Econometrics 3 (ECOM90013) Assignment 2

1. **Let $Y_1, Y_2, \dots, Y_n$ denote a simple random sample of size n from a Normal population with mean μ and variance 1. Consider the first observation Y_1 as an estimator for μ.**

   a. **(1 mark) Show that Y_1 is an unbiased estimator for μ.**

An estimator can be shown to be unbiased if its expectation is equal to the true parameter value. Given the distribution:

$$Y_1 \sim N(\mu, 1)$$

By definition:

$$E[Y_1] = \mu$$

Hence, Y_1 is an unbiased estimator of μ.

   b. **(2 marks) Find $\Pr(|Y_1 - \mu| \leq 1)$.**

Given we know $Y_1 \sim N(\mu, 1)$, we define the standard normal variable as the following:

$$Z = \frac{Y_1 - \mu}{\sqrt{1}} = \Pr(|Z| \leq 1)$$

From the standard normal table we know that:

$$\Pr(Z \leq 1) \approx 0.841, \quad \Pr(Z \leq -1) \approx 0.159$$

Therefore:

$$\Pr(-1 \leq Z \leq 1) = 0.841 - 0.1587 = 0.683$$

Hence:

$$\Pr(|Y_1 - \mu| \leq 1) = 0.683$$

There's about a 68% chance that the observed Y_1 is within 1 unit of the true mean μ.

   c. **(2 marks) Based on your answer to 1(b), is Y_1 a consistent estimator for μ? Explain your answer both theoretically and intuitively.**

Y_1 is not a consistent estimator for μ. Proving consistency requires demonstrating convergence in probability. To do this, we must show Y_1 is both an unbiased estimator of

μ and that its variance converges to zero as the number of observations approach infinity. We have already showed the prior (Q1a) but are unable to show the latter. Y_1's variance does not converge to zero as the number of observations approaches infinity because it's variance is static (independent of the number of observation) as it is a single element of the sequence Y_n. Typically, we apply the weak law of large numbers to establish consistency, but we're unable to do this because Y_n is just a single element of Y_n and therefore has a static variance.

More formally:

$$Var(Y_1) = 1$$

$$\operatorname*{plim}_{n\to\infty} Var(Y_1) = 1 \neq 0$$

Therefore:

$$\operatorname*{plim}_{n\to\infty} \Pr(|Y_1 - Y| \geq \varepsilon) = 1 \neq 0$$

Y_1 is not a consistent estimator for μ is the latter does not converge to the prior in probability.

2. The Constant Elasticity of Substitution (CES) production function is of the form

$$\boldsymbol{Q = A(\delta K^{1-\rho} + (1-\delta)L^{-\rho})^{-1/\rho}},$$

Where K and L are the factor inputs, capital and labour say, and the parameters of the function are $A > 0, 0 < \delta < 1$ and $-1 < \rho \neq 0$. In this model the elasticity of substitution can be shown to be $\epsilon = 1/(1+\rho)$. Suppose that you have an estimator for the parameters of the CES production function with joint limiting distribution of the form

$$\boldsymbol{\sqrt{n}\left(\begin{bmatrix}\hat{A}\\ \hat{\delta}\\ \hat{\rho}\end{bmatrix} - \begin{bmatrix}A\\ \delta\\ \rho\end{bmatrix}\right) \xrightarrow{d} N(0,\Sigma), \qquad \Sigma = \begin{bmatrix}\sigma_A^2 & \sigma_{A\delta} & \sigma_{A\rho}\\ \sigma_{A\delta} & \sigma_\delta^2 & \sigma_{\delta\rho}\\ \sigma_{A\rho} & \sigma_{\delta\rho} & \sigma_\rho^2\end{bmatrix}}$$

a) (1 mark) what is the marginal limiting distribution of $\hat{\rho}$?

The marginal limiting distribution of $\hat{\rho}$ is simply the portion of the distribution above which concerns purely $\hat{\rho}$:

$$\hat{\rho} \xrightarrow{d} N\left(\rho, \frac{\sigma_\rho^2}{n}\right)$$

ρ is $\hat{\rho}$'s true parameter value, and $\frac{\sigma_\rho^2}{n}$ is its asymptotic variance which shrinks as the number of observations increases.

b) **(4 marks) If $\widehat{\boldsymbol{\Sigma}}$ denotes a consistent estimator for $\boldsymbol{\Sigma}$, derive an operational 95% confidence interval for ε, where:**

$$\boldsymbol{\Sigma} = \begin{bmatrix} \widehat{\boldsymbol{\sigma}}_A^2 & \widehat{\boldsymbol{\sigma}}_{A\delta} & \widehat{\boldsymbol{\sigma}}_{A\rho} \\ \widehat{\boldsymbol{\sigma}}_{A\delta} & \widehat{\boldsymbol{\sigma}}_\delta^2 & \widehat{\boldsymbol{\sigma}}_{\delta\rho} \\ \widehat{\boldsymbol{\sigma}}_{A\rho} & \widehat{\boldsymbol{\sigma}}_{\delta\rho} & \widehat{\boldsymbol{\sigma}}_\rho^2 \end{bmatrix}$$

By 'operational' is meant that your answer cannot depend upon any unknown parameters. Be sure to include all steps of your derivation.

To derive confidence intervals, we need to apply the Delta Method to our distribution given ρ has been non-linearly transformed. The Delta Method states:

$$n^{\frac{1}{2}}[g(\hat{\theta}) - g(\theta)] \xrightarrow{d} N(0, [g'(\mu)]^2\sigma^2)$$

Simplifying to:

$$g(\hat{\theta}) \sim^a \ N(g(\theta), \frac{1}{n}[g'(\theta)]^2\sigma^2)$$

In this question, $g(\hat{\theta}) = (1+\hat{\rho})^{-1}$ and $g'(\theta) = (1+\rho)^{-2}$. Therefore:

$$(1+\hat{\rho})^{-1} \sim^a N((1+\rho)^{-1}, \frac{1}{n}(1+\rho)^{-4}\sigma^2)$$

$$(1+\hat{\rho})^{-1} \sim^a N\left((1+\rho)^{-1}, \frac{\sigma_\rho^2}{n(1+\rho)^4}\right)$$

Now we have an expression for the distribution of ε, we can derive confidence intervals. Specifically:

$$P\left(Z_{0.025} \leq \frac{g(\hat{\theta}) - g(\theta)}{\sigma_\theta} \leq Z_{0.975}\right) = 0.95$$

By symmetry of normal means around zero, if $Z_{0.975} \equiv Z$ then $Z_{0.025} \equiv -Z$.

$$-Z \leq \frac{g(\hat{\theta}) - g(\theta)}{sd(\hat{\theta})} \leq Z$$

Now, substituting in the relevant terms:

$$-Z \leq \frac{(1+\hat{\rho})^{-1} - (1+\rho)^{-1}}{\frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2}} \leq Z$$

Simplifying:

$$-Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} \leq (1+\hat{\rho})^{-1} - (1+\rho)^{-1} \leq Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2}$$

$$-Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1} \leq -(1+\rho)^{-1} \leq Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1}$$

$$-Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1} \leq -(1+\rho)^{-1} \leq Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1}$$

$$\frac{1}{-Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1}} - 1 \leq \rho \leq \frac{1}{Z * \frac{\sigma_\rho}{\sqrt{n}(1+\rho)^2} + (1+\hat{\rho})^{-1}} - 1$$

To make this confidence interval operational, we need to get rid of all appearances of ρ and replace them with $\hat{\rho}$ as we do not know the true value of this parameter. Therefore it cannot appear in the standard error (outer part of the inequality). To do this, we use to Weak Law of Large Numbers (WLLN), which tells us estimators converge to true parameters as the sample size gets large.

More formally, due to WLLN, we assume:

$$\hat{\rho} \xrightarrow{d} \rho$$

Furthermore, by Continuous Mapping Theorem (CMT) we can also assume the non-linear transformation of $\hat{\rho}$ converges to ρ in probability. That is:

$$g(\hat{\rho}) \xrightarrow{d} g(\rho)$$

Therefore, our operational confidence intervals are:

$$\frac{1}{-Z * \frac{\sigma_\rho}{\sqrt{n}(1+\hat{\rho})^2} + (1+\hat{\rho})^{-1}} - 1 \leq \rho \leq \frac{1}{Z * \frac{\sigma_\rho}{\sqrt{n}(1+\hat{\rho})^2} + (1+\hat{\rho})^{-1}} - 1$$