Lecture 4
Moving Average Models

Review: Autoregressive Model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Autoregressive model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \phi_1 Z_{t-1}$$

Prediction depends on the previous value of Z_{t-1}.

Review: Autoregressive Model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Autoregressive model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \phi_1 Z_{t-1}$$

$$Z_t - E(Z_t \mid \mathscr{Z}_{t-1}) = Z_t - \phi_1 Z_{t-1}$$

Review: Autoregressive Model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Autoregressive model for deviations from trend:

$$E(Z_t \mid \mathscr{I}_{t-1}) = \phi_1 Z_{t-1}$$

$$Z_t - E(Z_t \mid \mathscr{I}_{t-1}) = Z_t - \phi_1 Z_{t-1}$$

One-step-ahead prediction error $\rightarrow$ $U_t = Z_t - \phi_1 Z_{t-1}$

Review: Autoregressive Model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Autoregressive model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \phi_1 Z_{t-1}$$

$$\begin{aligned} {\color{red}U_t} &= Z_t - \phi_1 Z_{t-1} \\ Z_t &= \phi_1 Z_{t-1} + {\color{red}U_t} \end{aligned}$$

Review: Autoregressive Model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Autoregressive model for deviations from trend:

$$Z_t = \phi_1 Z_{t-1} + U_t$$

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \theta_1 \left(Z_{t-1} - E(Z_{t-1} \mid \mathscr{Z}_{t-2})\right)$$

Prediction depends on the previous *prediction error*.

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$\begin{aligned} E(Z_t \mid \mathscr{Z}_{t-1}) &= \theta_1 \left(Z_{t-1} - E(Z_{t-1} \mid \mathscr{Z}_{t-2})\right) \\ &= \theta_1 \, U_{t-1} \end{aligned}$$

Moving average model

Deterministic trend model:

$$Y_t = X_t' \beta + Z_t$$

Moving average model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \theta_1 \, U_{t-1}$$

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \theta_1 \, U_{t-1}$$

$$Z_t - E(Z_t \mid \mathscr{Z}_{t-1}) = Z_t - \theta_1 \, U_{t-1}$$

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$E(Z_t \mid \mathscr{Z}_{t-1}) = \theta_1\, U_{t-1}$$

$$Z_t - E(Z_t \mid \mathscr{Z}_{t-1}) = \qquad \theta_1\, U_{t-1}$$

$$U_t \quad = \quad -\,\theta_1\, U_{t-1}$$

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$U_t \qquad = Z_t - \theta_1 \, U_{t-1}$$

$$Z_t \; = \; U_t \; + \; \theta_1 U_{t-1}$$

Moving average model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

Moving average model for deviations from trend:

$$Z_t = U_t + \theta_1 U_{t-1}$$

AR(p) model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

AR(p) model for deviations from trend:

$$Z_t \ = \ \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} + U_t$$

MA(q) model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

MA(q) model for deviations from trend:

$$Z_t \;=\; U_t + \theta_1 U_{t-1} + \ldots + \theta_q U_{t-q}$$

ARMA(p,q) model

Deterministic trend model:

$$Y_t = X_t'\beta + Z_t$$

ARMA(p,q) model for deviations from trend:

$$\begin{aligned} Z_t &= \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} \\ &\quad + U_t + \theta_1 U_{t-1} + \ldots + \theta_q U_{t-q} \end{aligned}$$

ARMA(p,q) model selection

Same *practical* approach as AR(p) selection:

- check residual autocorrelation tests
- choose model with smallest AICc

Also *in theory* we can look at autocorrelations of the time series.

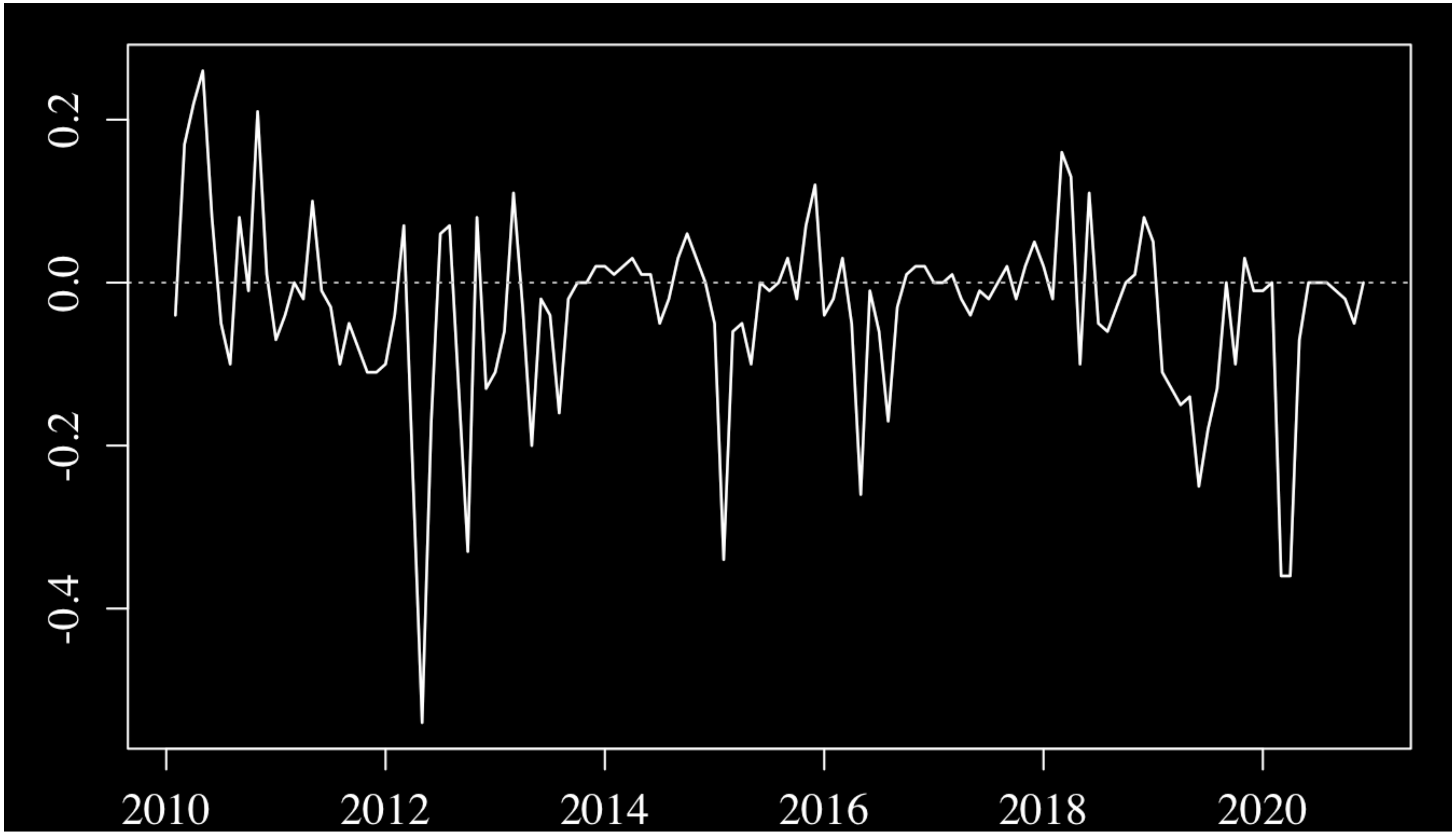
First difference of interest rates

(3 month Bank Accepted Bills)

```r
dt <- read.csv("BAB3mth.csv")
Y <- ts(dt$BAB3,
        start=c(2010,1),
        end=c(2025,6),
        frequency=12)
Y <- window(Y, end=c(2020,12))
DY <- diff(Y)
```

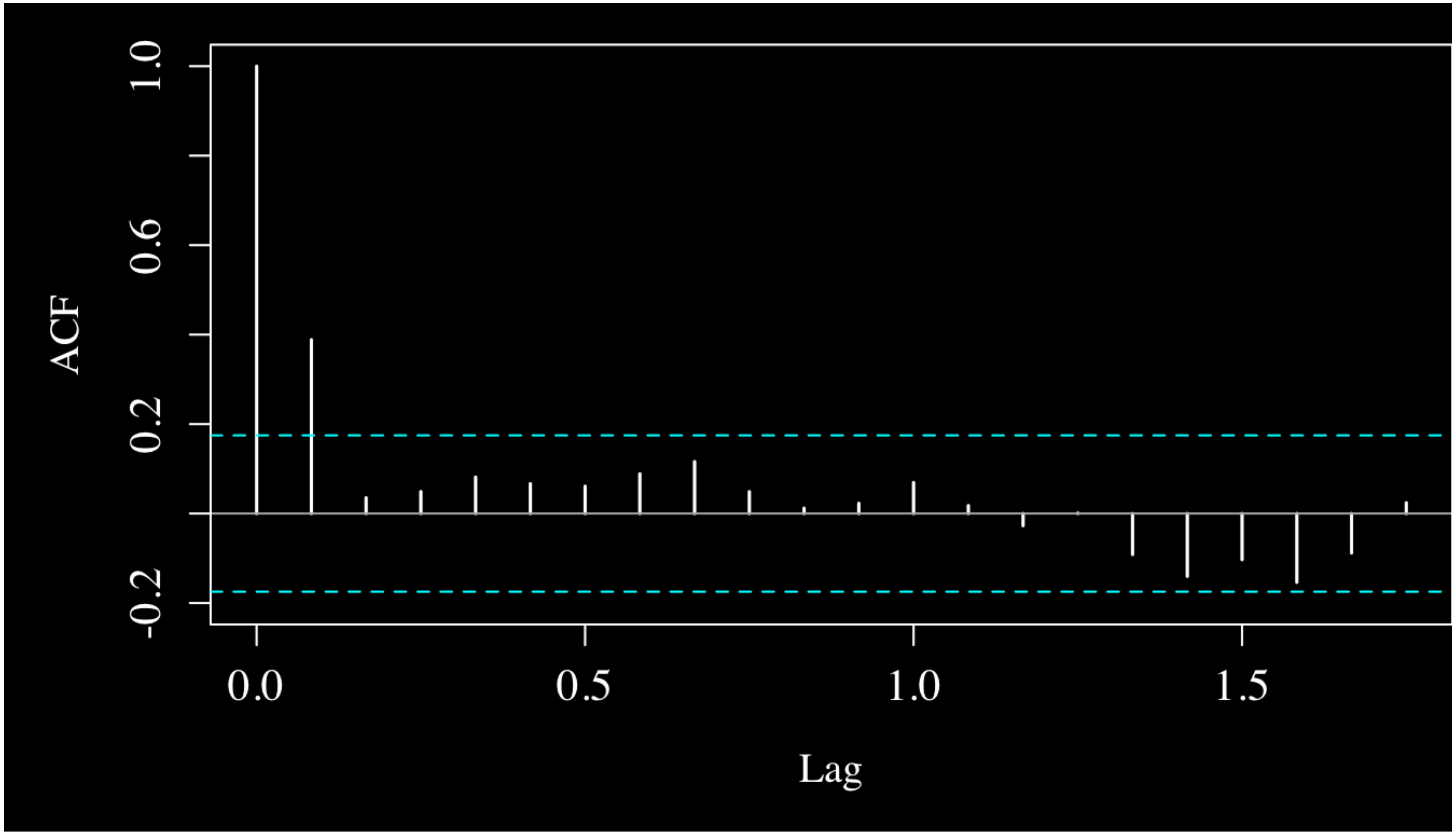
First difference of interest rates

(3 month Bank Accepted Bills)



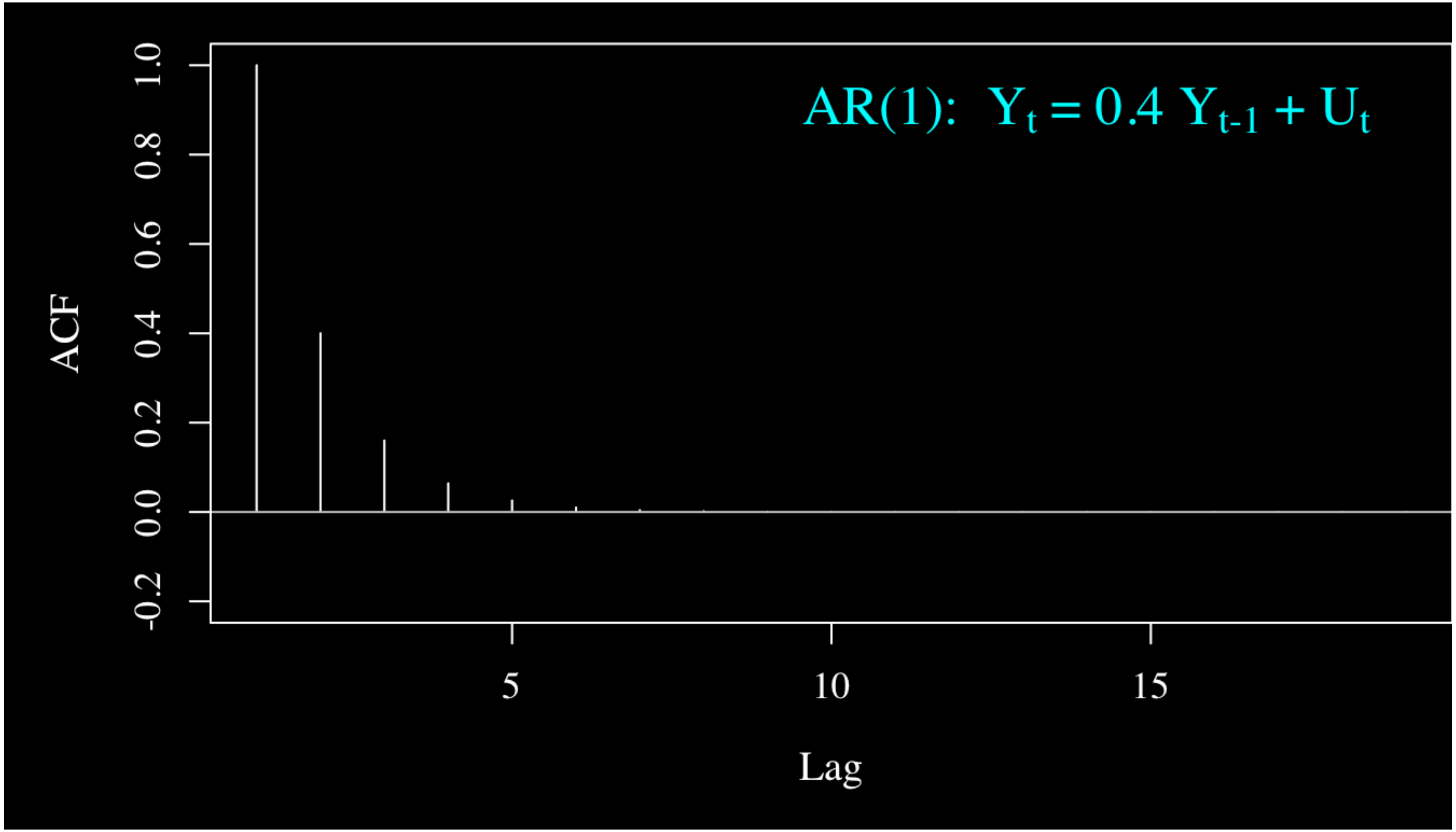
First difference of interest rates

```
acf(DY)
```



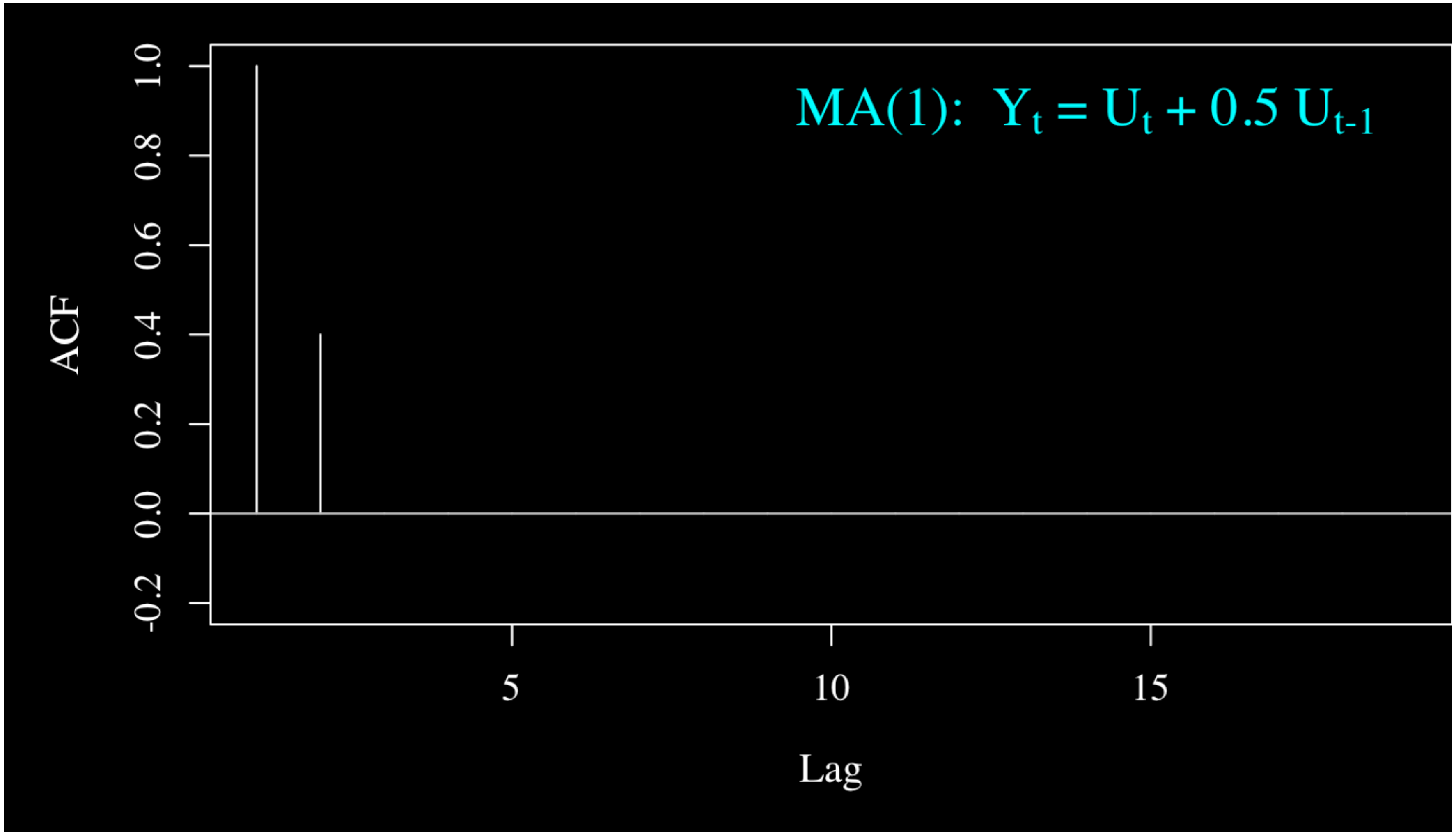
Theoretical autocorrelation function

```r
acf_AR1 <- ARMAacf(ar=0.4, lag.max=18)
```



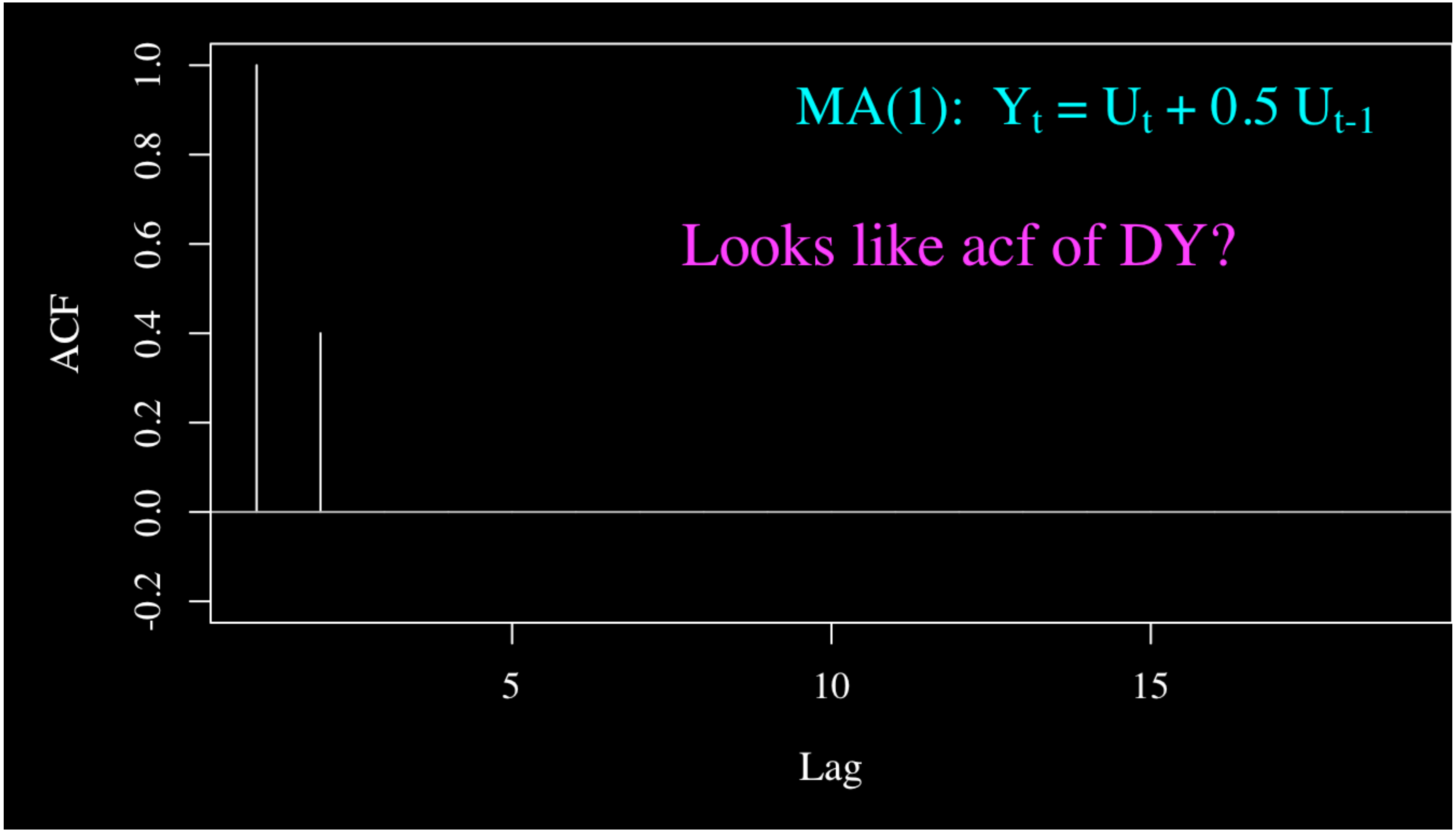
Theoretical autocorrelation function

```r
acf_MA1 <- ARMAacf(ma=0.5, lag.max=18)
```



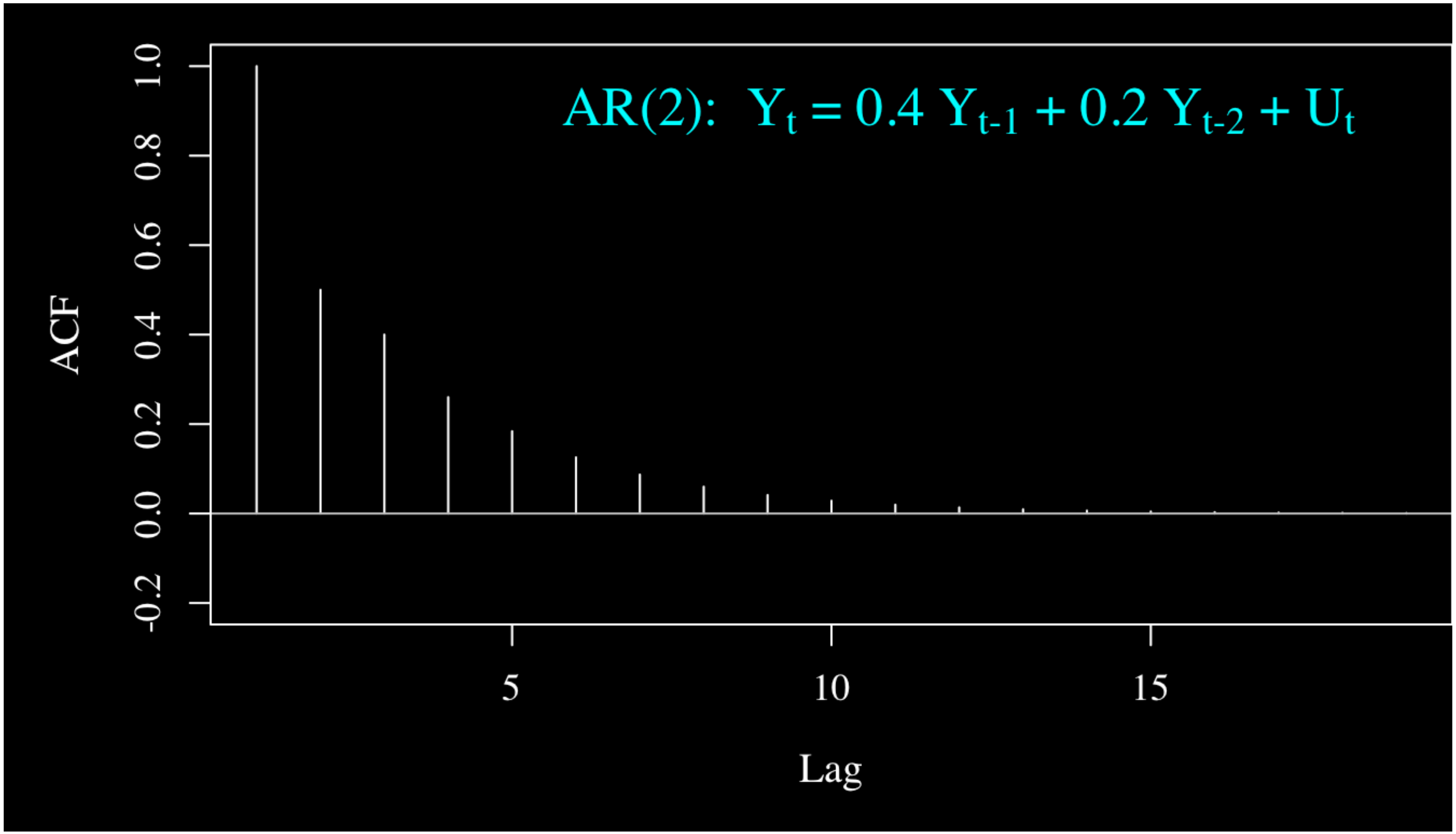
Theoretical autocorrelation function

```
acf_MA1 <- ARMAacf(ma=0.5, lag.max=18)
```



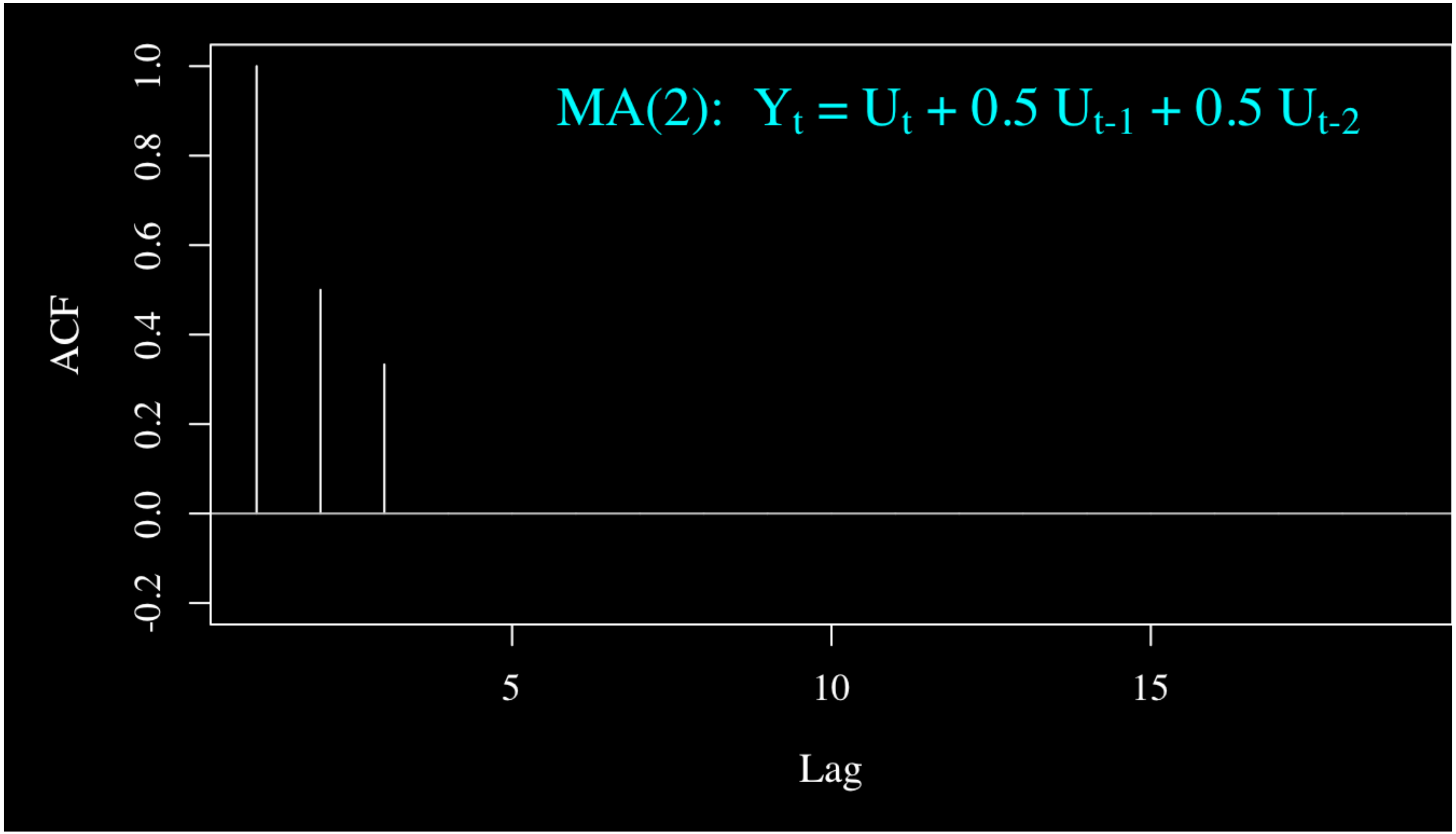
Some other theoretical acf's

```r
acf_AR2 <- ARMAacf(ar=c(0.4,0.2), lag.max=1
```



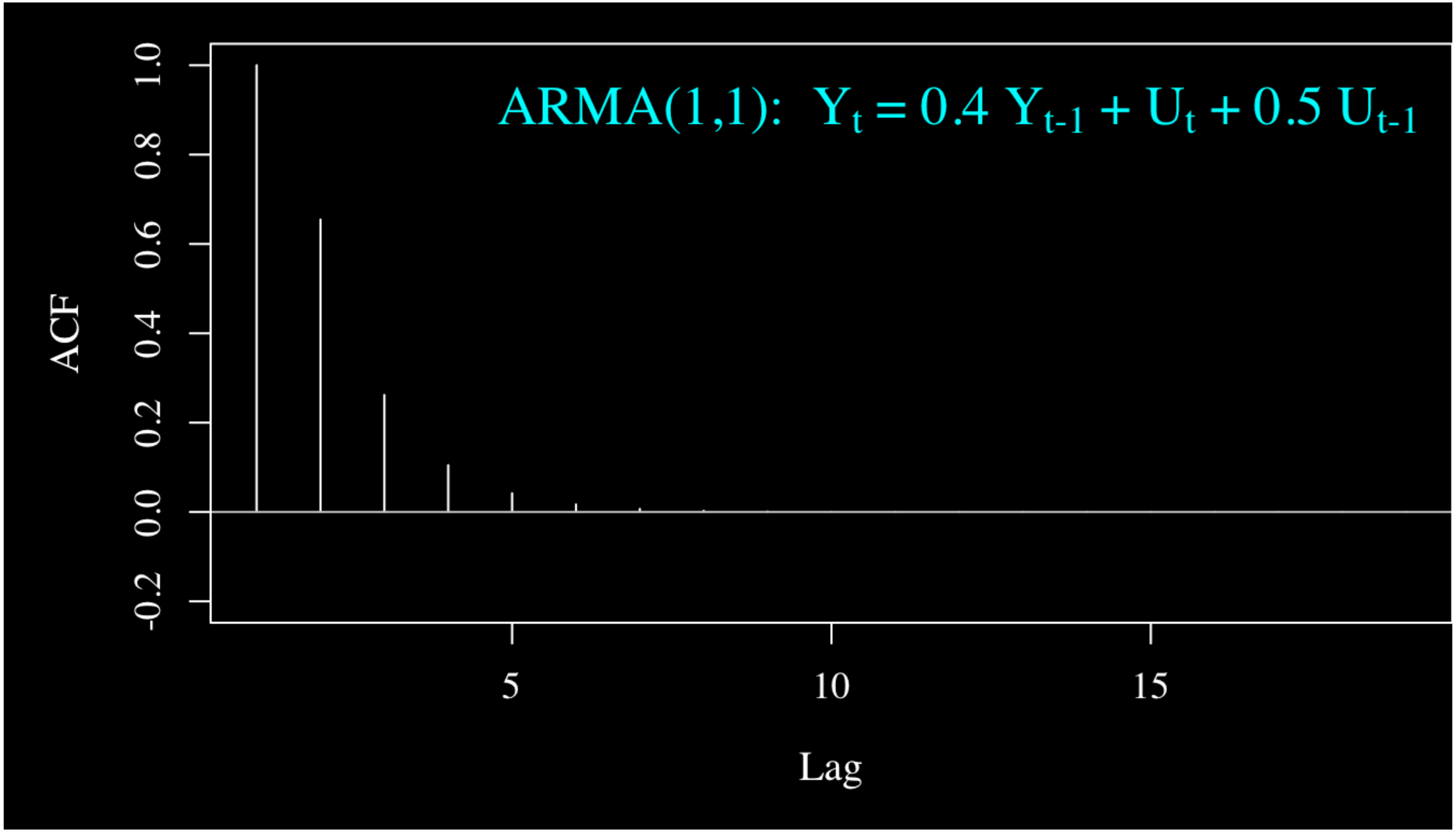
Some other theoretical acf’s

```
acf_MA2 <- ARMAacf(ma=c(0.5,0.5), lag.max=1
```



Some other theoretical acf's

```
acf_ARMA11 <- ARMAacf(ar=0.4, ma=0.5, lag.m
```



Partial autocorrelation function

Recall autocorrelation at lag k:

correlation between Y_t and Y_{t-k} for any $k = 1, 2, 3, \ldots$.

Partial autocorrelation at lag k:

correlation between Y_t and Y_{t-k} *after controlling for intermediate lags* $Y_{t-1}, \ldots, Y_{t-k+1}$.

Partial autocorrelation function

Partial autocorrelation at lag k:

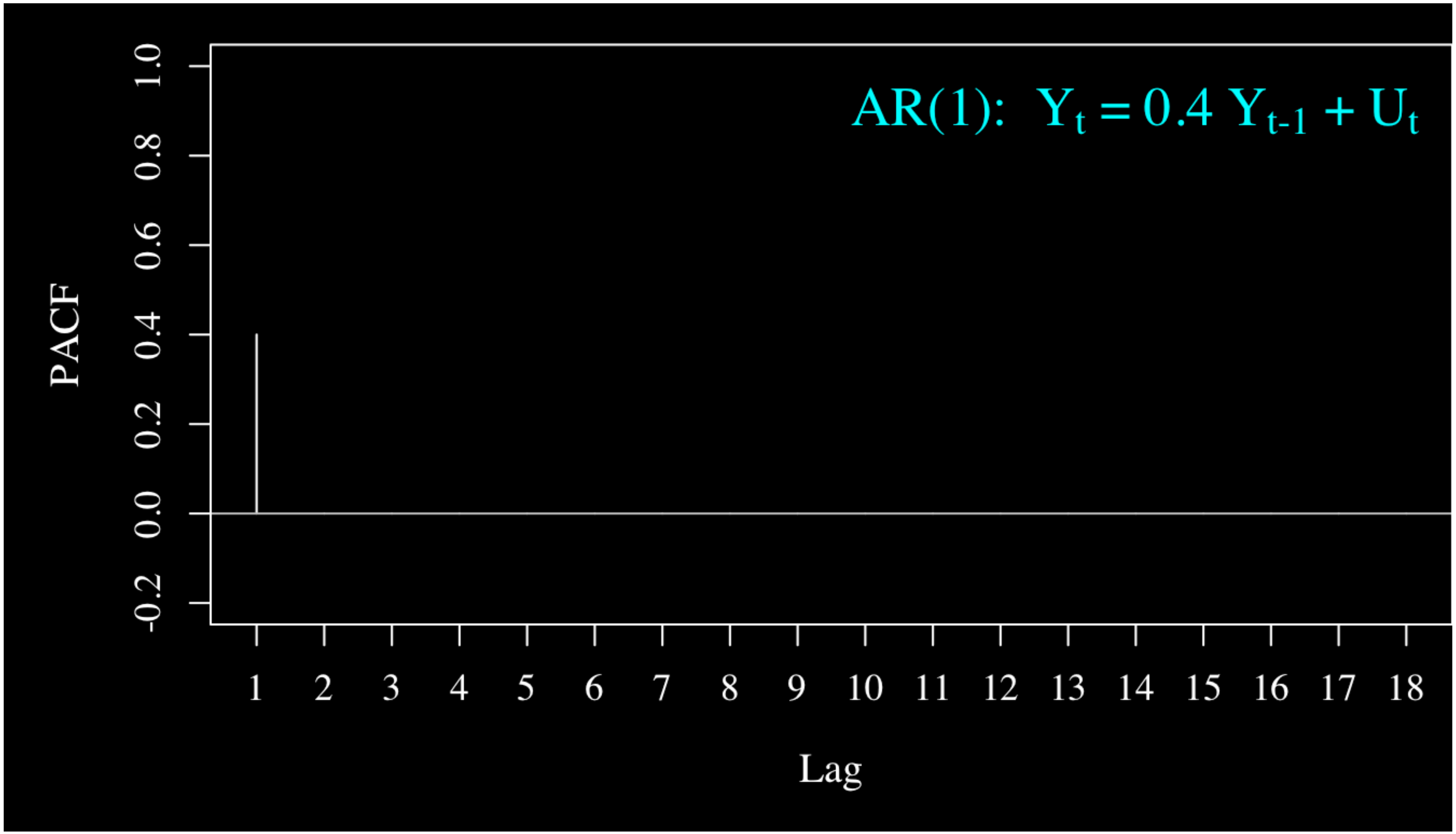
correlation between Y_t and Y_{t-k} *after controlling for intermediate lags* $Y_{t-1}, \ldots, Y_{t-k+1}$.

i.e. correlation between $Y^*_{t|k}$ and $Y^*_{t-k|k}$ where

$Y^*_{t|k}$ and $Y^*_{t-k|k}$ are residuals from regressions of Y_t and Y_{t-k} on $Y_{t-1}, \ldots, Y_{t-k+1}$.

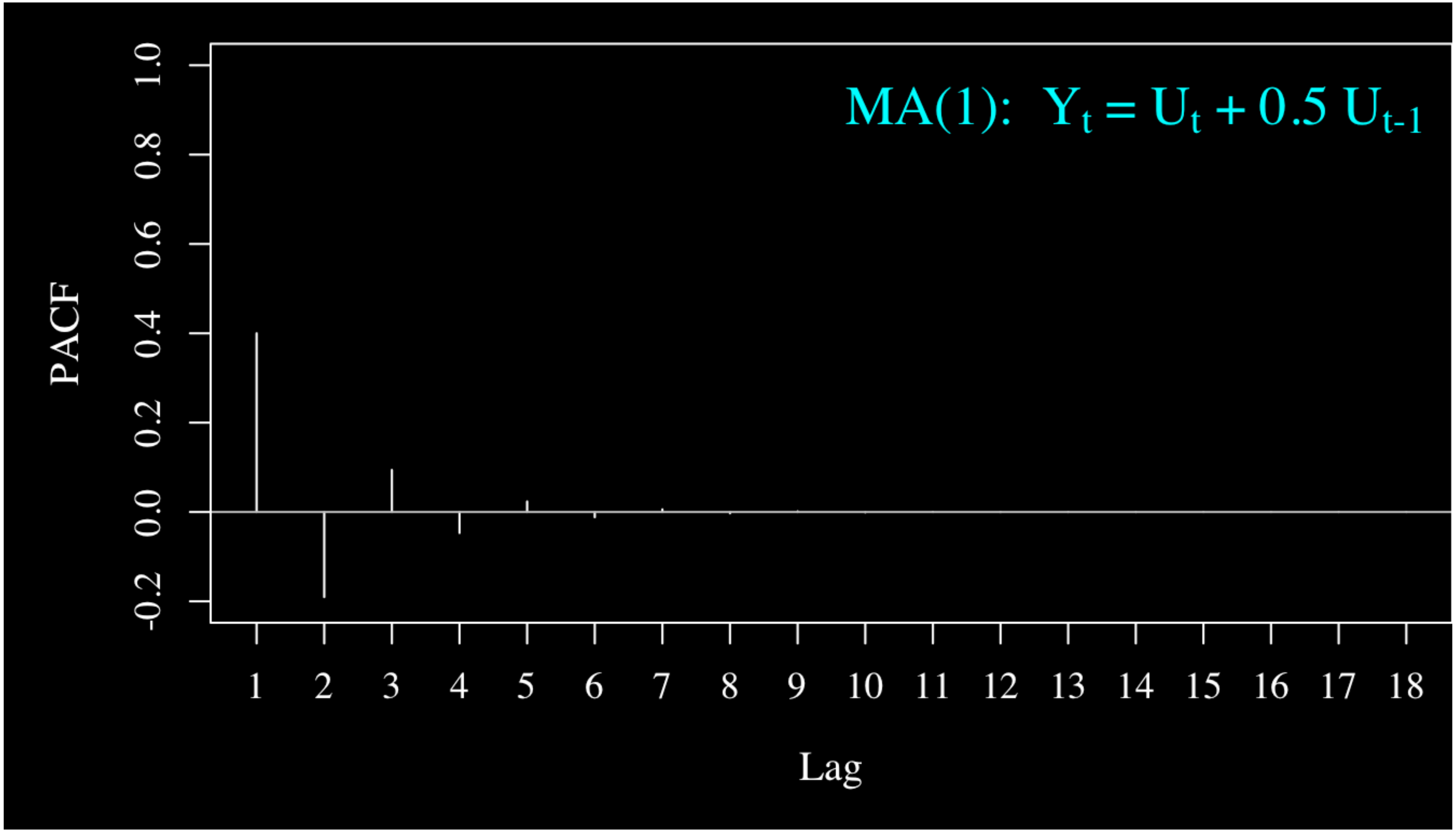
Theoretical partial autocorrelation function

```
pacf_AR1 <- ARMAacf(ar=0.4, pacf=TRUE, la
```



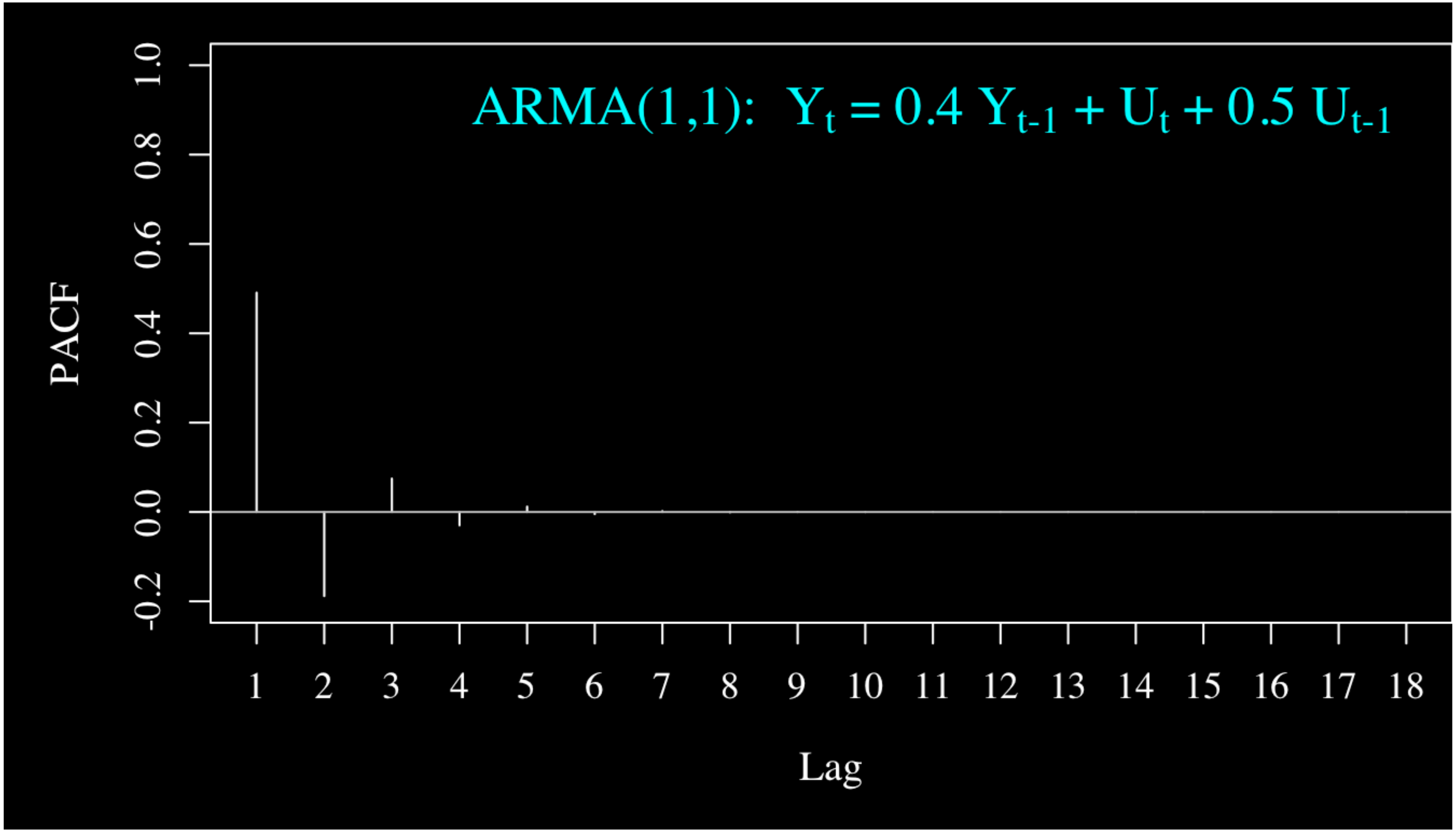
Theoretical partial autocorrelation function

```
pacf_MA1 <- ARMAacf(ma=0.5, pacf=TRUE, la
```



Theoretical partial autocorrelation function

```
pacf_ARMA11 <- ARMAacf(ar=0.2, ma=0.4, pacf=
```



Theoretical ACF and PACF patterns

As lag increases:

	ACF	PACF
$\mathrm{AR}(p)$	declines	zero after p lags
$\mathrm{MA}(q)$	zero after q lags	declines
$\mathrm{ARMA}(p, q)$	declines	declines

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\operatorname{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\operatorname{var}(U_t) = \sigma^2$ for every t.

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \qquad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

$$\mathrm{cov}(Z_t, Z_{t-1}) = \mathrm{cov}(U_t + \theta_1 U_{t-1},\ U_{t-1} + \theta_1 U_{t-2})$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \qquad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\operatorname{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\operatorname{var}(U_t) = \sigma^2$ for every t.

$$\begin{aligned}
\operatorname{cov}(Z_t, Z_{t-1}) &= \operatorname{cov}(U_t + \theta_1 U_{t-1},\ U_{t-1} + \theta_1 U_{t-2}) \\
&= \operatorname{cov}(U_t,\ U_{t-1}) + \theta_1 \operatorname{cov}(U_{t-1},\ U_{t-1}) \\
&\qquad + \theta_1 \operatorname{cov}(U_t,\ U_{t-2}) \\
&\qquad + \theta_1^2 \operatorname{cov}(U_{t-1},\ U_{t-2})
\end{aligned}$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

$$\begin{aligned}\mathrm{cov}(Z_t, Z_{t-1}) = \mathrm{cov}(U_t,\, U_{t-1}) &+ \theta_1 \mathrm{cov}(U_{t-1},\, U_{t-1}) \\ &+ \theta_1 \mathrm{cov}(U_t,\, U_{t-2}) \\ &+ \theta_1^2 \mathrm{cov}(U_{t-1},\, U_{t-2})\end{aligned}$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathbf{var}(U_t) = \sigma^2$ for every t.

$$\begin{aligned} \mathrm{cov}(Z_t, Z_{t-1}) = {} & \mathrm{cov}(U_t,\, U_{t-1}) + \theta_1 \mathbf{cov}(U_{t-1},\, U_{t-1}) \\ & + \theta_1 \mathrm{cov}(U_t,\, U_{t-2}) \\ & + \theta_1^2 \mathrm{cov}(U_{t-1},\, U_{t-2}) \end{aligned}$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

$$\begin{aligned}\mathrm{cov}(Z_t, Z_{t-1}) = \quad & 0 \qquad\qquad + \theta_1 \quad \sigma^2 \\ & + \theta_1 \quad 0 \\ & + \theta_1^2 \quad 0\end{aligned}$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

$\mathrm{cov}(Z_t, Z_{t-1}) = \theta_1 \sigma^2$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

$$\begin{aligned}\mathrm{cov}(Z_t, Z_{t-2}) = \mathrm{cov}(U_t,\, U_{t-2}) + \theta_1 \mathrm{cov}(U_{t-1},\, U_{t-2}) \\ + \theta_1 \mathrm{cov}(U_t,\, U_{t-3}) \\ + \theta_1^2 \mathrm{cov}(U_{t-1},\, U_{t-3}) \\ = 0\end{aligned}$$

Example: MA(1)

$$Z_t = U_t + \theta_1 U_{t-1}, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

- Recall $\mathrm{cov}(U_t, U_{t-k}) = 0$ for every $k = 1, 2, \ldots$.
- *Assume* $\mathrm{var}(U_t) = \sigma^2$ for every t.

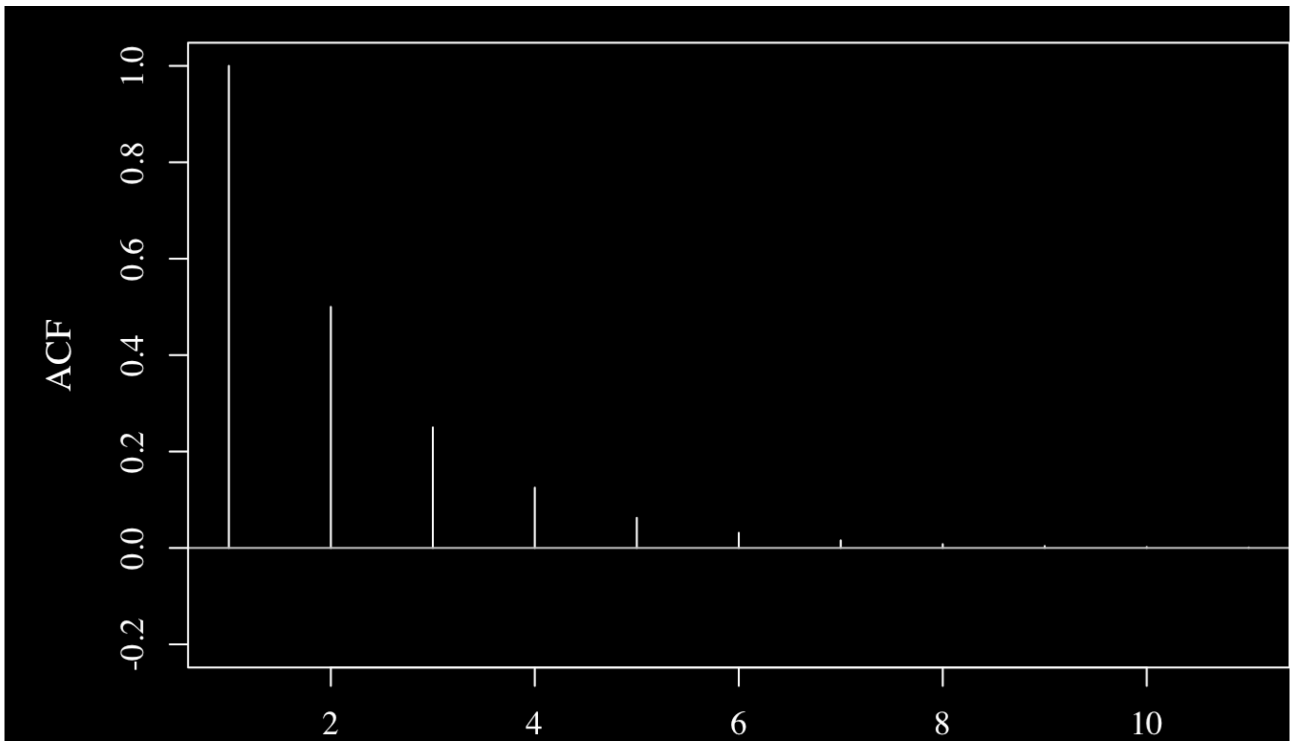
For any $k > 1$:

$$\begin{aligned}
\mathrm{cov}(Z_t, Z_{t-k}) &= \mathrm{cov}(U_t,\, U_{t-k}) + \theta_1 \mathrm{cov}(U_{t-1},\, U_{t-k}) \\
&\quad + \theta_1 \mathrm{cov}(U_t,\, U_{t-k-1}) \\
&\quad + \theta_1^2 \mathrm{cov}(U_{t-1},\, U_{t-k-1}) \\
&= 0
\end{aligned}$$

Example: AR(1)

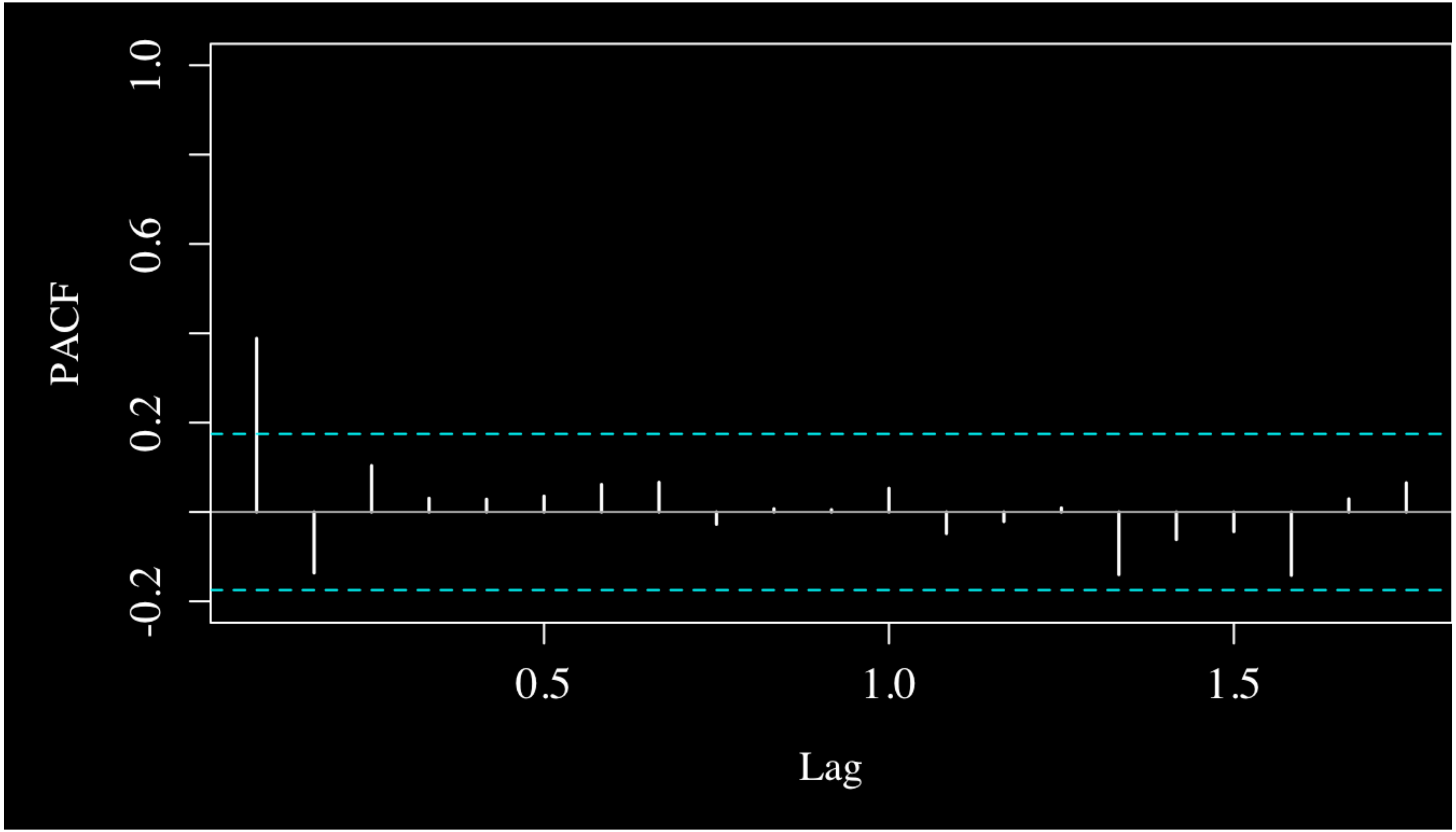
$$Z_t = \phi_1 Z_{t-1} + U_t, \quad U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

If $|\phi_1| < 1$ then $\text{cov}(Z_t, Z_{t-k}) = \phi_1^k$. (*Details later*)



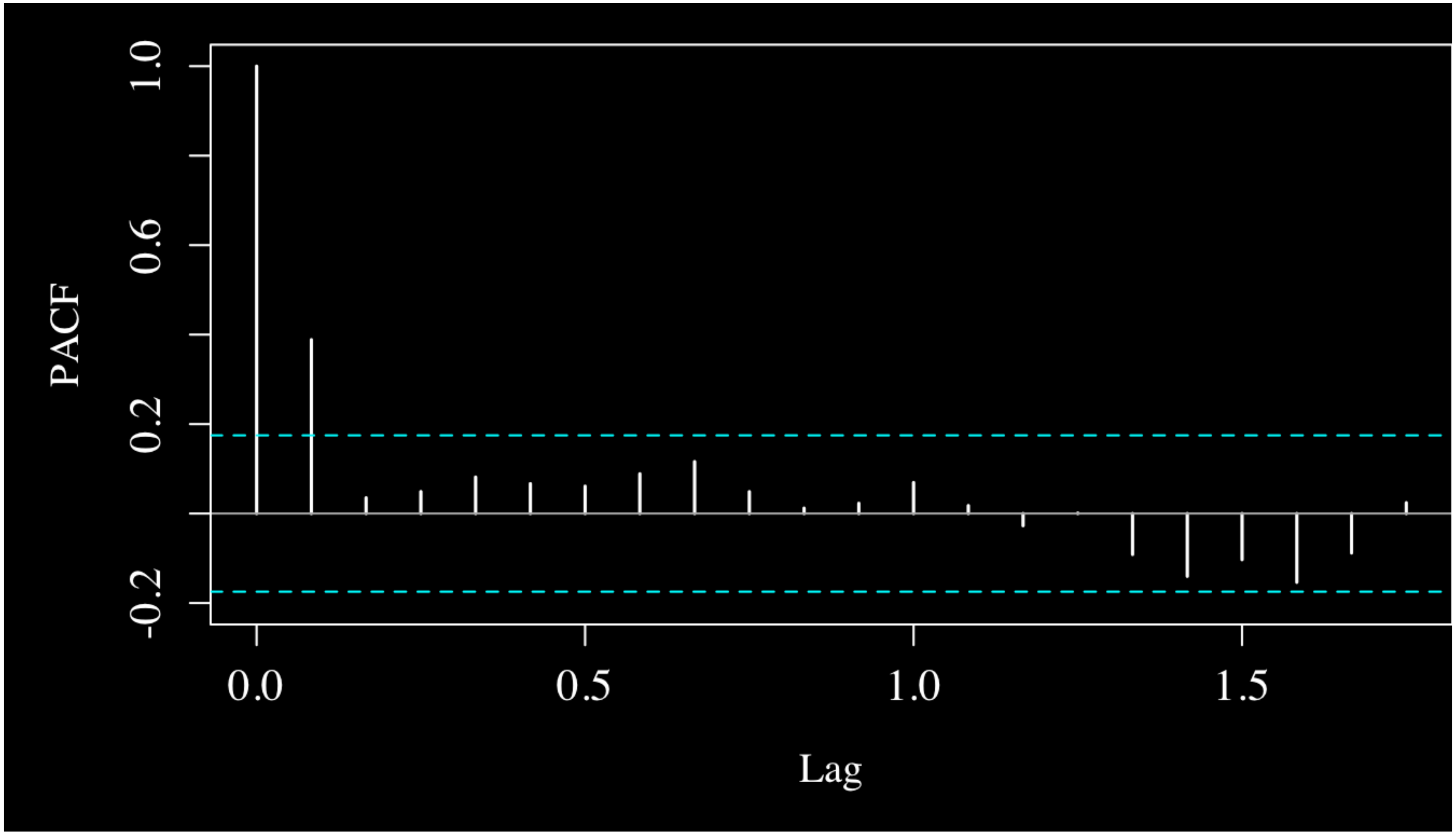
First difference of interest rates

```
acf(DY, type="partial")
```



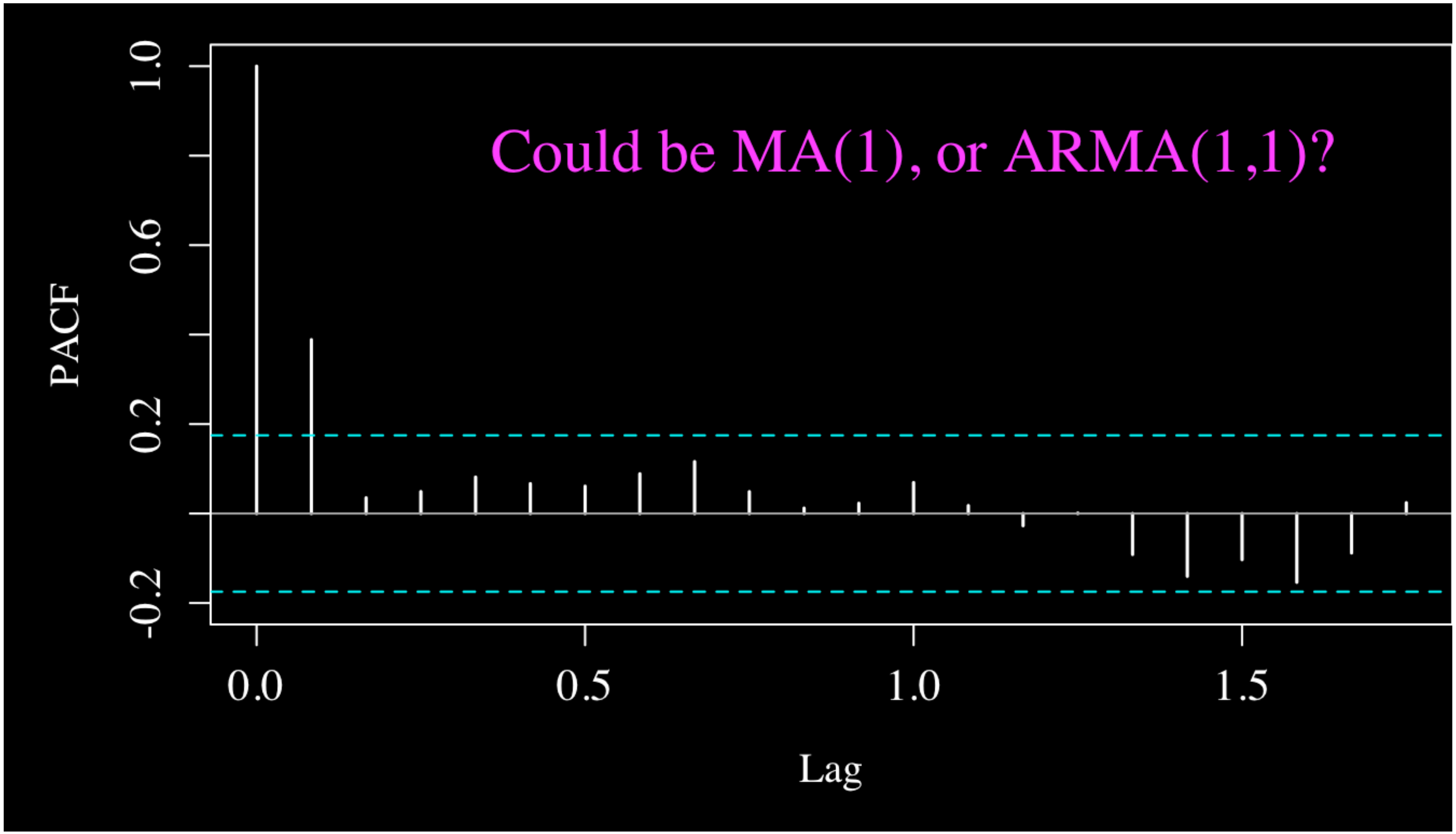
First difference of interest rates

```
acf(DY)
```



First difference of interest rates

```
acf(DY)
```



Model estimation: AR(1)

```r
AR1 <- Arima(DY, order=c(1,0,0))
print(AR1)
```

```
Series: DY
ARIMA(1,0,0) with non-zero mean

Coefficients:
         ar1     mean
      0.3858  -0.0318
s.e.  0.0801   0.0145
```

Model estimation: MA(1)

```r
MA1 <- Arima(DY, order=c(0,0,1))
print(MA1)
```

```
Series: DY
ARIMA(0,0,1) with non-zero mean

Coefficients:
         ma1     mean
      0.4434  -0.0320
s.e.  0.0762   0.0127
```

Model estimation: ARMA(1,1)

```r
ARMA11 <- Arima(DY, order=c(1,0,1))
print(ARMA11)
```

```
Series: DY
ARIMA(1,0,1) with non-zero mean

Coefficients:
         ar1     ma1    mean
      0.0558  0.3989  -0.032
s.e.  0.1915  0.1758   0.013
```

Residual autocorrelation testing

```r
LBp_AR1 <- checkresiduals(AR1)
```

```
	Ljung-Box test

data:  Residuals from ARIMA(1,0,0) with non-
zero mean
Q* = 19, df = 23, p-value = 0.7012

Model df: 1.   Total lags used: 24
```

Residual autocorrelation testing

```
LBp_MA1 <- checkresiduals(MA1)
```

```
	Ljung-Box test

data:  Residuals from ARIMA(0,0,1) with non-zero mean
Q* = 16.201, df = 23, p-value = 0.8465 ✓

Model df: 1.   Total lags used: 24
```

Residual autocorrelation testing

```r
LBp_ARMA11 <- checkresiduals(ARMA11)
```

```
	Ljung-Box test

data:  Residuals from ARIMA(1,0,1) with non-zero mean
Q* = 15.748, df = 22, p-value = 0.8282

Model df: 2.   Total lags used: 24
```

Model selection: AICc

```
          AICc
AR1    -219.24
MA1    -222.58
ARMA11 -220.54
```

- All three models pass autocorrelation tests.
- The AICc prefers the MA(1) model.

Summary

One step ahead prediction error:

$$U_t = Z_t - E(Z_t | \mathscr{Z}_{t-1})$$

$$\text{MA}(q): \qquad Z_t = U_t + \theta_1 U_{t-1} + \ldots + \theta_q U_{t-q}$$

$$\begin{aligned}\text{ARMA}(p,q): \qquad Z_t &= \phi_1 Z_{t-1} + \ldots + \phi_p Z_{t-p} \\ &\quad + U_t + \theta_1 U_{t-1} + \ldots + \theta_q U_{t-q}\end{aligned}$$

Select p, q using AICc and autocorrelation tests.

Summary

As lag increases:

	ACF	PACF
AR(p)	declines	zero after p lags
MA(q)	zero after q lags	declines
ARMA(p, q)	declines	declines