ECON30009/90080 – Tutorial 5 Solution

This Version: Semester 2, 2025

These questions are designed to give you practice solving the OLG model when government spending is exogenous.

Answers in red

Question 1: Government spending financed by a payroll tax

Consider the OLG model we discussed in class. There exists a government that needs to spend an exogenous amount $G_t = G$ each period. This government spending goes towards financing a public good that households get some utility from. Specifically let the preferences of a household be given by

$$U(c_t^y, c_{t+1}^o, G) = \ln c_t^y + \beta \ln c_{t+1}^o + \gamma \ln G$$

where $0 < \beta < 1$ and $\gamma > 0$. The government runs a balanced budget each period and fully finances its government spending within period with a payroll tax on firms, i.e., for each unit of labour employed, the firms pay $\tau_t^w w_t$ as tax revenue.

As per lecture slides, households work when young and retire when old. They inelastically supply 1 unit of labour when they are young, and receive wage and dividend income. Households when young can choose to save in an asset a_{t+1} and the gross return to a unit of savings is equal to $1 + r_{t+1}$. Households when old consume their savings. Firms use capital and labour in production and the per unit cost of these inputs is given by R_t and w_t, respectively. Output is produced via a Cobb-Douglas production function, $Y_t = zK_t^\alpha L_t^{1-\alpha}$ where $0 < \alpha < 1$.

a) Set up the household problem and derive the optimality conditions of the household.

The household's problem is given as:

$$\max_{c_t^y, c_{t+1}^o} \ln c_t^y + \beta \ln c_{t+1}^o + \gamma \ln G$$

s.t.

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t$$

Writing this as a Lagrangian, we have:

$$\mathcal{L} = \ln c_t^y + \beta \ln c_{t+1}^o + \gamma \ln G + \lambda_t \left[w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} \right]$$

Taking FOC, we have:

$$(c_t^y): \quad \frac{1}{c_t^y} = \lambda_t$$

$$(c_{t+1}^o): \quad \frac{\beta}{c_{t+1}^o} = \frac{\lambda_t}{1 + r_{t+1}}$$

$$(\lambda_t): \quad w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} = 0$$

The FOC wrt λ_t gives us one of the household's optimality conditions, the household's LBC. The other optimality condition is the household Euler's equation which can be found by combining the FOC wrt c_t^y and c_{t+1}^o:

$$\frac{1}{c_t^y} = \frac{\beta (1 + r_{t+1})}{c_{t+1}^o}$$

b) Set up the firm's problem and derive the optimality conditions of the firm. Explain how the firm's labour demand varies the payroll tax, **holding all else constant**.
The firm's profit maximization problem is given by:

$$\max_{K_t, L_t} zK_t^\alpha L_t^{1-\alpha} - (1 + \tau_t^w) w_t L_t - R_t K_t$$

Taking FOC we have:

$$(K_t): \quad \alpha z K_t^{\alpha-1} L_t^{1-\alpha} = R_t$$

$$(L_t): \quad (1-\alpha) z K_t^\alpha L_t^{-\alpha} = (1 + \tau_t^w) w_t$$

In per capita terms, these optimality conditions can be written as:

$$R_t = \alpha z k_t^{-(1-\alpha)} \quad \text{and} \quad w_t = \frac{1}{1 + \tau_t^w} (1-\alpha) z k_t^\alpha$$

Using the FOC wrt L_t, we can make L_t the subject of the equation to derive an expression for labour demand:

$$L_t = \left[\frac{(1-\alpha) z K_t^\alpha}{(1 + \tau_t^w) w_t} \right]^{1/\alpha}$$

The firm's labour demand is declining in the payroll tax rate, τ_t^w. Intuitively, a rise in the payroll tax makes hiring labour more expensive and thus the firm demands less labour in production, holding all else constant.

c) Write down the government budget constraint. Explain succinctly why we don't need to worry about the transversality condition in this case.
The government runs a balanced budget every period given by:

$$G_t = G = \tau_t^w w_t L_t$$

Since the government balances its budget every period, it incurs zero debt. Thus, the transversality condition is trivially satisfied in this problem.

d) Derive a transition equation for k_{t+1} in terms of pre-determined k_t, exogenous variables z and $g = G/N$, as well as parameters of the model.
In equilibrium, all markets clear, and we have $L_t = N, K_t = Na_{t+1}$. Re-writing the government budget constraint in per-capita terms where $g = G/N$, we have:

$$g = \tau_t^w w_t \implies \tau_t^w = g/w_t$$

We plug the form τ_t^w into the firm's optimality condition for labour and re-arrange the equation to get the following expression for w_t:

$$w_t = (1-\alpha)zk_t^\alpha - g$$

From the household problem we can plug the Euler equation into the LBC to get:

$$\begin{aligned} c_t^y &= \frac{1}{1+\beta}(w_t + \pi_t) \\ &= \frac{1}{1+\beta}\left[(1-\alpha)zk_t^\alpha - g\right] \end{aligned}$$

And using capital market clearing and the household's budget constraint when young, we can derive our transition equation:

$$\begin{aligned} k_{t+1} &= a_{t+1} = w_t + \pi_t - c_t^y \\ &= \frac{\beta}{1+\beta}\left[(1-\alpha)zk_t^\alpha - g\right] \end{aligned}$$

e) Does k_{t+1} vary positively or negatively with more government spending? Provide some intuition as to why k_{t+1} varies with g in that manner.
From our transition equation, we can see that k_{t+1} varies negatively with g. Intuitively, as government spending increases, more resources need to be allocated towards the provision of the public good. Since the government spending is financed by a payroll tax, more government spending raises the cost of labour, holding all else constant. To clear the labour market, the wage rate adjusts downwards so that supply is once again equal to demand. Since the wage rate is now lower, young households have less income to consume and save from. As such, both consumption of the young and investment are reduced to free up more resources for the increased government spending.

f) Is the welfare of the household necessarily lower with higher government spending? Explain how your answer depends on the size of γ.
The welfare of the household need not necessarily be lower with higher government spending, especially if γ is very large. Since government spending

goes towards the provision of a public good that households get utility from, the weight that households put on the utility from this public good relative to the weight they put on private consumption determines the extent to which their utility is increasing in the public good provision.

Question 2: Government spending financed by lump-sum tax on young

Suppose the government spends $G_t = G$ each period. Government spending in this case is wasteful. The government runs a balanced budget and finances its spending G_t in each period by issuing a lump-sum tax only on young households. The rest of the problem is standard. The household has log utility given by $U(c_t^y, c_{t+1}^o) = \ln c_t^y + \beta \ln c_{t+1}^o$. The household works when young and inelastically supplies one unit of labour. She/he collects wage income and dividend income when young and pays the lump-sum tax. The young household can invest an asset, a_{t+1} that has a return of $1 + r_{t+1}$ when old. Households when old consume their savings. Firms use capital and labour in production and the per unit cost of these inputs is given by R_t and w_t, respectively. Output is produced via a Cobb-Douglas production function, $Y_t = zK_t^\alpha L_t^{1-\alpha}$ where $0 < \alpha < 1$.

a) Write down the government budget constraint
Denote τ_t as the lump-sum tax on young households. Then the government budget constraint is given by:

$$G_t = N\tau_t$$

b) Set up the firm's problem and derive the firm's optimality conditions
The firm's problem is standard and given by:

$$\max_{K_t, L_t} zK_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t$$

Taking FOCs and writing the optimality conditions in per-capita terms, we have:

$$(1-\alpha) z k_t^\alpha = w_t$$
$$\alpha z k_t^{-(1-\alpha)} = R_t$$

c) Set up the household's problem and derive the household's optimality conditions
The household's problem is given by:

$$\max_{c_t^y, c_{t+1}^o} \ln c_t^y + \beta \ln c_{t+1}^o$$

s.t.

$$w_t + \pi_t = \tau_t + c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}}$$

We can write the Lagrangian as:

$$\mathcal{L} = \max \ln c_t^y + \beta \ln c_{t+1}^o + \lambda_t \left[w_t + \pi_t - \tau_t - c_t^y - \frac{c_{t+1}^o}{1+r_{t+1}} \right]$$

Taking FOCs: we have

$$\frac{1}{c_t^y} = \lambda_t$$

$$\frac{\beta}{c_{t+1}^o} = \frac{\lambda_t}{1+r_{t+1}}$$

and taking FOC wrt λ_t yields the lifetime budget constraint which forms one of the household's optimality conditions. The other optimality condition is the household's euler equation which can be derived by combining the FOCs wrt c_t^y and c_{t+1}^o:

$$\frac{1}{c_t^y} = \beta \frac{1+r_{t+1}}{c_{t+1}^o}$$

d) Solve for k_{t+1}, c_t^y and c_t^o in terms of k_t, z, g and parameters of the model, α, β.
Plugging the Euler equation into the LBC, we get:

$$c_t^y = \frac{1}{1+\beta} \left[w_t + \pi_t - \tau_t \right]$$

Then we know in equilibrium that:

$$\begin{aligned} k_{t+1} &= a_{t+1} = w_t + \pi_t - \tau_t - c_t^y \\ &= \frac{\beta}{1+\beta} \left[(1-\alpha)\, z k_t^\alpha - g \right] \end{aligned}$$

and

$$c_t^y = \frac{1}{1+\beta} \left[(1-\alpha)\, z k_t^\alpha - g \right]$$

and

$$c_t^o = R_t k_t = \alpha z k_t^\alpha$$

e) Suppose at date t, G increases permanently to G' where $G' > G$. What happens to investment per person k_{t+1}, and consumption per person (c_t^y and c_t^o) at date t. What happens to the growth path of k_t when government spending rises. How does that affect the path of consumption spending when young and old? Provide some intuition for your answer. At date t when government spending permanently increases to G', both investment k_{t+1} and consumption of the young c_t^y fall on impact. Consumption of the old is unaffected at date t. Going forward however, the economy is on a lower growth path of k_t, and so going forward, both the growth path of consumption spending of the young and old are on a lower trajectory.

Intuitively, the increase in government spending means that more resources need to be allocated towards government spending. Since output at date t is fixed (as K_t is predetermined and L_t is equal to the fixed supply of N households in equilibrium), this means that more output needs to be diverted away from private investment and consumption in order to accommodate the higher government spending. Because the tax is levied on young households, it is the consumption of the young (as opposed to consumption of the old) that falls on impact when more resources need to diverted towards government spending.