Week 6 Solutions

Semester 1, 2025

2. An estimator $\hat{\theta}$ is said to be consistent for a parameter θ iff $\hat{\theta} \xrightarrow{p} \theta$. Let $Y_1, Y_2, \ldots, Y_n$ denote a simple random sample from a population with probability density function

$$f(y) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the sample mean $\overline{Y}$ is a consistent estimator of $\theta/(\theta+1)$.
Hint: First derive the mean of the population and then remember that laws of large numbers are your friends.

Solution:
As an aside, this distribution is a Beta with parameters $\alpha = \theta$ and $\beta = 1$, where if $X \sim B(\alpha, \beta)$ then

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \alpha > 0, \beta > 0; 0 < x < 1.$$

Now,

$$\mathrm{E}[Y] = \int_0^1 \theta y^{\theta-1} \times y \, \mathrm{d}y = \theta \int_0^1 y^{\theta} \, \mathrm{d}y = \left. \frac{\theta y^{\theta+1}}{\theta+1} \right|_0^1 = \frac{\theta}{\theta+1}.$$

$$\mathrm{E}\left[Y^2\right] = \int_0^1 \theta y^{\theta-1} \times y^2 \, \mathrm{d}y = \theta \int_0^1 y^{\theta+1} \, \mathrm{d}y = \left. \frac{\theta y^{\theta+2}}{\theta+2} \right|_0^1 = \frac{\theta}{\theta+2}.$$

Therefore,

$$\mathrm{Var}[Y] = \mathrm{E}\left[Y^2\right] - (\mathrm{E}[Y])^2 = \frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2 = \frac{\theta}{(\theta+2)(\theta+1)^2}.$$

For the sample mean of a simple random sample we know that

$$\mathrm{E}\left[\overline{Y}\right] = \frac{1}{n} \sum_{j=1}^{n} \mathrm{E}[Y] = \frac{\theta}{\theta+1}$$

and

$$\mathrm{Var}\left[\overline{Y}\right] = \mathrm{E}\left[\frac{1}{n} \sum_{j=1}^{n} \mathrm{E}[Y]\right] = \frac{1}{n^2} \sum_{j=1}^{n} \mathrm{Var}[Y] = \frac{\theta}{n(\theta+2)(\theta+1)^2}.$$

We can establish mean square convergence, as

$$\lim_{n\to\infty} \frac{\theta}{\theta+1} = \frac{\theta}{\theta+1}$$

and

$$\lim_{n\to\infty} \frac{\theta}{n(\theta+2)(\theta+1)^2} = 0.$$

Hence, consistency is established as

$$\overline{Y} \overset{m.s.}{\to} \frac{\theta}{\theta+1} \Rightarrow \overline{Y} \overset{p}{\to} \frac{\theta}{\theta+1}.$$

3. Let $Y_1, Y_2, \ldots, Y_n$ denote a simple random sample of size n from a Normal population with mean μ and variance σ^2. Assuming that $n = 2k$ for some integer k, one possible estimator of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{2k} \sum_{j=1}^{k} (Y_{2j} - Y_{2j-1})^2.$$

   (a) Show that $\hat{\sigma}^2$ is an unbiased estimator for σ^2.

   Solution:

   By definition

$$\hat{\sigma}^2 = \frac{1}{2k}\sum_{j=1}^{k}(Y_{2j}-Y_{2j-1})^2 = \frac{1}{2k}\left[\underbrace{(Y_2-Y_1)^2+(Y_4-Y_3)^2+\cdots+(Y_n-Y_{n-1})^2}_{k \text{ terms}}\right].$$

   Observe that the $Y_1, Y_2, \ldots, Y_n$ are jointly normally distributed and mutually independent. For each pair (Y_{2j-1}, Y_{2j}), we have the marginal distribution

$$\begin{bmatrix} Y_{2j-1} \\ Y_{2j} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right),$$

   and hence that

$$\begin{aligned} Y_{2j} - Y_{2j-1} &= [-1, 1]\begin{bmatrix} Y_{2j-1} \\ Y_{2j} \end{bmatrix} \\ &\sim \mathrm{N}\left([-1,1]\begin{bmatrix} \mu \\ \mu \end{bmatrix} = -\mu+\mu = 0, [-1,1]\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2\sigma^2\right). \end{aligned}$$

   It follows that $\left((Y_{2j} - Y_{2j-1})/\sqrt{2\sigma^2}\right)^2 \sim \chi_1^2$ and that

$$\frac{(Y_2-Y_1)^2}{2\sigma^2} + \frac{(Y_4-Y_3)^2}{2\sigma^2} + \cdots + \frac{(Y_n-Y_{n-1})^2}{2\sigma^2} \sim \chi_k^2$$

   because the k terms in the sum are mutually independent. Thus,

$$\mathrm{E}\left[\hat{\sigma}^2\right] = \mathrm{E}\left[\frac{2\sigma^2}{2k}\sum_{j=1}^{k}\frac{(Y_{2j}-Y_{2j-1})^2}{2\sigma^2}\right] = \frac{\sigma^2}{k}\mathrm{E}\left[\chi_k^2\right] = \frac{k\sigma^2}{k} = \sigma^2.$$

   That is, $\hat{\sigma}^2$ is unbiased for σ^2.

(b) Show that $\hat{\sigma}^2$ is a consistent estimator for σ^2.

Solution:

$$\operatorname{Var}\left[\hat{\sigma}^2\right] = \operatorname{Var}\left[\frac{\sigma^2}{k}\sum_{j=1}^{k}\frac{(Y_{2j}-Y_{2j-1})^2}{2\sigma^2}\right] = \frac{\sigma^4}{k^2}\operatorname{Var}\left[\chi_k^2\right] = \frac{2k\sigma^4}{k^2} = \frac{2\sigma^4}{k}.$$

As $n \to \infty$, given that $k = n/2$, $k \to \infty$ and so $\lim_{n\to\infty}\operatorname{Var}\left[\hat{\sigma}^2\right] = 0$ for finite σ^2. Given that $\hat{\sigma}^2$ is unbiased for σ^2, we have established mean squared convergence and hence that $\hat{\sigma}^2$ is a consistent estimator for σ^2.

4. Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of independent random variables with $\mathrm{E}[Y_i] = \mu$ and $\operatorname{Var}[Y_i] = \sigma_i^2$. Notice that not all the σ_i^2's need be equal.

(a) What is $\mathrm{E}\left[\overline{Y}_n\right]$?

Solution:

As before $\mathrm{E}\left[\overline{Y}_n\right] = \frac{1}{n}\sum_{j=1}^{n}\mathrm{E}[Y_j] = \frac{n\mu}{n} = \mu$. So $\overline{Y}_n$ remains unbiased for μ in the presence of heteroskedasticity.

(b) What is $\operatorname{Var}\left[\overline{Y}_n\right]$?

Solution:

Similarly, noting that independence implies zero covariances,

$$\operatorname{Var}\left[\overline{Y}_n\right] = \operatorname{Var}\left[\frac{1}{n}\sum_{j=1}^{n}Y_j\right] = \frac{1}{n^2}\sum_{j=1}^{n}\operatorname{Var}[Y_j] = \frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2.$$

(c) Under what condition (on the σ_i^2's) can the following theorem be applied to show that $\overline{Y}_n$ is a consistent estimator for μ?

Theorem: An unbiased estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n\to\infty}\operatorname{Var}\left[\hat{\theta}_n\right] = 0.$$

Solution:

$$\operatorname{Var}\left[\overline{Y}_n\right] = \frac{1}{n^2}\sum_{j=1}^{n}\sigma_j^2 \le \frac{1}{n^2}\sum_{j=1}^{n}\sigma_{\max}^2 = \frac{n\sigma_{\max}^2}{n^2} = \frac{\sigma_{\max}^2}{n},$$

where $\sigma_{\max}^2 = \max_{j=1,\ldots,n}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$. All that is required for the theorem to be satisfied (to establish mean squared convergence) is that $\sigma_{\max}^2 = O(1)$, which means that $\sigma_{\max}^2 < \infty$, and hence that all variances are finite.

5. If $Y_1, Y_2, \ldots, Y_n$ denote a simple random sample of size n from a population with a gamma distribution with parameters α and β, show that $\overline{Y}$ converges in probability to some constant and find the constant, when

$$f(y \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/\beta}, \quad 0 < y < \infty.$$

Hint: Recall that

$$\int_0^\infty e^{-y/\beta}y^{\alpha-1}\,\mathrm{d}y = \beta^\alpha\Gamma(\alpha),$$

and explore the behaviour of $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$.

Solution:
Let us first establish the mean and variance of the population:

$$\begin{aligned}
\mathrm{E}[Y] &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} \times y \,\mathrm{d}y \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty e^{-y/\beta} y^{(\alpha+1)-1} \,\mathrm{d}y \\
&= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} && \text{(using the hint)} \\
&= \frac{\alpha\Gamma(\alpha)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} = \alpha\beta.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{E}\left[Y^2\right] &= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{(\alpha+2)-1} e^{-y/\beta} \,\mathrm{d}y \\
&= \frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha} \\
&= \alpha(\alpha+1)\beta^2.
\end{aligned}$$

Thus, $\mathrm{Var}[Y] = \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = \alpha(\alpha+1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$. We know that $\mathrm{E}\left[\overline{Y}\right] = \mathrm{E}[Y] = \alpha\beta$, so that the sample mean of a simple random sample is unbiased for the population mean. Moreover, we know that in this case $\mathrm{Var}\left[\overline{Y}\right] = n^{-1}\,\mathrm{E}[Y] = n^{-1}\alpha\beta^2$ and so $\lim_{n\to\infty} \mathrm{Var}\left[\overline{Y}\right] = 0$. We have established mean square convergence and hence that $\overline{Y} \xrightarrow{p} \alpha\beta = \mathrm{E}[Y]$.