < 1$



$\Delta y_t = y_t - y_{t-1} > 0$ for all t, so the dynamics is monotonic. Moreover, $\{y_t\}$ seems to approach 600.

$\Delta y_t = y_t - y_{t-1}$ alternates in sign, so the dynamics is oscillatory. Still, $\{y_t\}$ seems to approach 200.

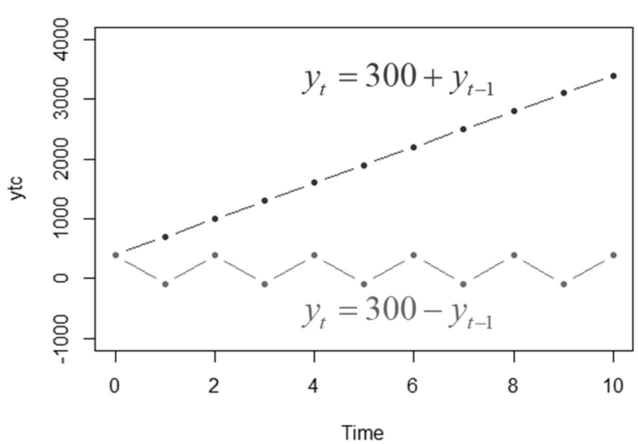
b) $a_1 = \pm 1.5 \quad \rightarrow \quad |a_1| > 1$



The dynamics of the first sequence is monotonic. However, $\{y_t\}$ does not converge to any finite value.

The dynamics of the second sequence is oscillatory, and $\{y_t\}$ does not converge.

c) $a_1 = \pm 1 \quad \rightarrow \quad |a_1| = 1$



The dynamics of the first sequence is monotonic, and $\{y_t\}$ does not converge.

The second sequence is oscillating between two values, so it is bounded, but it does not converge to any finite value.

Ex 2:

Do the difference equations in Ex 1 have equilibrium values? Are they stable?

a)

$$y_t = 300 + 0.5y_{t-1} \longrightarrow y^* = \frac{a_0 + x_t}{1 - a_1} = \frac{300}{1 - 0.5} = 600$$

$$y_t = 300 - 0.5y_{t-1} \longrightarrow y^* = \frac{a_0 + x_t}{1 - a_1} = \frac{300}{1 - (-0.5)} = 200$$

At the given initial value, $y_0 = 400$, these sequences (seem to) converge to 600 and 200, respectively (see Ex 1/a), so these difference equations (might be) are stable.

b)

$$y_t = 300 + 1.5y_{t-1} \longrightarrow y^* = \frac{a_0 + x_t}{1 - a_1} = \frac{300}{1 - 1.5} = -600$$

$$y_t = 300 - 1.5y_{t-1} \longrightarrow y^* = \frac{a_0 + x_t}{1 - a_1} = \frac{300}{1 - (-1.5)} = 120$$

At $y_0 = 400$ these sequences do not converge to any finite value (see Ex 1/b). Thus, these difference equations are unstable (though $y_t = y^*$ for all t if $y_0 = y^*$).

c)

$$y_t = 300 + y_{t-1} \longrightarrow y^* = 300 + y^* \rightarrow 0 \overset{?}{=} 300$$

This difference equation does not have an equilibrium value at all, so it is unstable.

$$y_t = 300 - y_{t-1} \longrightarrow y^* = \frac{a_0 + x_t}{1 - a_1} = \frac{300}{1-(-1)} = 150$$

At $y_0 = 400$ this sequence does not converge to any finite value (see Ex 1/c). Hence, this difference equation is unstable (though $y_t = y^*$ for all t if $y_0 = y^*$).

- The solution to a difference equation is a time-path of the dependent variable, i.e. $\{y_t\}$, that
    1. might depend on the $\{x_t\}$ sequence, the t time variable and possibly some initial value(s) of the $\{y_t\}$;
    2. satisfies the difference equation for all permissible values of t and $\{x_t\}$.

<u>Note</u>: The solution to a difference equation is a function rather than a number.

Ex 3:

Find the solution of the following first-order, non-homogeneous linear difference equation

$$y_t = 2 + y_{t-1}$$

With backward iteration,

$$y_t = 2 + y_{t-1} = 2 + (2 + y_{t-2}) = 2 + 2 + (2 + y_{t-3}) = \ldots = 2t + y_0$$

Since y_0 is unknown, we can replace it with an arbitrary constant c.

$\longrightarrow$ $\boxed{y_t = 2t + c}$ is the solution of the non-homogeneous equation.

This is indeed a solution, since it satisfies both requirements.

$\longleftarrow$ i. It is a simple linear function of t.

ii. Substituting this expression for y_t and y_{t-1} yields an identity

$$2t + c = 2 + 2(t-1) + c$$

This solution, however, is not unique because it depends on c.

- The general solution of a non-homogeneous difference equation is the sum of two components, y_t^p and y_t^h, where

  y_t^p is a particular solution, i.e. any solution of the non-homogeneous difference equation, and

  y_t^h is the homogeneous solution, i.e. the general solution of the homogeneous difference equation.

Ex 3 (cont.):

$y_t = 2 + y_{t-1}$ is a non-homogeneous difference equation ($a_0 + x_t = 2$).

The corresponding homogeneous equation is $y_t = y_{t-1}$ $\longrightarrow$ $y_t^h = c$

A particular solution is $y_t^p = 2t$ $\longleftarrow$ $2t = 2 + 2(t-1)$

$$\longrightarrow \quad y_t = 2t + c = y_t^p + y_t^h$$

Note: In this course we are not so much interested in how to solve difference equations in general, rather than in the properties of the solutions of homogeneous linear first- and second-order difference equations.

HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

- The general solution of a homogeneous first-order linear difference equation

$y_t = a_1 y_{t-1} \quad , \quad a_1 \neq 0$ takes the form $y_t = A\alpha^t$ (α: alpha)

Let's find the values of *A* and α that satisfy

$$y_t - a_1 y_{t-1} = 0 \longrightarrow A\alpha^t - a_1 A\alpha^{t-1} = 0$$

This equation is clearly satisfied by any arbitrary value of *A*, so we can cancel the (nonzero) common factor $A\alpha^{t-1}$.

$$\longrightarrow \alpha - a_1 = 0$$

This is the so called characteristic equation, and its solution

$\alpha = a_1$ is the characteristic root.

$\longrightarrow$ The homogeneous solution is $y_t = A\alpha^t = Aa_1^t$

- Under what conditions is the homogeneous first-order linear difference equation stable?

  First, notice that the equilibrium value is zero, since $y^* = a_1 y^* \rightarrow y^* = 0$

Therefore,

i. If $|\alpha| < 1$, α^t and also y_t converge to y^* as t approaches infinity, and the convergence is direct if $0 < \alpha < 1$, and oscillatory if $-1 < \alpha < 0$.

   $\longrightarrow$ The difference equation is stable.

ii. If $|\alpha| > 1$, α^t is increasing in absolute value as t approaches infinity. In particular, α^t and (given $A > 0$) y_t approach infinity if $\alpha > 1$ and oscillate explosively if $\alpha < -1$.

   $\longrightarrow$ The difference equation is unstable.

iii. If $\alpha = 1$, $\alpha^t = 1$ and $y_t = A$ for any t, so the solution does not converge to zero.

   If $\alpha = -1$, $\alpha^t = \pm 1$ and $y_t = \pm A$ depending on whether t is even or odd.

   $\longrightarrow$ The difference is unstable.

Alternatively, the stability condition can be given in terms of the coefficient of the homogeneous first-order difference equation, as well.

Namely, since $\alpha = a_1$, stability requires $-1 < a_1 < 1$

- Consider now a homogeneous second-order linear difference equation

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} \quad , \quad a_2 \neq 0$$

Like in the first-order case, the solution takes the form $y_t = A\alpha^t$

Its substitution into the equation yields

$$A\alpha^t - a_1 A\alpha^{t-1} - a_2 A\alpha^{t-2} = 0$$

and our task is to find the values of A and α that satisfy this equation.

Simplifying by the (nonzero) common factor $A\alpha^{t-2}$, we get the following quadratic characteristic equation:

$$\alpha^2 - a_1\alpha - a_2 = 0$$

From the quadratic formula the two characteristic roots are

$$\alpha_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

d: discriminant

and the two solutions are

$$y_{1t} = A\alpha_1^t \quad , \quad y_{2t} = A\alpha_2^t$$

However, since *A* can assume any value, these solutions are not unique, and it can be shown that any of their linear combination also solves the second-order homogeneous difference equation.

→ The general solution is

$$y_t = A_1\alpha_1^t + A_2\alpha_2^t$$

where A_1 and A_2 are two arbitrary constants.

The equilibrium value is again zero, since

$$y^* = a_1 y^* + a_2 y^* \quad \rightarrow \quad y^* = 0$$

Depending on the discriminant (d) of the characteristic equation, the two roots (α_1 and α_2) can be real or complex. There are three possibilities.

i. If $d > 0$, α_1 and α_2 are real and different from each other.

   The homogeneous equation is stable if $|\alpha_1| < 1$ and $|\alpha_2| < 1$, and unstable otherwise. In particular, y_t is explosive if $|\alpha_1| > 1$ or $|\alpha_2| > 1$.

   Hence, if α_1 and α_2 denote the larger root and the smaller root respectively, then stability requires $\alpha_1 < 1$ and $\alpha_2 > -1$.

$$\longrightarrow \quad \boxed{\alpha_1 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2} < 1} \quad \longrightarrow \quad \boxed{a_1 + \sqrt{a_1^2 + 4a_2} < 2}$$

$$\longrightarrow \quad \boxed{a_1^2 + 4a_2 < (2 - a_1)^2} \quad \longrightarrow \quad \boxed{a_1^2 + 4a_2 < 4 - 4a_1 + a_1^2} \quad \longrightarrow \quad \boxed{a_1 + a_2 < 1}$$

and

$$\boxed{\alpha_2 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2} > -1} \quad \longrightarrow \quad \boxed{a_1 - \sqrt{a_1^2 + 4a_2} > -2}$$

$$\longrightarrow \quad \boxed{(a_1 + 2)^2 > a_1^2 + 4a_2} \quad \longrightarrow \quad \boxed{a_1^2 + 4a_1 + 4 > a_1^2 + 4a_2} \quad \longrightarrow \quad \boxed{a_1 - a_2 > -1}$$

Ex 4:

Are the following homogeneous linear second-order difference equations stable? Find the characteristic roots, check whether the stability requirements given in terms of a_1, a_2 are satisfied, and illustrate the solution graphically.

a) $y_t = 0.4y_{t-1} + 0.1y_{t-2}$ $\longrightarrow$ $a_1 = 0.4$, $a_2 = 0.1$

The characteristic equation is

$\alpha^2 - 0.4\alpha - 0.1 = 0$ $\longrightarrow$ $d = a_1^2 + 4a_2 = 0.4^2 + 4 \times 0.1 = 0.56 > 0$

… so, there must be two distinct real roots.

$$\longrightarrow \quad \alpha_{1,2} = \frac{a_1 \pm \sqrt{d}}{2} = \frac{0.4 \pm \sqrt{0.56}}{2} = -0.174 \quad \text{and} \quad 0.574$$

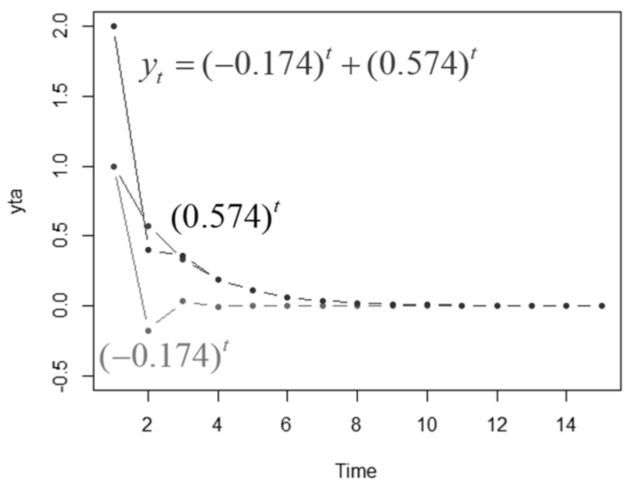
These characteristic roots satisfy $|\alpha_1| < 1$ and $|\alpha_2| < 1$, so this homogeneous linear second-order difference equation is stable.

$a_1 + a_2 = 0.4 + 0.1 = 0.5 < 1$ and $a_1 - a_2 = 0.4 - 0.1 = 0.3 > -1$

$\longrightarrow$ Both requirements of stability are satisfied.

→ The homogeneous solution is $y_t = A_1(-0.174)^t + A_2(0.574)^t$

The graph below illustrates the time path of this solution for $A_1 = A_2 = 1$ and $t = 0, \ldots, 15$.



The second root is the dominant (it is bigger in absolute value). Consequently, it has a bigger influence on y_t than the first root.

$\{y_t\}$ approaches the equilibrium value, $y^* = 0$, so this difference equation is clearly stable.

b) $y_t = 0.3y_{t-1} + 0.8y_{t-2}$ → $a_1 = 0.3$, $a_2 = 0.8$

Following the same steps as in part (a), we get

$\alpha^2 - 0.3\alpha - 0.8 = 0$ → $d = 0.3^2 + 4 \times 0.8 = 3.29 > 0$

… so, there are two distinct real roots.

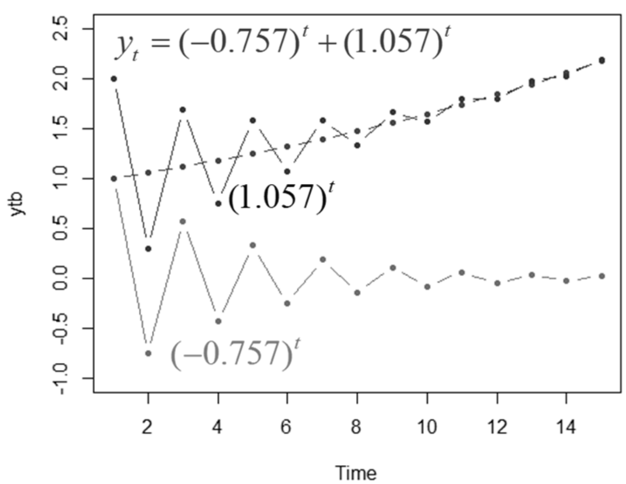
$$\alpha_{1,2} = -0.757 \quad \text{and} \quad 1.057 \longrightarrow y_t = A_1(-0.757)^t + A_2(1.057)^t$$

The first characteristic root is $|\alpha_1| < 1$ but the second is $|\alpha_2| > 1$, implying that this homogeneous second-order linear difference equation is unstable.

$$a_1 + a_2 = 0.3 + 0.8 = 1.3 \overset{?}{>} 1 \quad \text{and} \quad a_1 - a_2 = 0.3 - 0.8 = -0.5 > -1$$

$\longrightarrow$ The first requirement of stability is indeed violated.

$(A_1 = A_2 = 1)$



This time, due to the dominant second root, which is greater than one, $\{y_t\}$ explodes.

The first root is negative, and it is responsible for the oscillation.

However, its absolute value is less than one, so this oscillation is dampening.

ii. If $d = 0$, the two characteristic roots are real but equal to each other:

$$\alpha_1 = \frac{a_1}{2}$$

However, it can be shown that there is also a second homogeneous solution:

$$\alpha_2 = t\alpha_1^t$$

$$\longrightarrow \quad y_t = A_1\alpha_1^t + A_2\alpha_2^t = A_1\alpha_1^t + A_2 t\alpha_1^t = A_1\left(\frac{a_1}{2}\right)^t + A_2 t\left(\frac{a_1}{2}\right)^t$$

In this case, the second-order homogeneous difference equation is stable if $|\alpha_1| < 1$, and unstable otherwise. In particular, y_t is explosive if $|\alpha_1| > 1$.

Therefore, in terms of a_1, the second-order homogeneous linear difference equation with $d = 0$ is stable if

$$|a_1| < 2$$

Otherwise, it is unstable and y_t is explosive if $|a_1| > 2$.

(Ex 4)

c) $y_t = 0.8y_{t-1} - 0.16y_{t-2}$ $\longrightarrow$ $a_1 = 0.8$, $a_2 = -0.16$

$\alpha^2 - 0.8\alpha + 0.16 = 0$ $\longrightarrow$ $d = 0.8^2 - 4\times 0.16 = 0$

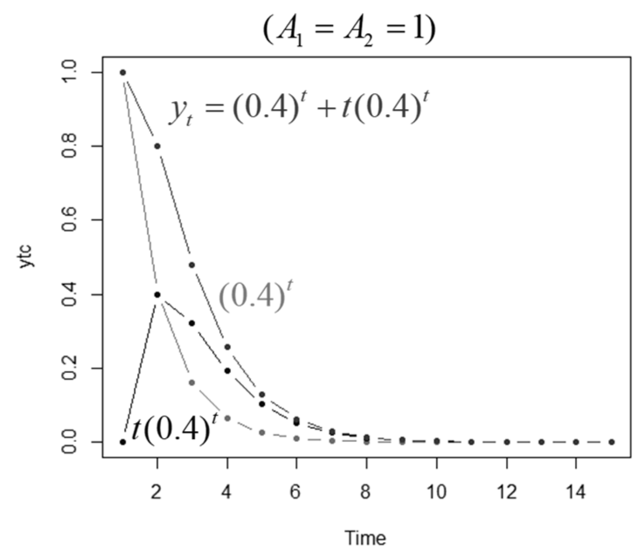
There is a single repeated characteristic root,

$\alpha_1 = 0.4$ $|\alpha_1| < 1$, so this second-order homogeneous linear difference equation is stable.

Alternatively, $|a_1| = 0.8 < 2$ also implies stability.

The second solution is

$\alpha_2 = t(0.4)^t$ $\longrightarrow$ $y_t = A_1(0.4)^t + A_2 t(0.4)^t$



The first solution is the dominant solution.

$\{y_t\}$ is monotonic and, as implied by stability, converges to the equilibrium value, $y^* = 0$.

d) $y_t = -2.01 y_{t-1} - 1.01 y_{t-2}$ $\longrightarrow$ a_1 = -2.01, a_2 = -1.01

$\alpha^2 + 2.01\alpha + 1.01 = 0$ $\longrightarrow$ $d = 2.01^2 - 4 \times 1.01 = 0$

Again, there is only a single repeated characteristic root.

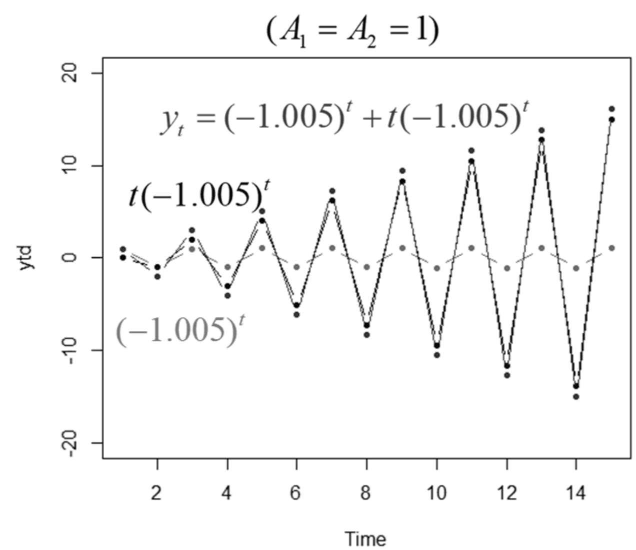
$\alpha_1 = -1.005$ $|\alpha_1| > 1$, so this second-order homogeneous linear difference equation is unstable.

Alternatively, $|a_1| = 2.01 > 2$ also implies instability.

The second solution is

$$\alpha_2 = t(-1.005)^t$$

$$\longrightarrow \quad y_t = A_1(-1.005)^t + A_2 t(-1.005)^t$$



Again, the first solution is the dominant solution, it is responsible for the oscillation and explosion of y_t.

iii. If $d = a_1^2 + 4a_2 < 0$, then $a_2 < 0$ and the two characteristic roots are complex conjugates

$$\alpha_{1,2} = \frac{a_1 \pm \sqrt{d}}{2} = \frac{a_1 \pm i\sqrt{-d}}{2}$$ where $$i = \sqrt{-1}$$ is an imaginary number.

Let r denote the common modulus (absolute value) of these roots.

$$\longrightarrow \quad r = \left| \alpha_{1,2} \right| = \left| \frac{a_1}{2} \pm i\frac{\sqrt{-d}}{2} \right| = \sqrt{\frac{a_1^2}{4} + \frac{-d}{4}}$$

$$= \sqrt{\frac{a_1^2}{4} - \frac{a_1^2 + 4a_2}{4}} = \sqrt{-a_2}$$

It can be shown (using de Moivre's theorem) that the homogeneous solution is

$$y_t = \beta_1 r^t \cos(\theta t + \beta_2)$$ where β_1 and β_2 are arbitrary constants,

and θ (theta) is a radian measure that satisfies $$\cos(\theta) = \frac{a_1}{2r}$$

Due to the cosine function, y_t has a wavelike pattern and the frequency of oscillation depends on θ. Moreover, since the cosine function is bounded, stability is determined solely by r^t.

Hence, the second-order homogeneous linear difference equation with $d < 0$ is stable if $0 < r < 1$, i.e.

$\boxed{-1 < a_2 < 0}$ and it is unstable otherwise.

The oscillations are of constant amplitude so that the solution is periodic if $r = 1$ (i.e. $a_2 = -1$), the oscillations are dampening if $r < 1$ (i.e. $-1 < a_2 < 0$), and they are explosive if $r > 1$ (i.e. $a_2 < -1$).

Recall, that this time $\boxed{d = a_1^2 + 4a_2 < 0} \longrightarrow \boxed{\frac{a_1^2}{4} < -a_2}$

so the previous stability condition implies

$$\boxed{a_2 > -1} \longrightarrow \boxed{-a_2 < 1} \longrightarrow \boxed{\frac{a_1^2}{4} < 1} \longrightarrow \boxed{|a_1| < 2}$$

This is the same requirement as in case ii ($d = 0$), however, this time it also means that the real part of the complex roots, i.e. $a_1/2$, must be less than 1 in absolute value.

(Ex 4)

e) $y_t = -0.81 y_{t-2}$ ⟶ $a_1 = 0$, $a_2 = -0.81$

$\alpha^2 + 0.81 = 0$ ⟶ $d = 0^2 - 4 \times 0.81 = -3.24$

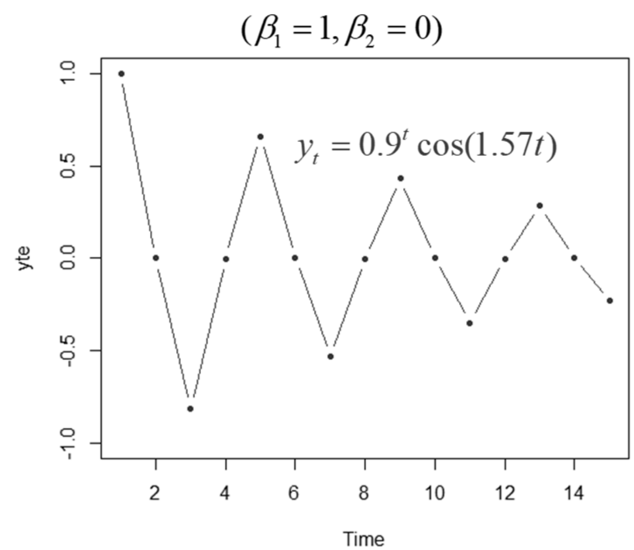
There are two distinct complex conjugate roots: $\alpha_{1,2} = \dfrac{a_1 \pm i\sqrt{-d}}{2} = \pm 0.9i$

$r = |\alpha_{1,2}| = \sqrt{-a_2} = \sqrt{0.81} = 0.9$ and $-1 < a_2 < 0$ and $|a_1| < 2$

⟶ These three equivalent conditions are met, so the equation is stable.

$\cos(\theta) = \dfrac{a_1}{2r} = \dfrac{0}{1.8} = 0$ ⟶ $\theta = \dfrac{\pi}{2} = 1.57$

$$y_t = \beta_1 r^t \cos(\theta t + \beta_2) = \beta_1 0.9^t \cos(1.57t + \beta_2)$$



$\{y_t\}$ converges to the equilibrium value, $y^* = 0$, and its time path is characterized by a dampening oscillation ($r < 1$).

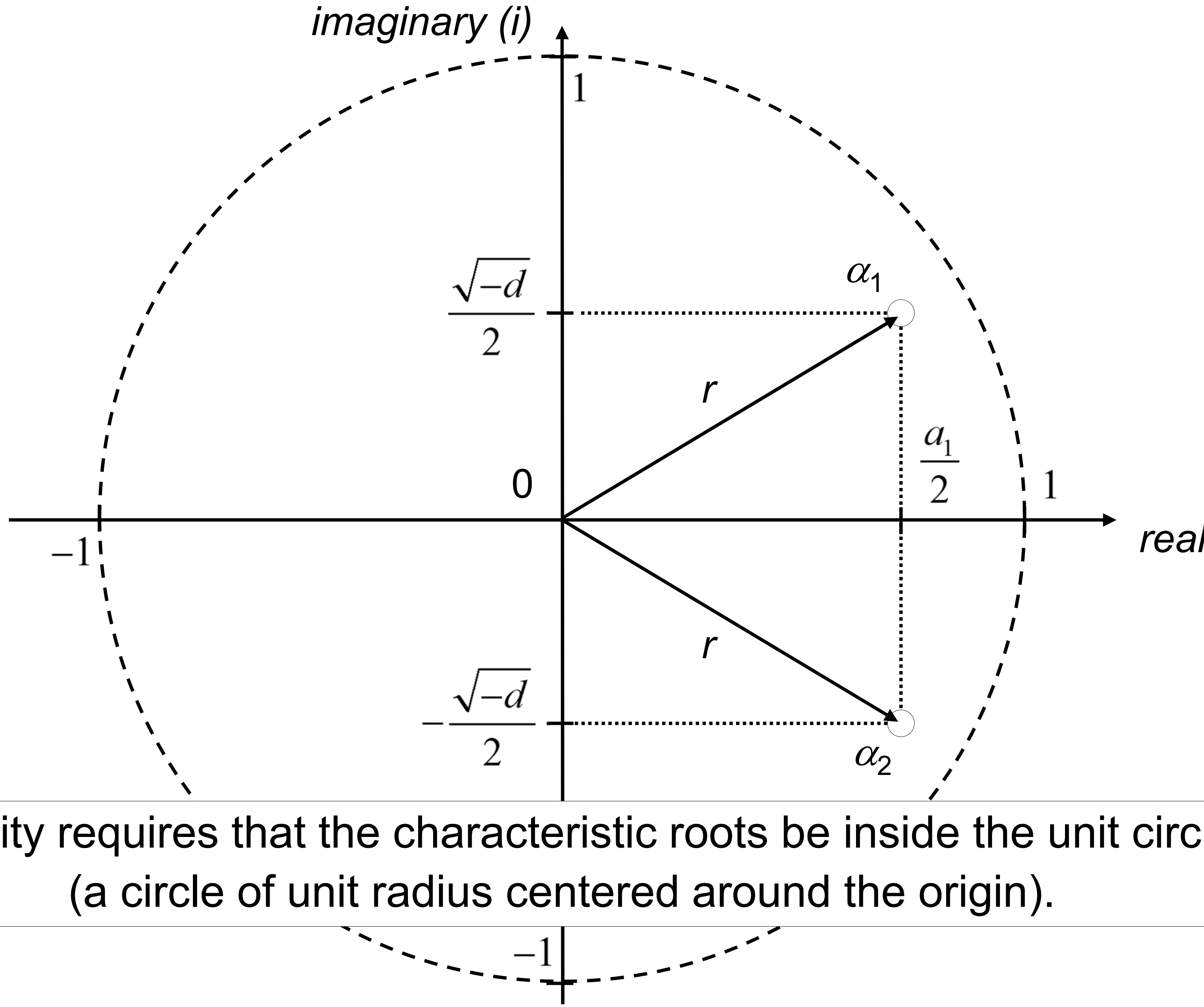
- Let us summarize the stability conditions of homogeneous second-order linear difference equations.

  The key term is the discriminant, whose sign / value distinguishes the three cases considered above from each other.

  In the first two cases, $d \geq 0$, the homogeneous solutions are real so in the real-imaginary coordinate system they are on the horizontal axis (their second coordinates are zero), and stability requires that they be smaller than one in absolute value.

  In the third case, $d < 0$, stability requires that the length of the vectors representing the complex roots, r, be smaller than one. Consequently, both the real and imaginary parts of these roots must be less than one in absolute value.

  In general, i.e., irrespectively of $d > 0$, $d = 0$ or $d < 0$, the second-order homogeneous difference equation is stable if in the real-imaginary coordinate system the characteristic roots lie within the unit circle.



Stability requires that the characteristic roots be inside the unit circle (a circle of unit radius centered around the origin).

- These results can be generalized as follows.

Consider the n^{th} order linear difference equation

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \ldots + a_n y_{t-n} + \varepsilon_t$$

$\longrightarrow$ The homogeneous equation is $y_t - a_1 y_{t-1} - \ldots - a_n y_{t-n} = 0$

and the trial solution is $y_t = A\alpha^t$

$\longrightarrow$ The characteristic equation is $\alpha^n - a_1\alpha^{n-1} - \ldots - a_n = 0$

This n^{th} order polynomial in α yields n solutions, i.e. characteristic roots,

and the homogeneous solution takes the form $y_t = \sum_{i=1}^{n} A_i \alpha_i^t$

The characteristic roots can be real numbers or complex conjugates, and stability requires that all of them lie inside the unit circle.

- Whether a difference equation is stable, can be decided in two ways: on the basis of the characteristic roots or of the coefficients.

  When the characteristic roots are unknown (because the difference equation is too complex to solve it manually and some appropriate software is not available) we can rely on the following three rules based on the coefficients:

  i-ii. A *sufficient* and a *necessary* condition of stability are

$$\sum_{i=1}^{n} |a_i| < 1 \quad \text{and} \quad \sum_{i=1}^{n} a_i < 1 \quad \text{respectively.}$$

This is *sufficient*, i.e. if it is satisfied the system is stable.

This is *necessary*, i.e. if it is violated the system is unstable.

  iii. At least one characteristic root is on the unit circle if

$$\sum_{i=1}^{n} a_i = 1$$

and in this case $\{y_t\}$ is called a unit-root process.

Ex 5:

Is the following homogeneous linear third-order difference equations stable?

$$y_t = 0.7y_{t-1} - 0.5y_{t-2} + 0.8y_{t-3} \longrightarrow a_1 = 0.7,\ a_2 = -0.5,\ a_3 = 0.8$$

a) Consider the stability conditions based on the coefficients.

$$\sum_{i=1}^{3} |a_i| = 2 > 1 \quad \text{and} \quad \sum_{i=1}^{3} a_i = 1 \longrightarrow$$

This difference equation is unstable and it has at least one unit root.

b) The characteristic equation is (see slide #29) $\alpha^3 - 0.7\alpha^2 + 0.5\alpha - 0.8 = 0$

The characteristic roots can be found with the *polyroot(z)* function of *R*, where *z* is the vector of polynomial coefficients in increasing order.

$\longrightarrow$ $z = [-0.8,\ 0.5,\ -0.7,\ 1]$ $\longrightarrow$ *polyroot(c(-0.8, 0.5, -0.7, 1))*

```
-0.15+0.8817596i -0.15-0.8817596i  1.00+0.0000000i
```

The first and second roots are complex conjugates and their modulus is smaller than one (*Mod(-0.15+0.8817596i)* returns 0.8944272), while the third root is a real unit root.

- Lag polynomials provide a concise way for writing difference equations.

← The homogeneous m^{th} order linear difference equation and its characteristic equation are

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} - \ldots - a_m y_{t-m} = 0$$

$$\alpha^m - a_1 \alpha^{m-1} - a_2 \alpha^{m-2} - \ldots - a_m = 0$$

$$\underbrace{\left(1 - a_1 L - a_2 L^2 - \ldots - a_m L^m\right)}_{A(L)} y_t = 0$$

$A(L)$ is an m^{th} order lag polynomial.

$$A(1) = 1 - a_1 - a_2 - \ldots - a_m$$

i.e., for $L = 1$, $A(1)$ is equal to the sum of the coefficients of the lag polynomial.

Suppose that $A(L) = 0$, i.e. $1 - a_1 L - a_2 L^2 - \ldots - a_m L^m = 0$

$$\longrightarrow \left(\frac{1}{L}\right)^m - a_1 \left(\frac{1}{L}\right)^{m-1} - a_2 \left(\frac{1}{L}\right)^{m-2} - \ldots - a_m = 0$$

This polynomial in $1/L$ has the same roots as the characteristic equation in α (see above).

Thus, the values of L that solve $A(L) = 0$ are the reciprocals of the α_i characteristic roots, i.e., $L_i = 1/\alpha_i$ ($i = 1, \ldots, m$).

For this reason, $A(L) = 0$ is called inverse characteristic equation and L_i ($i = 1, \ldots, m$) are the inverse characteristic roots.

$\downarrow$

The stability condition for an m^{th} order homogeneous difference equation can be stated in two alternative ways. Namely, stability requires that

i. All roots of the *characteristic equation* lie *inside* the unit circle.

ii. All roots of the *inverse characteristic equation* lie *outside* the unit circle.

<u>Note</u>: Following Enders (2015) we refer to the equation in α as *characteristic equation* and the one in $1/L$ as *inverse characteristic equation*.

Some authors and also *R*, however, call them the opposite way and hence state that stability requires that the characteristic roots be outside and the inverse characteristic roots be inside the unit circle.

For further information see

Enders (2015): § 1.2-1.4, 1.6