ECON30024
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Solution to Tutorial 10

1. (1) In the dynamic asset pricing models discussed in Topic 9, we focus on an individual consumer's problem. Which of the following statements regarding the consumer is FALSE?

    i. She aims to maximise her lifetime utility or expected lifetime utility.
    ii. She may need to transfer wealth across periods to smooth her consumption.
    iii. She is risk averse.
    iv. She aims to achieve maximum consumption in each period of life.
    v. She can transfer wealth by saving through holding a portfolio of assets.

    Answer: (d)

    (2) In the dynamic asset pricing model with multiple assets and multiple periods that we discussed in Topic 9, what decisions does the consumer need to make?

    i. How much to consume in every period of life.
    ii. How much to save in every period of life except the last period of life.
    iii. The allocation of total savings on each asset, i.e. an optimal portfolio choice, in every period of life except the last period of life.
    iv. All of the rest.

    Answer: (d)

    (3) Which of the following statements is FALSE regarding the intertemporal consumption Euler equation for the dynamic asset pricing model with multiple assets and multiple periods?

    i. The consumer's optimal consumption plan must satisfy this equation.
    ii. This equation links the returns on assets with consumption allocations.
    iii. This equation implies that the expected returns on every asset must be equal.
    iv. This equation implies that at the optimal consumption and portfolio plan a marginal increase in the investment in any asset must lead to the same increase in the expected utility of the consumer.
    v. This equation is in fact an NPV equation with a stochastic discount factor.

    Answer: (c)

    (4) Which of the following statements is TRUE regarding the consumption CAPM?

    i. The consumption CAPM is derived from the intertemporal consumption Euler equation.
    ii. In the consumption CAPM, the stochastic discount factor plays the role of the market return in the CAPM.

iii. In the consumption CAPM, the expected return on an asset that has zero correlation with the discount factor plays the role of ω in the Black CAPM.

iv. An asset's beta coefficient in the consumption CAPM measures its risk due to the comovement of its return with the stochastic discount factor.

v. All of the rest.

Answer: (e)

(5) Unlike the CAPM, the consumption CAPM does not involve a theory of portfolio selection by individual investors. True/False?

Answer: False. The consumption CAPM is derived from the intertemporal consumption Euler equation, which is derived from an individual consumer's optimal consumption and portfolio selection problem.

2. (a) The investor's utility maximisation problem is to choose a consumption plan (C_t, C_{t+1}) to maximisie her lifetime utility, where the consumption plan must satisfy her budget constraint.

Her lifetime utility function is given by

$$U(C_t, C_{t+1}) = u(C_t) + \delta u(C_{t+1}) = \frac{C_t^{1-\gamma}}{1-\gamma} + \delta \frac{C_{t+1}^{1-\gamma}}{1-\gamma}.$$

Her budget constraints in period t and $t+1$ are given by

$$C_t + S_t = W_t,$$
$$C_{t+1} = (1 + r_{t+1}) S_t,$$

respectively. Combining the two periodic budget constraints yields the investor's lifetime budget constraint:

$$C_{t+1} = (1 + r_{t+1})(W_t - C_t), \quad \text{equivalently}$$

$$C_t + \frac{C_{t+1}}{1 + r_{t+1}} = W_t.$$

Therefore, the investor's utility maximisation problem is formulated as

$$\max_{C_t, C_{t+1}} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \delta \frac{C_{t+1}^{1-\gamma}}{1-\gamma} \right\}$$

subject to

$$C_t + \frac{C_{t+1}}{1 + r_{t+1}} = W_t.$$

(b) The utility maximisation problem formulated above is constrained maximisation problem. We can use the constraint to substitute out C_{t+1} in the objective function to transform the problem into an unconstrained maximisation problem:

$$\max_{C_t} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \delta \frac{\big((1 + r_{t+1})(W_t - C_t)\big)^{1-\gamma}}{1-\gamma} \right\}$$

Now, C_t is the only unknown variable. The first-order condition with respect to C_t obtained by differentiating the objective function with respect to C_t:

$$C_t^{-\gamma} + \delta\big((1+r_{t+1})(W_t - C_t)\big)^{-\gamma}(1+r_{t+1})(-1) = 0.$$

Simplifying gives

$$C_t^{-\gamma} = \delta\big((1+r_{t+1})(W_t - C_t)\big)^{-\gamma}(1+r_{t+1}),$$

or equivalently,

$$\frac{C_t^{-\gamma}}{\delta C_{t+1}^{-\gamma}} = 1 + r_{t+1}.$$

This is the intertemporal consumption Euler equation.

Note that the left hand side of the equation above is indeed $MRS \equiv \frac{\partial U/\partial C_t}{\partial U/\partial C_{t+1}}$, because $U(C_t, C_{t+1}) = \frac{C_t^{1-\gamma}}{1-\gamma} + \delta \frac{C_{t+1}^{1-\gamma}}{1-\gamma}$ so that

$$\frac{\partial U}{\partial C_t} = u'(C_t) = C_t^{-\gamma}, \quad \frac{\partial U}{\partial C_{t+1}} = \delta u'(C_{t+1}) = \delta C_{t+1}^{-\gamma}.$$

Therefore, the Euler equation obtained here is a special case of the Euler equation we derived in the lecture for a general utility function.

(c) The optimal consumption plan is determined by two equations: the Euler equation and the lifetime budget constraint.

The Euler equation above can be rewritten as

$$\left(\frac{C_{t+1}}{C_t}\right)^{\gamma} = \delta(1 + r_{t+1})$$

$$\Rightarrow \frac{C_{t+1}}{C_t} = [\delta(1+r_{t+1})]^{\frac{1}{\gamma}}$$

$$\Rightarrow C_{t+1} = [\delta(1+r_{t+1})]^{\frac{1}{\gamma}} C_t$$

Plugging this expression into the lifetime budget constraint:

$$C_t + \frac{[\delta(1+r_{t+1})]^{\frac{1}{\gamma}} C_t}{1 + r_{t+1}} = W_t$$

$$\Rightarrow \left[1 + \delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}\right] C_t = W_t$$

$$\Rightarrow C_t^* = \frac{1}{1 + \delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}} W_t$$

Then

$$
\begin{aligned}
S_t^* &= W_t - C_t = \left[1 - \frac{1}{1+\delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}}\right] W_t \\
&= \frac{\delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}}{1+\delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}} W_t = \frac{1}{\frac{1}{\delta^{\frac{1}{\gamma}}(1+r_{t+1})^{\frac{1}{\gamma}-1}}+1} W_t \\
&= \frac{1}{1+\delta^{-\frac{1}{\gamma}}(1+r_{t+1})^{1-\frac{1}{\gamma}}} W_t \\
C_{t+1}^* &= (1+r_{t+1})S_t = \frac{1+r_{t+1}}{1+\delta^{-\frac{1}{\gamma}}(1+r_{t+1})^{1-\frac{1}{\gamma}}} W_t
\end{aligned}
$$

(d) The solution above shows that the optimal saving is given by

$$
S_t^* = \frac{1}{1+\delta^{-\frac{1}{\gamma}}(1+r_{t+1})^{1-\frac{1}{\gamma}}} W_t.
$$

Given that $-\frac{1}{\gamma} < 0$, a higher δ leads to a lower $\delta^{-\frac{1}{\gamma}}$, which leads to a higher S_t^*. That is, S_t^* increases with δ, consistent with the economic intuition: a higher δ implies that the investor puts a higher weight on second period utility, so she would save more in her first period of life.

To see how S_t^* depends on r_{t+1}, first note that S_t^* decreases with $(1+r_{t+1})^{1-\frac{1}{\gamma}}$. However, whether $(1+r_{t+1})^{1-\frac{1}{\gamma}}$ increases or decreases with r_{t+1} depends on the value of γ. Given that $\gamma \neq 1$, there are two cases:

(i) $0 < \gamma < 1$, then $1 - \frac{1}{\gamma} < 0$ such that $(1+r_{t+1})^{1-\frac{1}{\gamma}}$ decreases with r_{t+1}, and as a result, S_t^* increases with r_{t+1}.

(ii) $\gamma > 1$, then $1 - \frac{1}{\gamma} > 0$ such that $(1+r_{t+1})^{1-\frac{1}{\gamma}}$ increases with r_{t+1}, and as a result, S_t^* decreases with r_{t+1}.

Recall that γ is the coefficient of constant relative risk aversion, given the CRRA utility function $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$ (Topic 3 slides). The result above suggests if the investor is more risk averse ($\gamma > 1$), she tends to decrease her saving as the rate of return on saving increases.

3. With the CRRA utility function $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$,[1] the stochastic discount factor

$$
H \equiv \delta \frac{u'(C_{t+1})}{u'(C_t)} = \delta \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} = \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}.
$$

[1] The derivation in this question is not examinable. The purpose of this question is to show you how you can derive an empirically testable prediction from the theoretical model and to give you some intuition of the consumption CAPM.

Define the growth rate of consumption as $c \equiv \log\left(\frac{C_{t+1}}{C_t}\right)$, then

$$H = \delta\left(e^c\right)^{-\gamma} = \delta e^{-\gamma c} \approx \delta(1-\gamma c),$$

where the last approximation comes from the following approximation result (this result is widely used in transforming nonlinear equations into linear equations):

$$e^x \approx 1 + x \quad \text{if } x \text{ is not far from } 0.$$

The consumption CAPM prediction is given by

$$\mu_j - \mu_0 = \beta_{jH}\theta_H,$$

where $\beta_{jH} = \frac{cov(r_j,H)}{var(H)}$ and $\theta_H \equiv -\frac{var(H)}{E(H)}$. With $H = \delta(1-\gamma c)$,

$$E(H) = \delta(1-\gamma E(c)),\ var(H) = \delta^2\gamma^2 var(c),\ cov(r_j, H) = -\delta\gamma cov(r_j, c),$$

the prediction becomes

$$\begin{aligned}
\mu_j - \mu_0 &= \frac{cov(r_j,H)}{var(H)}\left(-\frac{var(H)}{E(H)}\right) \\
&= \left(\frac{-\delta\gamma cov(r_j,c)}{\delta^2\gamma^2 var(c)}\right)\left(-\frac{\delta^2\gamma^2 var(c)}{\delta(1-\gamma E(c))}\right) \\
&= \frac{cov(r_j,c)}{var(c)}\frac{\gamma var(c)}{1-\gamma E(c)} \\
&\equiv \beta_{jc}\theta_c,
\end{aligned}$$

where

$$\beta_{jc} = \frac{cov(r_j,c)}{var(c)},\ \theta_c = \frac{\gamma var(c)}{1-\gamma E(c)}.$$

Also, note that the asset whose return has zero correlation with H also has zero correlation with c because H is perfectly negatively correlated with c ($H = \delta(1-\gamma c)$). So μ_0 is the expected return on an asset whose return has zero correlation with c.

Therefore, with CRRA utility function, the consumption CAPM prediction is given by the equation above, in which the single systematic factor that predicts expected asset returns is the growth rate of aggregate consumption.

Note that the average consumption growth rate in the data is a small number, so for reasonable values of γ (i.e., reasonable degree of risk aversion), $1 - \gamma E(c) > 0$ such that $\theta_c > 0$. Therefore, the expected return on an asset μ_j increases with β_{jc}, which is consistent with the CAPM prediction – expected return increases with beta.

The consumption growth rate c indicates the state of the economy. The prediction above implies that an asset whose rate of return is highly positively correlated with the state of the economy and hence has higher positive β_{jc} needs to deliver a higher expected return to make investors/consumers willing to hold such an asset. The intuition is that such kind of asset is less valuable to consumers as their returns tend to be high (low) when the economy is doing well (bad). On the contrary, an asset whose rate of return is highly negatively correlated with the growth rate of the economy tend to have high returns when the economy is doing bad, so consumers would like to hold such kind of assets as an insurance against bad states of the economy although the expected return is low.