Lecture 6: Growth in the OLG model

ECON30009/90080 Macroeconomics
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Last class

- We looked at the set-up of an OLG model
- ... and characterized equilibrium in our simple OLG model
- This class, can we use our OLG model to talk about growth?

How does capital per capita evolve over time?

- Using our example from last class, we saw that we can express the evolution of capital per capita in $t+1$ as a function of capital per capita in t

$$k_{t+1} = \frac{\beta}{(1+\beta)} (1-\alpha)\, z_t k_t^{\alpha} = \psi(k_t)$$

We call this dynamical relationship between k_{t+1} and k_t a **transition** equation

- Further, we know that:
  - $\frac{\partial k_{t+1}}{\partial k_t} = \psi'(k_t) > 0$: k_{t+1} is increasing in k_t
  - $\frac{\partial^2 k_{t+1}}{\partial k_t^2} = \psi''(k_t) < 0$: Each additional unit of k_t leads to a smaller increment in k_{t+1} (diminishing marginal returns)

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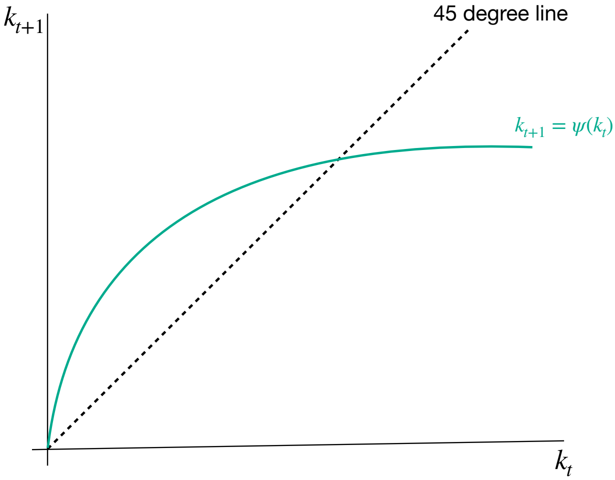
  - Production: $y_t = z k_t^{\alpha}$
  - Optimal consumption (young): $c_t^y = \frac{(1-\alpha)}{(1+\beta)} z_t k_t^{\alpha}$
  - Optimal consumption (retired): $c_t^o = \alpha z_t k_t^{\alpha}$

Transition Equation: why is it useful?

- Knowing the transition equation, $k_{t+1} = \psi(k_t)$, we know prices, investment per capita, output per capita and consumption per capita at each point in time.

- So we know how the economy is performing at each point in time in terms of prices and key aggregate variables.

Graphing the transition equation



- From the transition equation, we know:

$$\frac{\partial k_{t+1}}{\partial k_t} = \psi'(k_t) > 0$$

and

$$\frac{\partial^2 k_{t+1}}{\partial k_t^2} = \psi''(k_t) < 0$$

The Steady State

- The long-run equilibrium to which the economy converges to over time is also known as the **steady state**.
- Let $\bar{k}$ denote the steady-state capital-labour ratio.
- At steady state, capital-labour ratios are constant over time $k_{t+1} = k_t = \bar{k}$
- Because prices, r_t and w_t, and key aggregate outcomes like y_t, $c_t = c_t^y + c_t^o$, are functions of k_t, these variables are also unchanging at steady state
- Absent shocks, the steady state represents a fixed point in the economy

The Steady State

- Using the transition equation from our example and assuming $z_t = z$:

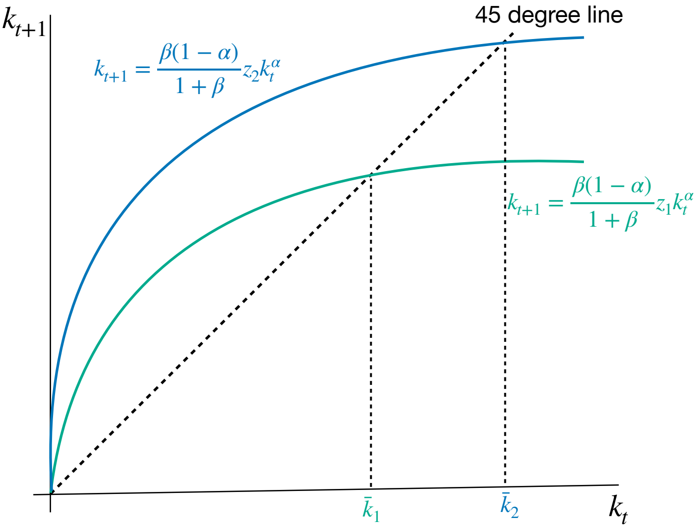
$$k_{t+1} = \frac{\beta}{(1+\beta)} (1-\alpha) z k_t^{\alpha}$$

- Solve for the steady state $\bar{k}$ by imposing $k_{t+1} = k_t = \bar{k}$ in the transition equation:

$$\bar{k} = \frac{\beta}{(1+\beta)} (1-\alpha) z \bar{k}^{\alpha} \quad \Longrightarrow \quad \bar{k} = \left[\frac{\beta(1-\alpha)}{(1+\beta)} z \right]^{1/(1-\alpha)}$$

- Question: how does $\bar{k}$ depend on z?

$\bar{k}$ is increasing in z



- For $z_2 > z_1$, $\bar{k}_2 > \bar{k}_1$
- Why is $\bar{k}$ increasing in z?

Equilibrium Dynamics

Suppose economy starts at some $k_0 < \bar{k}$, how does the economy grow over time?

- We can take the (natural) log of the transition equation:

$$\ln k_{t+1} = \ln \frac{\beta}{1+\beta} + \ln(1-\alpha) + \ln z + \alpha \ln k_t$$

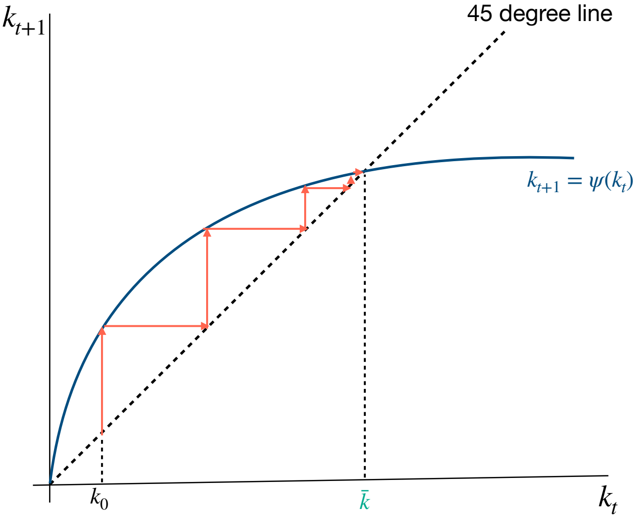
- Subtract $\ln k_t$ from both sides to get the growth rate of k between t and $t+1$:

$$g_{k,t} = \Delta \ln k_{t+1} = \ln \frac{\beta}{1+\beta} + \ln(1-\alpha) + \ln z - (1-\alpha) \ln k_t$$

- Growth rate of capital per capita, $g_{k,t}$, is negatively related to level of k_t.

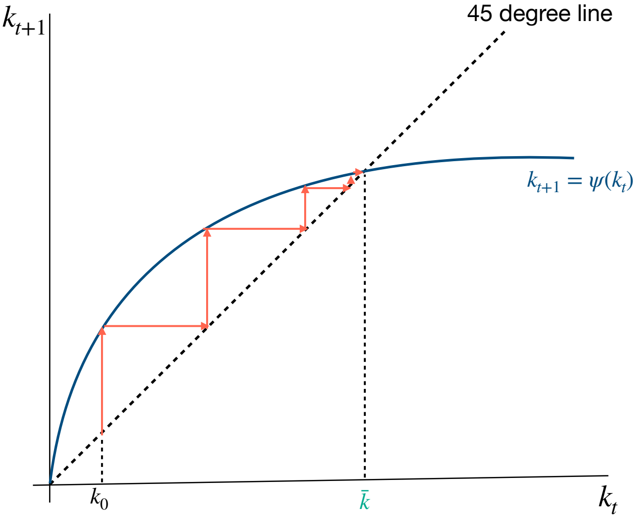
- Diminishing marginal returns $\implies$ the larger k_t is, the smaller the $\uparrow$ in k_{t+1}

Convergence to steady state



- Barring no shocks, initially k grows rapidly from k_0
- Over time, the growth rate of k becomes smaller as the economy converges to its steady state $\bar{k}$

Convergence to steady state



- What if $k_0 > \bar{k}$? What happens to economy over time and why?

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- Change in capital:

$$\begin{aligned} Na_{t+1} - Na_t &= K_{t+1} - K_t \\ &= I_t - K_t \end{aligned}$$

  - Above is consistent with the law of motion for capital:

$$K_{t+1} = (1-\delta)K_t + I_t$$

    with $\delta = 1$

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- Divide by N

$$k_{t+1} - k_t = i_t - k_t$$

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- For $k_0 > \bar{k}$, investment per person is too small relative to capital depreciation
- $\Delta k_t = k_{t+1} - k_t < 0$ if $i_t < k_t$
- As such, k shrinks over time until $k = \bar{k}$

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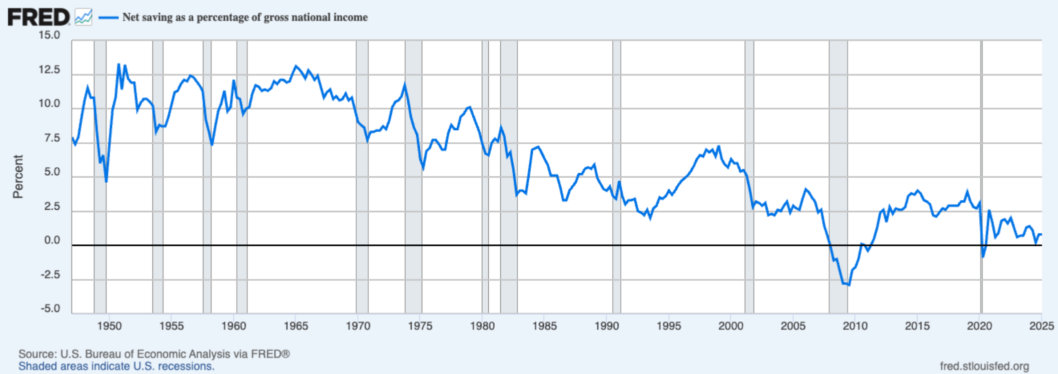
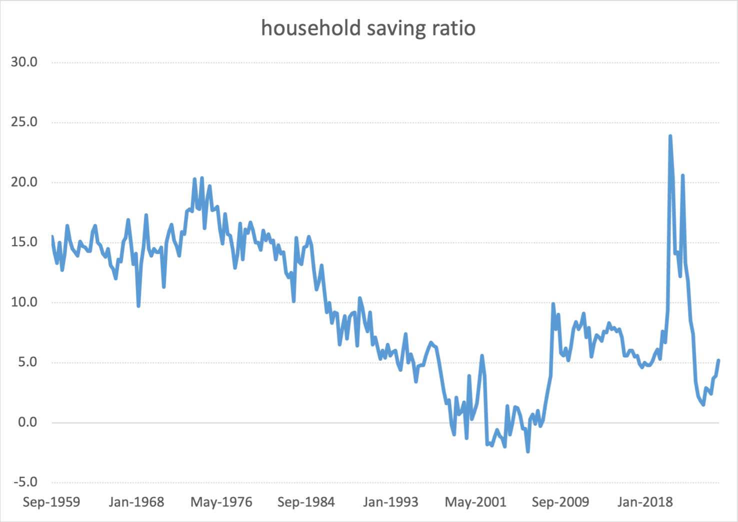
- Only at steady state, the savings rate is unchanging over time
- Also note that in a **closed economy**, investment can only be financed by saving.

Savings ratio in the US



Savings ratio in Australia



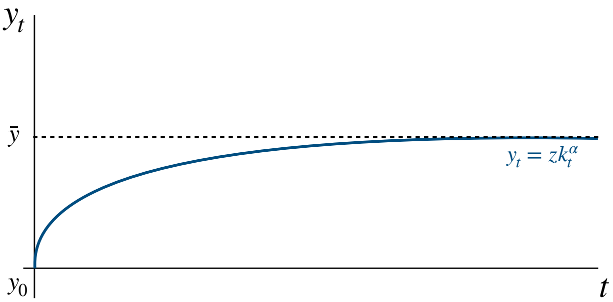
Source: ABS. Household savings ratio defined as proportion of disposable income saved

Simulating an economy's growth path

Simulating the Economy's Growth Path

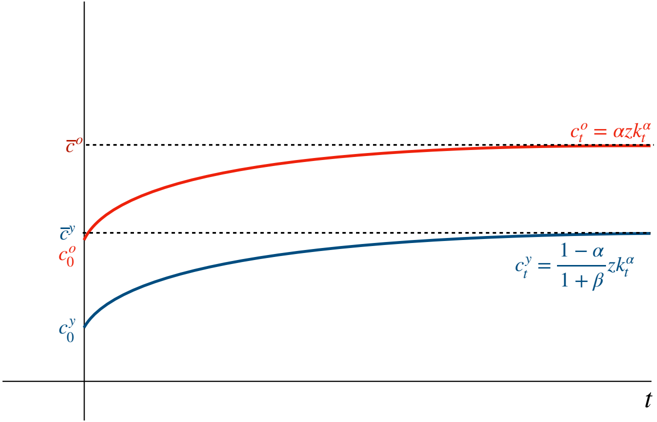
- Suppose the economy starts with $k_0 < \bar{k}$
- What happens to output per capita over time?
- What happens to consumption per capita over time?
- What happens to w_t and R_t over time?

Output per capita over time



- y_t initially grows rapidly
- Over time, growth in y_t slows as $y_t \to \bar{y} = z\bar{k}^\alpha$
- At some point, y_t converges to steady state output per-capita, $y_t = \bar{y}$

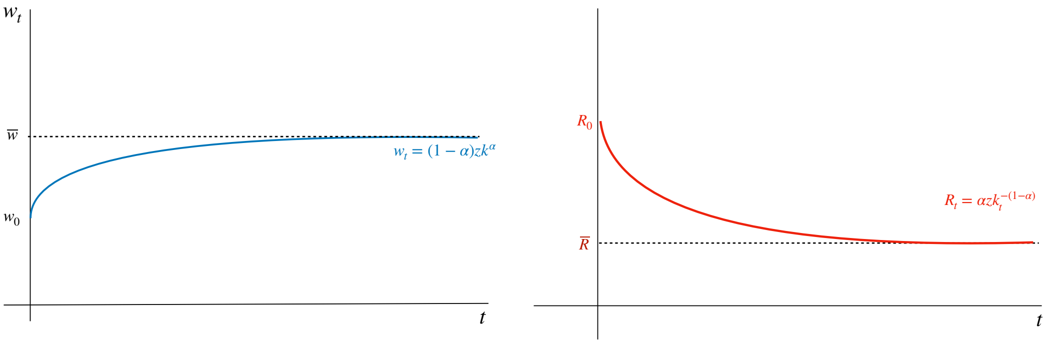
Consumption per capita over time



- $c_t = c_t^y + c_t^o$. Over time, c_t reaches its steady state level

Note: here I've drawn $c_t^o > c_t^y$, but you can find parameters where the reverse is true

Prices over time



Why is w_t initially increasing over time but R_t declining?

Starting from k_0, what drives growth in this example?

- Answer: Capital accumulation.

  - The **only source of output per capita growth** is an increasing capital stock per person and hence **an increasing capital-labour ratio**.

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  - But MPK is high and so each unit of output saved and invested in k yields a high return $\rightarrow k_1 > k_0$
  - A higher k_1 raises MPL and the **real wage** in period 1, which increases how much working generation can save, which in turn raises k_2.
  - However, each generation, the increase in the real wage becomes smaller and smaller.
  - And the return on your savings $R_t = 1 + r_t$ is also declining (less incentive to save)

Growth mechanics continued...

- Along the transition path, the **growth rate** of the real wage converges to zero.
- As $t \to \infty$, each generation saves the same amount when working and brings the same amount of capital into retirement.
- As a result, k_t (as well as all other endogenous variables) **remain constant** from one period to the next.
- **Output and output per capita are constant in the long run!**

Quick question: Why is output also constant? What did we assume?

But why do economists care about growth?

OLG model predictions: effects of growth on welfare

- Welfare is measured by lifetime utility $U(c_t^y, c_{t+1}^o)$.

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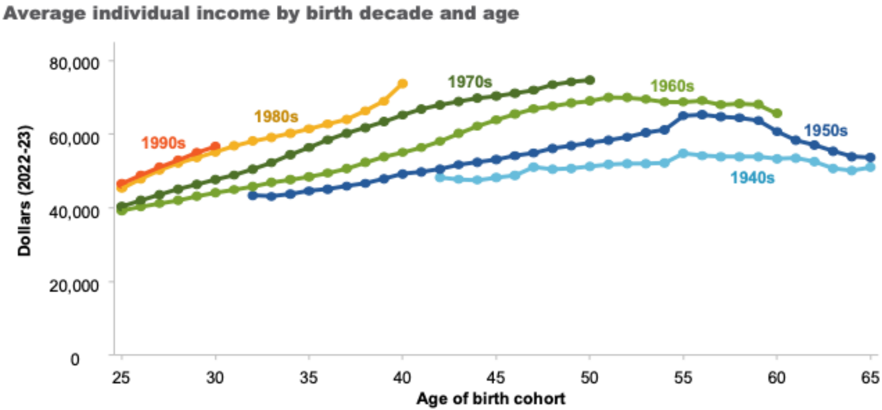
- If subsequent generations consume more in both periods of their life than their predecessors, **welfare unambiguously increases** from one generation to the next.

- In other words, starting from $k_0 < \bar{k}$, each generation is better off as the economy grows towards its steady state

So is each generation better off as economy grows?

Productivity Commission report: intergenerational mobility



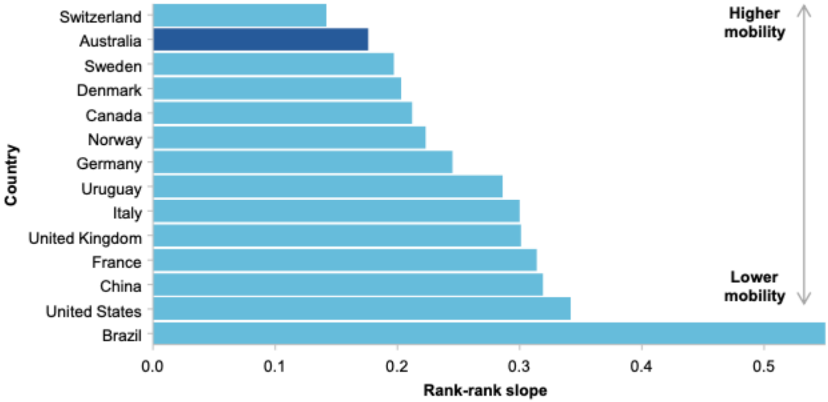
a. HILDA data shows similar trends, including the lack of growth in individual disposable incomes for those born in the 1990s. **b.** Using HILDA, when the income measure is equivalised household disposable income, the average incomes of those born in 1990s are materially higher than those born in the 1980s, which reflects the incomes of other household members increasing.

Source: Commission estimates using the preliminary version of the ATO Longitudinal Information Files Family (ALife-Family) dataset.

Productivity Commission report: across countries

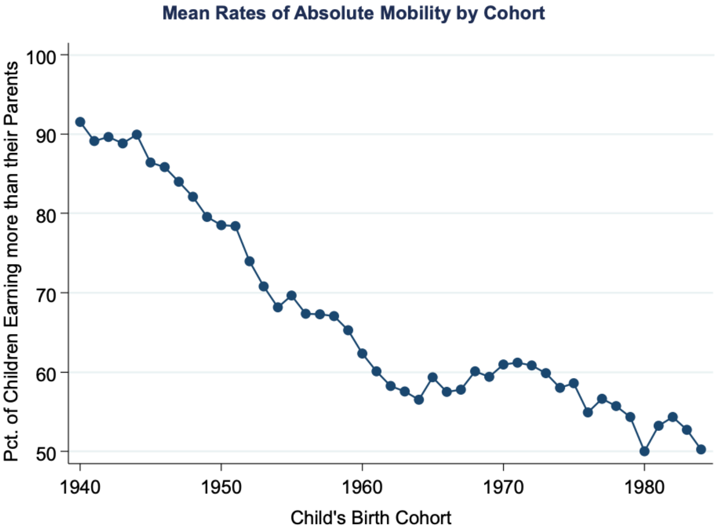
Figure 2 – Australia is one of the most mobile countries internationally, in terms of income rank

Rank-rank slope[a,b,c] for selected countries



a. The lower the rank-rank slope, the less a change in parents' income is passed on to their children, thus indicating higher mobility. **b**. For Australia, the rank-rank slope is for people born between 1976 and 1982. **c.** Where possible, the Commission has selected estimates for other countries that are comparable to the Commission's methodology.

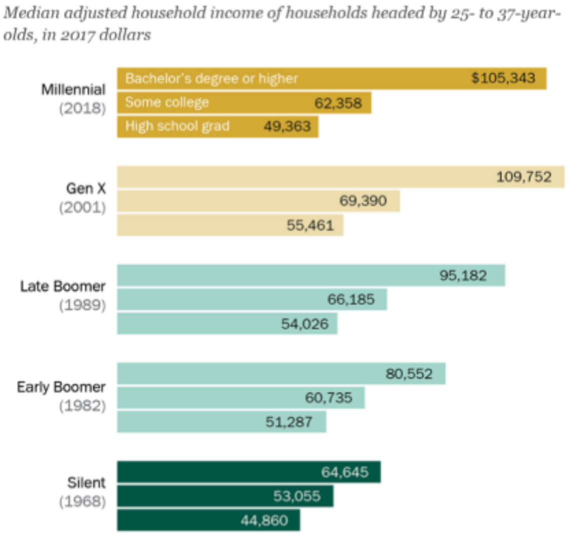
From the lens of our OLG model, why might some countries observe less upward generational mobility than others?



Source: Chetty et al (2016) "The Fading American Dream: Trends in Absolute Income Mobility Since 1940"

- In the US, mobility is declining
- If interested: read "The Fading American Dream: Trends in Absolute Income Mobility Since 1940" by Chetty et al (2016)

Across education groups in the US



Source: Pew Research Center "Millennial life: How young adulthood today compares with prior generations"

- Gaps between college educated and non-college educated growing
- Model assumption of representative households cannot account for differences

Roadmap

- Today we characterized what the growth path of the economy would look like in an OLG model
- Next class: Long run growth, welfare and pareto optimality