ECOM40006/ECOM90013 Econometrics 3
Department of Economics
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Week 7 Tutorial Exercise

Semester 1, 2025

Suppose that you estimate the following autoregressive model

$$y_t = \alpha + \rho y_{t-1} + u_t \tag{1}$$

by ordinary least squares when the true data generating process is given by

$$y_t = y_{t-1} + u_t. \tag{2}$$

where $y_0 = 0$ and the u_t are *iid* random variables with $\mathrm{E}[u_t] = 0$ and $\mathrm{E}[u_t^2] = \sigma^2$ for all $t = 1, \dots, T$.

(i) Show that

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix} = \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^T u_t \\ \sum_{t=1}^T y_{t-1} u_t \end{bmatrix}.$$

(ii) Show that, in terms of orders in probability

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix} = \begin{bmatrix} O_p(T) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) \end{bmatrix}^{-1} \begin{bmatrix} O_p(T^{1/2}) \\ O_p(T) \end{bmatrix}$$

and conclude that the quantity that might have a non-degenerate limiting distribution is

$$\begin{bmatrix} T^{1/2} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\rho} - 1 \end{bmatrix}.$$

(iii) Show that

$$\begin{aligned} &\begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} T & \sum_{t=1}^T y_{t-1} \\ \sum_{t=1}^T y_{t-1} & \sum_{t=1}^T y_{t-1}^2 \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \\ &\qquad \xrightarrow{d} \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 1 & \int_0^1 W(r)\,\mathrm{d}r \\ \int_0^1 W(r)\,\mathrm{d}r & \int_0^1 [W(r)]^2\,\mathrm{d}r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \end{aligned} \tag{3}$$

(iv) Show that

$$\begin{bmatrix} T^{-1/2} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} \sum_{t=1}^{T} u_t \\ \sum_{t=1}^{T} y_{t-1} u_t \end{bmatrix} \xrightarrow{d} \sigma \begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} W(1) \\ \frac{1}{2}\left\{[W(1)]^2 - 1\right\} \end{bmatrix} \tag{4}$$

(v) Combine the results of equations (3) and (4) to show that

$$\begin{bmatrix} T^{1/2}\hat{\alpha} \\ T(\hat{\rho} - 1) \end{bmatrix} \xrightarrow{d} \Delta^{-1} \begin{bmatrix} \sigma W(1) \cdot \int_0^1 [W(r)]^2 \, \mathrm{d}r - \dfrac{\sigma}{2}\left\{[W(1)]^2 - 1\right\} \cdot \int_0^1 W(r) \, \mathrm{d}r \\ \dfrac{1}{2}\left\{[W(1)]^2 - 1\right\} - W(1) \cdot \int_0^1 W(r) \, \mathrm{d}r \end{bmatrix}$$

where $\Delta = \int_0^1 [W(r)]^2 \, \mathrm{d}r - \left[\int_0^1 W(r) \, \mathrm{d}r\right]^2$.