Week 2 Tutorial Exercise

Semester 1, 2025

1. If you are at all interested in LaTeX, which you probably should be, give the Week 1 Exercise a whirl.

2. (a) Consider the model $y = x\beta + u$ where the observed variables y and x, and the unobserved disturbance u are all n-vectors. Show that the OLS estimator for β is
$$\hat{\beta} = \frac{x'y}{x'x}. \tag{1}$$
   (b) Suppose that, in (1), $y \sim \mathrm{N}(0, I_n)$ and that y and x are statistically independent. Then, show that
$$\alpha = \frac{x'y}{(x'x)^{1/2}} \sim \mathrm{N}(0,1)$$
   and that α is independent of x.
   (c) Further suppose that $x \sim \mathrm{N}(0, I_n)$. Since $\hat{\beta} = \alpha/(x'x)^{1/2}$, where $\alpha \sim \mathrm{N}(0,1)$ and is independent of x, and since $x'x \sim \chi^2_n$, deduce that $n^{1/2}\hat{\beta} \sim t_n$.

3. Suppose that $X = [X_1, X_2, X_3]' \sim \mathrm{N}(\mu, \Sigma)$. If
$$\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{bmatrix}$$
show that the conditional distribution of $[X_1, X_2]'$ given X_3 has mean vector
$$[\mu_1 + \rho^2(x_3 - \mu_3), \mu_2]'$$
and covariance matrix
$$\begin{bmatrix} 1 - \rho^4 & \rho \\ \rho & 1 \end{bmatrix}.$$

4. If X_1, X_2, and X_3 are iid. $\mathrm{N}(\mu, \Sigma)$ p-vectors, and if $Y_1 = X_1 + X_2$, $Y_2 = X_2 + X_3$, and $Y_3 = X_1 + X_3$, then obtain
   (a) the conditional distribution of Y_1 given Y_2; and
   (b) the conditional distribution of Y_1 given Y_2 and Y_3.

Hint: In lectures you were given a formula for K^{-1}, the inverse of a partitioned matrix K, where

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The formula that was given is very general and works even when B and C are not square. All it required was that A was non-singular. However, when B and C are also non-singular there is considerable simplification available. In our case there is even greater simplification arising from the facts that (i) $A = D$ and $B = C$, and (ii) that $A = \alpha B$ for some scalar constant α. Hence, we need the inverse of a matrix of the form

$$K = \begin{bmatrix} \alpha B & B \\ B & \alpha B \end{bmatrix},$$

with B non-singular. Now, it must be the case that $KK^{-1} = I$. If we partition K^{-1} conformably with K then we can write

$$K^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

say, and so $KK^{-1} = I$ implies a set of 4 equations in 4 unknowns (E, F, G, H):

$$\begin{aligned} \alpha BE + BG &= I, & \qquad \alpha BF + BH &= 0, \\ BE + \alpha BG &= 0, & \qquad BF + \alpha BH &= I, \end{aligned}$$

From the zero equations we see that

$$H = -\alpha F, \qquad\qquad E = -\alpha G,$$

which we can substitute back into the identity equations to obtain

$$\begin{aligned} -\alpha^2 BG + BG = I &\implies (1-\alpha^2)BG = I \implies G = (1-\alpha^2)^{-1}B^{-1}, \\ BF - \alpha^2 BF = I &\implies (1-\alpha^2)BF = I \implies F = (1-\alpha^2)^{-1}B^{-1}. \end{aligned}$$

We can substitute these expressions for G and H into those for E and F above to obtain

$$K^{-1} = \begin{bmatrix} -\frac{\alpha}{1-\alpha^2}B^{-1} & \frac{1}{1-\alpha^2}B^{-1} \\ \frac{1}{1-\alpha^2}B^{-1} & -\frac{\alpha}{1-\alpha^2}B^{-1} \end{bmatrix}$$

Note that K^{-1} retains the symmetry of K and, like K, all four partitions of K^{-1} are also non-singular.

5. R offers two commands `qqplot` for the construction of quantile-quantile plots: `qqnorm` and `qqplot`. Quantile-quantile plots, or Q-Q plots, as they are known, are a graphical device often used for comparing the quantiles of an empirical distribution with those of a theoretical distribution. Construct Q-Q plots comparing the following distributions against a t distribution with 5 degrees of freedom: N(0,1), t_{20}, t_{10}, t_5, t_2, and t_1. Describe the patterns that you observe and the lessons that you can take from the exercise.