FNCE90056: Investment Management

Lecture 8: Term Structure

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Introduction

Today

In the 1970s, the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME) pioneered the trading of interest rate futures. Pictured is the Roman goddess of grain, Ceres, atop the CBOT building.

- In this lecture, we will see that many transactions can be constructed using only the information in today's term structure.
- Forward rates
- Forward discount factor
- Expectations Hypothesis
- Modern theories of term structure
- The notation can get messy!

Forward Rates

Zero rates vs Forward rates vs Actual rate

- The T-year **Zero rate** is the yield on a T-year zero-coupon bond.
  - Zero rate from today to year T: $z_{0,T}$
- **Forward rates** are the interest rates for future transactions implied by today's term structure.
  - Forward rate from year t to year T: $f_{t,T}$
- **Actual rate** is the spot rate at year t from year t to T:
  - It t is not now, then it is a future spot rate.
  - Actual (future) rate from year t to year T: $r_{t,T}$
- By definition, $z_{0,T} = r_{0,T}$

Forward rates

- **Forward contracts** are the simplest form of derivative contract.
  - In a typical "spot" transaction, both the terms of the transaction are set, and the transaction is executed, today. E.g. I lend you \$50 today, at 5% interest payable in 1 year.
  - In a forward contract, the terms are set today, but the actual transaction occurs on some pre-specified future date. E.g. I will lend you \$50 in 2 years' time, at 5% interest payable 1 year later (in 3 years).

- **Forward rate agreements (FRAs)** are forward contracts written on interest rates. They allow borrowers and lenders to lock in future interest rates.

From today's term structure of spot interest rates, we can generate a family of forward rates, and vice versa.

A motivating example

It's currently year 0, and a firm wishes to borrow $100 from year 1 until year 2. They can structure this transaction in (at least) 2 ways:

1. Wait until year 1, and then borrow at the future spot rate, $r_{1,2}$.

   This involves interest rate risk: the future spot rate (at year 1) is uncertain from the perspective of someone today.

2. Lock in the forward borrowing rate, $f_{1,2}$, today for the future transaction.

   The forward rate is known in year 0, so this approach bears no interest rate risk.

No-arbitrage determination of forward rates

We will invoke no-arbitrage to show how spot rates are linked to forward rates. To do so, let's consider a different example than the one on the previous slide.

Example: A firm wishes to borrow \$100 today (at year 0) and repay at year 2. Again, we consider 2 alternatives:

1. **the simple way**:
   - at year 0: borrow \$100 at today's 2-year spot rate, $r_{0,2}$
   - at year 2: repay $100(1 + r_{0,2})^2$.
2. **the complicated way**:
   - at year 0: borrow \$100 at today's spot rate for 1 year, $r_{0,1}$
   - at year 1: repay $100(1 + r_{0,1})$
   - at year 0: enter into a FRA to borrow $100(1 + r_{0,1})$ at year 1 at the forward rate, $f_{1,2}$
   - at year 2: repay the amount owed on the second loan, which is $100(1 + r_{0,1}) \times (1 + f_{1,2})$.

No-arbitrage determination of forward rates

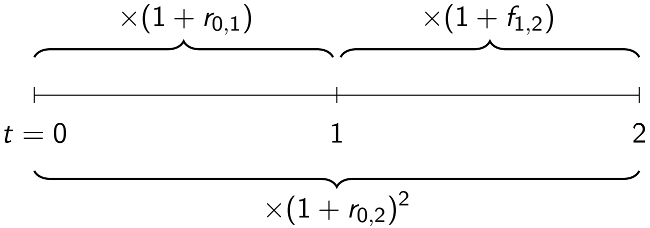
- In both cases, there is no uncertainty in the final amount owed.
- By no-arbitrage, it must be the case that the firm is indifferent between the two alternatives:

$$100(1 + r_{0,2})^2 = 100(1 + r_{0,1})(1 + f_{1,2}) \tag{1}$$

So the forward rate from year 1 to year 2 is given by:

$$f_{1,2} = \frac{(1 + r_{0,2})^2}{1 + r_{0,1}} - 1 \tag{2}$$

No-arbitrage determination of forward rates



Forward discount factor

Discount factor vs. Forward discount factor

Discount factor (Lecture 6):

$$\mathrm{DF}_{0,T} = \frac{1}{(1 + z_{0,T})^{T}} \tag{3}$$

Forward discount factor:

$$\mathrm{FDF}_{t,T} = \frac{1}{(1 + f_{t,T})^{T-t}} \tag{4}$$

Forward discount factor

- Let's analyse the 2 alternatives in the discount factor framework. (This avoids ambiguities arising from various compounding conventions. Once we establish a discount factor, we can immediately convert to the appropriate rate that corresponds to our desired compounding convention.)
- We know today's term structure. This means we know all of the corresponding discount factors, $\text{DF}_{0,T}$.

  (We have previously written this as $\text{DF}\,(0, T)$, or simply DF_T, but the current notation is meant to be a little more explicit.)
- Given the existence of forward rates, we see that there must be corresponding **forward discount factors**, which we will denote $\text{FDF}_{t,T}$.

Forward discount factor

By no-arbitrage, we can now state the following:

$$\mathrm{DF}_{0,T} = \mathrm{DF}_{0,t} \times \mathrm{FDF}_{t,T} \tag{5}$$

Or, equivalently,

$$\mathrm{FDF}_{t,T} = \frac{\mathrm{DF}_{0,T}}{\mathrm{DF}_{0,t}} \tag{6}$$

Properties of the forward discount factor

1. When $t = T$ then $\mathrm{FDF}_{t,T} = 1$.
2. Since (by no-arbitrage) we know that if $t \leq T$ then $\mathrm{DF}_{0,t} \geq \mathrm{DF}_{0,T}$, (6) implies that

$$\mathrm{FDF}_{t,T} \leq 1 \tag{7}$$

   This should look familiar: recall our no-arbitrage condition regarding discount factors from Lecture 6.
3. (7) and (4) imply that forward <u>rates</u> can't be negative:

$$f_{t,T} \geq 0 \tag{8}$$

Recovering forward rates (discrete compounding)

We compute forward rates from the forward discount factor in exactly the same way that we previously recovered zero rates from the contemporaneous discount factor.

- Under discrete compounding (n times per year), we know that

$$\mathrm{FDF}_{t,T} = \frac{1}{\left(1 + \frac{f_{t,T}}{n}\right)^{n(T-t)}} \tag{9}$$

- Solving for the forward rate gives

$$f_{t,T} = n\left(\left(\mathrm{FDF}_{t,T}\right)^{-\frac{1}{n(T-t)}} - 1\right) \tag{10}$$

If $n = 1$: $(\mathrm{FDF}_{t,T})^{-1} = \dfrac{\mathrm{DF}_{0,t}}{\mathrm{DF}_{0,T}} = \dfrac{\frac{1}{(1+z_{0,t})^{t}}}{\frac{1}{(1+z_{0,T})^{T}}} = \dfrac{(1+z_{0,T})^{T}}{(1+z_{0,t})^{t}}$

$$f_{t,T} = \left(\frac{(1+z_{0,T})^{T}}{(1+z_{0,t})^{t}}\right)^{\frac{1}{(T-t)}} - 1 \tag{11}$$

Recovering forward rates (continuous compounding)

Under continuous compounding, we know that

$$\text{FDF}_{t,T} = e^{-f_{t,T} \times (T-t)} \tag{12}$$

Solving for the forward rate gives

$$f_{t,T} = -\frac{\ln\left(\text{FDF}_{t,T}\right)}{T-t} \tag{13}$$

We get

$$f_{t,T} = -\frac{\ln\left(\frac{\text{DF}_{0,T}}{\text{DF}_{0,t}}\right)}{T-t} = -\frac{\ln\left(\text{DF}_{0,T}\right) - \ln\left(\text{DF}_{0,t}\right)}{T-t} = \frac{T \cdot z_{0,T} - t \cdot z_{0,t}}{T-t} \tag{14}$$

Linking Zero and Forward Rates

Example

Spot rates and forward rates reflect exactly the same information, and this slide shows how to move back and forth between the two.

Maturity	Zero	Forward
1	6.123	**6.123**
2	5.785	?
3	?	5.30
4	5.561	5.376
5	5.555	5.531
6	5.579	5.7
7	5.618	5.85
8	5.664	?
9	5.71	6.078
10	5.75371855	6.148

Let's fill in the question marks, assuming annual compounding...

Example

$$
\begin{aligned}
f_{1,2} &= \frac{(1+z_{0,2})^2}{1+z_{0,1}} - 1 = \frac{(1.05785)^2}{1.06123} - 1 \\
&= 0.0545
\end{aligned}
$$

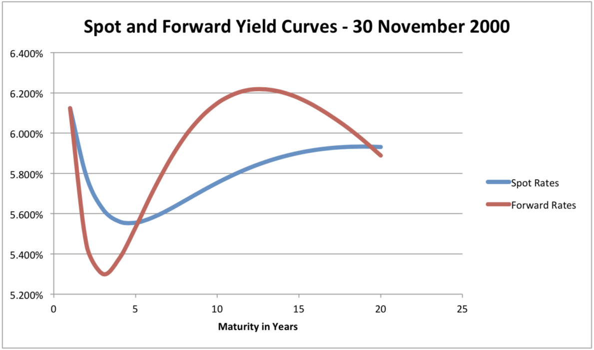
$$
\begin{aligned}
z_{0,3} &= ((1+z_{0,2})^2(1+f_{2,3}))^{1/3} - 1 = (1.05785^2 \times 1.0530)^{1/3} - 1 \\
&= 0.0562
\end{aligned}
$$

$$
\begin{aligned}
f_{7,8} &= \frac{(1+z_{0,8})^8}{(1+z_{0,7})^7} - 1 = \frac{1.05664^8}{1.05618^7} - 1 \\
&= 0.0598
\end{aligned}
$$

Graphical perspective

Plotting forward rates as a function of maturity gives the **forward curve**.

Maturity (Yrs.)	Spot Rates	Forward Rates
1	6.123%	6.123%
2	5.785%	5.447%
3	5.623%	5.300%
4	5.561%	5.376%
5	5.555%	5.531%
6	5.579%	5.700%
7	5.618%	5.852%
8	5.664%	5.981%
9	5.710%	6.078%
10	5.753%	6.148%
11	5.793%	6.193%
12	5.828%	6.216%
13	5.858%	6.217%
14	5.883%	6.203%
15	5.902%	6.174%
16	5.917%	6.133%
17	5.926%	6.082%
18	5.932%	6.025%
19	5.933%	5.960%
20	5.931%	5.889%



Observe that the position of the forward curve with respect to the spot curve is given by the slope of the spot curve.

Expectations Hypothesis

Expectations Hypothesis

- Originally formulated by Fisher (1896), Hicks (1939), and Lutz (1940).
- Relates current interest rates to expected future interest rates.

  Do today's forward rates PREDICT future spot rates?
- Intuition:
  - When lenders and borrowers decide between long-term or short-term bonds, they will compare the price of a long-term bond to the expected price on a roll-over strategy in short-term bonds.
  - This implies long-term rates and expected future short-term rates should be linked.
- How this linkage is modelled leads to a variety of different forms of the Expectations Hypothesis.

Expectations Hypothesis

- Assume that investors are risk neutral.
- Risk neutral investors care only about expected returns, ignoring risk.
- Consider 2 strategies to invest over 3 years:
  1. Strategy 1: Buy a 3-year zero today.
  2. Strategy 2: Buy a 2-year zero today and at the end of year 2, collect the par value, and then buy a 1-year zero.
- Strategy 1 is riskless and Strategy 2 is risky.

Expectations Hypothesis

Consider 2 strategies to invest over 3 years:

1. Strategy 1: Buy a 3-year zero today

$$S_1 = 100(1 + z_{0,3})^3 \tag{15}$$

2. Strategy 2: Buy a 2-year zero today and at the end of year 2, collect the par value, and then buy a 1-year zero.

$$S_2 = 100(1 + z_{0,2})^2(1 + r_{2,3}) \tag{16}$$

Today, $r_{2,3}$ is unknown and investors form expectation $\mathrm{E}[r_{2,3}]$ today:

$$E[S_2] = 100(1 + z_{0,2})^2(1 + E[r_{2,3}]) \tag{17}$$

Expectations Hypothesis

Under the Expectations Hypothesis,

$$S_1 = E[S_2] \tag{18}$$

$$(1 + z_{0,3})^3 = (1 + z_{0,2})^2(1 + E[r_{2,3}]) \tag{19}$$

By no-arbitrage (forward rates):

$$(1 + z_{0,3})^3 = (1 + z_{0,2})^2(1 + f_{2,3}) \tag{20}$$

Therefore $E[r_{2,3}] = f_{2,3}$.

More generally, under the Expectations Hypothesis, $f_{t,T} = E[r_{t,T}]$ for all $t \leq T$.

Weakness of the Expectations Hypothesis

- The Expectations Hypothesis assumes that investors are risk neutral.
- Risk neutrality is not the standard assumption in finance.
- If all investors were risk neutral then, for example, no homeowner would insure their home against fire.
- The standard assumption in finance is risk aversion.
- That is, investors regard risk as undesirable. Thus, bearing risk must be compensated by a higher expected return.
- When investors are risk-averse:

$$S_1 < E[S_2] \tag{21}$$

$$(1+z_{0,3})^3 < (1+z_{0,2})^2(1+E[r_{2,3}]) \tag{22}$$

$$f_{2,3} < E[r_{2,3}] \tag{23}$$

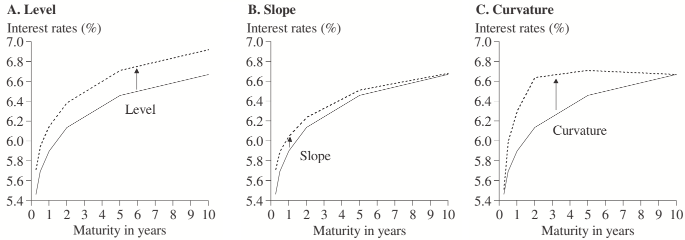
Term premium

- The most general form of the Expectations Hypothesis allows the expected returns on bonds of different maturities to differ by constants which can depend on maturity.
- This difference is known as a *term premium*.
- Early discussions of term premia tended to ignore the possibility that they might vary over time.
  - Hicks (1946) and Lutz (1940): Lenders prefer short maturities while borrowers prefer long maturities, so that long bonds should have higher average returns than short bonds.
  - *Preferred Habitats Hypothesis* (Modigliani and Sutch (1966)): Different lenders and borrowers might have different preferred horizons, or habitats, so that term premia might be negative as well as positive.

Modern theories of term structure

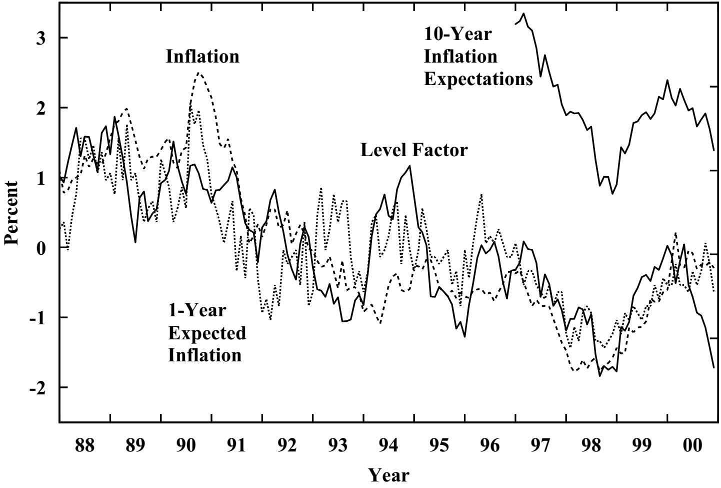
Modern theories

- Modern theories concerning the term structure of interest rates build models with time-varying term premia, or risk premia.
- These models link current and future interest rates such that no-arbitrage holds (Heath, Jarrow, & Morton, 1992). Some models also appeal to additional equilibrium restrictions (Cox, Ingersoll, & Ross, 1985).
- Empirical studies reveal that more than 99% of the movements of various Treasury bond yields are captured by three factors, which are often called "level," "slope," and "curvature" (Litterman & Scheinkman 1991).



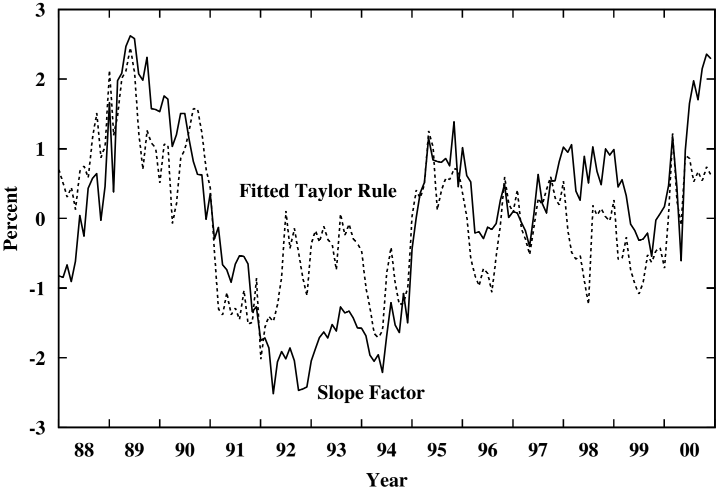
Level factor

The "level" factor seems to be linked to expectations about the central bank's inflation target:



Slope factor

The “slope” factor seems to be linked to the central bank’s policy decisions:



Monetary economics

- Study the role of central banks in the economy
- From the Federal Reserve Act of 1913, the Fed should seek "to promote effectively the goals of maximum employment, stable prices, and moderate long-term interest rates"
- A primary role of monetary economics is to understand optimal monetary policy
- First we should remember what tools the Fed has at its disposal...

Policy tools of the Fed

The Federal Reserve has 3 main tools of monetary policy:

- **Open Market Operations,** which are interventions in the market to buy/sell Treasury securities, to increase/decrease the money supply.
- **Reserve Requirements,** which are the amount of reserves that depository institutions (banks) are required to have at the Fed.
- **The Federal funds rate,** which is the rate at which the Federal Reserve lends to depository institutions approved by the FDIC (Federal Deposit Insurance Corporation).

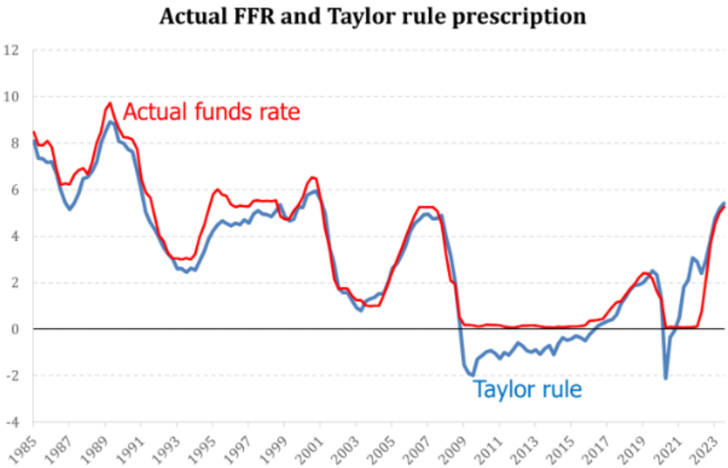
Taylor rules

Without going through the full history of monetary policy, Fed policy since Greenspan can be largely *described* by a monetary policy rule described by the Stanford economist John Taylor, now known as the "Taylor Rule":

$$r_t = \pi_t + \gamma(\pi_t - \bar{\pi}_t) + \lambda(y_t - \bar{y}_t) + \epsilon_t \tag{24}$$

- r_t: target interest rate, π_t: inflation rate, $\bar{\pi}_t$: desired rate of inflation, y_t: output, $\bar{y}_t$: potential output.
- The lefthand side represents the short-rate in the economy (federal funds). The Fed adjusts it based on inflation deviations and output deviations.
- The Fed funds rate increases if either inflation or real GDP are above their targets.
- This directly links monetary policy to the term structure.
- More sophisticated models link monetary policy to the bond market term premiums (Gallmeyer, Hollidield & Zin (2005)).

Taylor rules in action



Source: Prof. Llian Mihov's post at https://www.linkedin.com/pulse/taylor-rule-vs-fed-repost-ilian-mihov-ird8e/