11

Cointegration tests

The objective of cointegration testing is to determine if variables that haves stochastic trends share a common stochastic trend. There are two broad approaches:

Engle-Granger (EG) test

b/c we want WN resids, use "Model 1" of the ADF test.

- Effectively an (A)DF (unit root) test on the residuals as white noise residuals imply cointegration.

$$\boxed{\Delta e = \gamma e + \sum_{i=2}^{p} \beta_i \Delta e_{t-i+1} + \xi_t \qquad H_0: \gamma = 0 \qquad H_a: -2 < \gamma < 2}$$

e^t has a unit root, so ε_t is likely $I(1)$ & y_t, z_t are not $CI(1,1)$

e^t does not have a unit root $\therefore \varepsilon_t$ is likely $I(0)$ & y_t, z_t are $CI(1,1)$

- when testing, test both orderings of y_t & z_t, if at least 1 null hypothesis is rejected we can conclude cointegration.
- If two variables are $CI(1,1)$, the relationship is best captured by a VECM
- Note that EG tests are only useful for bivariate systems, and cannot be used to detect the number of cointegrating relationships - the cointegration rank.

Johansen (J) test

- Tests the number of endogenous variables (n), the number of different stochastic trends (k) they have, and the cointegration rank (r)

→ Suppose $n=3$ variables, which are pure random walks and are $I(1)$ with a stochastic trend.

$$\boxed{y_t = \sum_i \varepsilon_{1,t-i} \quad z_t = \sum_i \varepsilon_{2,t-i} \quad v_t = \sum_i \varepsilon_{3,t-i}}$$

when $n=2$ there are two possibilities:

i. The two stochastic trends are different, $k=2$.
↳ y & z are not $CI(1,1)$ → $r=0=2-2=n-k$

ii. The two stochastic trends are the same (up to a scalar)
↳ y & z are $CI(1,1)$ → $r=1=2-1=n-k$

when $n=3$ there are now three possibilities:

i. The three stochastic trends are different, $k=3$
↳ y, z & v are not $CI(1,1)$ → $r=0=3-3=n-k$

ii. There are two different stochastic trends
↳ y & z (or any combo) are $CI(1,1)$, but none are cointegrated w/ v individually.
↳ However together they are cointegrated, the third element of the cointegration vector is zero.
↳ y, z & v are $CI(1,1)$ → $r=1=3-2=n-k$

iii. There is only one stochastic trend, $k=1$
↳ All three variables have the same stochastic trend which drives the whole system → Two different scalars.
↳ y, z & v are $CI(1,1)$ → $r=2=3-1=n-k$

In general, for $n \geq 2$ of $I(1)$ variables, each can have their own stochastic trend. The cointegration rank must be between 1 and $n-1$. The value of r has useful modelling info:

i. If $r=0$ variables are not CI & ∴ no error correction in the system. The appropriate model is a first difference VAR.

ii. If $r=n$, there are n linearly independent stationary combos of the variables $(k=0)$. The appropriate model is a level VAR.

iii. If $1 \leq r \leq n-1$, there are n independent cointegration relations and $k=n-r$ different stochastic trends. The appropriate model is a VECM.

It can be shown r is equal to the number of non-zero characteristic roots, also called the eigenvalues, of your matrix.

↳ The J test estimates if the number of eigenvalues (r_0) found are statistically significant.

There are two kinds of J-tests:

i. Trace test

H_0: CI rank is $\leq r_0$

H_a: CI rank is $> r_0$

$$\boxed{\lambda_{trace}(r) = -T\sum_{i=r+1}^{n} \ln(1-\hat{\lambda}_i)}$$

ii. Maximum eigenvalue test

H_0: CI rank is r_0

H_a: CI rank is r_0+1

$$\boxed{\begin{aligned}\lambda_{eigen}(r, r+1) &= -T\ln(1-\hat{\lambda}_{r+1}) \\ &= \lambda_{trace}(r) - \lambda_{trace}(r+1)\end{aligned}}$$

Perform these tests sequentially, moving from $r_0=0$ to $r_0=n-1$ until we first fail to reject H_0