Lecture 11
Multivariate Models

Granger causality

Granger causality definitions

Suppose we have two time series, $Y_{1,t}$ and $Y_{2,t}$.

Denote the information sets

$$\mathscr{Y}_{1,t} = \{\, Y_{1,t}, Y_{1,t-1}, Y_{1,t-2}, \ldots, Y_{1,1} \,\}$$
$$\mathscr{Y}_{2,t} = \{\, Y_{2,t}, Y_{2,t-1}, Y_{2,t-2}, \ldots, Y_{2,1} \,\}$$

$$\mathscr{Y}_{t} = \mathscr{Y}_{1,t} \cup \mathscr{Y}_{2,t}$$

We have been considering forecasting based on

$$E(\, Y_{1,n+h} \,|\, \mathscr{Y}_{1,n} \,)$$

i.e. using only past values of $Y_{1,t}$

Granger causality definitions

Suppose we have two time series, $Y_{1,t}$ and $Y_{2,t}$.

Denote the information sets

$$\mathscr{Y}_{1,t} = \{\, Y_{1,t}, Y_{1,t-1}, Y_{1,t-2}, \ldots, Y_{1,1} \,\}$$
$$\mathscr{Y}_{2,t} = \{\, Y_{2,t}, Y_{2,t-1}, Y_{2,t-2}, \ldots, Y_{2,1} \,\}$$

$$\mathscr{Y}_t = \mathscr{Y}_{1,t} \cup \mathscr{Y}_{2,t}$$

We have been considering forecasting based on

$$E(Y_{1,n+h} \mid \mathscr{Y}_{1,n})$$

Now consider

$$E(Y_{1,n+h} \mid \mathscr{Y}_n)$$

Granger causality definitions

For any $\mathscr{F}_n$ define:

$$\text{MSE}(Y_{n+h} \mid \mathscr{F}_n) = E[\,(\,Y_{n+h} - E(Y_{n+h}|\mathscr{F}_n)\,)^2\,]$$

$E(Y_{n+h}|\mathscr{F}_n)$ is the MSE-optimal forecast of Y_{n+h} using the data in $\mathscr{F}_n$.

Compare

$\text{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{1,n})$ uses $Y_{1,t}$ only

and

$\text{MSE}(Y_{1,n+h} \mid \mathscr{Y}_n)$ uses $Y_{1,t}$ *and* $Y_{2,t}$.

Granger causality definitions

Compare

$$\mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{1,n}) \qquad \text{uses } Y_{1,t} \text{ only}$$

and

$$\mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{n}) \qquad \text{uses } Y_{1,t} \textit{ and } Y_{2,t}.$$

Definition. $Y_{2,t}$ "Granger causes" $Y_{1,t}$ if

$$\mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{n}) \;<\; \mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{1,n})$$

i.e. forecasts are improved by adding $Y_{2,t}$ to the model.

Granger causality definitions

Definition. $Y_{2,t}$ "Granger causes" $Y_{1,t}$ if

$$\mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_n) \; < \; \mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{1,n})$$

i.e. forecasts are improved by adding $Y_{2,t}$ to the model.

- This is not a definition of *causality*.
- Maybe better called "Granger predictability"…

Granger causality definitions

Definition. $Y_{2,t}$ "Granger causes" $Y_{1,t}$ if

$$\mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_n\ ) \ < \ \mathrm{MSE}(Y_{1,n+h} \mid \mathscr{Y}_{1,n})$$

- It is common to test for Granger causality with a (joint) test for the significance of all lags of $Y_{2,t}$ included in a model for $Y_{1,t}$.
- Practically we cancompare forecast properties of models for $Y_{1,t}$ with and without $Y_{2,t}$ included.

Vector Autoregression

Bivariate VAR(1)

Suppose we have two time series, $Y_{1,t}$ and $Y_{2,t}$.

Denote the information sets

$$\mathscr{Y}_{1,t} = \{\, Y_{1,t}, Y_{1,t-1}, Y_{1,t-2}, \ldots, Y_{1,1} \,\}$$
$$\mathscr{Y}_{2,t} = \{\, Y_{2,t}, Y_{2,t-1}, Y_{2,t-2}, \ldots, Y_{2,1} \,\}$$

$$\mathscr{Y}_{t} = \mathscr{Y}_{1,t} \cup \mathscr{Y}_{2,t}$$

Bivariate VAR(1)

Suppose we have two time series, $Y_{1,t}$ and $Y_{2,t}$.

Denote the information sets

$$\mathscr{Y}_{1,t} = \{\, Y_{1,t}, Y_{1,t-1}, Y_{1,t-2}, \ldots, Y_{1,1} \,\}$$
$$\mathscr{Y}_{2,t} = \{\, Y_{2,t}, Y_{2,t-1}, Y_{2,t-2}, \ldots, Y_{2,1} \,\}$$

$$\mathscr{Y}_t = \mathscr{Y}_{1,t} \cup \mathscr{Y}_{2,t}$$

Two equations, one lag:

$$E(\, Y_{1,t} \mid \mathscr{Y}_{t-1} \,) = \nu_1 + \Phi_{1,1} Y_{1,t-1} + \Phi_{1,2} Y_{2,t-1}$$
$$E(\, Y_{2,t} \mid \mathscr{Y}_{t-1} \,) = \nu_2 + \Phi_{2,1} Y_{1,t-1} + \Phi_{2,2} Y_{2,t-1}$$

Bivariate VAR(1)

Suppose we have two time series, $Y_{1,t}$ and $Y_{2,t}$.

Denote the information sets

$$\mathscr{Y}_{1,t} = \{\, Y_{1,t}, Y_{1,t-1}, Y_{1,t-2}, \ldots, Y_{1,1} \,\}$$
$$\mathscr{Y}_{2,t} = \{\, Y_{2,t}, Y_{2,t-1}, Y_{2,t-2}, \ldots, Y_{2,1} \,\}$$

$$\mathscr{Y}_t = \mathscr{Y}_{1,t} \cup \mathscr{Y}_{2,t}$$

Two equations, one lag:

$$E(\, Y_{1,t} \mid \mathscr{Y}_{t-1} \,) = \nu_1 + \Phi_{1,1} Y_{1,t-1} + \Phi_{1,2} Y_{2,t-1}$$

$$\text{if } \Phi_{1,2} \neq 0$$

$$\Rightarrow Y_{2,t} \text{ “Granger causes” } Y_{1,t}$$

Bivariate VAR(1)

Two equations, one lag:

$$E(\,Y_{1,t} \mid \mathscr{Y}_{t-1}\,) = \nu_1 + \Phi_{1,1} Y_{1,t-1} + \Phi_{1,2} Y_{2,t-1}$$
$$E(\,Y_{2,t} \mid \mathscr{Y}_{t-1}\,) = \nu_2 + \Phi_{2,1} Y_{1,t-1} + \Phi_{2,2} Y_{2,t-1}$$

Bivariate VAR(1)

Two equations, one lag:

$$E(\,Y_{1,t} \mid \mathscr{Y}_{t-1}\,) = \nu_1 + \Phi_{1,1} Y_{1,t-1} + \Phi_{1,2} Y_{2,t-1}$$

$$E(\,Y_{2,t} \mid \mathscr{Y}_{t-1}\,) = \nu_2 + \Phi_{2,1} Y_{1,t-1} + \Phi_{2,2} Y_{2,t-1}$$

In matrix form:

$$E\left(\begin{matrix} Y_{1,t} \\ Y_{2,t} \end{matrix} \middle| \mathscr{Y}_{t-1}\right) = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \end{pmatrix}$$

$$E(\,Y_t \mid \mathscr{Y}_{t-1}\,) = \nu + \Phi_1 Y_{t-1}$$

General K-variate VAR(p)

$k = 1, 2, \ldots, K$ equations, with p lags:

$$E(\, Y_{k,t} \mid \mathscr{Y}_{t-1} \,) = \nu_k + \sum_{j=1}^{p} \sum_{i=1}^{K} \Phi_{k,i}^{(j)} Y_{i,t-j}$$

- Each of the $k = 1, 2, \ldots, K$ equations includes p lags of each of the K variables as predictors.
- Same lag order p for each predictor.
- Each equation has the same predictors.

General K-variate VAR(p)

In matrix form:

$$E(Y_t|\mathscr{Y}_{t-1}) = \nu + \Phi_1 Y_{t-1} + \ldots + \Phi_p Y_{t-p}$$

where

$$Y_t = \begin{pmatrix} Y_{1,t} \\ \vdots \\ Y_{K,t} \end{pmatrix} \quad \nu = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_K \end{pmatrix} \quad \Phi_j = \begin{pmatrix} \Phi_{1,1}^{(j)} & \cdots & \Phi_{1,K}^{(j)} \\ \vdots & \ddots & \vdots \\ \Phi_{K,1}^{(j)} & \cdots & \Phi_{K,K}^{(j)} \end{pmatrix}$$

Vector Autoregression Implementation

VAR Model Specification

1. Choose variables
2. Check trends and stationarity
3. Select lag order p:
   - autocorrelation tests
   - minimise AIC

U.S. GDP, Taxes and Government Spending

Real per capita quarterly time series, 1980q1-2019q4:

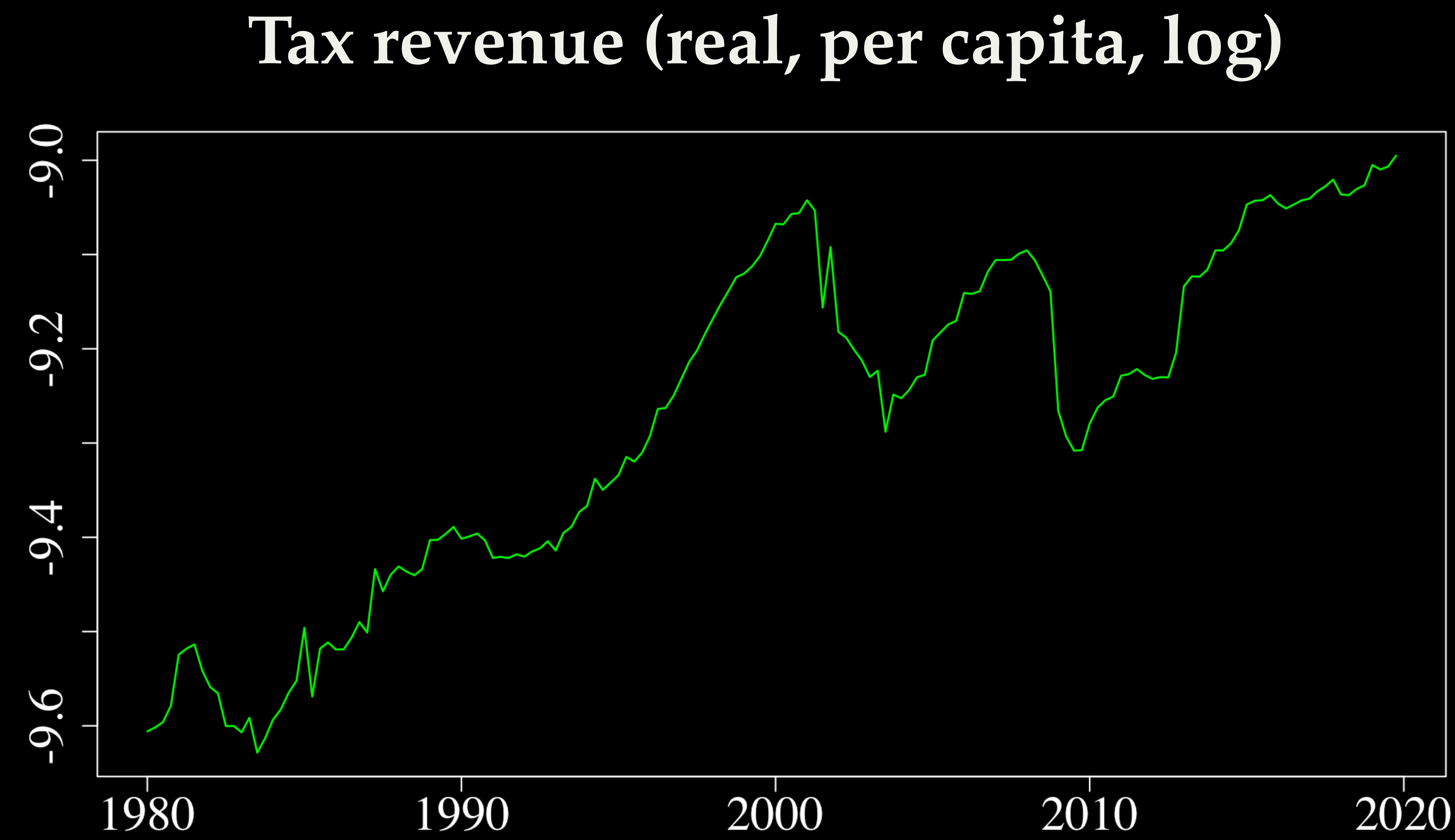
GDP_t : GDP

Tax_t : total tax revenue

Govt_t : total government spending

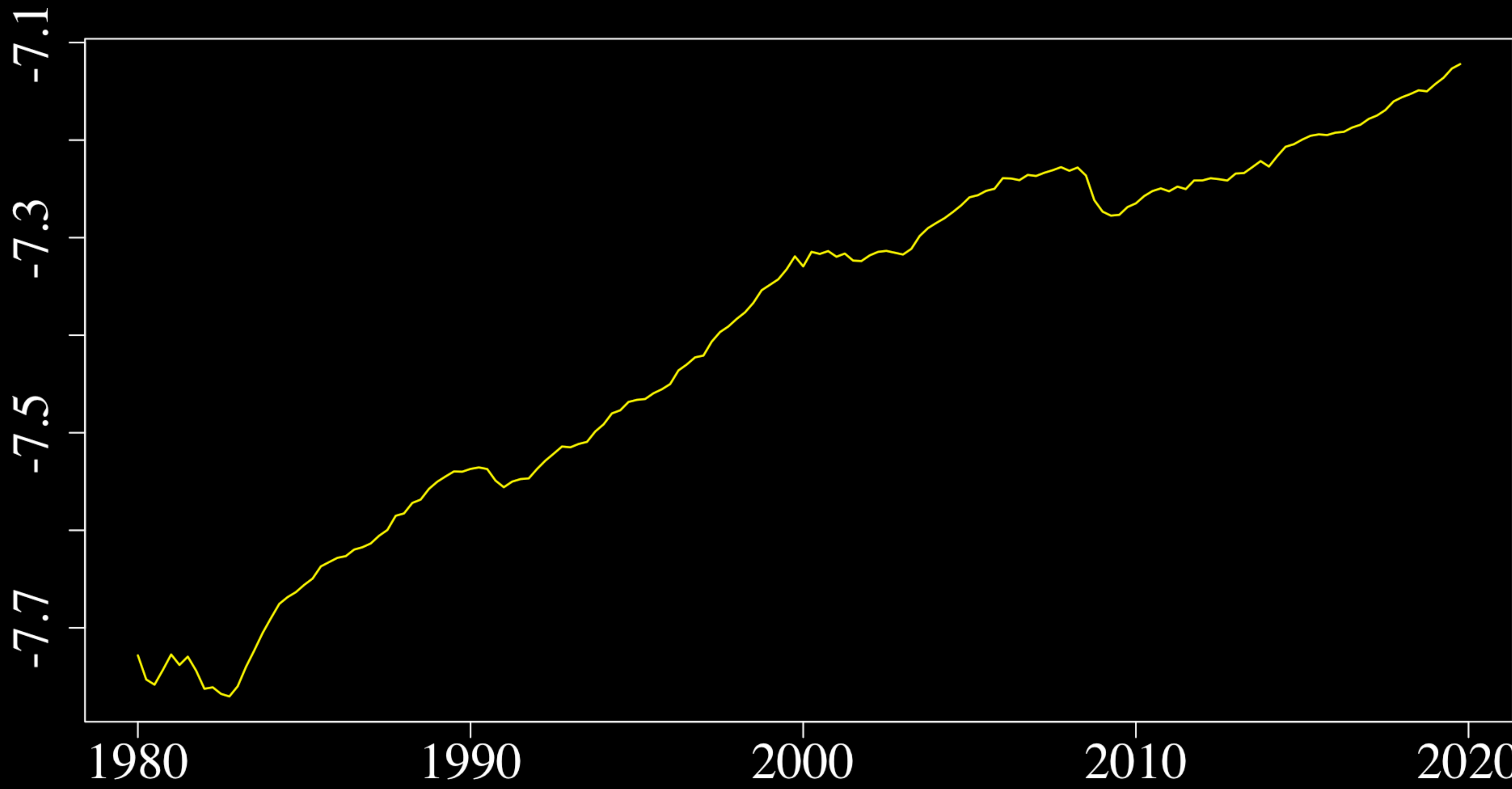
Mertens, K., and Ravn, M.O. (2014) A Reconciliation of SVAR and Narrative Estimates of Tax Multipliers, Journal of Monetary Economics, 68(S), S1–S19.

Woźniak, Tomasz (2025). bsvars: Bayesian Estimation of Structural Vector Autoregressive Models. R package version 4.0.

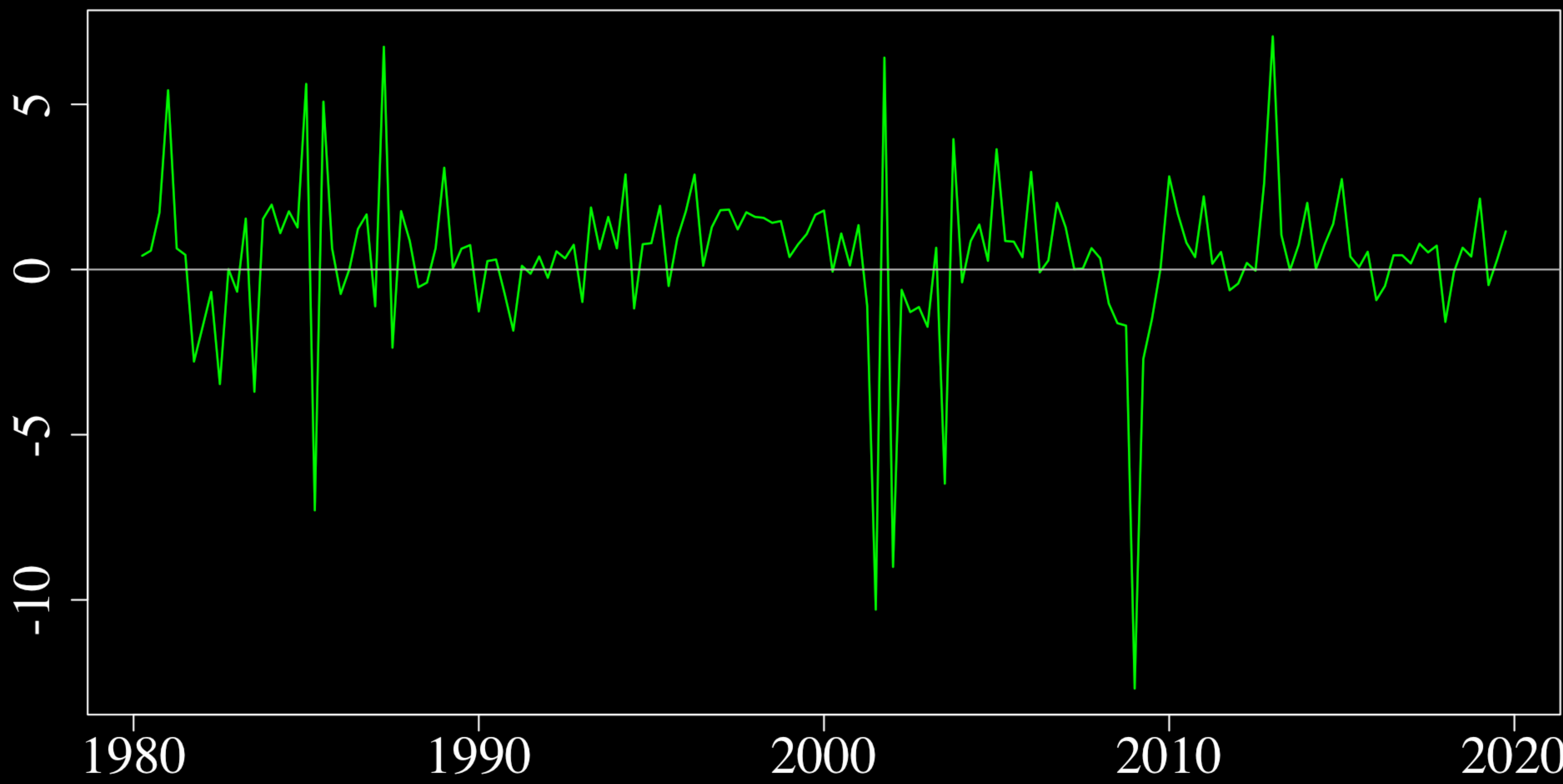

Tax revenue (real, per capita, log)
-9.0
-9.2
-9.4
-9.6
1980
1990
2000
2010
2020

Government expenditure (real, per capita, log)

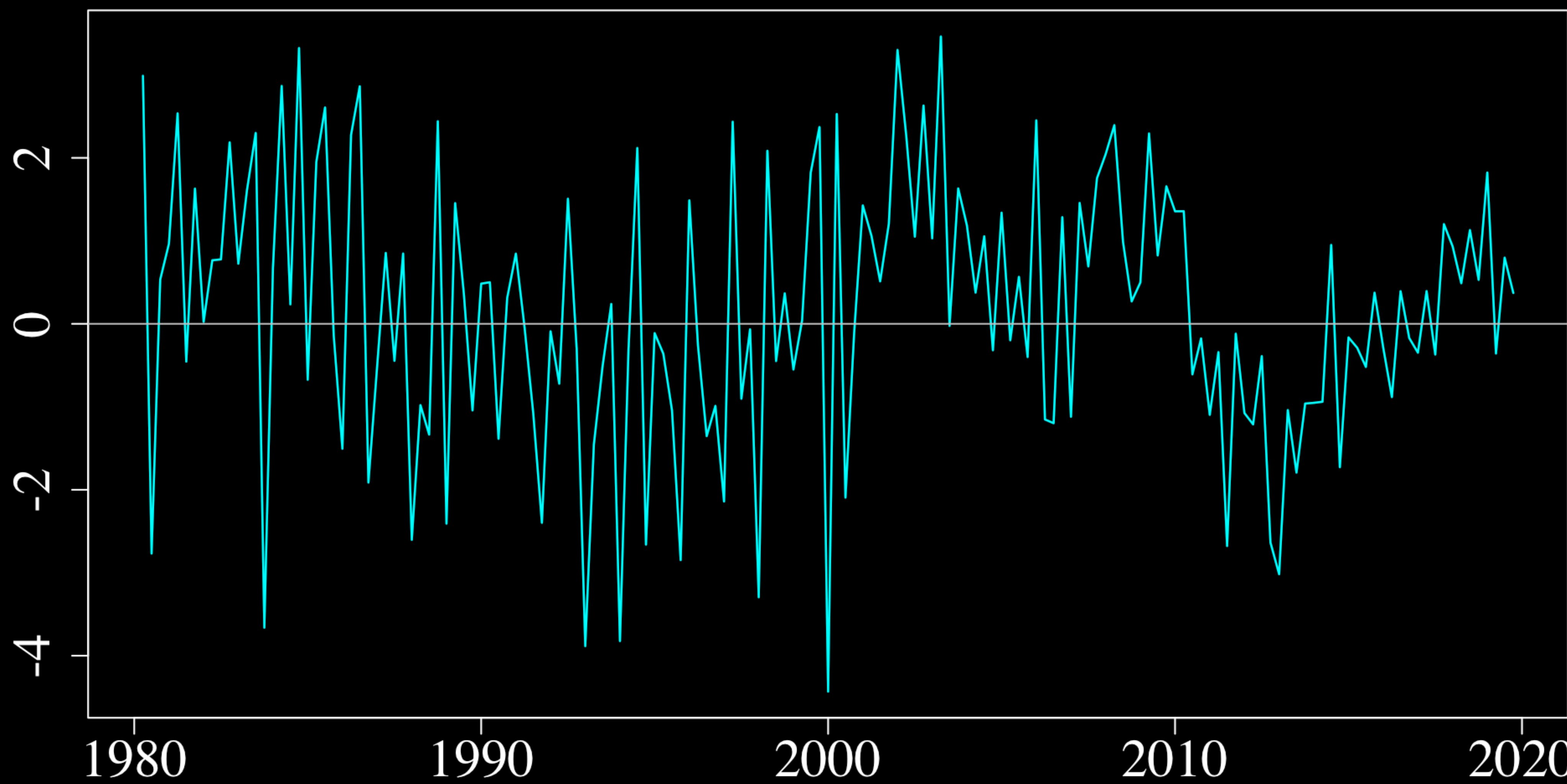
GDP (real, per capita, log)



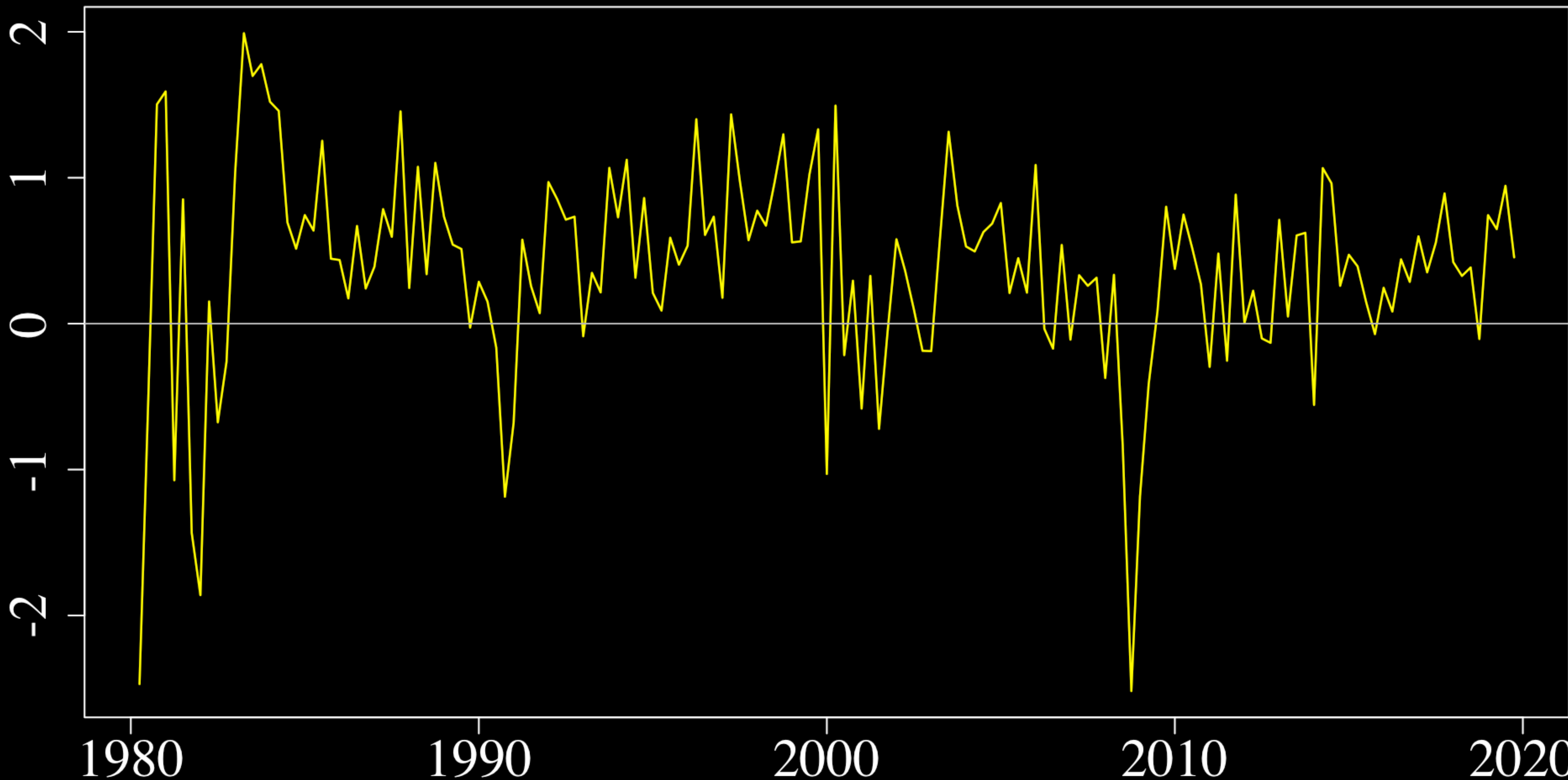
Δ Tax revenue (real, per capita, log)



Δ Govt expenditure (real, per capita, log)



Δ GDP (real, per capita, log)



Select VAR lag order

Select p to minimise AIC:

```r
library(vars)
DY <- data.frame(DTax=DTax, DGovt=DGovt,
                 DGDP=DGDP)
VARp <- VAR(DY, lag.max=8, ic="AIC")
print(VARp$p)
```

```
AIC(n)
     3
```

Select VAR lag order

Check residual autocorrelation of VAR(3):

```r
VARp <- VAR(DY, p=3)
Auto.test <- serial.test(VARp, lags.pt=12)
```

```
    Portmanteau Test (asymptotic)

data:  Residuals of VAR object VARp
Chi-squared = 85.618, df = 81, p-value =
0.3415 ✓
```

Tax equation

```
Coef.Tax <- VARp$varresult$DGDP$coefficient
```

```
 DTax.l1 DGovt.l1  DGDP.l1
  -0.017   -0.051    0.312
 DTax.l2 DGovt.l2  DGDP.l2
  -0.019   -0.038    0.218
 DTax.l3 DGovt.l3  DGDP.l3
  -0.050   -0.044    0.074
const
0.211
```

Tax equation

DTax.l1	DGovt.l1	DGDP.l1	$\leftarrow \; j = 1$
-0.017	-0.051	0.312	
DTax.l2	DGovt.l2	DGDP.l2	$\leftarrow \; j = 2$
-0.019	-0.038	0.218	
DTax.l3	DGovt.l3	DGDP.l3	$\leftarrow \; j = 3$
-0.050	-0.044	0.074	
$\uparrow$	$\uparrow$	$\uparrow$	
$\Delta\mathrm{Tax}_{t-j}$	$\Delta\mathrm{Govt}_{t-j}$	$\Delta\mathrm{GDP}_{t-j}$	

Tax equation

$$
\begin{aligned}
\widehat{E}(\,\Delta\mathrm{Tax}_t \mid \mathscr{Y}_{t-1}\,) = 0.211 \\
-0.017\,\Delta\mathrm{Tax}_{t-1} - 0.051\,\Delta\mathrm{Govt}_{t-1} \\
+0.312\,\Delta\mathrm{GDP}_{t-1} \\
-0.019\,\Delta\mathrm{Tax}_{t-2} - 0.038\,\Delta\mathrm{Govt}_{t-2} \\
+0.218\,\Delta\mathrm{GDP}_{t-2} \\
-0.050\,\Delta\mathrm{Tax}_{t-3} - 0.044\,\Delta\mathrm{Govt}_{t-3} \\
+0.074\,\Delta\mathrm{GDP}_{t-3}
\end{aligned}
$$

Government expenditure equation

```
DTax.l1 DGovt.l1  DGDP.l1
 -0.032   -0.040   -0.188
DTax.l2 DGovt.l2  DGDP.l2
 -0.079    0.148    0.030
DTax.l3 DGovt.l3  DGDP.l3
 -0.074    0.192   -0.036
```

$\leftarrow j = 1$

$\leftarrow j = 2$

$\leftarrow j = 3$

$\uparrow$ ΔTax_{t-j} $\uparrow$ ΔGovt_{t-j} $\uparrow$ ΔGDP_{t-j}

Government expenditure equation

$$
\begin{aligned}
\widehat{E}(\,\Delta\mathrm{Govt}_t \mid \mathscr{Y}_{t-1}\,) = 0.261 \\
-0.032\,\Delta\mathrm{Tax}_{t-1} - 0.040\,\Delta\mathrm{Govt}_{t-1} \\
-0.188\,\Delta\mathrm{GDP}_{t-1} \\
-0.079\,\Delta\mathrm{Tax}_{t-2} + 0.148\,\Delta\mathrm{Govt}_{t-2} \\
+0.030\,\Delta\mathrm{GDP}_{t-2} \\
-0.074\,\Delta\mathrm{Tax}_{t-3} + 0.192\,\Delta\mathrm{Govt}_{t-3} \\
-0.036\,\Delta\mathrm{GDP}_{t-3}
\end{aligned}
$$

GDP equation

```
DTax.l1 DGovt.l1  DGDP.l1
 -0.017   -0.051    0.312
DTax.l2 DGovt.l2  DGDP.l2
 -0.019   -0.038    0.218
DTax.l3 DGovt.l3  DGDP.l3
 -0.050   -0.044    0.074
```

$\leftarrow j = 1$

$\leftarrow j = 2$

$\leftarrow j = 3$

$\uparrow$ $\Delta\mathrm{Tax}_{t-j}$ $\uparrow$ $\Delta\mathrm{Govt}_{t-j}$ $\uparrow$ $\Delta\mathrm{GDP}_{t-j}$

GDP equation

$$
\begin{aligned}
\widehat{E}(\,\Delta\mathrm{GDP}_t \mid \mathscr{Y}_{t-1}\,) = 0.211 \\
-0.017\,\Delta\mathrm{Tax}_{t-1} - 0.051\,\Delta\mathrm{Govt}_{t-1} \\
+0.312\,\Delta\mathrm{GDP}_{t-1} \\
-0.019\,\Delta\mathrm{Tax}_{t-2} - 0.038\,\Delta\mathrm{Govt}_{t-2} \\
+0.218\,\Delta\mathrm{GDP}_{t-2} \\
-0.050\,\Delta\mathrm{Tax}_{t-3} - 0.044\,\Delta\mathrm{Govt}_{t-3} \\
+0.074\,\Delta\mathrm{GDP}_{t-3}
\end{aligned}
$$

Vector Autoregression Forecasting

Tax revenue growth forecasts

```r
VARpf <- predict(VARp, n.ahead=8)
DTaxf <- VARpf$fcst$DTax[,c("lower", "fcst"
```

```
 lower  fcst upper
-3.778 0.391 4.560
-4.037 0.417 4.872
-4.172 0.608 5.389
-4.485 0.304 5.093
-4.446 0.419 5.284
-4.506 0.383 5.272
-4.540 0.357 5.255
```

Government expenditure growth forecasts

```r
DGovtf <- VARpf$fcst$DGovt[,
                  c("lower", "fcst", "upper")]
```

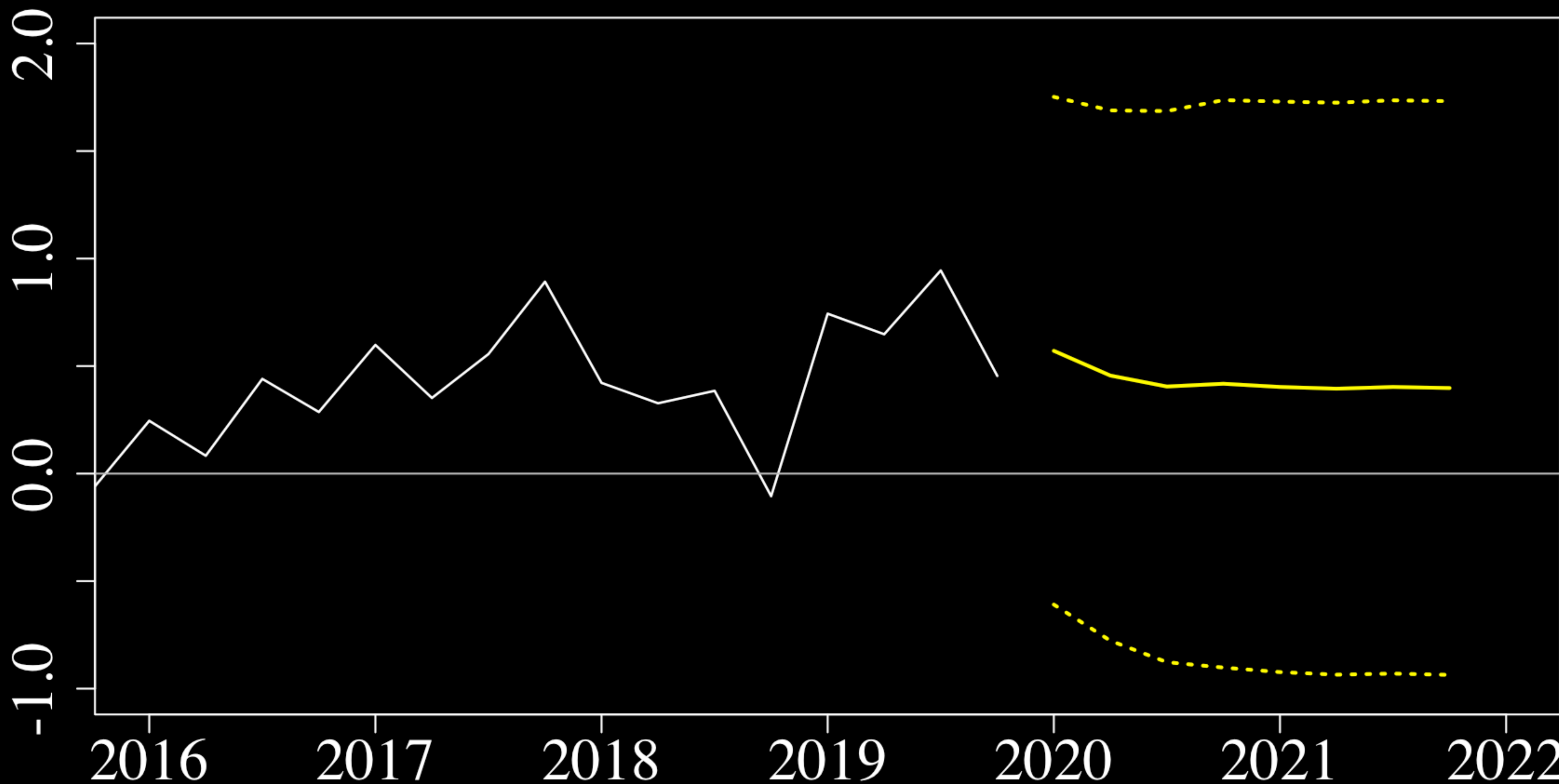
```
 lower  fcst upper
-2.783 0.190 3.163
-2.782 0.208 3.198
-2.907 0.137 3.182
-2.965 0.159 3.282
-2.995 0.143 3.282
-3.016 0.145 3.306
-3.010 0.162 3.334
```

GDP growth forecasts

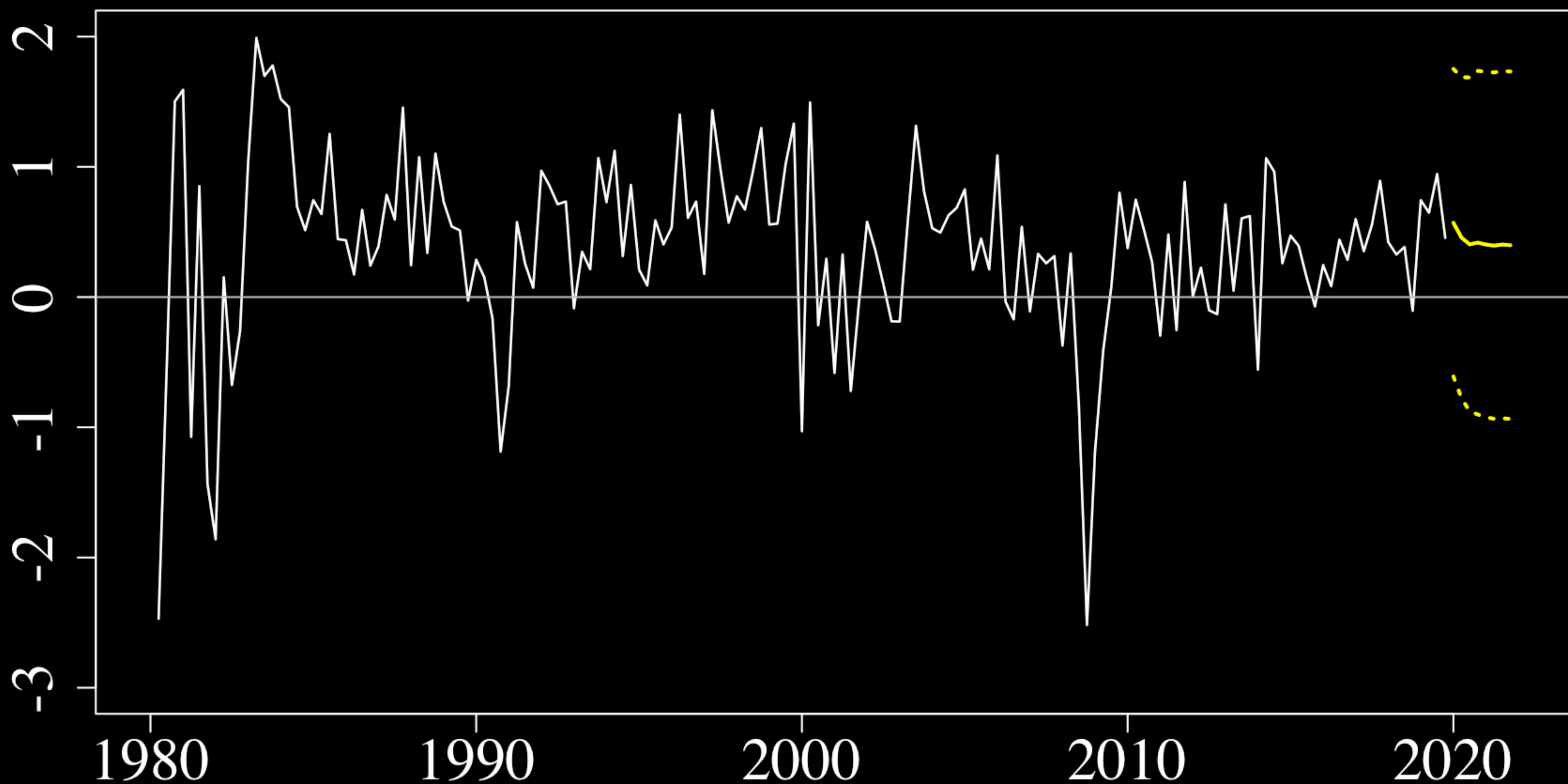
```r
DGDPf <- VARpf$fcst$DGDP[,
                   c("lower", "fcst", "upper")]
```

```
 lower  fcst upper
-0.609 0.571 1.752
-0.777 0.456 1.689
-0.877 0.405 1.686
-0.902 0.418 1.737
-0.923 0.403 1.730
-0.935 0.395 1.725
-0.930 0.403 1.736
```

GDP growth forecasts



GDP growth forecasts



Granger causality testing

Does economic growth forecast gov't finances?

```r
GCtest <- causality(VARp, cause="DGDP")
print(GCtest$Granger)
```

```
	Granger causality H0: DGDP do not Granger-cause DTax DGovt

data:  VAR object VARp
F-Test = 5.3586, df1 = 6, df2 = 438, p-value =
2.355e-05
```

$p < 0.05 \Rightarrow$ Yes!

Granger causality RMSE analysis

VARs for forecasting tax revenue growth:

1. DTax, DGovt, DGDP (*Trivariate*)
2. DTax, DGovt (*Bivariate*)
3. DTax, DGDP (*Bivariate*)
4. DTax (*Univariate AR*)

Granger causality RMSE analysis

VARs for forecasting tax revenue growth:

1. DTax, DGovt, DGDP
   $p = 3$
2. DTax, DGovt
   $p = 6$
3. DTax, DGDP
   $p = 2$
4. DTax (i.e. AR model)
   $p = 3$

Granger causality RMSE analysis

VARs for forecasting tax revenue growth:

1. DTax, DGovt, DGDP
   $p = 3$, $\text{RMSE}(\Delta\text{Tax}) = 3.706$
2. DTax, DGovt
   $p = 6$, $\text{RMSE}(\Delta\text{Tax}) = 3.734$
3. DTax, DGDP
   $p = 2$, $\text{RMSE}(\Delta\text{Tax}) = 3.730$
4. DTax (i.e. AR model)
   $p = 3$, $\text{RMSE}(\Delta\text{Tax}) = 3.775$

Granger causality RMSE analysis

VARs for forecasting tax revenue growth:

1. DTax, DGovt, DGDP
   $p = 3$, $\mathrm{RMSE}(\Delta\mathrm{Tax}) = 3.706$ ✓
2. DTax, DGovt
   $p = 6$, $\mathrm{RMSE}(\Delta\mathrm{Tax}) = 3.734$
3. DTax, DGDP
   $p = 2$, $\mathrm{RMSE}(\Delta\mathrm{Tax}) = 3.730$
4. DTax (i.e. AR model)
   $p = 3$, $\mathrm{RMSE}(\Delta\mathrm{Tax}) = 3.775$

Vector Autoregression

Impulse Responses

Impulse Response definitions

Consider: time series $Y_t = (Y_{1,t}, \ldots, Y_{K,t})'$

and model $E(\, Y_t \,|\, \mathscr{Y}_{t-1} \,)$.

Denote $\mathscr{Y}_n = \{Y_n \cup \mathscr{Y}_{n-1}\}$.

Usual forecasts based on observed data:

$$E(\, Y_{n+h} \,|\, \mathscr{Y}_n \,) = E(\, Y_{n+h} \,|\, Y_n \cup \mathscr{Y}_{n-1} \,)$$

Counterfactual with an "impulse" to Y_n:

$$E(\, Y_{n+h} \,|\, (Y_n + \delta) \cup \mathscr{Y}_{n-1} \,)$$

Impulse Response definitions

Usual forecasts based on observed data:

$$E(\,Y_{n+h}\,|\,\mathscr{Y}_n\,) = E(\,Y_{n+h}\,|\,Y_n \cup \mathscr{Y}_{n-1}\,)$$

Counterfactual with an "impulse" to Y_n:

$$E(\,Y_{n+h}\,|\,(Y_n + \delta) \cup \mathscr{Y}_{n-1}\,)$$

Impulse responses:

$$E(\,Y_{n+h}\,|\,(Y_n + \delta) \cup \mathscr{Y}_{n-1}\,) \\ -E(\,Y_{n+h}\,|\,Y_n \cup \mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Impulse Response definitions

Impulse responses:

$$E(\, Y_{n+h} \mid (Y_n + \delta) \cup \mathscr{Y}_{n-1} \,) \\ -E(\, Y_{n+h} \mid Y_n \cup \mathscr{Y}_{n-1} \,)$$

Change in forecasts due to δ change in Y_n.

Extends the regression interpretation:

$$E(Y_i|X_{1,i}, X_{2,i}) = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i}$$

Impulse Response definitions

Impulse responses:

$$E(\,Y_{n+h}\mid (Y_n+\delta)\cup \mathscr{Y}_{n-1}\,) \\ -E(\,Y_{n+h}\mid Y_n\cup \mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Extends the regression interpretation:

$$E(Y_i|X_{1,i},X_{2,i}) = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i}$$

β_1 : change in $E(Y_i|X_{1,i},X_{2,i})$ due to

$+1$ change in $X_{1,i}$

Impulse Response definitions

Impulse responses:

$$E(\, Y_{n+h} \mid (Y_n + \delta) \cup \mathscr{Y}_{n-1} \,) \\ -E(\, Y_{n+h} \mid Y_n \cup \mathscr{Y}_{n-1} \,)$$

Change in forecasts due to δ change in Y_n.

Extends the regression interpretation:

$$E(Y_i | X_{1,i}, X_{2,i}) = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i}$$

β_1 : change in $E(Y_i | X_{1,i}, X_{2,i})$ due to
$+1$ change in $X_{1,i}$ (holding $X_{2,i}$ constant).

Impulse Response definitions

Impulse responses:

$$E(\,Y_{n+h}\,|\,(Y_n+\delta)\cup\mathscr{Y}_{n-1}\,)$$
$$-E(\,Y_{n+h}\,|\,Y_n\cup\mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Extends the regression interpretation:

$$E(\,Y_i\,|\,X_{1,i}+1,X_{2,i}\,)-E(\,Y_i\,|\,X_{1,i},X_{2,i}\,)$$
$$=\beta_0+\beta_1(X_{1,i}+1)+\beta_2X_{2,i}$$
$$-(\,\beta_0+\beta_1X_{1,i}+\beta_2X_{2,i}\,)$$

Impulse Response definitions

Impulse responses:

$$E(\,Y_{n+h} \mid (Y_n + \delta) \cup \mathscr{Y}_{n-1}\,)$$
$$-E(\,Y_{n+h} \mid Y_n \cup \mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Extends the regression interpretation:

$$E(\,Y_i \mid X_{1,i} + 1, X_{2,i}\,) - E(\,Y_i \mid X_{1,i}, X_{2,i}\,)$$
$$= \beta_0 + \beta_1(X_{1,i} + 1) + \beta_2 X_{2,i}$$
$$-(\,\beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i}\,)$$
$$= \quad \beta_1$$

Impulse Response definitions

Impulse responses:

$$E(\, Y_{n+h} \mid (Y_n + \delta) \cup \mathscr{Y}_{n-1} \,) \\ -E(\, Y_{n+h} \mid Y_n \cup \mathscr{Y}_{n-1} \,)$$

Change in forecasts due to δ change in Y_n.

How to define the "impulse" δ?

There is HUGE literature on this…

Impulse Response definitions

Impulse responses:

$$E(\,Y_{n+h}\,|\,(Y_n+\delta)\cup\mathscr{Y}_{n-1}\,)$$
$$-E(\,Y_{n+h}\,|\,Y_n\cup\mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Example:

$$Y_t=\begin{pmatrix}\Delta\mathrm{Tax}_t\\ \Delta\mathrm{Govt}_t\\ \Delta\mathrm{GDP}_t\end{pmatrix}$$

Impulse Response definitions

Impulse responses:

$$E(\,Y_{n+h} \mid (Y_n + \delta) \cup \mathscr{Y}_{n-1}\,)$$
$$-E(\,Y_{n+h} \mid Y_n \cup \mathscr{Y}_{n-1}\,)$$

Change in forecasts due to δ change in Y_n.

Example:

$$Y_n = \begin{pmatrix} \Delta\mathrm{Tax}_n \\ \Delta\mathrm{Govt}_n \\ \Delta\mathrm{GDP}_n \end{pmatrix} \quad \delta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Impulse Response definitions

Example:

$$Y_n = \begin{pmatrix} \Delta \mathrm{Tax}_n \\ \Delta \mathrm{Govt}_n \\ \Delta \mathrm{GDP}_n \end{pmatrix} \quad \delta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Impulse: $+1$ unit to GDP growth

Impulse Response definitions

Example:

$$Y_n = \begin{pmatrix} \Delta \mathrm{Tax}_n \\ \Delta \mathrm{Govt}_n \\ \Delta \mathrm{GDP}_n \end{pmatrix} \qquad \delta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Impulse: $+1$ unit to GDP growth

How are forecasts revised in response to this impulse to GDP growth?

Impulse Response Application

```
VAR_TGY <- VAR(DY, p=3)
IRF <- irf(VAR_TGY,
                impulse="DGDP",
                ortho=FALSE,
                response=c("DTax"),
                n.ahead=8)
```

← VAR(3) for growth in Tax, Govt, GDP

Impulse Response Application

```
VAR_TGY <- VAR(DY, p=3)
IRF <- irf(VAR_TGY,
           impulse="DGDP",
           ortho=FALSE,
           response=c("DTax"),
           n.ahead=8)
```

Impulse to ΔGDP

Impulse Response Application

```
VAR_TGY <- VAR(DY, p=3)
IRF <- irf(VAR_TGY,
           impulse="DGDP",
           ortho=FALSE,
           response=c("DTax"),
           n.ahead=8)
```

$\leftarrow +1$ unit (more soon.

Impulse Response Application

```
VAR_TGY <- VAR(DY, p=3)
IRF <- irf(VAR_TGY,
                  impulse="DGDP",
                  ortho=FALSE,
                  response=c("DTax"),
                  n.ahead=8)
```

Response of
Tax forecasts
to the impulse

Impulse Response Application

Responses of ΔTax_{n+h} forecasts to $+1$ impulse to ΔGDP_n (with 95% CIs):

	Lower	IRF	Upper
t=n	0.000	0.000	0.000
t=n+1	0.457	1.014	1.625
t=n+2	0.542	1.114	1.713
t=n+3	-0.452	0.229	0.782
t=n+4	0.131	0.630	0.982
t=n+5	-0.096	0.290	0.537
t=n+6	-0.063	0.217	0.444

$\leftarrow$ 1-step-ahead forecast increased by 1.014

Impulse Response Application

Responses of ΔTax_{n+h} forecasts to $+1$ impulse to ΔGDP_n (with 95% CIs):

	Lower	IRF	Upper
t=n	0.000	0.000	0.000
t=n+1	0.457	1.014	1.625
t=n+2	0.542	1.114	1.713
t=n+3	-0.452	0.229	0.782
t=n+4	0.131	0.630	0.982
t=n+5	-0.096	0.290	0.537
t=n+6	-0.063	0.217	0.444

$\leftarrow$ 95% CI excludes zero

$\Rightarrow$ significant

Impulse Response Application

Responses of $\Delta \mathrm{Tax}_{n+h}$ forecasts to $+1$ impulse to $\Delta \mathrm{GDP}_n$ (with 95% CIs):

	Lower	IRF	Upper
t=n	0.000	0.000	0.000
t=n+1	0.457	1.014	1.625
t=n+2	0.542	1.114	1.713
t=n+3	-0.452	0.229	0.782
t=n+4	0.131	0.630	0.982
t=n+5	-0.096	0.290	0.537
t=n+6	-0.063	0.217	0.444

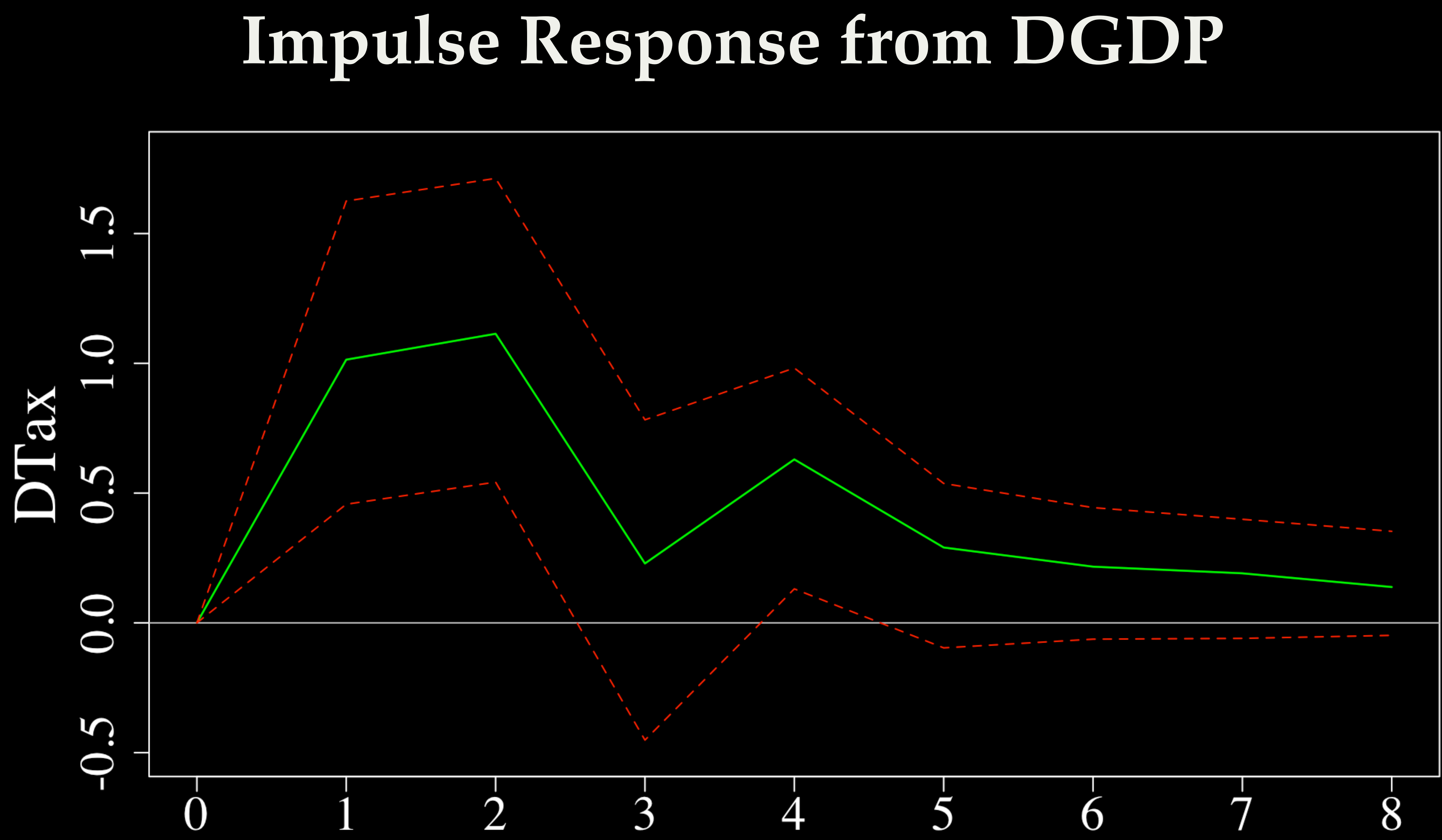
A 1 unit increase in GDP growth increases the 1-quarter-ahead forecast of Tax growth by 1.014.

Impulse Response Application

Responses of ΔTax_{n+h} forecasts to $+1$ impulse to ΔGDP_n (with 95% CIs):

	Lower	IRF	Upper
t=n	0.000	0.000	0.000
t=n+1	0.457	1.014	1.625
t=n+2	0.542	1.114	1.713
t=n+3	-0.452	0.229	0.782
t=n+4	0.131	0.630	0.982
t=n+5	-0.096	0.290	0.537
t=n+6	-0.063	0.217	0.444

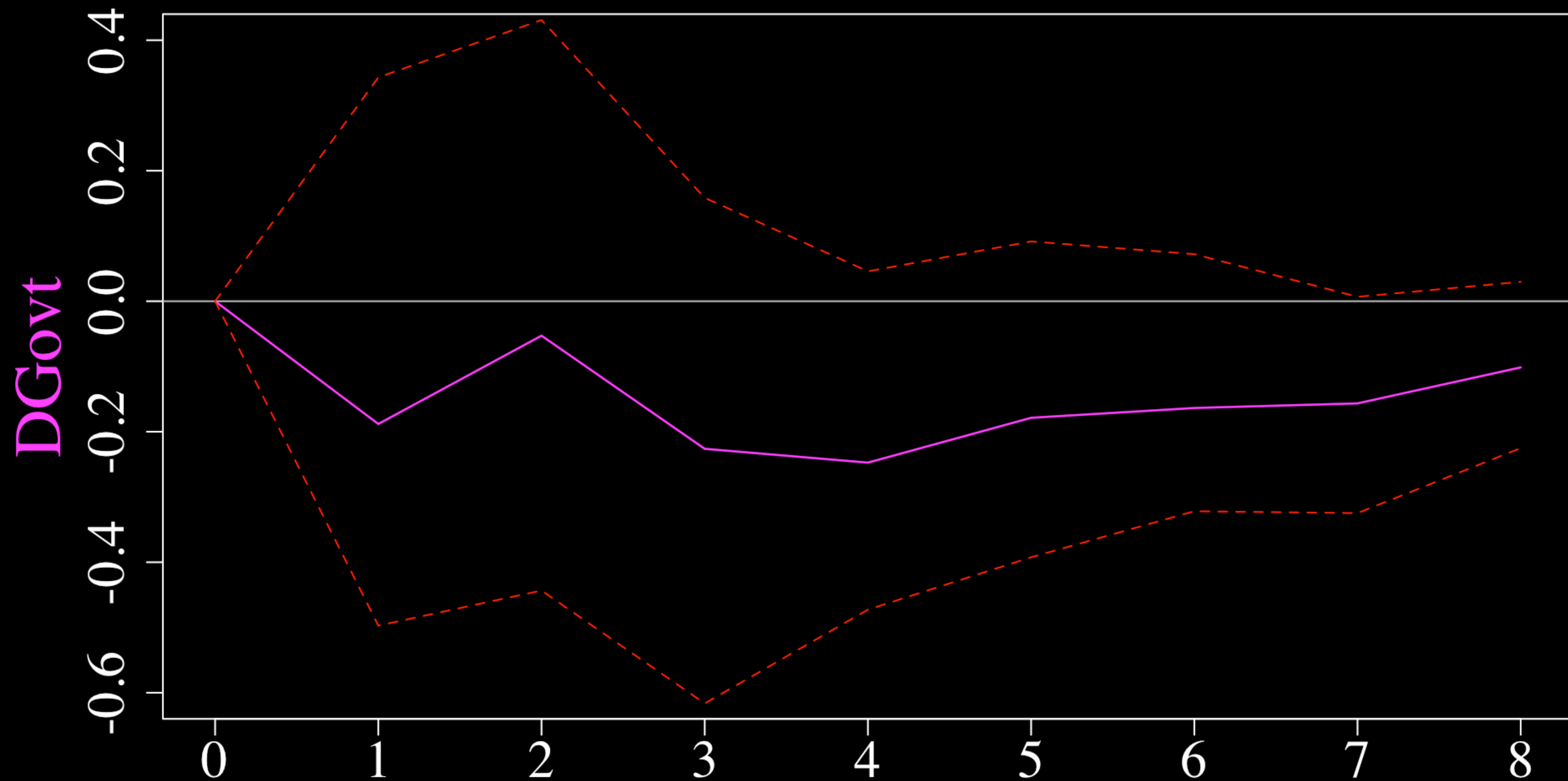
A 1 unit increase in GDP growth increases the 2-quarter-ahead forecast of Tax growth by 1.114.


Impulse Response from DGDP
DTax
1.5
1.0
0.5
0.0
-0.5
0
1
2
3
4
5
6
7
8

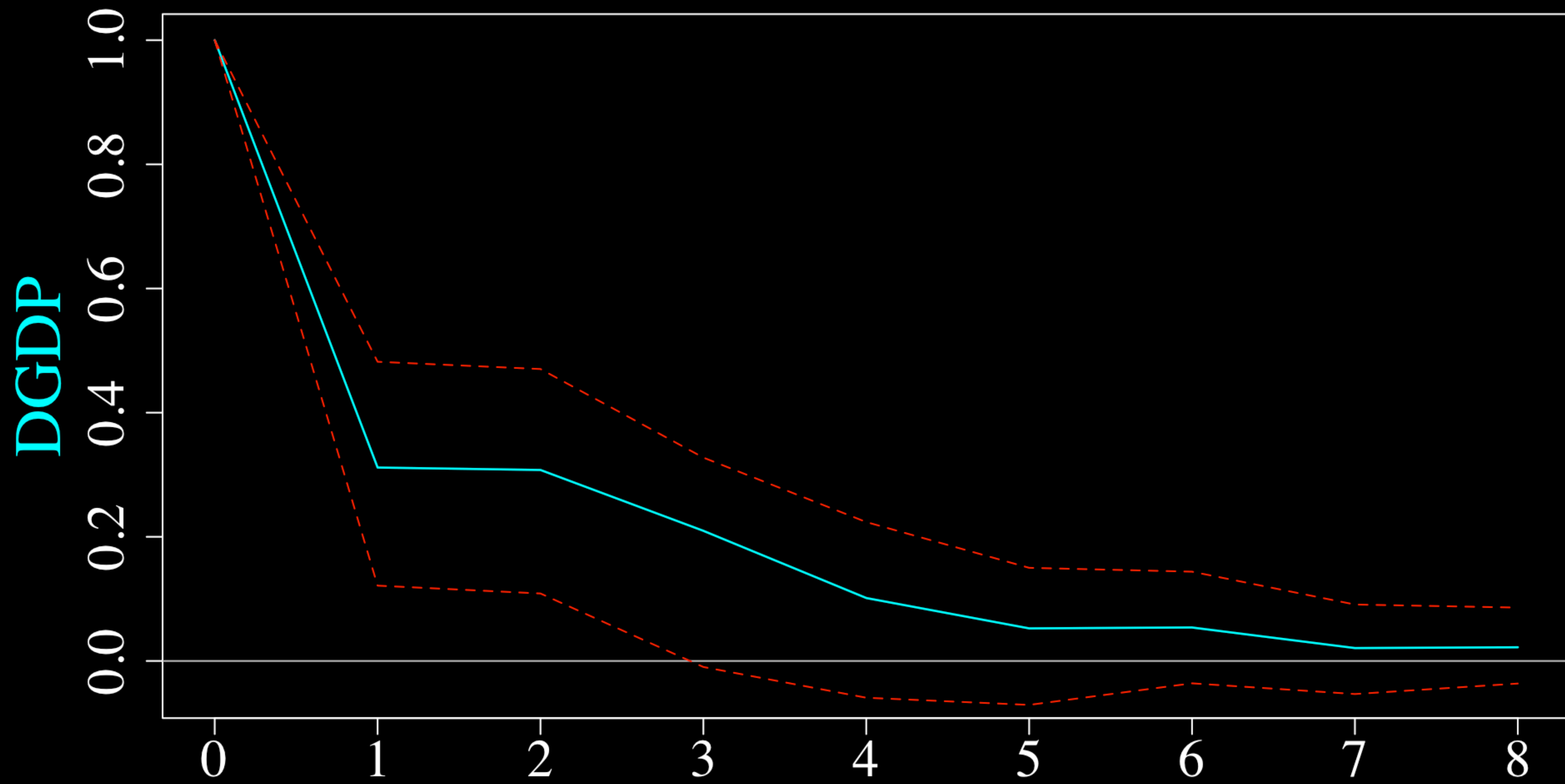
Impulse Response Application

```r
IRF <- irf(VAR_TGY,
           impulse="DGDP",
           ortho=FALSE,
           response=c("DTax","DGovt", "DGDP"),
           n.ahead=8)
```

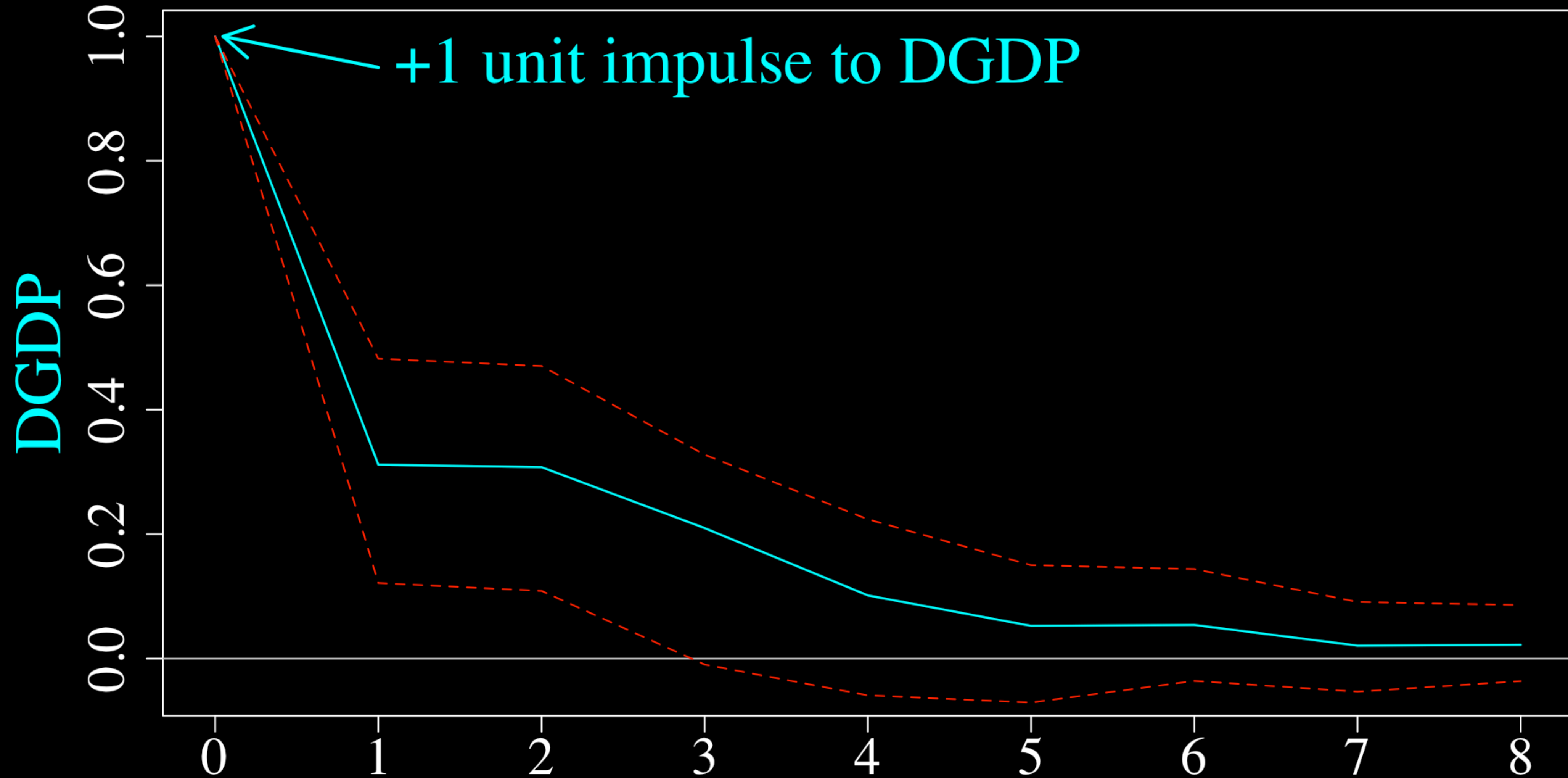
Impulse Response from DGDP



Impulse Response from DGDP



Impulse Response from DGDP



Summary

- Vector autoregression provides a natural extension of univariate AR models.
- Model selection: AIC and residual autocorrelation tests
- Granger "causality": adding a variable to the model improves forecasts of another variable.
- Impulse responses: how do forecasts change in response to an "impulse" to a variable in the model?