Question 3.

(a) For a time series Y_t with $t = 1, \ldots, n$, and $\mathscr{Y}_{t-1} = \{Y_{t-1}, \ldots, Y_1\}$, why do we use the conditional expectation $E(Y_t|\mathscr{Y}_{t-1})$ for one-step-ahead forecasting?

(b) Define the one-step-ahead prediction error $U_t = Y_t - E(Y_t|\mathscr{Y}_{t-1})$. Show that

(i) $E(U_t|\mathscr{Y}_{t-1}) = 0$

(ii) $E(U_t) = 0$

(iii) $E(U_t U_{t-j}) = 0$ for all $j = 1, 2, \ldots$.

(c) What is the implication of your answer to part (b) for practical time series model specification?

(d) Define and compare the concepts of *recursive* and *direct* forecasting for two-step-ahead forecasting.

(e) Are the one-step-ahead prediction errors U_t defined in part (b) necessarily stationary? If so, justify this. If not, what else is required for U_t to be stationary?

(f) Suppose $Y_t = U_t + \theta_1 U_{t-1}$ is an MA(1) time series where U_t is a stationary prediction error. Derive $E(Y_t)$, $\text{var}(Y_t)$, $\text{cov}(Y_t, Y_{t-1})$ and hence the first order autocorrelation $\text{cor}(Y_t, Y_{t-1})$. Are these expressions sufficient to conclude that Y_t is stationary?

(g) Use the expression for the first order autocorrelation $\rho_1 = \text{cor}(Y_t, Y_{t-1})$ in the previous part to work out the range of possible values for ρ_1 that can arise from an MA(1) model.

In case it's helpful, the quadratic formula for x that solves $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Suppose a time series produce a first order autocorrelation of 0.8. It is possible that an MA(1) model is appropriate for this time series?