Topic 6. Factor Models and the Arbitrage Pricing Theory (APT)

ECON30024 Economics of Finanical Markets

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Outline

1. Introduction
2. The arbitrage principle
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5. Multifactor models and the APT
6. A comparison with the CAPM

Required reading: Chap. 8 of Bailey

Further reading: Chap. 7 of Bailey

1. Introduction

- The **arbitrage pricing theory (APT)** is a more general theory of expected returns or prices of assets.
  - The APT combines **factor models** with **the arbitrage principle** to yield predictions on expected returns or prices.
  - Factor models postulate that asset returns are linear functions of a small number of 'factors'.
- The CAPM prediction is identical to the APT prediction in a single-factor model in which the excess return on the market portfolio serves as the factor.
- We first formally define the arbitrage principle and describe the factor models, then combine them to yield the APT predictions.

2. The Arbitrage Principle

- In Topic 1 we state that the arbitrage principle refers to the absence of arbitrage opportunities.

- A formal definition of an **arbitrage portfolio**: An arbitrage portfolio $(y_1, y_2, \ldots, y_n)$ of n assets, where $y_j \equiv p_j x_j$ denote the outlay (i.e. expenditure) on asset j, satisfies

  1) **zero initial outlay**:

  $$y_1 + y_2 + \ldots + y_n = 0, \text{ with } y_j \neq 0 \text{ for at least two assets} \quad (1)$$

  2) **risk-free**:

  $$r_1 y_1 + r_2 y_2 + \ldots + r_n y_n \geq 0, \text{ in every state}$$

- The arbitrary principle asserts that in equilibrium all **arbitrage portfolios** yield a zero payoff in every possible state of the world.

– That is, for any arbitrage portfolio $(y_1, y_2, \ldots, y_n)$, we have

$$r_1 y_1 + r_2 y_2 + \ldots + r_n y_n = 0 \text{ in every state} \tag{2}$$

– Does the arbitrage principle imply that every portfolio with a zero initial outlay has a zero payoff in all states?

- Eq. (2) implies that there must exist some relationship among the returns on assets $1, 2, \ldots, n$ in order to rule out the opportunity for arbitrage profits.
- However, by itself the arbitrage principle provides few testable predictions. It can be made empirically relevant when applied to a model of asset prices or returns.
- One such application is the arbitrage pricing theory (APT).

3. Factor Models

3.1 The CAPM and a single-factor model

- Recall that the CAPM prediction is given by

$$\mu_j = r_0 + \beta_j(\mu_M - r_0), \quad j = 1, 2, \ldots, n$$

- This implies that r_j can be written as

$$r_j = r_0 + \beta_j(r_M - r_0) + \varepsilon_j, \quad j = 1, 2, \ldots, n, \tag{3}$$

  where ε_j denotes an unobserved random shock to asset j's return, uncorrelated with the market return ($E(\varepsilon_j | r_M) = 0$).

- Eq. (3) is a single-factor model which specifies the rate of return on any asset as a linear function of a single factor, $r_M - r_0$.

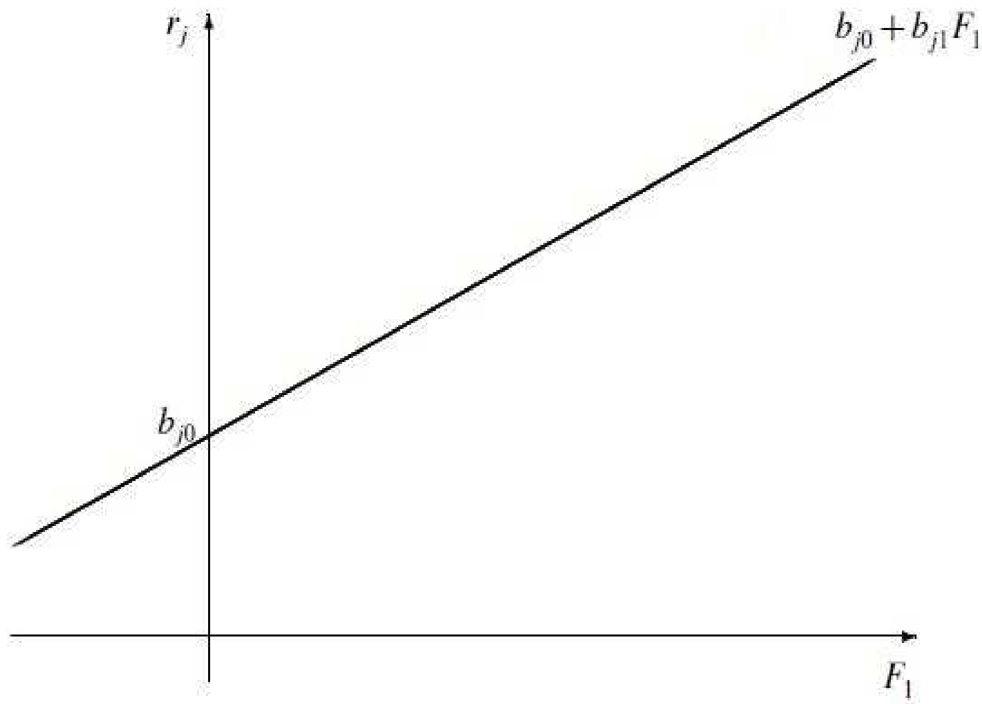
- More generally, a **single-factor model** postulates that returns on assets can be expressed as linear functions of a single factor:

$$r_j = b_{j0} + b_{j1}F_1 + \varepsilon_j, \quad j = 1, 2, \ldots, n. \tag{4}$$

  - F_1 denotes the single factor.
  - The parameter b_{j1} is referred to as the **'factor loading'** of asset j, measuring the sensitivity of r_j to variations in F_1.
  - The unobserved random shock ε_j has zero mean, and zero correlation with F_1.
  - ε_j is often referred to as the '**idiosyncratic** component' of asset j's return, capturing the asset-specific sources of risk that are not accounted for by the common factor F_1.

- The single-factor model is illustrated in Figure 1 below.

Figure 1. A single-factor model



  - Depending on realisations of ε_j, the observed values of F_1 and r_j would result in a scatter of points around the line.

3.2 Multifactor models

- For most applications the single-factor model is too restrictive; several factors are allowed to affect the rates of return on assets.
- The generalization to the **multifactor model** takes the form

$$r_j = b_{j0} + b_{j1}F_1 + b_{j2}F_2 + \ldots + b_{jK}F_K + \varepsilon_j, \quad j = 1, 2, \ldots, n, \quad (5)$$

  where K is the number of distinct factors, which is small relative to n.

- Unlike the CAPM, factor models are **reduced form** models to explain asset prices or returns, not underpinned by a theory of investor behaviour.

- The selection of factors is often ad hoc. The criterion is to choose variables that are considered most likely to influence asset prices.
- Three categories of factors
  - **Macroeconomic factors**: GDP growth rate, unemployment rate, inflation rate, interest rate, etc.
  - **Fundamental factors**: observable asset specific fundamentals, such as industrial classification, market capitalization, book value, and earnings
  - **Statistical factors**: rates of return on portfolios of assets, such as the market return.
- One widely discussed multifactor model is the Fama and French three-factor model: size of firms, book-to-market values, and excess market return (Topic 7).

4. The APT in a Single-factor Model

- The APT is most straightforward to comprehend in a single-factor model.
- We now apply the arbitrage principle to a single-factor model to derive the prediction of the APT.
- First, assume that asset returns follow a single-factor model, as formulated in Eq. (4):

$$r_j = b_{j0} + b_{j1} F_1 + \varepsilon_j, \quad j = 1, 2, \ldots, n \tag{4}$$

- Consider an arbitrage portfolio $(y_1, y_2, \ldots, y_n)$. By definition, the portfolio requires zero initial outlay:

$$y_1 + y_2 + \ldots + y_n = 0. \tag{1}$$

- Recall that the arbitrage principle implies that in equilibrium the arbitrage portfolio $(y_1, y_2, \ldots, y_n)$ yields a zero payoff in every state:

$$r_1 y_1 + r_2 y_2 + \ldots + r_n y_n = 0. \tag{2}$$

- As r_j is given by (4), we can rewrite (2) as:

$$\begin{aligned}
0 &= r_1 y_1 + r_2 y_2 + \ldots + r_n y_n \\
&= (b_{10} + b_{11} F_1 + \varepsilon_1) y_1 + (b_{20} + b_{21} F_1 + \varepsilon_2) y_2 + \ldots \\
&\quad + (b_{n0} + b_{n1} F_1 + \varepsilon_n) y_n \\
&= (b_{10} y_1 + b_{20} y_2 + \ldots + b_{n0} y_n) + (b_{11} y_1 + b_{21} y_2 + \ldots + b_{n1} y_n) F_1 \\
&\quad + (\varepsilon_1 y_1 + \varepsilon_2 y_2 + \ldots + \varepsilon_n y_n) \qquad (2')
\end{aligned}$$

- For (2') to hold for any values of F_1 and ε_j's, we must have:

$$b_{11}y_1 + b_{21}y_2 + \ldots + b_{n1}y_n = 0 \tag{6}$$

$$\varepsilon_1 y_1 + \varepsilon_2 y_2 + \ldots + \varepsilon_n y_n = 0. \tag{7}$$

  – Eq. (6) eliminates the **systematic risk** of the portfolio – the variations in the payoff of the portfolio that is caused by variations in the common factor F_1.

  – Eq. (7) eliminates the **unsystematic risk** of the portfolio – the variations in the payoff due to idiosyncratic risks of each asset.

  – If n is large, a well diversified portfolio, $(y_1, y_2, \ldots, y_n)$, can make (7) approximately hold.

- With (6) and (7) satisfied, then (2') reduces to

$$b_{10}y_1 + b_{20}y_2 + \ldots + b_{n0}y_n = 0 \tag{8}$$

where the portfolio $(y_1, \ldots, y_n)$ satisfy (1) and (6):

$$y_1 + y_2 + \ldots + y_n = 0 \tag{1}$$

$$b_{11}y_1 + b_{21}y_2 + \ldots + b_{n1}y_n = 0 \tag{6}$$

- As n is large, for (8) to holds, the coefficients b_{j0}'s must satisfy

$$b_{j0} = \lambda_0 + \theta b_{j1}, \quad \text{for all } j = 1, \ldots, n \tag{9}$$

where λ_0 and θ are some constant numbers.

- Substituting (9) into the single-factor model (4):

$$r_j = \lambda_0 + \theta b_{j1} + b_{j1}F_1 + \varepsilon_j = \lambda_0 + (\theta + F_1)b_{j1} + \varepsilon_j.$$

– Taking expectation of the equation above

$$\mu_j = \lambda_0 + [\theta + E(F_1)]b_{j1} \qquad j = 1, 2, \ldots, n$$

– Define $\lambda_1 \equiv \theta + E(F_1)$, then

$$\mu_j = \lambda_0 + \lambda_1 b_{j1} \quad j = 1, 2, \ldots, n \tag{10}$$

- If a risk-free asset 0 is present, applying (10) to asset 0 gives

$$r_0 = \lambda_0 + \lambda_1 b_{01} = \lambda_0 \qquad (b_{01} = 0)$$

Then (10) becomes

$$\mu_j = r_0 + b_{j1}\lambda_1, \quad j = 1, 2, \ldots, n \tag{11}$$

or equivalently,

$$\mu_j - r_0 = b_{j1}\lambda_1 \quad j = 1, 2, \ldots, n. \tag{12}$$

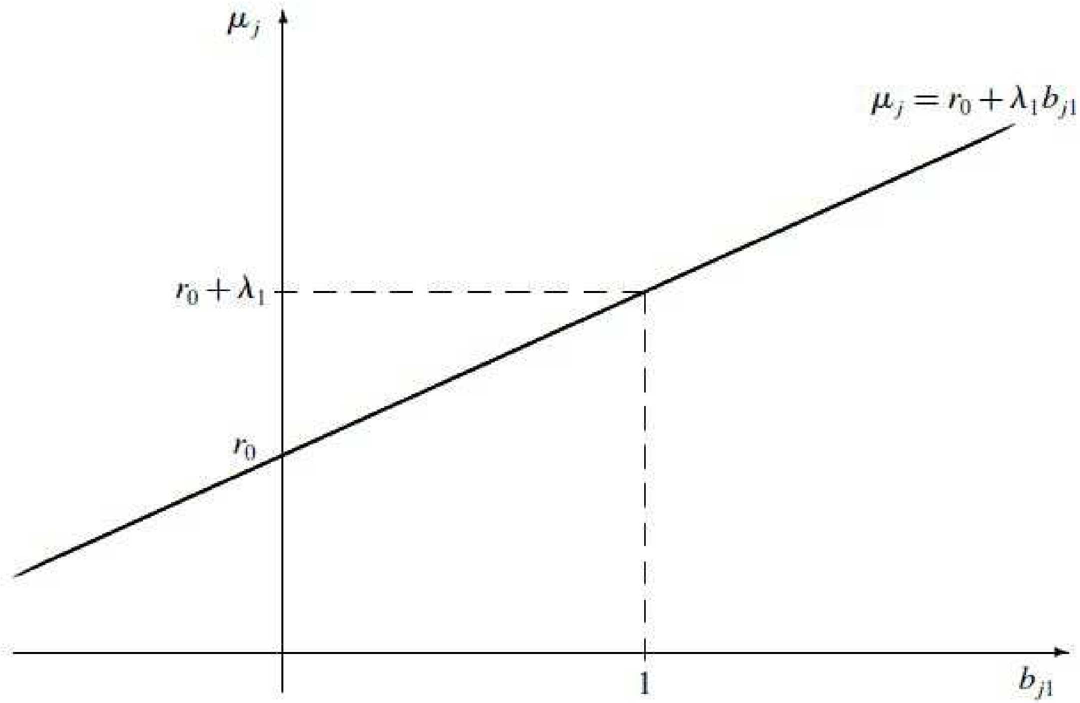
Eq. (11) or (12) is the APT prediction in a single-factor model.

– λ_1 is interpreted as the **risk premium** associated with the common or systematic factor F_1.

– The factor loading b_{j1} reflects the sensitivity of asset j's return to variations in F_1, so b_{j1} measures the systematic risk of asset j.

- In the single-factor model, the APT has a simple graphical representation, as shown in Figure 2.

Figure 2. The APT in a single-factor model



- The APT predicts that for all assets, (b_{j1}, μ_j)'s locate on the straight line. Otherwise, it is possible to construct portfolios that yield arbitrage profits.

- Back to the CAPM:

$$\mu_j - r_0 = \beta_j(\mu_M - r_0) \qquad j = 1, 2, \ldots, n$$

  - Compare it with the APT prediction in a single-factor model:

$$\mu_j - r_0 = b_{j1}\lambda_1 \quad j = 1, 2, \ldots, n. \tag{12}$$

  - $\lambda_1 = \mu_M - r_0$, which is the risk premium on the single factor, $r_M - r_0$.
  - $b_{j1} = \beta_j$, measuring the systematic risk of asset j.
  - Hence, if asset returns are explained by a single-factor model, where the single factor is $r_M - r_0$, the prediction of the APT is identical with that of the CAPM.

5. Multifactor Models and the APT

- The analysis in Section 4 can be extended to a multifactor model to derive the APT predictions.

- In a multifactor model with K factors, the arbitrage principle implies that there exist λ_0, λ_1, ..., λ_K such that

$$\mu_j = \lambda_0 + b_{j1}\lambda_1 + b_{j2}\lambda_2 + \ldots + b_{jK}\lambda_K \qquad j = 1, 2, \ldots, n$$

- In the presence of a risk-free asset, $\lambda_0 = r_0$, so we have

$$\mu_j - r_0 = b_{j1}\lambda_1 + b_{j2}\lambda_2 + \ldots + b_{jK}\lambda_K \qquad j = 1, 2, \ldots, n \quad (13)$$

  – λ_1, λ_2, ..., λ_K can be interpreted as risk premiums associated with the K systematic factors.

  – The factor loading b_{jk} measures the risk of asset j arising from variations in the systematic factor k.

- The APT does not specify what the λ's are exactly. In some cases, they can be explicitly determined.

- An example: the APT when factors are portfolio returns

  – For convenience, suppose there are just two factors, excess returns on portfolio A and B:

  $$F_1 = r_A - r_0,\ F_2 = r_B - r_0.$$

  and returns on assets follow a two-factor model:

  $$r_j = b_{j0} + b_{j1}F_1 + b_{j2}F_2 + \varepsilon_j, \quad j = 1, 2, \ldots, n$$

  – The APT prediction in a two-factor model: there exists λ_1 and λ_2 such that

$$\mu_j = r_0 + b_{j1}\lambda_1 + b_{j2}\lambda_2.$$

– On the other hand, taking expectation of the two-factor model gives

$$\mu_j = b_{j0} + b_{j1}E(F_1) + b_{j2}E(F_2)$$

– Comparing the two equations above, we must have

$$b_{j0} = r_0,\ \lambda_1 = E(F_1) = \mu_A - r_0,\ \lambda_2 = E(F_2) = \mu_B - r_0.$$

– So the APT prediction becomes

$$\mu_j = r_0 + b_{j1}(\mu_A - r_0) + b_{j2}(\mu_B - r_0) \quad j = 1, 2, \ldots, n$$

– This provides a testable prediction of the APT.

6. A Comparison with the CAPM

- In the 1960s, Treynor, Sharpe, Lintner, and Mossin developed the CAPM to determine the theoretically appropriate rate of return on an asset given the level of risk assumed.

- Thereafter, in 1976, Stephen Ross developed the APT as an alternative to the CAPM.

- The CAPM and the APT are two influential asset pricing theories, both providing benchmarks for fair rates of return on assets in efficient asset markets.

- They both predict the risk premium on an asset should be a linear function of the asset's beta(s), which measure the asset's risk arising from the comovement of its return with some systematic factors.

- However, there are important differences between these two theories.
- The APT requires less assumptions. From our earlier discussion, we can see that the APT uses the following underlying assumptions.
  - Asset markets are frictionless
  - Asset returns have means and variances that are finite.
  - Asset returns can be explained linearly by factors that are systematic.
  - Investors can build a portfolio of assets where unsystematic risk can be eliminated through diversification.
  - No arbitrage opportunity exists among well-diversified portfolios.

- The CAPM does try to explain the underlying causes of asset prices or expected returns, whereas the APT does not.
  - The APT does not require individual investors to hold efficient portfolios and market clearing.
- The CAPM is a single factor model, whereas the APT is a mulifactor model.
  - The APT is more flexible and more general than the CAPM; a number of variables that capture systematic risk to asset returns can be included as factors.
  - However, the APT does not say what the factors are or why they are economically or behaviorally relevant; factors included are subjective choices in applications of the APT.

Review questions

1. What is the formal definition of an arbitrage portfolio?
2. What does the arbitrage principle imply about the payoffs of arbitrage portfolios?
3. Is this statement correct: the arbitrage principle asserts that every portfolio with a zero initial outlay has a zero payoff in all states? Why?
4. Why does the CAPM imply a single-factor model on asset returns? What is the factor?
5. Write down the equation for the single-factor model, and interpret each term. In particular, what does b_{j1} capture?
6. Graphically illustrate the single-factor model.
7. Write down the equation for the multifactor model and interpret each term. What does b_{jk} capture?
8. Is there a theory of investor behaviour underlying the choices of factors in factor models?

9. What kind of variables are often used as factors in empirical applications?
10. Roughly understand the derivation of the APT prediction in the context of a single-factor model.
11. What is the APT prediction in a single-factor model? Interpret each item.
12. Graphically illustrate the APT prediction in a single-factor model.
13. How does the CAPM prediction compares with the APT prediction in a single-factor model?
14. Write down the APT prediction in a multifactor model, and interpret each term.
15. If portfolio returns are used as factors? What are the risk premiums associated with such factors?
16. What are the underlying assumptions of the APT?
17. What are the similarities and differences between the APT and CAPM?