Lecture 12: The Effects of Social Security in a Life-cycle Model with non-zero population growth

ECON30009/90080 Macroeconomics

Semester 2, 2025

Announcements

- Next week: there are no tutorial assignments posted
- Your tutors will instead use the tutorial sessions as their consultation hours. If your tutorial falls on Thursday 11 Sep, you may attend any of the other tutorials.
- MST on Thursday 11 Sep:
  - ECON30009: in-class. Bring your student ID.
  - ECON90080: Rm 315, FBE Bldg. Bring your student ID.

PAYG social security with population growth

- We showed a Fully-Funded social security policy is budget neutral. This conclusion will still hold with population growth.
- Intuitively because fully-funded social security implies that the government is just saving on the household's behalf
- Now we will instead focus on the effects of a PAYG policy with population growth
- When PAYG was first introduced in many OECD countries, many of those economies were experiencing fast growth in their working age populations.

Adding population growth

- Assume population grows at a constant rate, such that:

$$N_{t+1} = (1 + n)N_t$$

- Before adding social security to the model, let's see what the social planner would choose when population growth is not zero

- Apart from population growing at rate n, we will make the same assumptions as the example we have typically used in class

Too much or too little savings?

- The market economy without government and with population growing at rate n observed the following transition equation (See Tutorial 3 Q2!):

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)zk_t^{\alpha}$$

- which in steady state means that the market economy observes:

$$\bar{k}^M = \left[\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)z\right]^{1/(1-\alpha)}$$

and the associated rate of return on capital $\bar{R}^M = MPk$ in steady state is:

$$\begin{aligned}\bar{R}^M &= \alpha z(\bar{k}^M)^{-(1-\alpha)} \\ &= \frac{\alpha(1+\beta)}{\beta(1-\alpha)}(1+n)\end{aligned}$$

Too much or too little savings?

- We want to know if the market economy without government is saving too much or too little relative to what a social planner would choose in steady state
- Two ways to do this:
  - Compare $\bar{k}^M$ vs. $\bar{k}^{SP}$
  - Compare $\bar{R}^M$ to MPk^{SP} in social planner's solution.

Pareto optimal $\bar{k}$

- Social planner wants to make households happy (maximize lifetime utility)

- Subject to a resource constraint:

$$N_t c_t^y + N_t c_t^o + K_{t+1} = z K_t^\alpha N_t^{1-\alpha}$$

Pareto optimal $\bar{k}$

- Social planner wants to make households happy (maximize lifetime utility)

- Subject to a resource constraint:

$$N_t c_t^y + N_t c_t^o + K_{t+1} = z K_t^\alpha N_t^{1-\alpha}$$

- Dividing by N_t:

$$c_t^y + c_t^o + \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = z k_t^\alpha$$

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- which is same as:

$$c_t^y + c_t^o + k_{t+1}(1+n) = z k_t^\alpha$$

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- which is same as:

$$c_t^y + c_t^o + k_{t+1}(1+n) = z k_t^\alpha$$

- In steady state:

$$\bar{c}^y + \bar{c}^o + \bar{k}(1+n) = z \bar{k}^\alpha$$

Pareto optimal $\bar{k}$

- We want to find the long-run equilibrium the planner would choose
- This means solving the following problem:

$$\mathcal{L} = \max \ln \bar{c}^y + \beta \ln \bar{c}^o + \lambda \left[z\bar{k}^\alpha - \bar{c}^y - \bar{c}^o - \bar{k}(1+n) \right]$$

Pareto optimal $\bar{k}$

$$\mathcal{L} = \max \ln \bar{c}^y + \beta \ln \bar{c}^o + \lambda \left[z\bar{k}^\alpha - \bar{c}^y - \bar{c}^o - \bar{k}(1+n) \right]$$

- Planner's optimality conditions:
  - Optimal LR allocations across generations:

$$\frac{1}{\bar{c}^y} = \frac{\beta}{\bar{c}^o}$$

  - Optimal gross investment:

$$\alpha z {k^{\bar{S}P}}^{\alpha-1} = (1+n)$$

  - Allocations are feasible (resource constraint)

$$z\bar{k}^\alpha = \bar{c}^y + \bar{c}^o + \bar{k}(1+n)$$

Pareto optimal $\bar{k}$

$$\mathcal{L} = \max \ln \bar{c}^y + \beta \ln \bar{c}^o + \lambda \left[z\bar{k}^\alpha - \bar{c}^y - \bar{c}^o - \bar{k}(1+n) \right]$$

- Optimal gross investment:

$$\underbrace{\alpha z \bar{k}^{\alpha-1}}_{MPk^{SP}} = (1+n)$$

- which in turn implies that in an economy with constant population growth $n \neq 0$, pareto-optimal $\bar{k}$ is:

$$\bar{k}^{SP} = \left[\frac{\alpha z}{1+n} \right]^{1-\alpha}$$

Market economy MPk vs. Social planner's MPk

- In the market economy without government, we observed that in steady state

$$\bar{R}^M = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}(1+n)$$

  and we know that the rental rate of capital is equal to MPK in equilibrium in the market economy.

- From the social planner's problem we have:

$$\underbrace{\alpha z \bar{k}^{\alpha-1}}_{MPk^{SP}} = (1+n)$$

Market economy MPk vs. Social planner's MPk

- If $\bar{R}^M < MPk^{SP}$: there is overaccumulation of capital.

- The economy is saving too much and this causes the rate of return on capital in the market economy to be lower than the socially optimal level of MPk.

- In the simple model we wrote down, this occurs when:

$$\bar{R}^M = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}(1+n) < 1+n \quad \text{if} \quad \frac{\alpha(1+\beta)}{\beta(1-\alpha)} < 1$$

Market economy MPk vs. Social planner's MPk

- If $\bar{R}^M > MPk^{SP}$: there is underaccumulation of capital.
- The economy is saving too little and this causes the rate of return on capital in the market economy to be higher than the socially optimal level of MPk.
- In the simple model we wrote down, this occurs when:

$$\bar{R}^M = \frac{\alpha(1+\beta)}{\beta(1-\alpha)}(1+n) > 1+n \quad \text{if} \quad \frac{\alpha(1+\beta)}{\beta(1-\alpha)} > 1$$

PAYG social security under population growth

Decentralized economy with PAYG social security

- At any point t, the ratio of young to old is given by $N_t/N_{t-1} = 1 + n$
- Government levies tax s on each young household
- And gives each old household $(1 + n)s$
- Government's budget is balanced as total tax revenue equals total transfers:

$$N_t s = N_{t-1}(1 + n)s$$

Decentralized economy with PAYG social security

Household constraints

- Budget constraint of young

$$c_t^y + a_{t+1} + s = w_t + \pi_t$$

- Budget constraint of old:

$$c_{t+1}^o = (1 + r_{t+1})a_{t+1} + (1 + n)s$$

- LBC

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t - s + \frac{1 + n}{1 + r_{t+1}}s$$

Decentralized economy with PAYG social security

Household optimality

- Euler:

$$c_{t+1}^{o} = \beta\left(1 + r_{t+1}\right) c_t^{y}$$

- LBC

$$c_t^{y} + \frac{c_{t+1}^{o}}{1 + r_{t+1}} = w_t + \pi_t - s + \frac{1+n}{1 + r_{t+1}} s$$

- Plug Euler into LBC:

$$c_t^{y} = \frac{1}{(1+\beta)}\left[w_t + \pi_t - s + \frac{1+n}{1+r_{t+1}} s\right]$$

Decentralized economy with PAYG social security

Equilibrium

- Capital market clearing:

$$K_{t+1} = N_t a_{t+1}$$

- In eqm:

$$k_{t+1} = \frac{1}{1+n}\left\{\frac{\beta}{1+\beta}(1-\alpha)\,zk_t^{\alpha} - \frac{1}{1+\beta}\left[\beta + \frac{1+n}{1+r_{t+1}}\right]s\right\}$$

- As before, introduction of PAYG social security shifts transition curve down (can show this numerically)

Decentralized economy with PAYG social security

Welfare

- Welfare can actually be higher if population is growing fast enough.

- In particular, if $1 + n > 1 + r_{t+1}$, welfare is higher

- From LBC, if $1 + n > 1 + r_{t+1}$, then lifetime income is higher

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t + \left[\frac{1+n}{1 + r_{t+1}} - 1\right] s$$

- Higher lifetime income means more resources to consume from: $c_t^y, c_{t+1}^o \uparrow$

Decentralized economy with PAYG social security

Welfare

- Why can households be better off with PAYG social security when $1 + n > 1 + r_{t+1}$?

Decentralized economy with PAYG social security

Welfare

- Why can households be better off with PAYG social security when $1 + n > 1 + r_{t+1}$?

- Growth path of k_t is lower with PAYG yet welfare can be higher.

Decentralized economy with PAYG social security

Welfare

- Why can households be better off with PAYG social security when $1 + n > 1 + r_{t+1}$?

- Growth path of k_t is lower with PAYG yet welfare can be higher.

- Let's look at budget constraint of old again:

$$c^o_{t+1} = (1 + r_{t+1})a_{t+1} + (1 + n)s$$

Decentralized economy with PAYG social security

Welfare

- Why can households be better off with PAYG social security when $1 + n > 1 + r_{t+1}$?

- Growth path of k_t is lower with PAYG yet welfare can be higher.

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$$c^o_{t+1} = (1 + r_{t+1})a_{t+1} + (1 + n)s$$

- Note that in the market economy with no govt, individuals' only source of income was private savings.

Decentralized economy with PAYG social security

Welfare

- Why can households be better off with PAYG social security when $1+n > 1+r_{t+1}$?

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- Let's look at budget constraint of old again:

$$c^o_{t+1} = (1+r_{t+1})a_{t+1} + (1+n)s$$

- Note that in the market economy with no govt, individuals' only source of income was private savings.

- With PAYG, old also get income from the transfer, and this s transfer yields a higher return than private savings **if** $1+n > 1+r_{t+1}$

Decentralized economy with PAYG social security

Welfare

- When $1 + n > 1 + r_{t+1}$, individuals when old get a bigger "return" from s than from private savings.

- PAYG in this case helps to resolve a missing "market" problem. Savings decision of $t - 1$ generation affects K_t (and its MP) which t generation have to work with

- How?: Government provides insurance in old age: individuals don't need to save as much

- Saving less and consuming more led market economy closer to social planner's solution

Decentralized economy with PAYG social security

Welfare

- What if $1 + n < 1 + r_{t+1}$?
- Note this is the case where the market economy without social security was already under-accumulating capital in steady state
- What will the introduction of PAYG do in this case? Make households better off or worse off?

Decentralized economy with PAYG social security

Welfare

- What if $1 + n < 1 + r_{t+1}$?
- Note this is the case where the market economy without social security was already under-accumulating capital in steady state
- What will the introduction of PAYG do in this case? Make households better off or worse off?

Ageing populations?

Non-constant n

- A note: not attractive to have $n < 0$ forever in our model.
- Why? This would mean there's no one in the economy at some point (asymptotically the economy approaches zero population).
- We can consider variations in population growth n_t
- We can ask what happens if n_t persistently < 0 but not permanently < 0.
- We can ask how the aggregate outcomes of this economy at date t is affected when $n_t > 0$ or $n_t < 0$

Decentralized economy with PAYG social security

- At any point t, the ratio of young to old is given by $N_t/N_{t-1} = 1 + n_t$
- Government levies tax s on each young household
- And gives each old household $(1 + n_t)s$
- Note if $n_t < 0$, we are implicitly assuming that in that period t, the government gives a smaller transfer to old households to balance the budget

$$N_t s = N_{t-1}(1 + n_t)s$$

If you have a shrinking population: either you have to reduce the benefit to the old OR raise the amount that you tax from the young.

Decentralized economy with PAYG social security

Household constraints

- Budget constraint of young

$$c_t^y + a_{t+1} + s = w_t + \pi_t$$

- Budget constraint of old:

$$c_{t+1}^o = (1 + r_{t+1})a_{t+1} + (1 + n_t)s$$

- LBC

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t - s + \frac{1 + n_t}{1 + r_{t+1}}s$$

- Note $-s + \frac{1+n_t}{1+r_{t+1}}s$ becomes more negative as n_t gets smaller: lifetime income is smaller, *holding all else constant*

Decentralized economy with PAYG social security

Rest of problem is similar (follows same steps).

Equilibrium

- Capital market clearing:

$$K_{t+1} = N_t a_{t+1} \implies \frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} = a_{t+1} \implies k_{t+1}(1 + n_{t+1}) = a_{t+1}$$

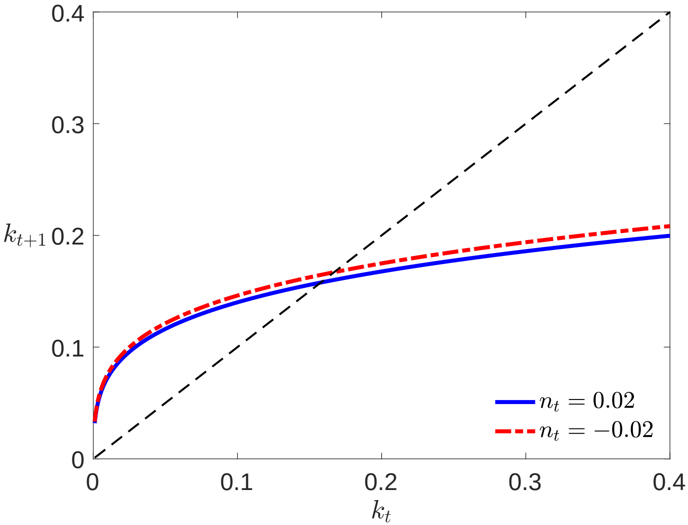
- In eqm:

$$k_{t+1} = \frac{1}{1 + n_{t+1}} \left\{ \frac{\beta}{1 + \beta} (1 - \alpha) z k_t^{\alpha} - \frac{1}{1 + \beta} \left[\beta + \frac{1 + n_{t+1}}{1 + r_{t+1}} \right] s \right\}$$

- As before, we need to solve this numerically. Let's ask given a k_t, what happens to k_{t+1} if $n_t = n_{t+1} > 0$ vs. $n_t = n_{t+1} < 0$

k_{t+1} under shrinking and growing population

Suppose $\beta = 0.95, s = 0.1, \alpha = 0.2, z = 1$



- For given k_t, why is k_{t+1} higher if $n_t = n_{t+1} < 0$ relative to $n_t = n_{t+1} > 0$?

k_{t+1}, c_t^y, c_t^o under shrinking and growing population

Suppose economy currently at $k_t = 0.15$, what are the outcomes if $n_t = n_{t+1} = 0.02$ vs. $n_t = n_{t+1} = -0.02$?

	k_{t+1}	c_t^y	c_t^o
$n_t = n_{t+1} = 0.02$	0.156	0.289	0.239
$n_t = n_{t+1} = -0.02$	0.162	0.288	0.235

Note that changes in k_{t+1} affect $R_{t+1} = (1 + r_{t+1})$ which affects c_t^y

$$c_t^y = \frac{1}{(1+\beta)}\left[w_t + \pi_t - s + \frac{1+n_t}{1+r_{t+1}}s\right]$$

k_{t+1}, c_t^y, c_t^o under shrinking and growing population

Suppose economy currently at $k_t = 0.15$, what are the outcomes if $n_t = n_{t+1} = 0.02$ vs. $n_t = n_{t+1} = -0.02$?

	k_{t+1}	c_t^y	c_t^o
$n_{t+1} = 0.02$	0.156	0.289	0.239
$n_{t+1} = -0.02$	0.162	0.288	0.235

We already saw that if $n_t < 0$, old household gets smaller transfer.

Wrapping up

- We have seen how fiscal policies can be incorporated into the model:
  - looked at spending and tax policies
  - looked at transfers
- After MST: we will start thinking about short-run fluctuations in the economy.