ECON30009/90080 – TUTORIAL 3

This Version: Semester 2, 2025

Note: these questions are designed to give you some practice solving the life-cycle OLG model but with slightly different assumptions relative to those we discussed in lectures.

Question 1: Solving for equilibrium in the OLG model

The following question asks you to consider a variation of the life-cycle OLG model with a minimal reference level of consumption over the lifetime. That is, we will assume that individuals must consume at least $\bar{c}$ each period and get utility only if their consumption in each period exceeds this minimum amount. Specifically, assume that households in each generation have the following preferences:

$$U(c_t^y, c_{t+1}^o) = \ln(c_t^y - \bar{c}) + \beta \ln\left(c_{t+1}^o - \bar{c}\right)$$

where $\bar{c} \geq 0$ is a parameter representing the minimum consumption level a household must have. β is a parameter representing the household's discount factor, i.e., the weight they put on consumption when old. Specifically, $0 < \beta < 1$. There are N households in every generation and there is no population growth. The household lives for two periods, and can choose to save in an asset a_{t+1} which gives gross return of $1 + r_{t+1}$ in period $t+1$. Individuals when young supply one unit of labour inelastically and earn a wage w_t per unit of labour supplied. Individuals when old retire and do not have any labour income. They instead consume their savings.

In addition to households in the economy, firms produce output according to a Cobb-Douglas production function $Y_t = zK_t^\alpha L_t^{1-\alpha}$ where α is a parameter that takes values between 0 and 1, and z represents TFP which we will assume is exogenous and constant for simplicity. In every period, firms rent capital at rate R_t and hire labour at rate w_t. Capital used in production depreciates at rate $\delta = 1$. This implies $K_{t+1} = I_t$.

ANSWERS IN RED

a) Set up the household's problem

$$\max_{c_t^y, c_{t+1}^o} \ln(c_t^y - \bar{c}) + \beta \ln\left(c_{t+1}^o - \bar{c}\right)$$

s.t.

$$c_t^y + \frac{c_{t+1}^o}{1 + r_{t+1}} = w_t + \pi_t$$

Also acceptable if you write the Lagrangian directly:

$$\mathcal{L} = \ln\left(c_t^y - \bar{c}\right) + \beta \ln\left(c_{t+1}^o - \bar{c}\right) + \lambda_t \left[w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}}\right]$$

b) Derive the household's optimality conditions.
Writing the problem as a Lagrangian (see above) and taking FOCs, we have:

$$(c_t^y): \quad \frac{1}{c_t^y - \bar{c}} = \lambda_t$$

$$(c_{t+1}^o): \quad \frac{\beta\left(1 + r_{t+1}\right)}{c_{t+1}^o - \bar{c}} = \lambda_t$$

$$(\lambda_t): \quad w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1 + r_{t+1}} = 0$$

Using the FOC wrt c_t^y and c_{t+1}^o, we can derive the household's Euler equation:

$$\beta\left(1 + r_{t+1}\right)\left[c_t^y - \bar{c}\right] = c_{t+1}^o - \bar{c}$$

and the other household optimality condition is the household's lifetime budget constraint which is what we found in the FOC wrt λ_t.

c) Set up the firm's problem.
The firm's profit maximization problem is given by:

$$\max_{K_t, L_t} z K_t^\alpha L_t^{1-\alpha} - R_t K_t - w_t L_t$$

d) Derive the firm's optimality conditions. [Note: you may find it useful at this stage to write things in terms of $k_t = K_t / L_t$]
Taking FOC we have the firm's optimality conditions:

$$(L_t): \quad (1-\alpha) z K_t^\alpha L_t^{-\alpha} = w_t \implies (1-\alpha) z k_t^\alpha = w_t$$

$$(K_t): \quad \alpha z K_t^{\alpha-1} L_t^{1-\alpha} = R_t \implies \alpha z k_t^{-(1-\alpha)} = R_t$$

e) State the markets which clear in equilibrium [Hint: there are 3 different markets that firms and households participate in]
The labour market, asset market and goods market clear in equilibrium.

f) Using your equilibrium conditions, derive a transition equation for k_{t+1} in terms of k_t, z, α, β and $\bar{c}$.
We can first plug in the household's Euler equation into the household's lifetime budget constraint to get a relationship between c_t^y and the objects that the household takes as given:

$$\frac{1}{1+\beta}\left[w_t + \pi_t - \bar{c} - \frac{\bar{c}}{1 + r_{t+1}}\right] = c_t^y - \bar{c}$$

Since labour market clearing implies $L_t = N$ and capital market clearing implies $K_{t+1} = Na_{t+1}$ which in turn implies $k_{t+1} = a_{t+1}$, we can use the budget constraint of the young to write k_{t+1} in terms of w_t, π_t, c_t^y and $\bar{c}$

$$k_{t+1} = w_t + \pi_t - [c_t^y - \bar{c}] - \bar{c}$$

Using the fact that $R_t = 1 + r_t$ in equilibrium and what we know about w_t, π_t, R_t from the firm's optimality conditions as well the form of c_t^y from the household's problem, we have:

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha) z k_t^{\alpha} - \left[\frac{\beta}{1+\beta} - \frac{1}{1+\beta}\frac{k_{t+1}^{1-\alpha}}{\alpha z}\right]\bar{c}$$

g) Assume $\alpha = 0.5$,$z = 2$, $\beta = 0.95$. For the following values of k_t contained in Table 1, use the transition equation you derived in f) to compute the corresponding values of k_{t+1} for $\bar{c} = 0.05$. Plot how k_{t+1} varies with k_t. You can do this part in Excel or any spreadsheet/database management software you prefer.
It is useful to note that when $\alpha = 0.5$,$z = 2$, $\beta = 0.95$, the transition equation simplifies to:

$$k_{t+1} = \frac{0.95}{1+0.95}k_t^{0.5} - \left[\frac{0.95}{1+0.95} - \frac{1}{1+0.95}k_{t+1}^{0.5}\right]\bar{c}$$

This is a **non-linear** equation so you can only solve this numerically. Nonetheless, this equation implicitly characterizes how k_{t+1} evolves with respect to k_t and parameters of the model. To see how things evolve, Table 1 provides the numerical solution to the problem for $\bar{c} = 0.05$. The graph in part j) shows how k_{t+1} varies with k_t.

k_t	k_{t+1}	c_t^y	c_t^o	$y_t = c_t^y + c_t^o + k_{t+1}$	$y_t = zk_t^{\alpha}$
0.01	0.03	0.07	0.1	0.2	0.2
0.02	0.05	0.09	0.14	0.28	0.28
0.04	0.08	0.12	0.2	0.4	0.4
0.06	0.10	0.14	0.24	0.49	0.49
0.08	0.12	0.16	0.28	0.57	0.57
0.10	0.14	0.18	0.32	0.63	0.63
0.12	0.15	0.19	0.35	0.69	0.69
0.14	0.17	0.21	0.37	0.75	0.75
0.16	0.18	0.22	0.4	0.8	0.8
0.18	0.19	0.23	0.42	0.85	0.85
0.20	0.21	0.24	0.45	0.89	0.89

Table 1: Equilibrium values of key aggregate outcomes

h) Still assuming $\alpha = 0.5$,$z = 2$, $\beta = 0.95$ and $\bar{c} = 0.05$, compute the values of c_t^y, c_t^o given the values of k_t contained in Table 1. Also compute total

expenditure in the economy as given by $y_t = c_t^y + c_t^o + k_{t+1}$.
Note the equations characterizing c_t^y, c_t^o in terms of parameters of the model, z and k_t is given by:

$$c_t^y = \frac{1}{1+\beta}(1-\alpha)zk_t^{\alpha} + \left[\frac{\beta}{1+\beta} - \frac{1}{1+\beta}\frac{1}{\alpha z k_{t+1}^{-(1-\alpha)}}\right]\bar{c}$$

and

$$c_t^o = R_t k_t = \alpha z k_t^{\alpha}$$

When $\alpha = 0.5$,$z = 2$, $\beta = 0.95$, the equations characterizing c_t^y and c_t^o are given by:

$$c_t^y = \frac{1}{1+0.95}k_t^{0.5} + \left[\frac{0.95}{1+0.95} - \frac{1}{1+0.95}k_{t+1}^{0.5}\right]\bar{c}$$

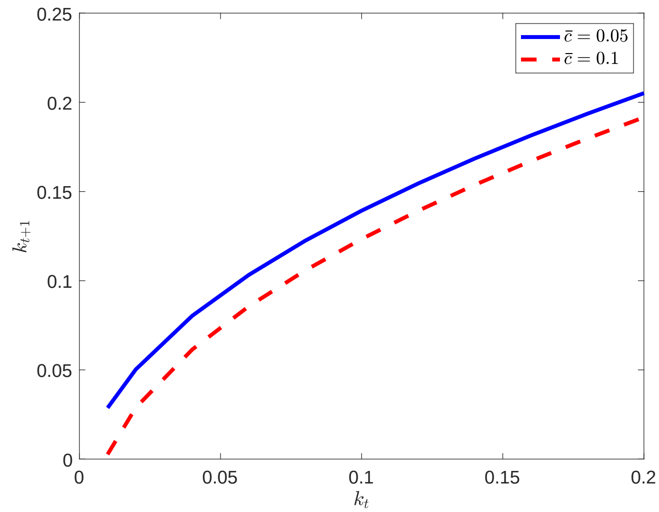
and

$$c_t^o = k_t^{0.5}$$

For $\bar{c} = 0.05$, Table 1 shows the corresponding values of c_t^y and c_t^o.

i) Verify that the goods market clears in equilibrium by computing output supplied as $y_t = zk_t^{\alpha}$. Is total output supplied equal to output demanded?
See Table 1. Total expenditure (output demanded) = output supplied.

j) Finally repeat the exercise in g and compute the values of k_{t+1} but for $\bar{c} = 0.1$. Plot your answer on the same graph where you plotted how k_{t+1} varies with k_t when $\bar{c} = 0.05$. Provide some intuition as to why the two graphs differ.
k_{t+1} is on a lower growth path when $\bar{c}$ is higher. Intuitively, a higher $\bar{c}$ implies that individuals must devote more resources to consumption than to savings as they only get utility if their consumption exceeds the minimum amount $\bar{c}$. Consequently, there is less investment in the economy and capital grows at slower rate.



Question 2: The life-cycle model with population growth

Consider the life-cycle model we discussed in class. In particular, households have preferences

$$U(c_t^y, c_{t+1}^o) = \ln c_t^y + \beta \ln c_{t+1}^o$$

And firms have the following production function

$$Y_t = zK_t^\alpha L_t^{1-\alpha}$$

Everything is the same as the example we discussed in Lecture 5 *except* population grows at a constant rate such that $N_{t+1} = (1+n)N_t$.

a) Derive the household optimality conditions

The household's problem can be written as:

$$\mathcal{L} = \ln c_t^y + \beta \ln c_{t+1}^o + \lambda_t \left[w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1+r_t} \right]$$

Taking FOC we have:

$$(c_t^y): \quad \frac{1}{c_t^y} = \lambda_t$$

$$(c_{t+1}^o): \quad \frac{\beta(1+r_t)}{c_{t+1}^o} = \lambda_t$$

$$(\lambda_t): \quad w_t + \pi_t - c_t^y - \frac{c_{t+1}^o}{1+r_t} = 0$$

Combining the FOCs wrt c_t^y and c_{t+1}^o, we get our Euler equation:

$$\frac{1}{c_t^y} = \frac{\beta\,(1+r_t)}{c_{t+1}^o}$$

and the LBC is the other household optimality condition

b) Derive the firm's optimality conditions
The firm's profit maximization problem is given by:

$$\max_{K_t, L_t} zK_t^{\alpha} L_t^{1-\alpha} - R_t K_t - w_t L_t$$

and the firm's optimality conditions are given by its FOCs:

$$(L_t): \quad (1-\alpha) z k_t^{\alpha} = w_t$$

$$(K_t): \quad \alpha z k_t^{-(1-\alpha)} = R_t$$

c) Derive the transition equation. How does k_{t+1} depend on population growth rate n? Provide some intuition.
Plugging the household Euler equation into the LBC, we get:

$$c_t^y = \frac{1}{1+\beta}\left[w_t + \pi_t\right]$$

which using what we know about w_t and π_t from the firm's optimality conditions, we get:

$$c_t^y = \frac{1}{1+\beta}(1-\alpha) z k_t^{\alpha}$$

From labour market clearing, we have $L_t = N_t$. From asset market clearing, we have that

$$\begin{aligned}
K_{t+1} &= N_t a_{t+1} \\
\frac{K_{t+1}}{N_{t+1}} \frac{N_{t+1}}{N_t} &= a_{t+1} \\
k_{t+1}(1+n) &= a_{t+1}
\end{aligned}$$

From the household's budget constraint when young, we know

$$a_{t+1} = w_t + \pi_t - c_t^y$$

Thus, using this information we have that:

$$\begin{aligned}
k_{t+1} &= \frac{1}{1+n}\left[a_{t+1}\right] \\
&= \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha) z k_t^{\alpha}
\end{aligned}$$

k_{t+1} is declining in population growth rate n. Intuitively, when n is higher, a given amount of capital tomorrow K_{t+1} must be shared among a larger group of individuals. Consequently k_{t+1} is declining in n.