FNCE90056: Investment Management

Lecture 2: Modern Portfolio Theory

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Introduction

Last week

- Trade-off between risk and reward
- Reward: expected returns
- Risk: standard deviation of returns
- Risk aversion, utility, and optimal allocation (1 risky, 1 risk-free)
- Normal distribution $\rightarrow$ analytically convenient for analysing returns of a trading strategy and making statistical inference
- Deviations from normality happen, mainly excess kurtosis and small negative skewness

This week

- Portfolios with multiple risky assets
  - Covariance and correlation
  - Diversification
  - Two risky assets case
- Minimum variance efficient frontier
  - Mean-variance analysis without a risk-free asset
  - Mean-variance analysis with a risk-free asset

Portfolios with Multiple Risky Assets

What is portfolio?

A **portfolio** is any collection of investments

- A share of Qantas is a (simple) portfolio
- 12 shares of Qantas, 15 shares of Microsoft, and 8 Government bonds is a portfolio
- A house, an education, 11 shares of Qantas, a lottery ticket, and some US currency is also a portfolio

Portfolio returns and variances

Just as with assets, portfolios have uncertain returns:

$$r_p(s) = \sum_i \left(w_i \times r_i(s) \right) \tag{1}$$

portfolio's return in state s

weight of asset i in portfolio
(decided before s is known)

asset i's return in state s

Taking expectations, we get:

$$\mathbb{E}[\tilde{r}_p] = \sum_i (w_i \times \mathbb{E}[\tilde{r}_i]) \tag{2}$$

What about σ_p^2? With only 2 assets, we have:

$$\sigma_p^2 = w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \sigma_{ab} \tag{3}$$

Mean & SD of portfolio return with 2 assets: formulas

- The **return of a 2-asset portfolio**:

$$R_p = w_A R_A + (1 - w_A) R_B$$

- The **expected return of a 2-asset portfolio**:

$$E[R_p] = w_A E[R_A] + (1 - w_A) E[R_B]$$

  using $E[X + Y] = E[X] + E[Y]$ and $E[cX] = cE[X]$ for a constant c.

- The **standard deviation (or volatility) of a 2-asset portfolio**:

$$\sigma_p = \sqrt{w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 w_A (1 - w_A) Cov(R_A, R_B)}$$

  using: $Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]$,
  $Var[cX] = c^2 Var[X]$, and $Cov[cX, dY] = cdCov[X, Y]$.
  If we know Corr., instead of Cov.: $Cov(R_A, R_B) = Corr(R_A, R_B)\sigma_A \sigma_B$

Covariance and Correlation

Covariance

Covariance, σ_{ij}, measures the linear relationship between variables i and j.

$$\sigma_{ij} = \sum_{s} \text{Prob}\,(s) \times (r_i\,(s) - \mathbb{E}\,[r_i]) \times (r_j\,(s) - \mathbb{E}\,[r_j])$$

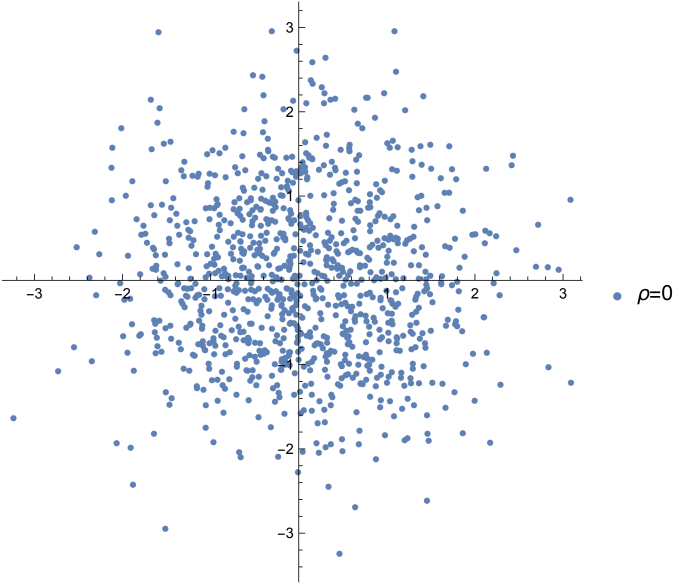
- Inutitively: "When variable i is above its average, is variable j also above its average (and by how much)?"
- What if $j = i$? $\implies \sigma_{ij} = \sigma_{ii} = \sigma_i^2$ (this is simply the **variance**)
- Covariance has squared units, like variance. The values are hard to interpret. To address that, we standardise $\rightarrow$

Define **correlation** as

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \times \sigma_j}$$

- Correlation is dimensionless (i.e. has no units) and bounded: $-1 \leq \rho_{ij} \leq 1$
- $+1$: always together, -1: always moving opposite, 0: no linear pattern.

σ_{ij} and ρ_{ij} measure linear association



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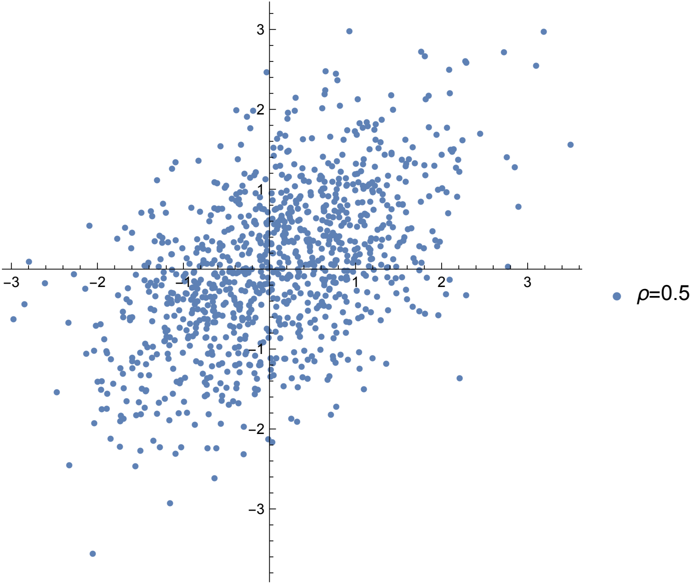

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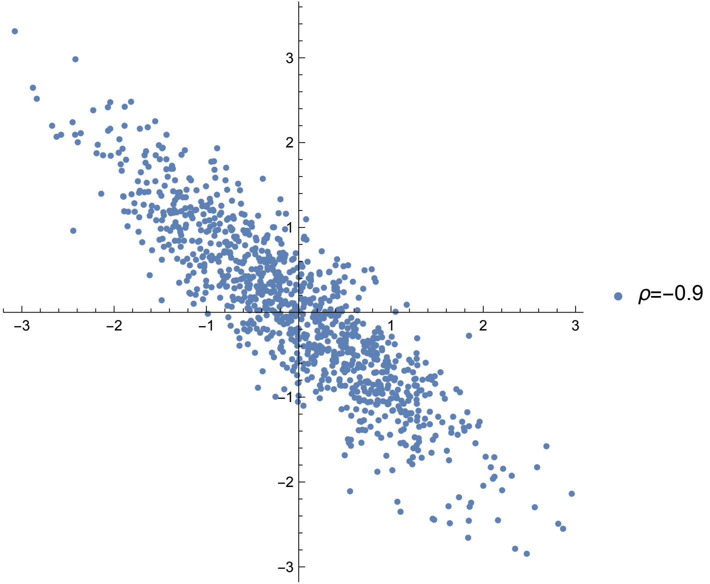


Joint distribution of multiple returns

- With the correlations among multiple returns, we need a joint (or multivariate) distribution to capture that.

- A joint-normal distribution of two returns

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \sim N \left[\begin{bmatrix} \bar{R}_1 \\ \bar{R}_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right] \tag{1}$$

- A joint-normal distribution of N returns
  - Define the $N \times 1$ column vector: $R_{t+1} = \left(R_{t+1}^1, R_{t+1}^2, \cdots, R_{t+1}^N\right)'$, and we assume

$$R_{t+1} \sim \text{iid} N(\mu, \Sigma) \tag{2}$$

  - Here, μ is a $N \times 1$ expectation vector: $\mu = (\mu_1, \mu_2, \cdots, \mu_N)'$, and Σ is a $N \times N$ variance-covariance (VC) matrix.

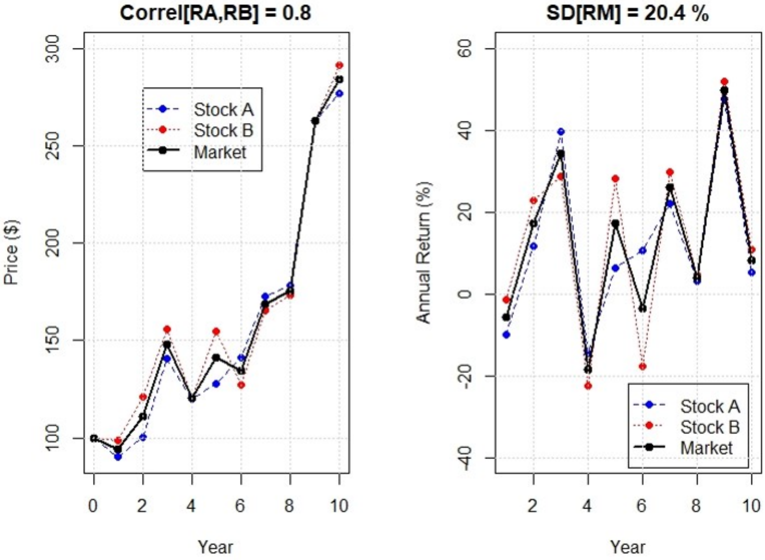
$$\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix} \tag{3}$$

Diversification

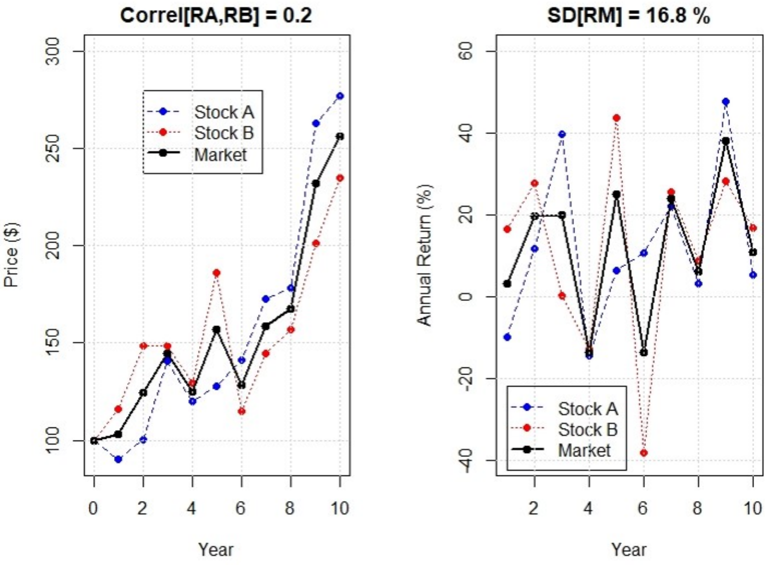
Portfolio diversification reduces risk

- **"Market risk"** or **"systematic risk"** is the risk of the entire market losing value, due to an economic downturn such as a recession.
  - Systematic risk is multi-dimensional, e.g., multiple factor models
  - One can eliminate exposure to some specific systematic risks (e.g., market risk in CAPM), but not to all systematic risk unless she choose to hold only risk-less asset
- **"Idiosyncratic risk"** is the risk due to factors which are specific to individual assets, such as the CEO leaving.
- We can reduce idiosyncratic risk by diversifying our portfolio and buying more and more stocks. **Diversification** does not eliminate market risk, but the overall risk still goes down.

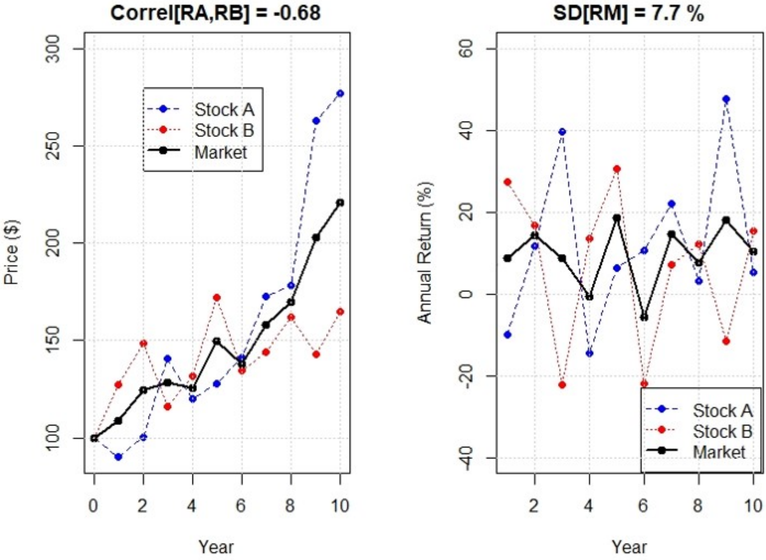
Even adding highly correlated stocks reduces risk



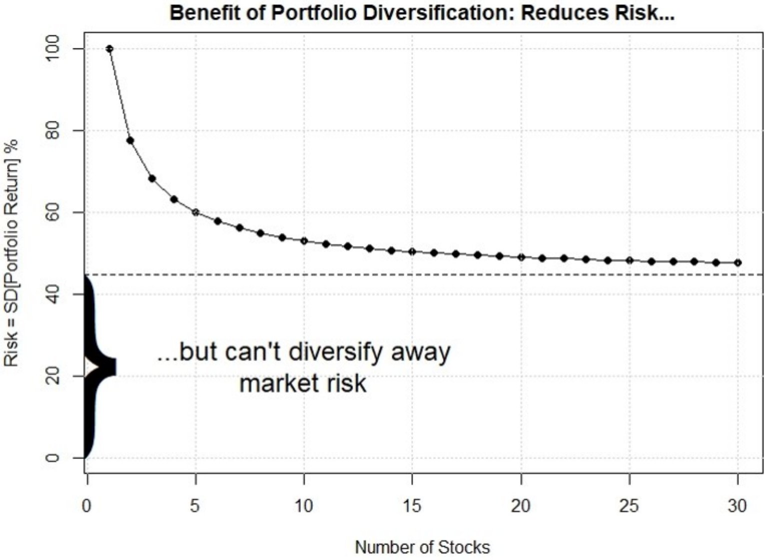
The lower the correlation, the greater the risk reduction



Adding very negatively correlated stock reduces risk more



Portfolio diversification reduces risk



Diversification: Two Risky Assets

- First consider two risky assets R_1 and R_2 with weights w_1 and w_2
- The return on a portfolio containing two risky assets is

$$R_p = w_1 R_1 + w_2 R_2 = w_1 R_1 + (1 - w_1) R_2 \tag{4}$$

- The expected return (denoted by $\bar{R}$) is also linear in portfolio weights

$$\bar{R}_p = w_1 \bar{R}_1 + (1 - w_1) \bar{R}_2 \Rightarrow w_1 = \frac{\bar{R}_p - \bar{R}_2}{\bar{R}_1 - \bar{R}_2} \tag{5}$$

- The variance of the portfolio return is :

$$\begin{aligned} \sigma_p^2 &= \mathrm{Var}\left(w_1 R_1 + (1 - w_1) R_2\right) \\ &= w_1^2 \mathrm{Var}(R_1) + (1 - w_1)^2 \mathrm{Var}(R_2) + 2w_1 (1 - w_1) \mathrm{Cov}(R_1, R_2) \\ &= w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1 (1 - w_1) \sigma_{12} \\ &= w_1^2 \sigma_1^2 + (1 - w_1)^2 \sigma_2^2 + 2w_1 (1 - w_1) \sigma_1 \sigma_2 \rho_{12} \end{aligned} \tag{6}$$

where $\rho_{12} = \mathrm{Corr}(R_1, R_2)$.

Diversification: Two Risky Assets

- Since $\sigma_{12} = \sigma_1\sigma_2\rho_{12} \leq \sigma_1\sigma_2$ given $-1 \leq \rho_{12} \leq 1$, we have

$$\begin{aligned}\sigma_p^2 &= w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_1\sigma_2\rho_{12} \\ &\leq w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_1\sigma_2 1 \\ &= (w_1\sigma_1 + (1-w_1)\sigma_2)^2\end{aligned} \tag{7}$$

$$\sigma_p^2 \leq (w_1\sigma_1 + (1-w_1)\sigma_2)^2, \text{with equality only when} \rho_{12} = 1. \tag{8}$$

- Taking square root to obtain

$$\sigma_p \leq w_1\sigma_1 + (1-w_1)\sigma_2; \text{for all } 0 \leq w_1 \leq 1 \tag{9}$$

$$\sigma_p \leq \sigma_1, \text{if } \sigma_1 = \sigma_2 \tag{10}$$

- This illustrates the power of diversification to reduce portfolio risk.
  - The standard deviation (volatility) of a portfolio is always less than the weighted summation of individual stock's volatility

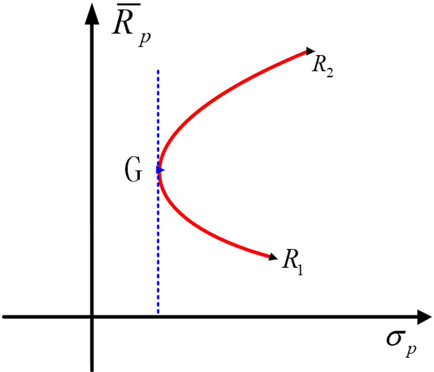
Diversification: Two Risky Assets

- With following two expressions that links $\bar{R}_p, w_1, \sigma_p$

$$\sigma_p^2 = w_1^2\sigma_1^2 + (1 - w_1)^2\,\sigma_2^2 + 2w_1\,(1 - w_1)\,\sigma_{12}$$

$$w_1 = \frac{\bar{R}_p - \bar{R}_2}{\bar{R}_1 - \bar{R}_2}$$

- We plot the minimum variance efficient frontier ($\bar{R}_p, \sigma_p$ relationship)



Diversification: Two Risky Assets

- We can find the global minimum variance portfolio G (a special w_1) by taking derivative w.r.t w_1

$$\begin{aligned}\frac{d\sigma_p^2}{dw_1} &= 2w_1\sigma_1^2 - 2\left(1-w_1\right)\sigma_2^2 + 2\left(1-2w_1\right)\sigma_{12} \\ &= 2w_1\left[\sigma_1^2+\sigma_2^2-2\sigma_{12}\right] - 2\left[\sigma_2^2-\sigma_{12}\right] = 0\end{aligned} \tag{12}$$

- So, we get the weight w_1 of the global minimum variance portfolio G

$$w_1 = \frac{\sigma_2^2-\sigma_{12}}{\sigma_1^2+\sigma_2^2-2\sigma_{12}}$$

- Special cases with specific value of correlation coefficient ρ_{12}:
  - When $\sigma_{12}=0$, $w_1=\sigma_2^2/\left(\sigma_1^2+\sigma_2^2\right)$ and $w_2=\sigma_1^2/\left(\sigma_1^2+\sigma_2^2\right)$.
  - When $\sigma_{12}=\sigma_1\sigma_2\left(\rho_{12}=1\right)$, portfolio variance can go to zero

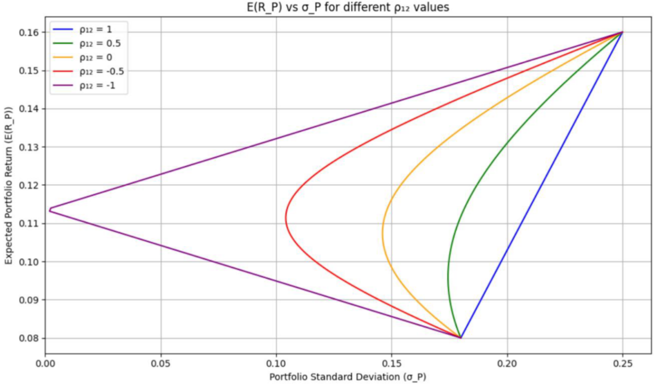
$$w_1=\frac{-\sigma_2}{\sigma_1-\sigma_2}, \quad w_2=\frac{\sigma_1}{\sigma_1-\sigma_2}$$

  - When $\sigma_{12}=-\sigma_1\sigma_2\left(\rho_{12}=-1\right)$, portfolio variance can go to zero

$$w_1=\frac{\sigma_2}{\sigma_1+\sigma_2}, \quad w_2=\frac{\sigma_1}{\sigma_1+\sigma_2}$$

Two Risky Assets: A numerical Example

- Consider a numerical example with $\bar{R}_1 = 0.08$, $\bar{R}_2 = 0.16$, $\sigma_1 = 0.18$, $\sigma_2 = 0.25$
- We consider 5 cases with: 1) $\rho_{12} = 1$; 2) $\rho_{12} = 0.5$; 3) $\rho_{12} = 0$; 4) $\rho_{12} = -0.5$; 5) $\rho_{12} = -1$
- We plot $\bar{R}_p$ against σ_p by increasing w_1 from 0 to 1 in each case



Simple Example with Zero Covariance

Simple example with zero covariance

- You give me \$1 (as an investment)
  - Deal A: I'll give you \$1.05 tomorrow
  - Deal B: I'll give you \$1.225 tomorrow if rainfall exceeds 0.2mm (Prob = 0.5), \$0.90 otherwise
  - Deal C: $\frac{1}{2}$ Deal A + $\frac{1}{2}$ Deal B

Outcome	Return A	Return B	Return C
Rain	0.05	0.225	0.1375
No Rain	0.05	-0.10	-0.025

- Expected returns: $E(r_a) = 0.05, E(r_b) = 0.0625, E(r_c) = 0.05625$
- Risk: $\sigma_a^2 = 0$, $\sigma_b^2 = 0.0264$, and $\sigma_c^2 = 0.5(0.1375 - 0.05625)^2 + 0.5(-0.025 - 0.05625)^2 = 0.0066$

Simple example with zero covariance

How are A and B related?

Use covariance formula:

$$
\begin{aligned}
\sigma_{ij} &= \sum_s \text{Prob}(s) \times (r_i(s) - \mathbb{E}[r_i]) \times (r_j(s) - \mathbb{E}[r_j]) \\
\sigma_{ab} &= 0.5 \times (0.05 - 0.05)(0.225 - 0.0625) \\
&\quad +0.5 \times (0.05 - 0.05)(-0.10 - 0.0625) \\
&= 0
\end{aligned}
$$

Covariance between r_a and r_b is 0 because whatever r_b is, r_a is at its mean; thus, no relation. Covariance of a constant with anything is always zero!

What is the variance of Deal C?

$$
\begin{aligned}
\sigma_c^2 &= (w_a^2 \times \sigma_a^2) + (2 \times w_a \times w_b \times \sigma_{ab}) + (w_b^2 \times \sigma_b^2) \\
&= (0.5^2 \times 0) + (2 \times 0.5 \times 0.5 \times 0) + (0.5^2 \times 0.0264) \\
&= 0.0066
\end{aligned}
$$

as before.

Is there a "best" choice?

Figure out the risk-return tradeoff for every w_b using $U = E(r_p) - 0.5\gamma\sigma_p^2$:

w_b	$\mathbb{E}[r_p]$	σ_p^2	$U_p, \gamma = 2$	$U_p, \gamma = 1/2$
0.0	0.0500	0.0000	0.050000	0.050000
0.1	0.0513	0.0003	0.050986	0.051184
0.2	0.0525	0.0011	0.051444	0.052236
0.3	0.0538	0.0024	0.051373	0.053156
0.4	0.0550	0.0042	0.050775	0.053944
0.5	0.0563	0.0066	0.049648	0.054600
0.6	0.0575	0.0095	0.047994	0.055123
0.7	0.0588	0.0129	0.045811	0.055515
0.8	0.0600	0.0169	0.043100	0.055775
0.9	0.0613	0.0214	0.039861	0.055903
1.0	0.0625	0.0264	0.036094	0.055898

If we can borrow and sell shares *short*, we can have $w_b < 0$ or $w_b > 1$.

In practice you would either use Excel to maximise U, or you would use the optimisation (calculus) procedure as in lecture 1.

Example with non-zero covariance

Example with non-zero covariance

Previously, we had $\sigma_a^2 = 0$ and $\sigma_{ab} = 0$ (i.e. A is risk-free).

Now, assume two correlated risky assets:

Asset	$\mathbb{E}[r]$	Variance	Covariance with other asset
A	0.05	0.0100	0.0050
B	0.0625	0.0264	0.0050

Let's think about a portfolio that is 1/3 in A and 2/3 in B

- $\mathbb{E}[r_p] = (1/3 \times 0.05) + (2/3 \times 0.0625) = 0.0583$
- $\sigma_p^2 = (1/3)^2 \times 0.01 + (2/3)^2 \times 0.026 + 2 \times (1/3) \times (2/3) \times 0.005 = 0.0151$

Positive covariance makes this portfolio have more risk than if A & B were uncorrelated.

Best portfolio of risky assets?

For every possible w_b, compute U and choose the highest:

w_b	$E(r_p)$	σ_p^2	$\gamma = 2$	$\gamma = 1/2$
0.0	0.0500	0.0100	0.040000	0.047500
0.1	0.0513	0.0093	0.041986	0.048934
0.2	0.0525	0.0091	0.043444	0.050236
0.3	0.0538	0.0094	0.044373	0.051406
0.4	0.0550	0.0102	0.044775	0.052444
0.5	0.0563	0.0116	0.044648	0.053350
0.6	0.0575	0.0135	0.043994	0.054123
0.7	0.0588	0.0159	0.042811	0.054765
0.8	0.0600	0.0189	0.041100	0.055275
0.9	0.0613	0.0224	0.038861	0.055653
1.0	0.0625	0.0264	0.036094	0.055898

Minimum Variance Frontier

Minimum variance portfolios with N risky assets

In his 1955 Ph.D. dissertation, Harry Markowitz asked the following: "What is the lowest variance portfolio with a particular expected return?"

We find the **minimum variance frontier / efficient frontier** by solving the following problem for different values of $\bar{r}$:

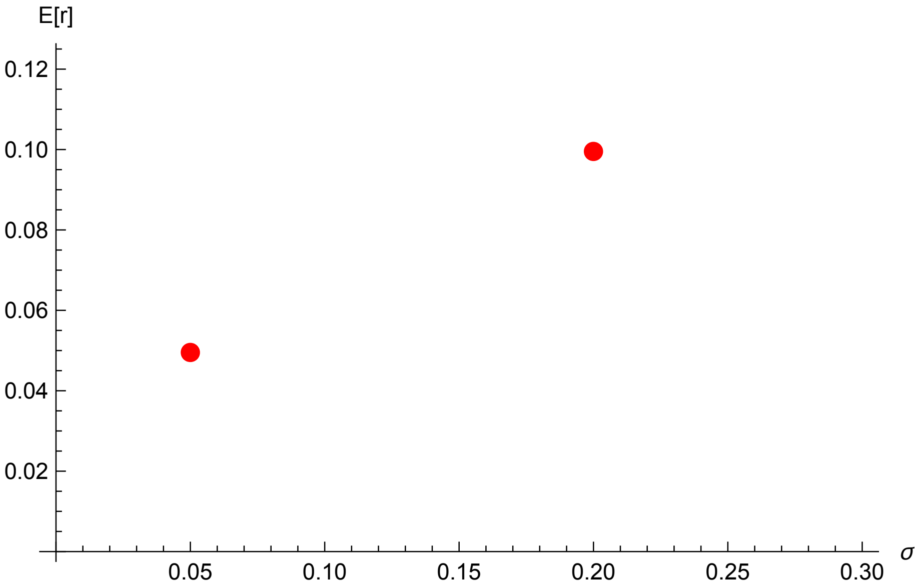
$$\min_{w_1, w_2, \ldots, w_{N-1}} \sigma_p^2 = \sum_{i=1}^{N} \left[\sum_{j=1}^{N} (w_i \times w_j \times \sigma_{ij}) \right]$$

$$\text{subject to } \bar{r} = \mathbb{E}\left[r_p\right] = \sum_{i} \left(w_i \times \mathbb{E}\left[r_i\right]\right)$$

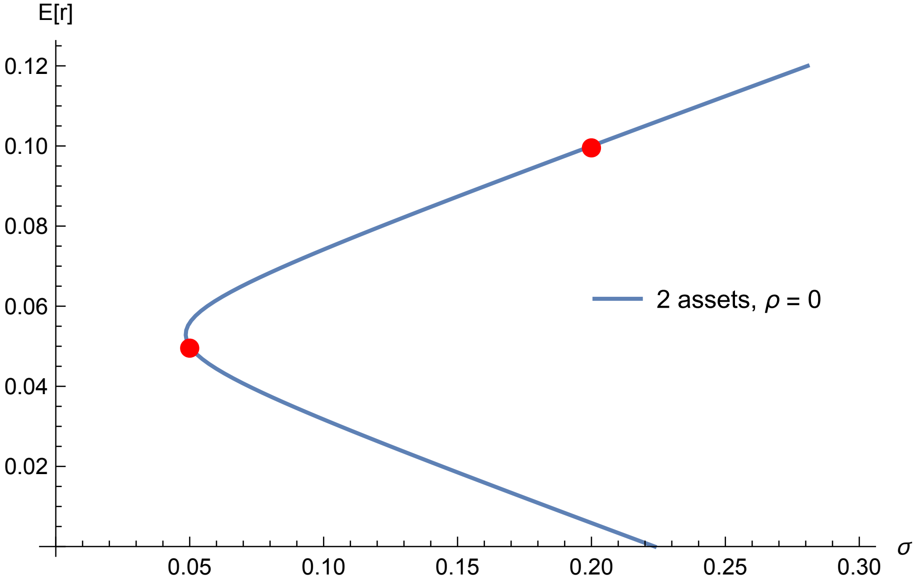
$$\text{and } \sum_{i=1}^{N} w_i = 1,$$

Excel can do this pretty easily.

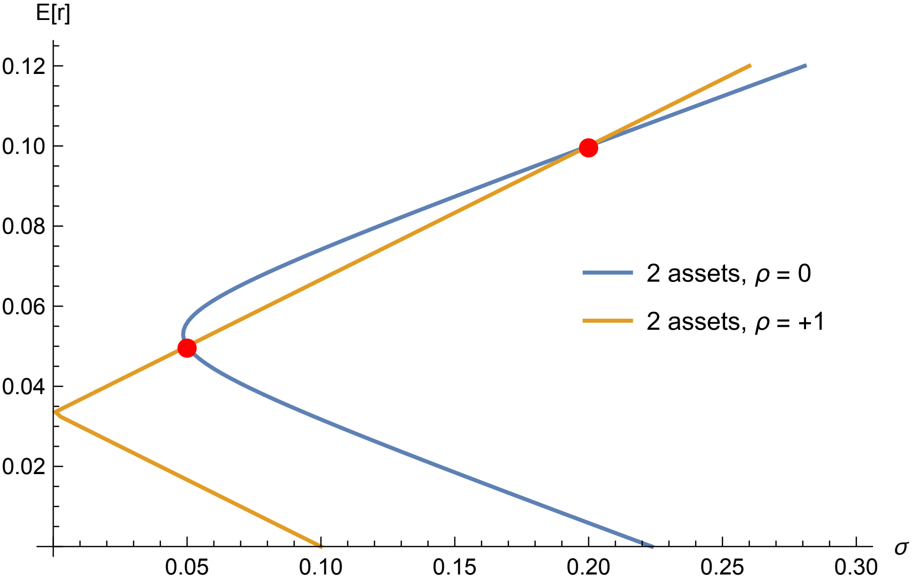
Minimum variance (efficient) frontier



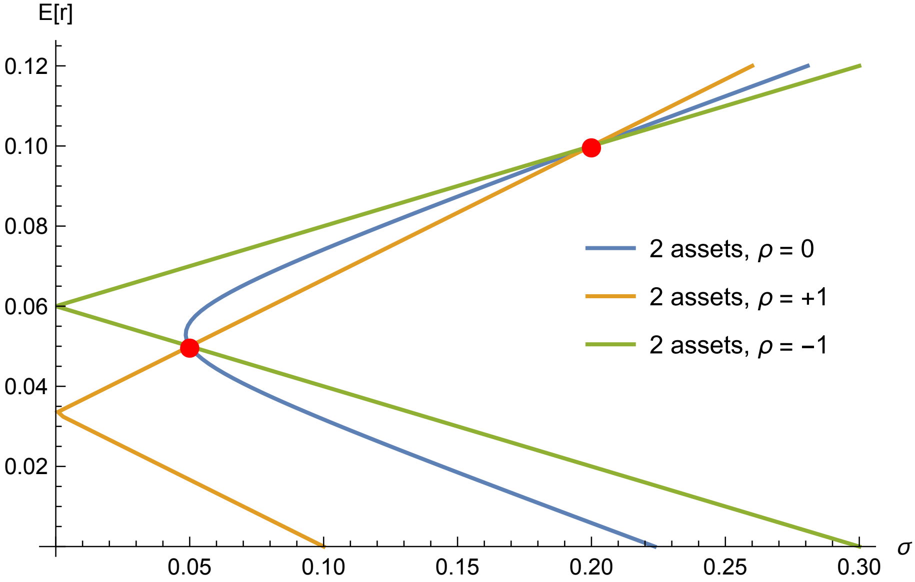
Minimum variance (efficient) frontier



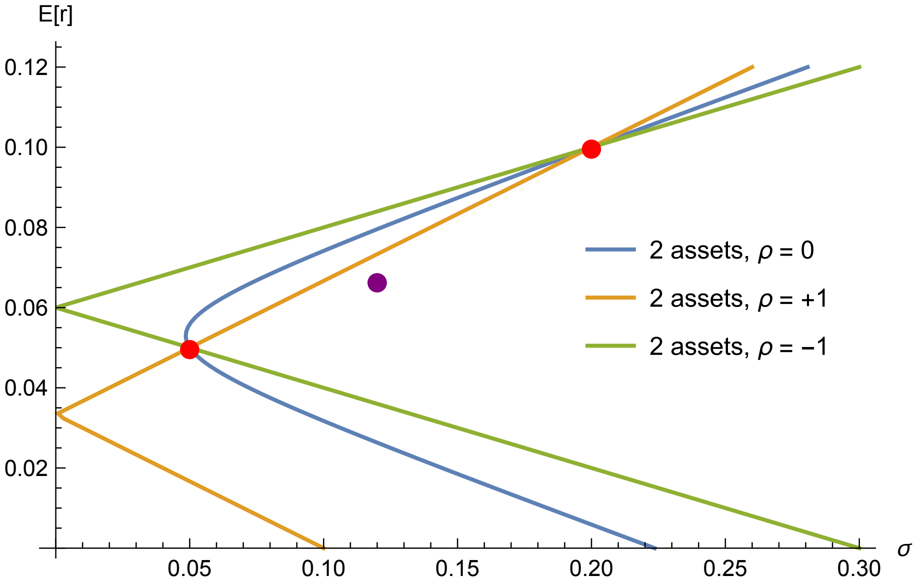
Minimum variance (efficient) frontier



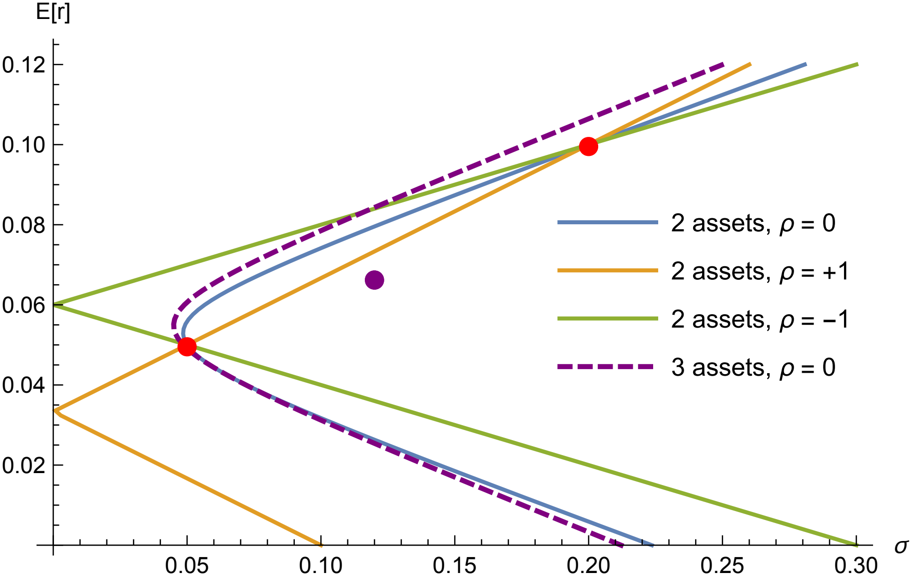
Minimum variance (efficient) frontier



Minimum variance (efficient) frontier



Minimum variance (efficient) frontier



What happens if N is large?

For any desired $\mathbb{E}[r]$, we can construct the efficient frontier by finding the smallest feasible σ.

- With $N+1$ assets, we can always do at least as well as with N assets.
- We can also find the **global minimum variance** risky portfolio, though that may not be the optimal portfolio.

How many terms are in the variance/covariance calculation when $N = 100$?

$$\sigma_p^2 = \sum_{i=1}^{100} \left[\sum_{j=1}^{100} (w_i \times w_j \times \sigma_{ij}) \right]$$

- That's $N^2 = 100^2 = 10,000$ terms to be summed up!
- The number of individual asset variances in the portfolio's variance calculation is N. The number of covariances is $(N^2 - N)/2$.

Is there a more compact way to compute the portfolio's variance? It would be a help to picture the problem in matrix form.

The matrix approach

Let's picture $\sigma_p^2 = \sum_{i=1}^{N} \left[\sum_{j=1}^{N} (w_i \times w_j \times \sigma_{ij}) \right]$ like this:

$$\begin{bmatrix} w_1 & \cdots & w_N \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{N-1,1} & \sigma_{N-1,2} & \cdots & \sigma_{N-1,N} \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_{N-1} \\ w_N \end{bmatrix}$$

With two assets this setup gives:

$$\sigma_p^2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{1,2} + w_2^2\sigma_2^2$$

For larger cases, Excel has the MMULT and TRANSPOSE functions for doing all this multiplication and addition using just a single cell formula!

Mean-variance Efficient Frontier without a Risk-less Asset

- Suppose there are N assets. Let $\bar{R} = \left(\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_N\right)'$ denote an $N \times 1$ vector of expected returns, i.e. $\bar{R}_i = E(R_i)$, for $i = 1, 2, \cdots, N$
- Let Σ denote the variance-covariance matrix of returns.
  - Assume Σ is non-singular (cannot replicate a riskless asset).
- **The optimization problem**: we choose weights ω to minimize portfolio variance

$$\min_{\omega} \frac{1}{2}\sigma_p^2 = \frac{1}{2}\omega' \Sigma \omega \tag{13}$$

- subject to

$$\begin{aligned} \mathbf{1}'\omega &= \omega_1 + \omega_2 + \cdots \omega_N = 1 \\ E\left(R^P\right) &= \bar{R}'\omega = \omega_1 \bar{R}_1 + \omega_2 \bar{R}_2 + \cdots \omega_N \bar{R}_N = \mu \end{aligned} \tag{14}$$

Step-by-step derivation

Mean-variance Efficient Frontier without a Risk-less Asset

- Through solving above maximization, we can get a relationship between targeted portfolio level return $\mu = \bar{R}_p$ and portfolio σ_p

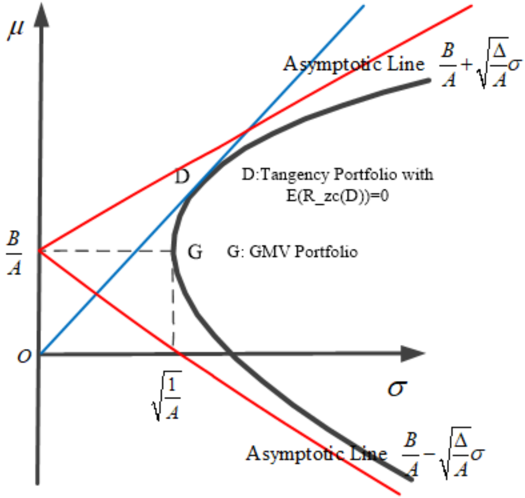
$$\begin{bmatrix} \mu_1 & w_1^\star & \sigma_1 \\ \mu_2 & w_2^\star & \sigma_2 \\ \mu_3 & w_2^\star & \sigma_3 \\ \vdots & \vdots & \vdots \\ \mu_N & w_N^\star & \sigma_N \end{bmatrix}$$

- If we plot μ against σ, it is a **hyperbola** relationship

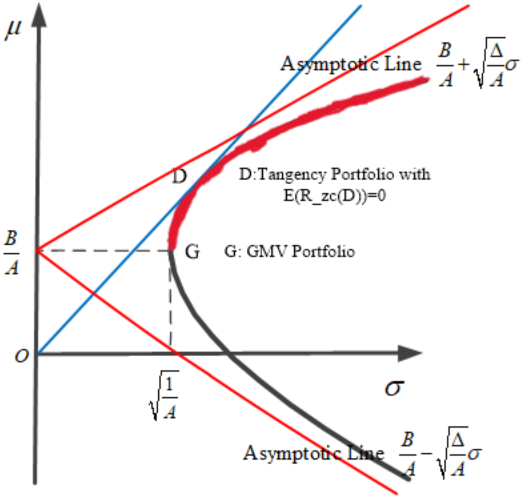
$$\frac{\sigma_p^2}{1/A} - \frac{\left[\mu - \frac{B}{A}\right]^2}{\Delta/A^2} = 1 \tag{15}$$

Check derivations and the value of A, B, Δ here!

The mean-variance efficient set



The mean-variance efficient set

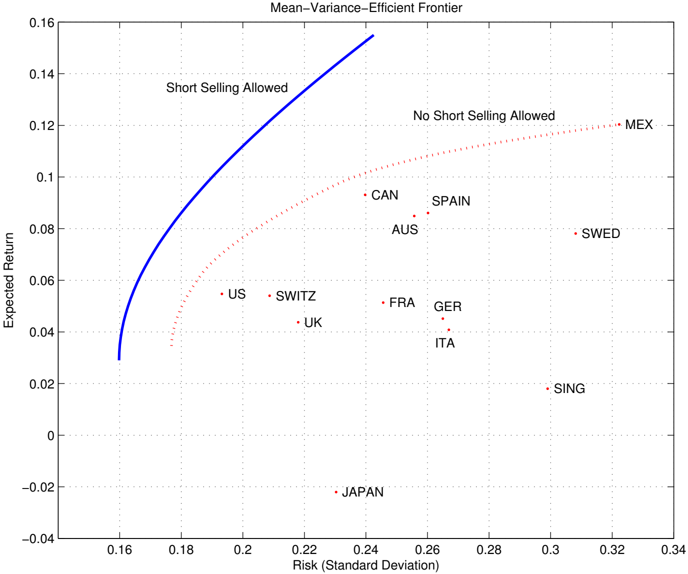


Example: international portfolio selection

- As an example, let us look at the minimum variance frontier from investing in international stock markets.
- The following table, estimated from ETF returns over 1-Apr-1996 to 11-Oct-2011, illustrates the required information:

	Expected Return	Return Std. Dev.	Correlation Coefficients												
			US	AUS	CAN	MEX	JAPAN	SING	FRA	GER	ITA	SPAIN	SWED	SWITZ	UK
US	5.5%	19.3%	1.00												
AUS	8.5%	25.6%	0.70	1.00											
CAN	9.3%	24.0%	0.75	0.75	1.00										
MEX	12.0%	32.2%	0.72	0.65	0.64	1.00									
JAPAN	-2.2%	23.0%	0.53	0.55	0.52	0.48	1.00								
SING	1.8%	29.9%	0.59	0.59	0.56	0.56	0.53	1.00							
FRA	5.1%	24.6%	0.79	0.73	0.72	0.66	0.56	0.60	1.00						
GER	4.5%	26.5%	0.79	0.70	0.73	0.68	0.54	0.61	0.90	1.00					
ITA	4.1%	26.7%	0.70	0.70	0.65	0.61	0.48	0.55	0.86	0.82	1.00				
SPAIN	8.6%	26.0%	0.68	0.67	0.64	0.64	0.46	0.55	0.83	0.81	0.82	1.00			
SWED	7.8%	30.8%	0.74	0.69	0.71	0.65	0.49	0.58	0.83	0.83	0.77	0.76	1.00		
SWITZ	5.4%	20.9%	0.70	0.64	0.64	0.57	0.50	0.53	0.79	0.79	0.74	0.74	0.72	1.00	
UK	4.4%	21.8%	0.79	0.72	0.71	0.68	0.54	0.62	0.83	0.81	0.76	0.73	0.75	0.75	1.00

Example: international portfolio selection



Adding constraints on assets' weights

- Optimization can lead to some really extreme portfolio weights or positions. Reasonable? Feasible even?
- Might constrain the portfolio weights. Why?
  - No short sales.
  - Volatility limits.
  - Target CAPM β: $\sum_{i}^{N} w_i \beta_i = \beta$
  - Minimize benchmark tracking error.
  - Limit exposure to certain asset classes due to transaction costs, liquidity, or uncertainty about estimates.
- Every constraint shrinks the set of feasible portfolios. It is important to understand how much is potentially being sacrificed. To do this you look at the benefit of relaxing the constraint by one unit— this is called the "shadow price" of the constraint.

Mutual Fund theorem

1. The mutual fund theorem of Tobin (1958) says that
   - all minimum-variance portfolios can be obtained by mixing just two minimum-variance portfolios in different proportions. Check proof here!

- Assume we know two portfolios on the frontier with weights $\underline{w}_A$ and $\underline{w}_B$
  - The expected returns of these two portfolios are

$$\bar{R}_A = \underbrace{\bar{\mathbf{R}}'}_{1\times N} \underbrace{\underline{w}_A}_{N\times 1} ; \bar{R}_B = \underbrace{\bar{\mathbf{R}}'}_{1\times N} \underbrace{\underline{w}_B}_{N\times 1}$$

  - The corresponding minimized standard derivations are σ_A and σ_B
- Then, according to Mutual Fund theorem, any other portfolios w on the frontier can be a combination of A and B

$$\underline{w}_{\text{new}} = \theta \underline{w}_A + (1-\theta)\,\underline{w}_B$$

- We can obtain any other portfolios w on the frontier by changing the value of θ, just as what we did in the two risky asset case (page 19)

Constructing the frontier using two efficient portfolios

Let's construct an efficient portfolio with expected return equal to r using two know efficient portfolios (A and B):

- If we allocate fraction θ of our capital to (known) portfolio A and the rest to (known) portfolio B, the new portfolio's weights are:

$$\underline{w}_{\text{new}} = \theta \underline{w}_A + (1 - \theta)\, \underline{w}_B \tag{16}$$

  So the expected return on the new portfolio is:

$$\begin{aligned} r = \underline{w}_{\text{new}}^{\mathsf{T}} \underline{\mu} &= \theta \bar{R}_A + (1 - \theta)\, \bar{R}_B \\ \implies \theta &= \frac{r - \bar{R}_B}{\left(\bar{R}_A - \bar{R}_B\right)} \end{aligned} \tag{17}$$

  Plug this value for θ into (16) to obtain $\underline{w}_{\text{new}}$.
- Calculate the portfolio's variance by plugging $\underline{w}_{\text{new}}$ into (6).
- Repeat for any desired value(s) of r...

Which two efficient portfolios?

- Usually, the two frontier portfolios we use are: global minimum variance portfolio and the tangency portfolio.
- The asset weights corresponding to the **global minimum variance portfolio** are

$$\underline{w}_{\text{global min var}} = \frac{\Sigma^{-1}\underline{1}}{\underline{1}^{\mathsf{T}}\Sigma^{-1}\underline{1}} \tag{18}$$

- The asset weights corresponding to the **tangency portfolio**, which is the frontier portfolio with the highest Sharpe ratio, are

$$\underline{w}_{\text{tangency}} = \frac{\Sigma^{-1}\left(\underline{\mu} - \underline{r_f}\right)}{\underline{1}^{\mathsf{T}}\Sigma^{-1}\left(\underline{\mu} - \underline{r_f}\right)} \tag{19}$$

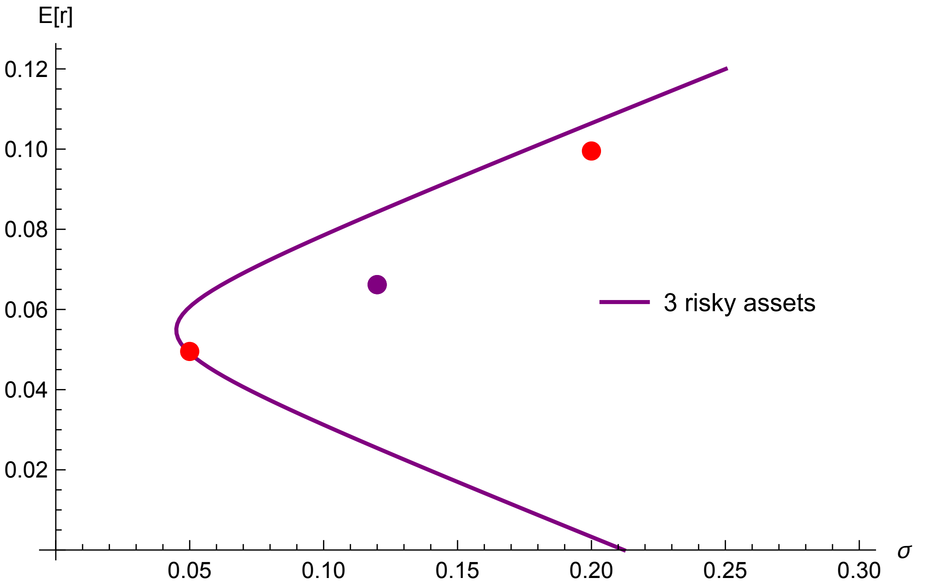
No need to memorize these two equations!

Capital Allocation Line Revisited

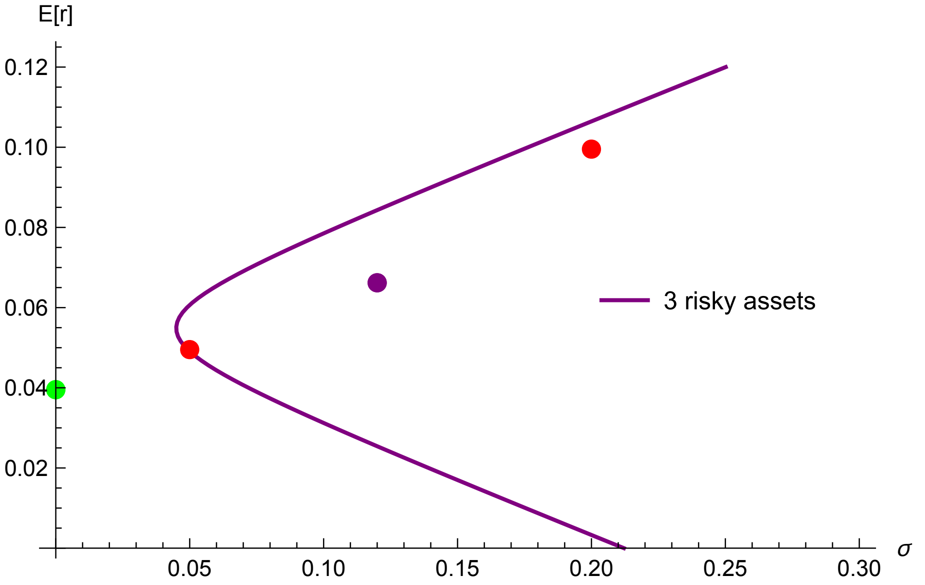
Minimum variance frontier with a risk-free asset

- So far, we have a set of minimum variance portfolios with multiple risky assets.
- Let's add in a risk-free asset.
- Covariance between the risk-free asset and any other portfolio is 0.
- We use the same procedures to obtain the minimum variance frontier
  - For each targeted expected return $\bar{R}_p$, we find the optimal portfolio weights $w^\star$ that yields the minimum variance σ_p^2
  - We then can obtain a relationship between targeted expected return $\bar{R}_p$ and the minimum variance σ_p^2
  - Plotting targeted expected return $\bar{R}_p$ against σ_p to obtain the minimum variance frontier

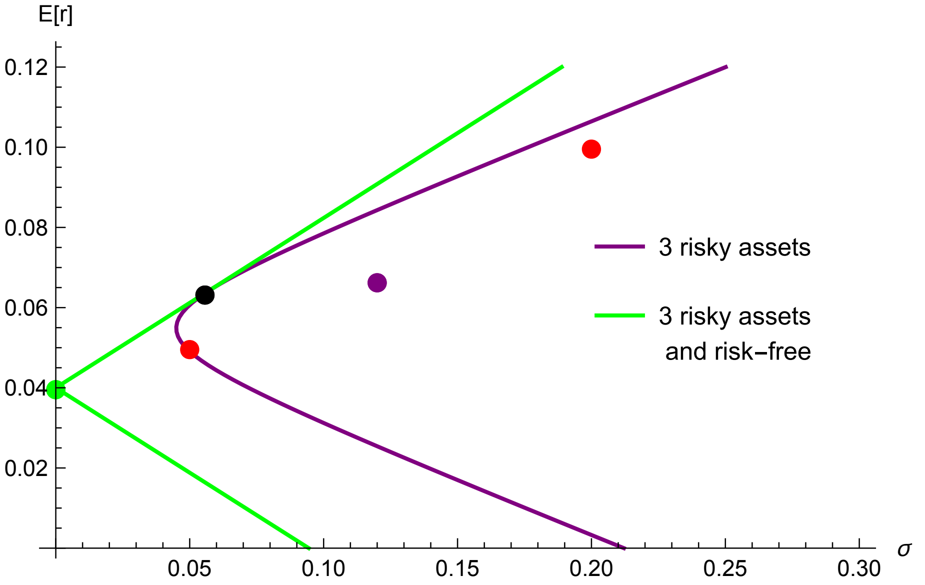
Minimum variance frontier with a risk-free asset, rf = 4%



Minimum variance frontier with a risk-free asset, rf $= 4\%$



Minimum variance frontier with a risk-free asset, rf $= 4\%$



Mean-Variance Efficient Frontier with a Risk-less Asset

- Now we add one more risk-less asset, denote its weight by ω_0, while keep all other N risky assets with weights ω $(N \times 1)$
- The mean-variance **optimization problem** becomes choosing ω, ω_0 to

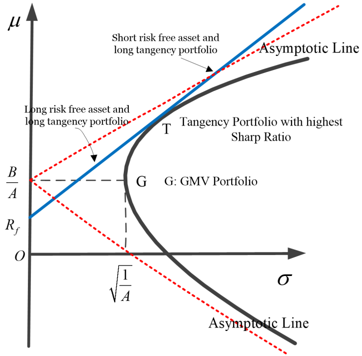
$$\min_{\omega,\omega_0} \frac{1}{2}\sigma_p^2 = \frac{1}{2}\omega'\Sigma\omega$$

subject to

$$\omega_0 + \mathbf{1}'\omega = 1$$
$$\omega_0 R_f + \bar{R}'\omega = \mu$$

Step-by-step derivation

Mean Variance Efficient Frontier with a Risk-free asset



- This new efficient frontier is also called the capital market line

$$\mathbb{E}\left[R_p\right] = R_f + \left(\frac{\mathbb{E}\left[R_{\text{Tangency}}\right] - R_f}{\sigma_{\text{Tangency}}}\right) \times \sigma_p$$

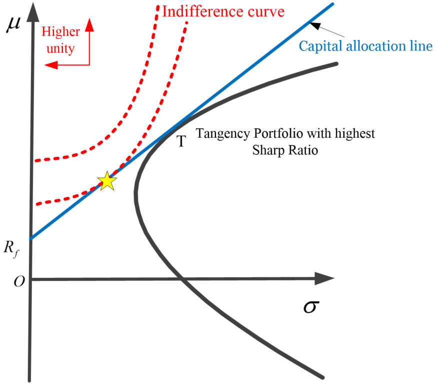
- The tangency portfolio T is the portfolios of N risky assets with highest Sharpe ratio

Two-fund separation theorem

- Given the linear Mean-Variance Frontiers, we have the **Two-fund separation theorem** (mutual fund theorem in the case with R_f):
  - All investors hold a combination of the safe asset and a unique mutual fund containing risky asset (i.e., the "tangency portfolio").
  - Extreme cases: 100% risk-free, 100% risky (= tangency portfolio).
- Two-fund separation theorem allows us to solve the investors' wealth allocation decision on $N+1$ assets (N risky, 1 risk-free) in two steps:
  - **Mean-variance optimization**: for a targeted expected return, identify the portfolio that minimizes variance for that given expected return.
  - **Capital allocation problem**: use the MVEF and investors' preferences to pick the optimal risk/return combination along the frontier.

Investors' capital allocation problem

- By choosing the optimal mix w, investor is choosing a combination of $E(R), \sigma$ on the Capital allocation Line to maximize the utility
- The utility function can be described by indifference cure of $E(R), \sigma$, the tangent point will pin down w.



Conclusions

Summary

- Portfolios can help us achieve better risk-return outcomes than individual investments.
- Even if we have a good investment, it can usually be made better by mixing it with other investments.
- The best portfolio depends on what actions you are allowed to take.
- Start with risky asset allocations, then use the capital allocation line.
- Key insights of portfolio theory
  - In stead of evaluating an asset alone, evaluate it within the portfolio!
  - Demand for an asset not only depends on its own risk-return feature, but its correlation with other assets within the portfolio!

Appendix

Mean-variance analysis with many assets: notations

- $\bar{R}$ $(N \times 1)$ is the vector of mean returns.
- ω $(N \times 1)$ is the vector of portfolio weights.
- $\mathbf{1}$ $(N \times 1)$ is a vector of ones.

$$\bar{R} = \begin{bmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_N \end{bmatrix} ; \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix} ; \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

- Σ $(N \times N)$ is the (symmetric) variance-covariance matrix of returns.

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix}$$

- $\omega^\star$ $(N \times 1)$ is the optimal vector of portfolio weights.
- μ is a scalar.

Mean-variance analysis with many assets: notations

- The portfolio P return

$$R^P = \omega_1 R_1 + \omega_2 R_2 + \cdots \omega_N R_N = [R_1, \cdots, R_N] \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix} = R'\omega \quad (20)$$

- The portfolio P expected return

$$E\left(R^P\right) = \omega_1 \bar{R}_1 + \omega_2 \bar{R}_2 + \cdots \omega_N \bar{R}_N = \bar{R}'\omega \quad (21)$$

- The total portfolio weights

$$\omega_1 + \omega_2 + \cdots \omega_N = \mathbf{1}'\omega \quad (22)$$

Mean-variance analysis with many assets

- We solve the optimization problem using Lagrangian approach with 3 steps:
  - **Step 1**: set up the Lagrangian function
  - **Step 2**: take first order conditions
  - **Step 3**: substitute the results from step 2 into the constraints to solve the Lagrangian multiplier, and obtain the final solution of the problem.
- After we solve the optimization problem, we will have a optimal portfolio $\omega^\star$ for each targeted expected return level μ, and this optimal portfolio will have its own variance $Var^\star/\sigma_p^\star$.
- When we change the targeted expected return μ, the optimal portfolio $\omega^\star$ and its $Var^\star$ will change accordingly.
- **Our goal** is to get all the $\left(\mu, \omega^\star, Var^\star/\sigma_p^\star\right)$ combinations $\Rightarrow$ derive the mean-variance efficient set (frontier)!

Derive the portfolio level variance in matrix

- Why is $\omega' \Sigma \omega$ the portfolio level variance?
- Let's look at the two assets case: $R_p = w_1 R_1 + w_2 R_2$, the portfolio variance

$$\begin{aligned} \sigma_p^2 &= \operatorname{Var}(w_1 R_1 + w_2 R_2) \\ &= w_1^2 \operatorname{Var}(R_1) + w_2^2 \operatorname{Var}(R_2) + 2 w_1 w_2 \operatorname{Cov}(R_1, R_2) \\ &= w_1^2 \sigma_1^2 + \omega^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12} \end{aligned} \tag{23}$$

- If we write it in matrix form, we have

$$\begin{aligned} \sigma_p^2 &= w_1^2 \sigma_1^2 + \omega^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12} \\ &= \left[\begin{array}{cc} w_1 & w_2 \end{array} \right] \left[\begin{array}{cc} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{array} \right] \left[\begin{array}{c} w_1 \\ w_2 \end{array} \right] \\ &= \omega' \Sigma \omega \end{aligned} \tag{24}$$

  Where we have $\omega = \left[\begin{array}{c} w_1 \\ w_2 \end{array} \right]$, and $\Sigma = \left[\begin{array}{cc} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{array} \right]$ in this case.
- It is straightforward to generalize this to the N asset case.

Derive the portfolio level variance in matrix

- Why is $\omega'\Sigma\omega$ the portfolio level variance?
- In the case with N assets, the portfolio level variance

$$\begin{aligned}
\sigma_p^2 \equiv V\left[R^p\right] = V\left[\sum_{i=1}^{N}\omega_i R_i\right] &= \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i\omega_j \operatorname{Cov}\left[R_i, R_j\right] \\
&= \sum_{i=1}^{N}(\omega_i)^2\, V\left[R_i\right] + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\omega_i\omega_j \operatorname{Cov}\left[R_i, R_j\right] \\
&= \left[\omega_1, \cdots, \omega_N\right]\begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix}
\end{aligned} \tag{25}$$

- Basic rules of matrix calculus: Let a and x be vectors and A be a symmetric matrix in (ii). Then:
  - (i) $\dfrac{\partial a'x}{\partial x} = \dfrac{\partial x'a}{\partial x} = a$; (ii) $\dfrac{\partial x'Ax}{\partial x} = 2Ax$

The Lagrangian approach

- We use Lagrangian approach to solve constrained maximization problem
- **Step 1**: set up the following Lagrangian function

$$\min_{\omega} L = \frac{1}{2}\omega'\Sigma\omega + \gamma_1\left(1 - \mathbf{1}'\omega\right) + \gamma_2\left(\mu - \bar{R}'\omega\right) \tag{26}$$

- in which the scalars γ_1 and γ_1 denote the Lagrangian multipliers.
- **Step 2**: take first-order conditions with respect to ω:

Check Matrix calculus here!

$$\Sigma\omega^* = \gamma_1\mathbf{1} + \gamma_2\bar{R} \tag{27}$$

- Premultiply both sides by Σ^{-1}

$$\omega^* = \gamma_1\Sigma^{-1}\mathbf{1} + \gamma_2\Sigma^{-1}\bar{R} \tag{28}$$

which is the solution to the optimization problem. But here, we still don't know γ_1 and γ_2 yet.

Solving for Lagrangian multipliers

- **Step 3**: We can find γ_1 and γ_2 by substituting ω^* into the two constraints:

$$\mathbf{1}'\omega = 1 \Rightarrow \gamma_1 \mathbf{1}'\Sigma^{-1}\mathbf{1} + \gamma_2 \mathbf{1}'\Sigma^{-1}\bar{R} = 1 \tag{29}$$

$$\bar{R}'\omega = \mu \Rightarrow \gamma_1 \bar{R}'\Sigma^{-1}\mathbf{1} + \gamma_2 \bar{R}'\Sigma^{-1}\bar{R} = \mu \tag{30}$$

- If we define A, B, C and Δ (all 1×1 scalar) are functions of Σ and $\bar{R}$:

$$\begin{aligned} A &= \mathbf{1}'\Sigma^{-1}\mathbf{1};\;\; B = \bar{R}'\Sigma^{-1}\mathbf{1} = \mathbf{1}'\Sigma^{-1}\bar{R} \\ C &= \bar{R}'\Sigma^{-1}\bar{R};\;\; \Delta = AC - B^2 \end{aligned} \tag{31}$$

- Above equation system can be written as

$$\begin{aligned} \gamma_1 A + \gamma_2 B &= 1 \\ \gamma_1 B + \gamma_2 C &= \mu \end{aligned} \tag{32}$$

- With is two equations, we can solve two unknowns γ_1 and γ_2.

$$\gamma_1 = \frac{C - \mu B}{\Delta} \text{ and } \gamma_2 = \frac{\mu A - B}{\Delta} \tag{33}$$

The final solution: how is ω^* related to target μ

- Given the results in (37), we can plug γ_1 and γ_2 into the expression of ω^* in eq. (28), and rewrite ω^* as a function of targeted expected return μ, as follows:

$$
\begin{aligned}
\omega^* &= \gamma_1 \Sigma^{-1}\mathbf{1} + \gamma_2 \Sigma^{-1}\bar{R} \\
&= \frac{C - \mu B}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{\mu A - B}{\Delta}\Sigma^{-1}\bar{R} \\
&= \frac{C}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{-B}{\Delta}\Sigma^{-1}\bar{R} + \frac{-\mu B}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{\mu A}{\Delta}\Sigma^{-1}\bar{R} \\
&= \frac{1}{\Delta}\left[C\left(\Sigma^{-1}\mathbf{1}\right) - B\left(\Sigma^{-1}\bar{R}\right)\right] + \frac{1}{\Delta}\left[A\left(\Sigma^{-1}\bar{R}\right) - B\left(\Sigma^{-1}\mathbf{1}\right)\right]\mu \\
&= g + h\mu
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
g &= \frac{1}{\Delta}\left[C\left(\Sigma^{-1}\mathbf{1}\right) - B\left(\Sigma^{-1}\bar{R}\right)\right] \\
h &= \frac{1}{\Delta}\left[A\left(\Sigma^{-1}\bar{R}\right) - B\left(\Sigma^{-1}\mathbf{1}\right)\right]
\end{aligned}
\tag{35}
$$

The final solution: how is ω^* related to target μ

- Given the results in (37), we can plug γ_1 and γ_2 into the expression of ω^* in eq. (28), and rewrite ω^* as a function of targeted expected return μ, as follows:

$$\omega^* = g + h\mu$$

- **Implications**: any optimal (minimized variance) portfolio having an targeted expected return μ can be characterized by equation (35).
  - For each target μ, we have an optimal portfolio ω^*
- On the other hand, any portfolio that can be represented by (35) is a optimal portfolio.
- The set of all optimal portfolios is called the **mean-variance efficient set**
  - A set of all (μ, ω^*) combinations, when we change target μ, the optimal portfolio ω^* will change accordingly

The mean-variance efficient frontier

- Based on the (μ, ω^*) relationship, we would like to characterize the relationship between optimal portfolio variance and target μ.
  - to find all the optimal mean-variance combinations which is our final goal of the mean-variance analysis !
- Denote the return of the optimal (frontier) portfolio ω^* to be R^*, the variance (σ^2) of this portfolio is

$$\begin{aligned}
\mathrm{Var}\left(R^*\right) &= \omega^{*\prime}\Sigma\omega^* \\
&= \omega^{*\prime}\left(\gamma_1\Sigma\Sigma^{-1}\mathbf{1} + \gamma_2\Sigma\Sigma^{-1}\bar{R}\right) \\
&= \gamma_1\omega^{*\prime}\mathbf{1} + \gamma_2\omega^{*\prime}\bar{R} \\
&= \gamma_1 + \gamma_2\mu
\end{aligned} \tag{36}$$

The mean-variance efficient frontier

- Recall our solution to γ_1 and γ_2

$$\gamma_1 = \frac{C - \mu B}{\Delta} \text{ and } \gamma_2 = \frac{\mu A - B}{\Delta} \tag{37}$$

- Substituting for γ_1 and γ_2 from equation (37):

$$\sigma^2(R^*) = \mathrm{Var}(R^*) = \frac{A\mu^2 - 2\mu B + C}{\Delta} \tag{38}$$

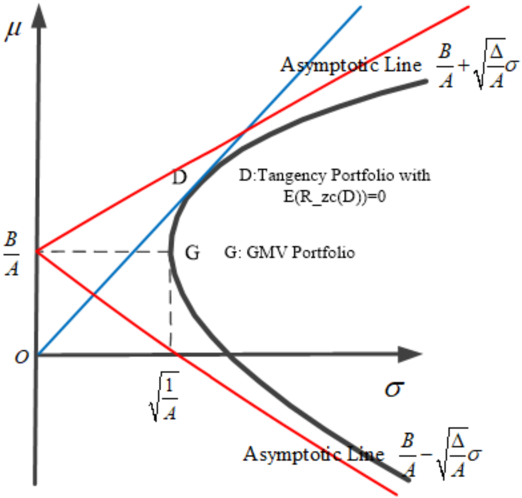
- This is a parabola in (σ^2, μ) and a **hyperbola** in (σ, μ) because Graph

$$\frac{\sigma^2(R_*)}{1/A} - \frac{\left[\mu - \frac{B}{A}\right]^2}{\Delta/A^2} = 1 \tag{39}$$

with asymptotes $\mu = B/A \pm \sqrt{\Delta/A}\sigma(R_p)$

- Therefore, we can easily plot the (σ, μ) relationship in figure, each dot along the line corresponds to a portfolio $\omega^\star$ for a given target μ.

The mean-variance efficient set



The Mutual Fund Theorem

- The mutual fund theorem of Tobin (1958) says that
  - all minimum-variance portfolios can be obtained by mixing just two minimum-variance portfolios in different proportions.
- To show this, recall the minimum variance portfolios are given by

$$\begin{aligned}
\omega^* &= \gamma_1 \Sigma^{-1}\mathbf{1} + \gamma_2 \Sigma^{-1}\bar{R} \\
&= \gamma_1 \underbrace{\mathbf{1}'\Sigma^{-1}\mathbf{1}}_{A} \left(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \right) + \gamma_2 \underbrace{\mathbf{1}'\Sigma^{-1}\bar{R}}_{B} \left(\frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} \right) \\
&= \gamma_1 A \omega_G + \gamma_2 B \left(\frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} \right) \\
&= \underbrace{\gamma_1 A}_{\alpha} \omega_G + \underbrace{\gamma_2 B}_{\beta} \omega_D = \alpha \omega_G + \beta \omega_D
\end{aligned} \tag{40}$$

- Where we define a new portfolio D with

$$\omega_D = \frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} \tag{41}$$

The Mutual Fund Theorem

- Note that $\gamma_1 A + \gamma_2 B = \alpha + \beta = 1$. So any portfolio on the frontier can be formed by a linear combination of these two portfolios, G and D.
- What is portfolio D? we use a 3 step procedure to find it
- Step 1: calculate the mean and variance of its return, and verify that D is on the efficient frontier
  - Given the weight $\omega_D = \dfrac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}}$, calculate the expected portfolio return by

$$E(R_D) = \bar{R}'\omega_D = \frac{\bar{R}'\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} = \frac{C}{B} \tag{42}$$

The Mutual Fund Theorem

- What is portfolio D? we use a 3 step procedure to find it
- Step 1: calculate the mean and variance of its return, and verify that D is on the efficient frontier
  - With ω_D and $E(R_D)$, D is on the efficient frontier because ω_D and $E(R_D)$ satisfy the first-order condition (ω_D is already optimal weight!)

$$\omega^* = \gamma_1 \Sigma^{-1}\mathbf{1} + \gamma_2 \Sigma^{-1}\bar{R} = \frac{C - \mu B}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{\mu A - B}{\Delta}\Sigma^{-1}\bar{R}$$

$$\begin{aligned}\omega_D = \frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} &= \frac{C - E(R_D)B}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{E(R_D)A - B}{\Delta}\Sigma^{-1}\bar{R} \\ &= \frac{C - \frac{C}{B}B}{\Delta}\Sigma^{-1}\mathbf{1} + \frac{\frac{C}{B}A - B}{\Delta}\Sigma^{-1}\bar{R} = \frac{\Sigma^{-1}\bar{R}}{B} = \frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}}\end{aligned}$$

  - Then we can calculate the variance / standard deviation of portfolio D

$$\sigma^2(R_D) = \omega_D'\Sigma\omega_D = \frac{\bar{R}'\Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}\bar{R}}\Sigma\frac{\Sigma^{-1}\bar{R}}{\mathbf{1}'\Sigma^{-1}\bar{R}} = \frac{\bar{R}'\Sigma^{-1}\Sigma\Sigma^{-1}\bar{R}}{\left(\mathbf{1}'\Sigma^{-1}\bar{R}\right)^2} = \frac{C}{B^2} \tag{43}$$

The Mutual Fund Theorem

- Then we show that D is the tangent point by following 2 steps
- Step 2: drive the slope of the tangency line at D $(\mu = E(R_D), \sigma = \sigma(R_D))$
  - calculate $\frac{d\mu}{d\sigma}$ using the $\mu - \sigma$ relationship in eq.(39), and evaluate $\frac{d\mu}{d\sigma}$ at D

$$\frac{dE(R_D)}{d\sigma(R_D)} = \frac{\sigma(R_D)\Delta}{AE(R_D) - B} \tag{44}$$

- Step 3: compute the slope of origin$-D$ line, i.e., $\frac{E(R_D)}{\sigma(R_D)}$

$$\underbrace{\frac{dE(R_D)}{d\sigma(R_D)}}_{\text{Slope of tangency line at } D} = \frac{\sqrt{\frac{C}{B^2}}\Delta}{\frac{AC-B^2}{B}} = \sqrt{C} = \underbrace{\frac{E(R_D)}{\sigma(R_D)}}_{\text{Slope of } OD} \tag{45}$$

- Therefore, portfolio D is the tangent point of the line from the origin and MVEF, we show this graphically later.

D is here

The Mutual Fund Theorem

- Recall that we have any optimal portfolio ω^* can be written as a linear combination of G and D portfolio

$$\omega^* = \underbrace{\gamma_1 A}_{\alpha} \omega_G + \underbrace{\gamma_2 B}_{\beta} \omega_D = \alpha\omega_G + \beta\omega_D \tag{46}$$

where $\gamma_1 A + \gamma_2 B = \alpha + \beta = 1$, which means

$$\omega^* = \alpha\omega_G + (1 - \alpha)\,\omega_D \tag{47}$$

- In fact, we can span the MVEF with any two efficient portfolios, not just these two, i.e., G and D.
- **Theorem**: We can use any two portfolios on the frontier to span all others.
- **Proof**: From equation (40), if ω_1 and ω_2 are the portfolio weights corresponding to two portfolios on the frontier, we have

$$\begin{aligned} \omega_1 &= a\omega_G + (1 - a)\omega_D \\ \omega_2 &= b\omega_G + (1 - b)\omega_D \end{aligned} \tag{48}$$

The Mutual Fund Theorem

- We can solve this system to obtain

$$\begin{aligned} \omega_G &= g\omega_1 + (1-g)\omega_2 \\ \omega_D &= d\omega_1 + (1-d)\omega_2 \end{aligned} \tag{49}$$

- Consequently, any other portfolio ω_3 which is also on the frontier, can be written as

$$\omega_3 = c\omega_G + (1-c)\omega_D = \widetilde{c}\omega_1 + (1-\widetilde{c})\omega_2 \tag{50}$$

- where $\widetilde{c}$ can be computed by substituting ω_G and ω_D by ω_1 and ω_2.

The Lagrangian approach and solution

- **Step 1**: set up the Lagrangian function,

$$\min L = \frac{1}{2}\omega'\Sigma\omega + \gamma\left[\mu - R_f - \left(\bar{R} - R_f\mathbf{1}\right)'\omega\right] \tag{51}$$

- **Step 2**: Take first-order condition with respect to ω:

$$\frac{\partial L}{\partial \omega} = \Sigma\omega - \gamma\left(\bar{R} - R_f\mathbf{1}\right) = 0 \tag{52}$$

so we have optimal portfolio weight ω^*

$$\omega^* = \gamma\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right) \tag{53}$$

- **Step 3**: Plugging ω^* into the constraint (**??**) to solve γ, then we have the final solution for ω^*

$$\gamma = \frac{\mu - R_f}{\left(\bar{R} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)} = \frac{\mu - R_f}{E} \tag{54}$$

where we define $E = \left(\bar{R} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)$ is a scalar.

Property of the optimal solution

- Again, given the optimal portfolio ω^* for each target μ, our final goal is to find all the optimal mean-variance combinations.

- The variance of this portfolio is

$$\sigma_p^2 = \omega^{*\prime}\Sigma\omega^* = \gamma^2 \left(\bar{R} - R_f \mathbf{1}\right)' \Sigma^{-1}\Sigma\Sigma^{-1} \left(\bar{R} - R_f \mathbf{1}\right) = \gamma^2 E \qquad (55)$$

  Thus,

$$\sigma_p^2 = \frac{(\mu - R_f)^2}{E} \Longrightarrow \sigma_p = \frac{|\mu - R_f|}{\sqrt{E}} \qquad (56)$$

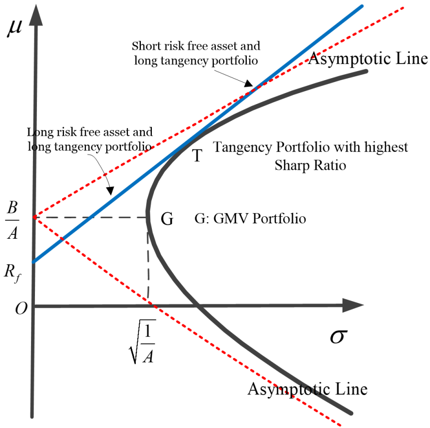
  and

$$|\mu - R_f| = \sqrt{E}\sigma_p, \qquad (57)$$

- So now the mean-variance frontier is a straight line in (μ, σ).
  - For $\mu = R_f$ we have $Std(R^*) = 0$ so the line goes through $(0, R_f)$
  - The Sharpe ratio of the tangency portfolio is $\sqrt{E}$.

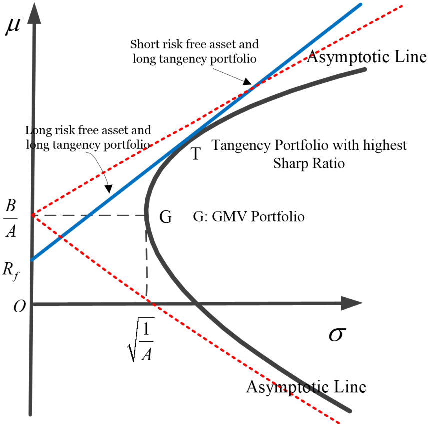
Mean-Variance Frontiers with and without a risk-free asset

- We again can depict this relationship in a graph.



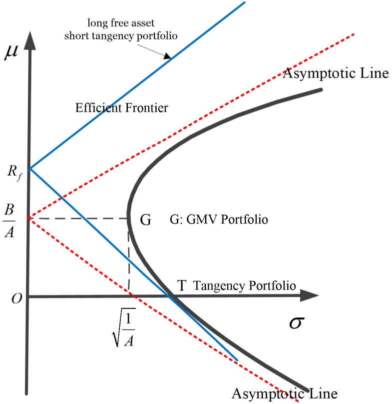
- The tangent portfolio must have the highest Sharpe Ratio

Mean Variance Efficient Frontier Realistic Case: $R_f < \dfrac{B}{A}$



- Tangency portfolio is efficient.
- Aggregate demand for tangency portfolio is positive.
- Aggregate demand for risk-free asset can be positive, negative, or zero.

Mean Variance Efficient Frontier Unrealistic Case: $R_f > \dfrac{B}{A}$



- Tangency portfolio is inefficient.
- Aggregate demand for tangency portfolio is negative ($\neq$ positive aggregate supply)
- Aggregate demand for risk-free asset is positive.

The tangency portfolio

- The tangency portfolio is
  - Located on the efficient MV frontier with a risk-free asset.
  - Located on the efficient MV frontier without a risk-free asset.
  - Thus, the tangency portfolio only allocates (i.e, $\omega_0 = 0$) wealth to risky assets.
- Obtain tangency portfolio by solving for the scalar γ which ensures that $\mathbf{1}'\omega^T = 1$ (all wealth allocated to risky assets) using ω^* in Eq.(50)

$$\begin{aligned} \mathbf{1}'\omega^T &= \gamma \mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right) = 1 \\ \gamma &= \frac{1}{\mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)} \\ \omega^T &= \gamma\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right) = \frac{\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}{\mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)} \end{aligned} \tag{58}$$

- By setting $\mathbf{1}'\omega^T = 1$, we impose $\omega_0 = 0$ in $\omega_0 + \mathbf{1}'\omega = 1$
- Note that here ω^T is the tangency portfolio, not transpose of ω

Optimal Portfolio and the Tangency Portfolio

- To see the relationship between any optimal portfolio ω^* and the tangency portfolio ω^T, we plug γ (eq (54)) into equation (50) to obtain

$$\omega^* = \frac{(\mu - R_f)\,\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}{\left(\bar{R} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)} \tag{59}$$

- and we can decompose this portfolio in two terms:

$$\omega^* = \underbrace{\frac{(\mu - R_f)\,\mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}{\left(\bar{R} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}}_{c}\underbrace{\frac{\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}{\mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}}_{\omega^T} = c\omega^T \tag{60}$$

$$c = \frac{(\mu - R_f)\,\mathbf{1}'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)}{\left(\bar{R} - R_f\mathbf{1}\right)'\Sigma^{-1}\left(\bar{R} - R_f\mathbf{1}\right)} \tag{61}$$

- which is a scalar that depends on the target mean μ, and ω^T is the weight of the tangency portfolio.

Matrix Differentiation

- If $\boldsymbol{\beta}$ and $\mathbf{a}$ are both $k \times 1$ vectors then, $\dfrac{\partial \boldsymbol{\beta}'\mathbf{a}}{\partial \boldsymbol{\beta}} = \dfrac{\partial \mathbf{a}'\boldsymbol{\beta}}{\partial \boldsymbol{\beta}} = \mathbf{a}$.

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \end{pmatrix}'; \mathbf{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \end{pmatrix}'$$

$$\boldsymbol{\beta}'\mathbf{a} = \beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_k a_k = \mathbf{a}'\boldsymbol{\beta}$$

- **Proof**:

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\beta}}\left(\boldsymbol{\beta}'\mathbf{a}\right) &= \frac{\partial}{\partial \boldsymbol{\beta}}\left(\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_k a_k\right) \\
&= \begin{bmatrix} \frac{\partial}{\partial \beta_1}\left(\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_k a_k\right) \\ \frac{\partial}{\partial \beta_2}\left(\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_k a_k\right) \\ \vdots \\ \frac{\partial}{\partial \beta_k}\left(\beta_1 a_1 + \beta_2 a_2 + \ldots + \beta_k a_k\right) \end{bmatrix} \\
&= \mathbf{a}
\end{aligned}$$

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Matrix Differentiation

- Let $\boldsymbol{\beta}$ be a $k \times 1$ vector and $\mathbf{A}$ be a $k \times k$ symmetric matrix then

$$\frac{\partial \boldsymbol{\beta}' \mathbf{A} \boldsymbol{\beta}}{\partial \boldsymbol{\beta}} = 2\mathbf{A}\boldsymbol{\beta}.$$

- **Proof**: By means of proof, say $\boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, then

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{\beta}} \left(\boldsymbol{\beta}' \mathbf{A} \boldsymbol{\beta} \right) &= \frac{\partial}{\partial \boldsymbol{\beta}} \left(\beta_1^2 a_{11} + 2a_{12}\beta_1\beta_2 + \beta_2^2 a_{22} \right) \\
&= \begin{bmatrix} \frac{\partial}{\partial \beta_1} \left(\beta_1^2 a_{11} + 2a_{12}\beta_1\beta_2 + \beta_2^2 a_{22} \right) \\ \frac{\partial}{\partial \beta_2} \left(\beta_1^2 a_{11} + 2a_{12}\beta_1\beta_2 + \beta_2^2 a_{22} \right) \end{bmatrix} \\
&= \begin{bmatrix} 2\beta_1 a_{11} + 2a_{12}\beta_2 \\ 2\beta_1 a_{12} + 2a_{22}\beta_2 \end{bmatrix} \\
&= 2\mathbf{A}\boldsymbol{\beta}
\end{aligned}$$

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