ECON30024
Shuyun May Li

Solution to Tutorial 4

1. Quiz 4 questions

(1) Suppose an investor has initial wealth A to invest in $n \geq 2$ assets. The prices of the assets are given by p_1, p_2, ..., p_n, respectively. How to represent an arbitrary portfolio of the n assets that the investor may choose to hold?

(a) $(x_1, x_2, \ldots, x_n)$, where x_j denotes the amount of asset j the investor chooses to hold.

(b) $(p_1x_1, p_2x_2, \ldots, p_nx_n)$, where p_jx_j denotes the investor's expenditure on asset j.

(c) $(a_1, a_2, \ldots, a_n)$, where $a_j \equiv \frac{p_jx_j}{A}$ denotes the proportion of wealth the investor chooses to invest in asset j.

(d) (σ_P, μ_P), where μ_P and σ_P denote the mean and standard deviation of the rate of return on the portfolio that the investor chooses to hold.

(e) All of the above.

<u>Answer</u>: (e)

(2) An investor chooses a portfolio comprising one risky asset with expected rate of return $\mu_Z = 0.15$, and standard deviation of return $\sigma_Z = 0.40$, and lending or borrowing at a risk-free rate, $r_0 = 7\%$. Let q denote the proportion of the portfolio invested in the risky asset, μ_P the expected rate of return on the portfolio, and σ_P the standard deviation of the rate of return on the portfolio. Assume that the investor acts to maximise the mean-variance objective function

$$G(\mu_P, \sigma_P) = \mu_P - 0.5\sigma_P^2.$$

What is the optimal value of q, denoted as q^*, that maximises the investor's objective?

(a) 1

(b) 0

(c) 0.5

(d) 1.5

(e) 0.8

<u>Answer</u>: 0.5

First, express μ_P and σ_P^2 as functions of q:

$$\mu_P = (1-q)r_0 + q\mu_Z = r_0 + q(\mu_Z - r_0) = 0.07 + 0.08\,q$$
$$\sigma_P^2 = q^2\sigma_Z^2 = q^2(0.4)^2 = (0.16)q^2$$

To find the optimal value of q, we solve the following maximisation problem:

$$\max_q \; G(\mu_P, \sigma_P) = \mu_P - 0.5\sigma_P^2 = (0.07 + 0.08\, q) - 0.5(0.16)q^2,$$

The first-order condition is given by

$$0.08 - (0.5)(0.16)2q = 0$$

$$\Rightarrow \quad q^* = \frac{0.08}{0.16} = 0.5$$

(3) Following the question above, which of the following statement is TRUE?

(a) If the investor becomes more risk averse, the new optimal value of q will be higher than q^*.

(b) If r_0 increases to 10%, the new optimal value of q will become negative.

(c) If r_0 falls, the new optimal value of q will be lower than q^*.

(d) If $\sigma_Z = 0.5$, the new optimal value of q will be higher than q^*.

(e) If μ_Z rises, the new optimal value of q could be bigger than 1.

Answer: (e)

As q is the proportion of wealth invested in the risky asset, the optimal value of q decreases with r_0, σ_Z and the degree of risk aversion of the investor. Given that $\mu_Z > r_0$, the optimal value of q is always positive, so (b) is wrong. The optimal value of q can be greater than 1 is μ_Z is sufficiently high.

(4) Asset 1 and Asset 2 have returns as below:

State	r_1	r_2	prob
A	2	6	0.25
B	3	9	0.5
C	4	12	0.25

What is the correlation coefficient of r_1 and r_2? Consider a portfolio that consists of Asset 1 and 2. What is the minimum variance of this portfolio?

(a) The correlation coefficient is 1, and the minimum variance is 1.5.

(b) The correlation coefficient is 1, and the minimum variance is 0.5.

(c) The correlation coefficient is 0, and the minimum variance is 4.5.

(d) The correlation coefficient is 1.5, and the minimum variance is 0.5.

(e) The correlation coefficient is 0.5, and the minimum variance is 1.

Answer: (b)

It's easy to see from the table that the two assets are perfectly positively correlated, i.e., their correlation coefficient is 1. In this case, the minimum risk portfolio is simply the asset that has a smaller variance. Asset 1 has a smaller variance among the two, and its variance is given by

$$\sigma_1^2 = var(r_1) = E(r_1^2) - (E(r_1))^2,$$

where

$$\begin{aligned} E(r_1) &= (0.25)(2) + (0.5)(3) + (0.25)(4) = 3 \\ E(r_1^2) &= (0.25)(4) + (0.5)(9) + (0.25)(16) = 9.5. \end{aligned}$$

So $\sigma_1^2 = 9.5 - (3)^2 = 0.5$, and hence the variance of the minimum risk portfolio is 0.5. That is, the minimum variance of a portfolio of the two assets is 0.5.

(5) Which of the the following statements is TRUE?

(a) A portfolio located on the portfolio frontier with two risky assets is an efficient portfolio of the two risky assets.

(b) By investing in a properly chosen portfolio of two risky assets, we can always achieve a lower risk than holding any of the two risky assets alone.

(c) The optimal portfolio chosen by an investor should have minimum variance among all portfolios that the investor can afford.

(d) The minimum risk portfolio of two risky assets can have zero risk.

(e) None of the rest.

<u>Answer</u>: (d)

(a) is false because the efficient frontier with two risky assets is the upward sloping part of the portfolio frontier with two risky assets.

(b) is false because if the two risk assets are perfectly positively correlated, any portfolio of them cannot achieve risk reduction.

(c) is false because the optimum portfolio maximises the mean-variance objective while the minimum risk portfolio minimises the variance of portfolios, so they may not coincide with each other.

(d) is true because the minimum risk portfolio of two risky assets that are perfectly negatively correlated has zero variance.

2. Given values: $r_0 = 0.07$, $\mu_Z = 0.15$, $\sigma_Z = 0.4$.

(a) To plot the portfolio frontier, first, express μ_P and σ_P as functions of q:

$$\begin{aligned} \mu_P &= (1-q)r_0 + q\mu_Z = r_0 + q(\mu_Z - r_0) = 0.07 + (0.15 - 0.07)q = 0.07 + 0.08\,q \\ \sigma_P^2 &= q^2\sigma_Z^2 \Rightarrow \ \sigma_P = q\sigma_Z = 0.4\,q \end{aligned}$$

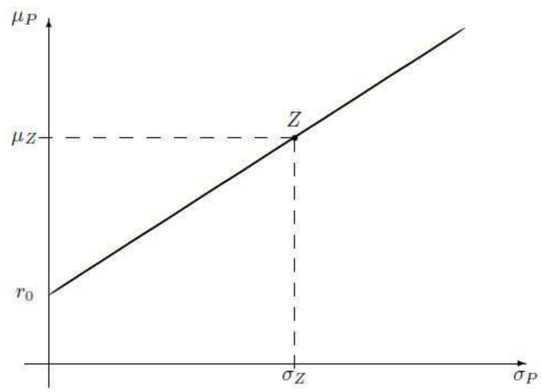
Both μ_P and σ_P are linear functions of q, so μ_P must be a linear function of σ_P, i.e., the portfolio frontier must be a straight line in the (σ_P, μ_P) space.

As the portfolio frontier is a straight line, to plot it, we just need to find two points in the (σ_P, μ_P) space that lie on the portfolio frontier. Let q take on two different values, we can find the two points:

$$q = 0 \;\Rightarrow\; \mu_P = 0.07 = r_0,\ \sigma_P = 0$$
$$q = 1 \;\Rightarrow\; \mu_P = 0.15 = \mu_Z,\ \sigma_P = 0.4 = \sigma_Z$$

The two points are $(0, r_0)$ and (σ_Z, μ_Z), which represent the risk-free asset and the risky asset, respectively. The straight line connecting these two points can be extended to the right, as q can be greater than 1 to allow for borrowing. Hence the portfolio frontier is as shown in Figure 1.

Figure 1. Portfolio frontier with a risky asset and a risk-free asset



Alternatively, we can combine the two expressions for μ_P and σ_P to find the relationship between μ_P and σ_P:

$$\mu_P = 0.07 + 0.08\,q = 0.07 + (0.08)\left(\frac{\sigma_P}{0.4}\right) = 0.07 + 0.2\,\sigma_P$$

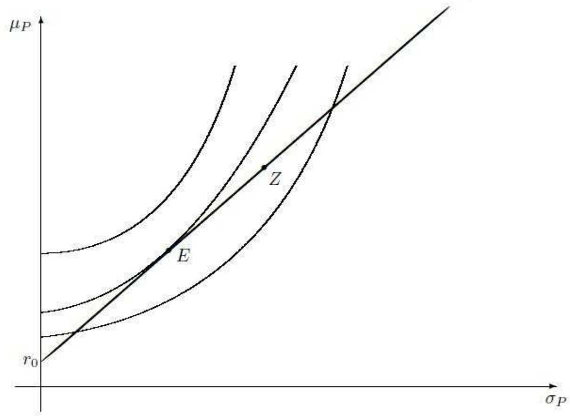
Let σ_P take on two different values:

$$\sigma_P = 0 \;\Rightarrow\; \mu_P = 0.07 = r_0,$$
$$\sigma_P = \sigma_Z = 0.4 \;\Rightarrow\; \mu_P = 0.15 = \mu_Z.$$

Again, we get two points in the (σ_P, μ_P) space. As borrowing is allowed, σ_P can be greater than σ_Z. Then we can also draw the portfolio frontier as shown in the figure above.

(b) Sketch the indifference curves of $G(\mu_P, \sigma_P)$ in Figure 1, we obtain Figure 2. The optimum portfolio is the point at which the portfolio frontier is tangent to an indifferent curve, represented by point E.

Figure 2. The optimum portfolio of a risky asset and a risk-free asset



Recall that from Quiz 4, question (2), the optimal value of q is $q^* = 0.5$. The corresponding optimal portfolio (σ_E, μ_E) is given by

$$\sigma_E = 0.4\,q^* = 0.2$$
$$\mu_E = 0.07 + 0.08\,q^* = 0.11.$$

Alternatively, we can solve the following maximisation problem

$$\max_{\mu_P,\sigma_P} \; G(\mu_P, \sigma_P) = \mu_P - 0.5\sigma_P^2,$$

subject to

$$\mu_P = 0.07 + 0.2\,\sigma_P,$$

which is equivalent to

$$\max_{\sigma_P} \; (0.07 + 0.2\,\sigma_P) - 0.5\sigma_P^2.$$

The first-order condition with respect to σ_P is given by

$$0.2 - (0.5)(2)\sigma_P = 0 \;\Rightarrow\; 0.2 - \sigma_P = 0$$

$$\Rightarrow \sigma_P^* = 0.2, \quad \mu_P^* = 0.07 + 0.2\,\sigma_P^* = 0.11.$$

Note that (σ_P^*, μ_P^*) is exactly the optimal portfolio (σ_E, μ_E) we found earlier.

(c) With $r_0 = 0.11$ now,

$$\mu_P = r_0 + q(\mu_Z - r_0) = 0.11 + 0.04q,$$

and σ_P is still $0.4q$. So the relationship between μ_P and σ_P becomes

$$\mu_P = 0.11 + (0.04)\left(\frac{\sigma_P}{0.4}\right) = 0.11 + 0.1\,\sigma_P,$$

which defines the new portfolio frontier. Note that the slope of the new portfolio frontier is smaller than the previous portfolio frontier.

Re-solve the maximisation problem

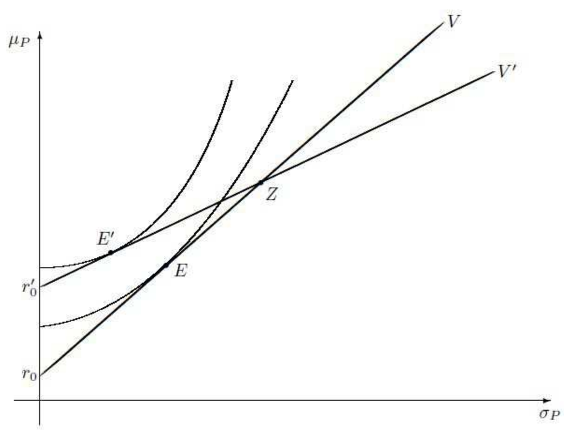
$$\max_{\mu_P,\sigma_P} \ \mu_P - 0.5\sigma_P^2,$$

subject to

$$\mu_P = 0.11 + 0.1\,\sigma_P,$$

and we obtain $\sigma_P^* = 0.1$ and $\mu_P^* = 0.12$. The corresponding $q^* = \sigma_P^*/0.4 = 0.25$. We see that an increase in the risk-free rate leads to a lower proportion of wealth invested in the risky asset.

Sketch the new portfolio frontier and the the new optimal portfolio in Figure 3:

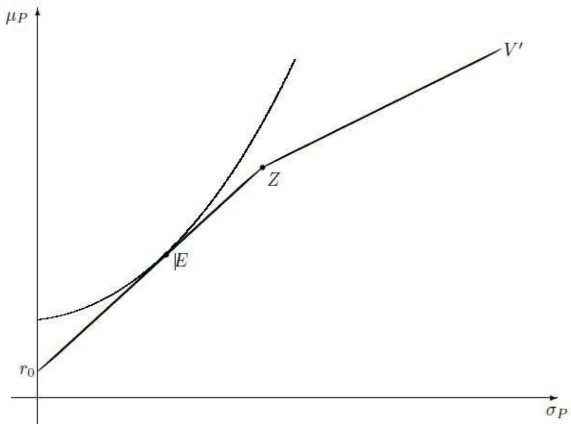


In Figure 3, the lines V and V' represent the original and the new portfolio frontiers, respectively, and the points E and E' represent the original and new optimal portfolios. Note that both portfolio frontiers pass through the risky asset, but the new portfolio frontier with a higher risk-free rate is flatter with a higher intercept. It's clear that an increase in the risk-free rate (with other things unchanged) leads to an optimal portfolio with higher return and lower risk than the original optimal portfolio.

(d) As the risk-free rates for borrowing and lending are different, the portfolio frontier has two segments, as shown in Figure 4: (i) the existing line r_0Z for lending ($q \in [0,1]$ so that the investor holds a positive amount of the risk-free asset), and (ii) the flatter line to the right of Z, for which the investor is a borrower ($q > 1$ so that the investor holds a negative amount of the risk-free asset).

However, this does not change the optimal portfolio of the investor. Given that her objective function is unchanged and her lending rate is still 7%, the optimum portfolio must remain at E, with $q = 0.5$. There is no reason for the investor to become a borrower at 11%, when she chose to lend at 7%.

Figure 4. Portfolio frontier and optimum portfolio with a higher borrowing rate



4. From the description, you have mean-variance objective. First, you need to specify a specific mean-variance objective function, which is what you try to maximise by choosing an optimal portfolio of the two stocks. For example, you can specify it as

$$G(\mu_P, \sigma_P^2) = \mu_P - \alpha\sigma_P^2$$

choose a value for α that you think best describes your degree of risk aversion.

Second, you need to form beliefs about the two stocks' rates of return one month later, denoted as r_1 and r_2. Since you only care about the mean and variance of the rate of return on your investment, denoted as r_P, you only need to form beliefs about the means and variances of r_1 and r_2, as well as the correlation between r_1 and r_2 (recall the expressions of $E(r_P)$ and $var(r_P)$ in equations (2) and (3) in Topic 4).

How to form these beliefs? One way is to collect past monthly data on the prices of the two stocks, and calculate their monthly rates of return. This gives you a sample of observations on the monthly rates of return for the two stocks. Then calculate the sample mean and sample variance of each stock's rate of return and the sample correlation coefficient between the two stocks' rates of return. These sample statistics can be used as your beliefs or estimates for the means and variances of r_1 and r_2 and their correlation ρ_{12}.

Now you are ready to do the analysis. Recall the portfolio selection problem is to choose a portfolio on the portfolio frontier to maximise the mean-variance objective, and the portfolio frontier is defined by equations (2) and (3), or by equation (4) in Topic 4. So you can solve a maximisation problem

$$\max_{a_1} \; G(\mu_P, \sigma_P^2), \quad \text{s.t. (2) and (3)}$$

or

$$\max_{\mu_P} \; G(\mu_P, \sigma_P^2), \quad \text{s.t. (4)}$$

Both problems are univariate maximisation problems, which can be solved as we did in Q4 in Exercise_Topic4.