ECOM40006/ECOM90013 Econometrics 3
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Week 8 Tutorial Exercise Solutions

Semester 1, 2025

3. The generalised least squares (GLS) estimator for the model

$$y = X\beta + u, \qquad \mathrm{E}\left[u \mid X\right] = 0, \quad \mathrm{Var}\left[u \mid X\right] = \Omega,$$

is given by

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

Show that this estimator is consistent for β and derive its asymptotic distribution. How do you need to modify the standard assumptions in order to enable this?

Solution:
The easiest thing to do here is to pre-multiply the model by $\Omega^{-1/2}$ to obtain

$$\Omega^{-1/2}y = \Omega^{-1/2}X\beta + \Omega^{-1/2}u \quad \text{or} \quad y^* = X^*\beta + u^*,$$

in an obvious notation. We note that $\mathrm{E}\left[u^* \mid X^*\right] = 0$ and $\mathrm{Var}\left[u^* \mid X^*\right] = I_n$, which satisfies the typical results for OLS. We need to also assume that

$$\operatorname*{plim}_{n\to\infty} n^{-1}(X^*)'X^* = Q^*,$$

say, where Q^* is a finite positive definite matrix. Note that this assumption, in terms of the original quantities is

$$\operatorname*{plim}_{n\to\infty} n^{-1}X'\Omega^{-1}X = Q^*.$$

The various derivations are identical to those in the asymptotics handout, with the exception that they are performed using the starred version of the variables, and won't be repeated here. Note that application of the Liapunov CLT requires the assumption that the third moments of the elements of u^* are uniformly bounded. Given these assumptions we find that $\tilde{\beta}$ is consistent and that

$$\tilde{\beta} \underset{a}{\sim} \mathrm{N}\left(\beta, n^{-1}(X'\Omega^{-1}X)^{-1}\right).$$

4. Consider a simple random sample of size n from a population with Bernoulli distribution with parameter $0 < \pi < 1$, so that

$$\mathcal{L}(\pi) = \prod_{i=1}^{n}(1-\pi)^{1-y_i}\pi^{y_i}.$$

Find the maximum likelihood estimator of π. Verify that the second order condition for a maximum is satisfied.

Solution:
The log-likelihood is

$$\begin{aligned}\ln \mathcal{L}(\pi) &= \sum_{i=1}^{n} [(1-y_i)\ln(1-\pi) + y_i \ln \pi] \\ &= \ln(1-\pi)\sum_{i=1}^{n}(1-y_i) + \ln \pi \sum_{i=1}^{n} y_i \\ &= n(1-\overline{y})\ln(1-\pi) + n\overline{y}\ln \pi, \quad \overline{y} = n^{-1}\sum_{i=1}^{n} y_i.\end{aligned}$$

Then

$$\mathcal{S}(\pi) = \frac{\mathrm{d}\mathcal{L}(\pi)}{\mathrm{d}\pi} = -\frac{n(1-\overline{y})}{1-\pi} + \frac{n\overline{y}}{\pi}$$

and

$$\mathcal{H}(\pi) = \frac{\mathrm{d}\mathcal{S}(\pi)}{\mathrm{d}\pi} = \frac{\mathrm{d}^2\mathcal{L}(\pi)}{\mathrm{d}\pi^2} = -\frac{n(1-\overline{y})}{(1-\pi)^2} - \frac{n\overline{y}}{\pi^2}.$$

The first order condition is that $\mathcal{S}(\hat{\pi}) = 0$, which implies that

$$-n(1-\overline{y})\hat{\pi} + n\overline{y}(1-\hat{\pi}) = 0 \implies \hat{\pi} = \overline{y}.$$

Observe that, because (i) $0 < \pi < 1$, (ii) $n > 0$, and (iii) $0 \le \overline{y} \le 1$ (why?) it follows that $\mathcal{H}(\pi) < 0$ for all possible π and all possible samples. Consequently, $\overline{y}$ is a *maximum* likelihood estimator.

Note: There are alternative representations of the score. for example

$$\begin{aligned}\mathcal{S}(\pi) &= \frac{\mathrm{d}}{\mathrm{d}\pi}\sum_{i=1}^{n}[(1-y_i)\ln(1-\pi) + y_i\ln\pi] = \sum_{i=1}^{n}\left[-\frac{(1-y_i)}{1-\pi} + \frac{y_i}{\pi}\right] \\ &= \sum_{i=1}^{n}\frac{y_i - \pi}{\pi(1-\pi)} = \frac{n(\overline{y}-\pi)}{\pi(1-\pi)}.\end{aligned}$$

They all yield the same mle when the FOC is solved. The preferred choice really boils down to which you find easiest to differentiate to check the second order condition.

5. Consider a simple random sample of size n from a population with exponential density function

$$f(y;\lambda) = \lambda\exp\{-\lambda y\}, \qquad \lambda > 0, y \ge 0.$$

Find the maximum likelihood estimator of λ. Verify that the second order condition for a maximum is satisfied.

Solution:
The likelihood is

$$\mathcal{L}(\lambda) = \lambda^n \exp\left\{-\lambda\sum_{j=1}^{n} y_i\right\}, \qquad \lambda > 0, y_i > 0, i = 1, \ldots, n.$$

so the log-likelihood is

$$\ln \mathcal{L}(\lambda) = n \ln(\lambda) - \lambda \sum_{j=1}^{n} y_i,$$

the score is

$$\frac{\mathrm{d}\mathcal{L}(\lambda)}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{n} y_i,$$

and the second derivative of the log-likelihood is

$$\frac{\mathrm{d}^2\mathcal{L}(\lambda)}{\mathrm{d}\lambda^2} = -\frac{n}{\lambda^2} < 0$$

for all $n > 0$ and all $\lambda > 0$, so the first-order condition will yield a maximum. This first-order condition is

$$\frac{\mathrm{d}\mathcal{L}(\hat{\lambda})}{\mathrm{d}\lambda} = 0 \implies \frac{n}{\hat{\lambda}} - \sum_{j=1}^{n} y_i = 0 \implies \frac{n}{\hat{\lambda}} = \sum_{j=1}^{n} y_i \implies \hat{\lambda} = \frac{n}{\sum_{j=1}^{n} y_i} = 1/\overline{y}.$$

6. Consider a simple random sample of size n from a Poisson population with probability mass function

$$f(y;\lambda) = \frac{\exp\{-\lambda\}\lambda^y}{y!}, \qquad \lambda > 0, y = 0, 1, 2, 3, \ldots$$

Find the maximum likelihood estimator of λ. Verify that the second order condition for a maximum is satisfied.

Solution:
The likelihood is

$$\mathcal{L}(\lambda) = \frac{\exp\{-n\lambda\}\lambda^{\sum_{j=1}^{n} y_j}}{\prod_{j=1}^{n} y_j!}, \qquad \lambda > 0, y = 0, 1, 2, 3, \ldots$$

so the log-likelihood is

$$\ln \mathcal{L}(\lambda) = -n\lambda + \ln(\lambda) \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} \ln(y_j!)$$

the score is

$$\frac{\mathrm{d}\mathcal{L}(\lambda)}{\mathrm{d}\lambda} = -n + \lambda^{-1} \sum_{j=1}^{n} y_i,$$

and the second derivative of the log-likelihood is

$$\frac{\mathrm{d}^2\mathcal{L}(\lambda)}{\mathrm{d}\lambda^2} = -\lambda^{-2} \sum_{j=1}^{n} y_i < 0$$

for all $n > 0$ and all $\lambda > 0$, as $\sum_{j=1}^{n} y_i > 0$, so the first-order condition will yield a maximum. This first-order condition is

$$\frac{\mathrm{d}\mathcal{L}(\hat{\lambda})}{\mathrm{d}\lambda} = 0 \implies -n + \hat{\lambda}^{-1} \sum_{j=1}^{n} y_i = 0 \implies n\hat{\lambda} = \sum_{j=1}^{n} y_i \implies \hat{\lambda} = \frac{1}{n} \sum_{j=1}^{n} y_i = \overline{y}.$$

7. Consider a simple CAPM model of the form

$$y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, \dots, n,$$

where y_i and x_i are the excess returns in a particular sector and the whole market, respectively. Assume that the model satisfies the assumptions of a classical linear regression model and, specifically that, given the x's, the disturbances have zero mean, are homoskedastic, and are independent of one another. Now assume that the disturbances have a t_5 distribution, scaled to have unit variance, so that the density of ϵ_i is of the form

$$f(\epsilon_i) = \frac{c_5}{\sigma}\left(1 + \frac{\epsilon_i^2}{5\sigma^2}\right)^{-3}$$

where c_5 is a scaling constant (that does not depend on σ) so that $\int f(\epsilon_i)\,\mathrm{d}\epsilon_i = 1$. The log=likelihood is given by

$$\ln \mathcal{L}_n(\alpha, \beta, \sigma^2) = \sum_{i=1}^{n} \ln f(\epsilon_i) = n \ln c_5 - \frac{n}{2}\ln \sigma^2 - 3\sum_{i=1}^{n} \ln\left(1 + \frac{(y_i - \alpha - \beta x_i)^2}{5\sigma^2}\right).$$

Derive the various scores that will require solution in order to estimate this model.

Solution:
Given that we have the log-likelihood function already, all that is required is partially differentiate with respect to each of the coefficients. Thus, remembering that $\epsilon_i = y_i - \alpha - \beta x_i$,

$$\begin{aligned}
\frac{\partial \ln \mathcal{L}}{\partial \alpha} &= \sum_{i=1}^{n} \frac{6\epsilon_i}{5\sigma^2 + \epsilon_i^2}, \\
\frac{\partial \ln \mathcal{L}}{\partial \beta} &= \sum_{i=1}^{n} \frac{6\epsilon_i x_i}{5\sigma^2 + \epsilon_i^2}, \\
\frac{\partial \ln \mathcal{L}}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{3}{\sigma^2}\sum_{i=1}^{n} \frac{\epsilon_i^2}{5\sigma^2 + \epsilon_i^2}.
\end{aligned}$$

8. Stan the Statistician wishes to test the null hypothesis that the mean of some population is zero against the alternative that it is three. If the population is $\mathrm{N}(\mu, 1)$ and Stan uses the statistic $z = \sqrt{n}(\overline{y} - \mu_0)$, using data from a simple random sample of size n, as his test statistic, find the size and power of his test if he uses as his acceptance region $\{z : z \leq 2\}$. Use the probability tables provided in the Normality handout when completing your calculations. Note that there has been some adjustment to those tables so make sure to use the version of the handout dated 10 April 2021 (which is on the LMS).

Solution:
We see that $z \overset{H_0}{\sim} \mathrm{N}(0, 1)$, that is $\mu_0 = 0$, and so

$$\begin{aligned}
\text{Size} = \Pr(\text{Type I Error}) &= \Pr(z > 2 \mid H_0 \text{ true}) = 1 - \Pr(z \leq 2 \mid \mu = 0) \\
&= 1 - (\Pr(z \leq 0 \mid \mu = 0) + \Pr(0 < z \leq 2 \mid \mu = 0)) \\
&= 1 - (0.5 + 0.4772) = 0.0228.
\end{aligned}$$

and

$$\text{Power} = 1 - \Pr(\text{Type II Error}) = \Pr(z > 2 \mid H_1 \text{ true}) = \Pr(z > 2 \mid \mu = 3).$$

Now, $z \overset{H_1}{\sim} \mathrm{N}(3, 1)$, and so

$$\begin{aligned}\text{Power} &= \Pr(z > 2 \mid \mu = 3) = \Pr(Z > -1 \mid Z \sim \mathrm{N}(0,1)) = \Pr(Z < 1 \mid Z \sim \mathrm{N}(0,1)) \\ &= 0.5 + \Pr(0 < Z < 1 \mid Z \sim \mathrm{N}(0,1)) = 0.5 + 0.3413 = 0.8413.\end{aligned}$$