Phys Enph 479/879 Assignment #3: Simple Harmonic Oscillators and Planar Three-Body Dynamics

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(Dated: October 1, 2020)

This assignment discusses the use of a form Velocity Verlet method, the leapfrog method, as a ordinary differential equation solver. It is used to simulate the dynamics of a three body problem, where animation software is used to visualise the interactions of the system.

I. INTRODUCTION

A. Symplectic Integrators

Symplectic integrators are commonly used for the integration of Hamiltonian systems, the help to conserve certain properties of the hamiltonian, specifically conserving the energy of the system. Making them incredibly useful for simulations of dynamic systems with long periods such as a many bodied problem. The integrators we have used before such as Euler forward step method and Runge Kutta 4 are both not symplectic integrators and so if we use them in our simulations we cannot expect the energy of the system to be totally conserved.

II. THEORY AND EQUATIONS

A. Runge Kutta 4

The Runge-Kutta family is a a series of iterative methods for producing approximate solutions to differential equations. It contains the Euler forward step method in its simplest case (for a single slope) but as the number of slopes in the weighted average increases the absolute error to the analytical solution decreases. Euler only calculates a single slope per interval, but Runge-Kutta 4 (RK4) the fourth degree method, calculates four different slopes per step and finds the next anticipated position as a weighted average of each of the slopes.

$$k_1 = hf(t_n, y_n), \tag{1a}$$

$$k_2 = hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \tag{1b}$$

$$k_3 = hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \tag{1c}$$

$$k_4 = hf(t_n + h, y_n + k_3). \tag{1d}$$

Where h is the step size, $f(t_n, y_n)$ is the derivative of the solution at time t_n. These four components create the basis for the complete iterative function:

$$y_{n+1} = y_n \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2a}$$

$$t_{n+1} = t_n + h. \tag{2b}$$

We can call this function multiple times to create the approximate solution to the differential equation. By reducing the step size (h) the approximation converges on the analytical solution. This is the first of our own ordinary differential equation solvers that we be using in this assignment.

B. Leapfrog

Leapfrog is the second method we will be using to solve ordinary differential equations, specifically of the form:

$$\frac{dv}{dt} = a(x), \frac{dv}{dt} = v, \tag{3}$$

A variation of the velocity Verlet integration method, Leapfrog is a symplectic integrator method meaning that it will conserve the energy of the system by preserving the area in phase space. Leapfrog is the second order method compared to Euler, but is stable for oscillatory motion for a constant time step. The algorithm for Leapfrog,to second order accuracy, is as follows:

$$x_{1/2} = x_0 + v_0 \frac{h}{2} \tag{4a}$$

$$v_1 = v_0 + a(x_{1/2})h \tag{4b}$$

$$x1 = x_{1/2} + v1 \frac{h}{2} \tag{4c}$$

C. Classical Simple Harmonic Oscillator

In a single dimension the simple harmonic oscillator takes the form of:

$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 x(t), \tag{5}$$

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Where we also choose $\omega_0 = \sqrt{k/m}$ and $m = k = w_0 = 1$, we can write it as two first order coupled ordinary differential equations:

$$\frac{dx}{dt} = v(t), \frac{dv}{dt} = -x = a(t). \tag{6}$$

For the initial conditions of $x(0) = 1, v(0) = 0$, this becomes a well know problem with an established solution:

$$x(t) = \cos\omega_0 t, v(t) = -\omega_0 \sin\omega_0 t. \tag{7}$$

The total energy of the system (Kinetic energy + Potential energy) should be conserved by our numeric integrator if the leapfrog method is used. We know for a simple harmonic oscillator that the sum of kinetic energy and potential energy considering $(k = m = 1)$ is as follows:

$$E = 0.5v^2 + 0.5x^2 \tag{8}$$

D. Planar Three Body Problem

The planar three body problem is a series of coupled masses or bodies that interact with each other in a two dimensional plane. For three masses (m_1, m_2, m_3) the equations of motion are given:

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i, \frac{d\boldsymbol{v}_i}{dt} = -Gm_j\frac{\boldsymbol{r_{ij}}}{r_{ij}^3} - Gm_k\frac{\boldsymbol{r_{ik}}}{r_{ik}^3}. \tag{9a}$$

$$i(=1,2,3) \neq j \neq k, \tag{9b}$$

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j, r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|, \boldsymbol{r}_{ij} = -\boldsymbol{r}_{ji}. \tag{9c}$$

These sets of equations for three bodies in two dimensions, produces a set of twelve coupled ordinary differential equations that we will attempt to solve using both Leapfrog and Runge Kutta 4 integration methods.

III. IMPLEMENTATION

A. Solving the Simple Harmonic Oscillator

Using the leapfrog method we will solve the case for the simple harmonic oscillator 6. Considering we have a known solution 7 for initial conditions $(x(0) = 1, v(0) = 0)$ we can compare how our Leapfrog method compares to the analytical solution using the absolute (Δx) and relative errors (δx), defined as follows:

$$\Delta x = x_0 - x, \tag{10a}$$

$$\delta x = \frac{x_0 - x}{x} = \frac{\Delta x}{x}. \tag{10b}$$

Where x_0 is the inferred value and x the true value. We can iterate over decreasing the step size until we find a relative error less than six orders of magnitude in the total energy of the system. This relative error is tested for 50 cycles of the oscillator. With the relative error we can plot this a function of time to see its temporal evolution.

We can then create phase space graphs of our numerical solutions (Plotting $v - x$) together with the Energy of the system for both the Leapfrog method and the Runge Kutta 4 method. For a fixed period of oscillation of 800 periods we can vary the step sizes to values of $h = 0.02T_0$, $0.04T_0$ and $0.1T_0$.

B. Three Body Problem Initial Positions and Velocities

For the three body problem the first variables we need to find are the starting positions and velocities, these can be arbitrary coordinates and velocities but finding the solution to Euler's quintic equation is suggested. For masses $m_1 = 1, m_2 = 2, m_3 = 3$ we seek the single real root of the following equation:

$$\lambda^5(m_2 + m_3) + \lambda^4(2m_2 + 3m_3) + \lambda^3(m_2 + 3m_3) + \lambda^2(-3m_1 - m_2) + \lambda(-3m_1 - 2m_2) + (-m_1 - m_2) = 0 \tag{11}$$

The roots can be found by using the numerical python (numpy) root solver. The distance in the x axis between mass 1 and mass 2 $(a = x_3 - x_2)$ is then characterised as:

$$a^3 = \frac{1}{\omega^2}[m_2 + m_3 - \frac{m_1(1+2\lambda)}{\lambda^2(1+\lambda)^2}] \tag{12}$$

The initial positions then become:

$$x_2 = \frac{1}{\omega^2 a^2}[\frac{m_1}{\lambda^2} - m_3], \tag{13a}$$

$$x_1 = x_2 - \lambda a, \tag{13b}$$

$$x_3 = -\frac{m_1 x_1 + m_2 x_2}{m_3}(C - M), \tag{13c}$$

and velocities:

$$v_{1y} = \omega x_1, \tag{14a}$$

$$v_{2y} = \omega x_2, \tag{14b}$$

$$v_{3y} = \omega x_3. \tag{14c}$$

We can now solve the three body problem using both leapfrog and RK4 using a fixed step size of $h = 0.001$ and a total time of three periods (19000 steps). We can slightly perturb the frequency (w_0) by $\pm\delta$ where $\delta = 10^{-9}$ and investigate the results. We can use the animation functionality of the package matplotlib to show the time evolution of the system for all of the masses, we can create circles to represent point masses and move them with respect to their current x and y positions. Each active mass will have a vector arrow that will point in the direction of the velocity of said particle, this will show its relative motion and

as and a path of smaller circles will be plotted to show its general trajectory.
Using the Python ODE package solver (odeint) we can compare our two methods against an optimized routine to comment on the accuracy of each method over the same time period, we can also pass through an absolute error and relative error to the function.

C. Velocity Flip

If we doubled the length of the simulation to 6 periods using the frequency $\omega_0 = 1+\delta$ where $\delta = 10^{-9}$, its possible to flip the direction of the velocities after half the time has elapsed an investigate what happens to both the positions of the masses and also the energies of the system.

D. Choosing Initial Conditions

Using the most accurate ordinary differential equation solver the initial conditions of for the bodies can be set, i.e the coordinate positions and the vecloity direction and magnitudes. For masses $m_1 = m_2 = m_3 = 1/3$ with $\omega_0 \equiv 3.3$ the first set of initial conditions are:

$$x_1 = -0.30805788, v_{1y} = -1.015378093, \tag{15a}$$
$$x_2 = 0.15402894, y_2 = -0.09324743, \tag{15b}$$
$$v_{2x} = 0.963502817, v_{2y} = 0.507689046, \tag{15c}$$
$$x_3 = x_2, y_3 = -y_2, \tag{15d}$$
$$v_{3x} = -v_{2x}, v - 3y = v_{2y}, \tag{15e}$$
$$\tag{15f}$$

The second set of initial conditions suggested for masses $m_1 = m_2 = m_3 = 1$ and $\omega \equiv 2.47$ are:

$$x_1 = 0.97000436, y_1 = -0.24308753, \tag{16a}$$
$$v_{3x} = -0.93240737, v_{3y} = -0.86473146, \tag{16b}$$
$$v_{1x} = -v_{3x}/2, v_{1y} = -v_{3y}/2, \tag{16c}$$
$$x_2 = -x_1, y_2 = -y_1, \tag{16d}$$
$$v_{2x} = v_{1x}, v_{2y} = v_{1y}, \tag{16e}$$
$$\tag{16f}$$

IV. RESULTS

V. CONCLUSIONS

A. Simple Harmonic Oscillator

Solving the simple harmonic oscillator using the leapfrog technique the step size that produces a relative error in the energy smaller than 6 orders of magnitude.

$$h \equiv 0.001$$

Figure 1 shows the relative error over a time period, it can be seen that the relative error does not exceed 10^{-7} on this $\log_{10}$ scaled plot. The relative error oscillates seemingly randomly but the peak is almost always constant.

B. Phase Space and Energy Diagrams

Figures 2 shows the RK4 and Leapfrog phase space plots (x vs v) and energy evolution for varying step sizes, it shows that as the step size increases the Leapfrog method cycles at the same values of x and v better than the RK4 method shown by the thickness of the ring. As the step size increases for Leapfrog the energy of the system oscillation minimum and maximum values start to widen but is always conserved. For the RK4 method the energy of the system gradually decreases over time as step size increases, which is as expected due to the integrator method not being symplectic as discussed previously and in the case of $h = 0.1T_0$, the system energy exponentially tends to zero very early on.

C. Three Body Problem

Figures 3 shows the stable solution for the RK4 method that behaves for all values of ω_0. The energy of the system is constant in all cases and the bodies all orbit the centre of mass located at the origin. If we slightly increase the maximum time period, we begin to see the effect of slightly perturbing the frequency as shown in figures 4, here the smallest change to ω_0 has extremely varying effects to the dynamics, demonstrating the sensitivity of system. The same effect can be seen for the Leapfrog method in graphs 5 and 6. Using the scipy package, Odeint, to find an optimized solution to the ordinary differential equations we find that all three cases of w_0 should be the same, which is what would be expected since the perturbation of the system is so small (10^-9), figures 7 show that the orbits of the bodies should remain in the stable case, this tells us that both our integration methods are flawed as over the same time period neither of them are close to reproducing the exact solution in terms of dynamics or energy.

D. Reversing Direction

Figures 8 shows how the dynamics of the system changes when the velocities are reversed halfway into the simulation. Firstly the most notable thing is that the energy of the system reverses as well, changing from unstable and tending back towards a constant energy. The contour plots show the bodies travelling in the opposite direction to the initial conditions in their last frame, this flip can be best seen in the animation.

E. Setting the Initial Conditions

Two sets of initial conditions (15) were suggested and running the simulation for the first set shows a stable solution that traces out a heart shape, where each of the bodies follow the same path, this solution is stable which can be

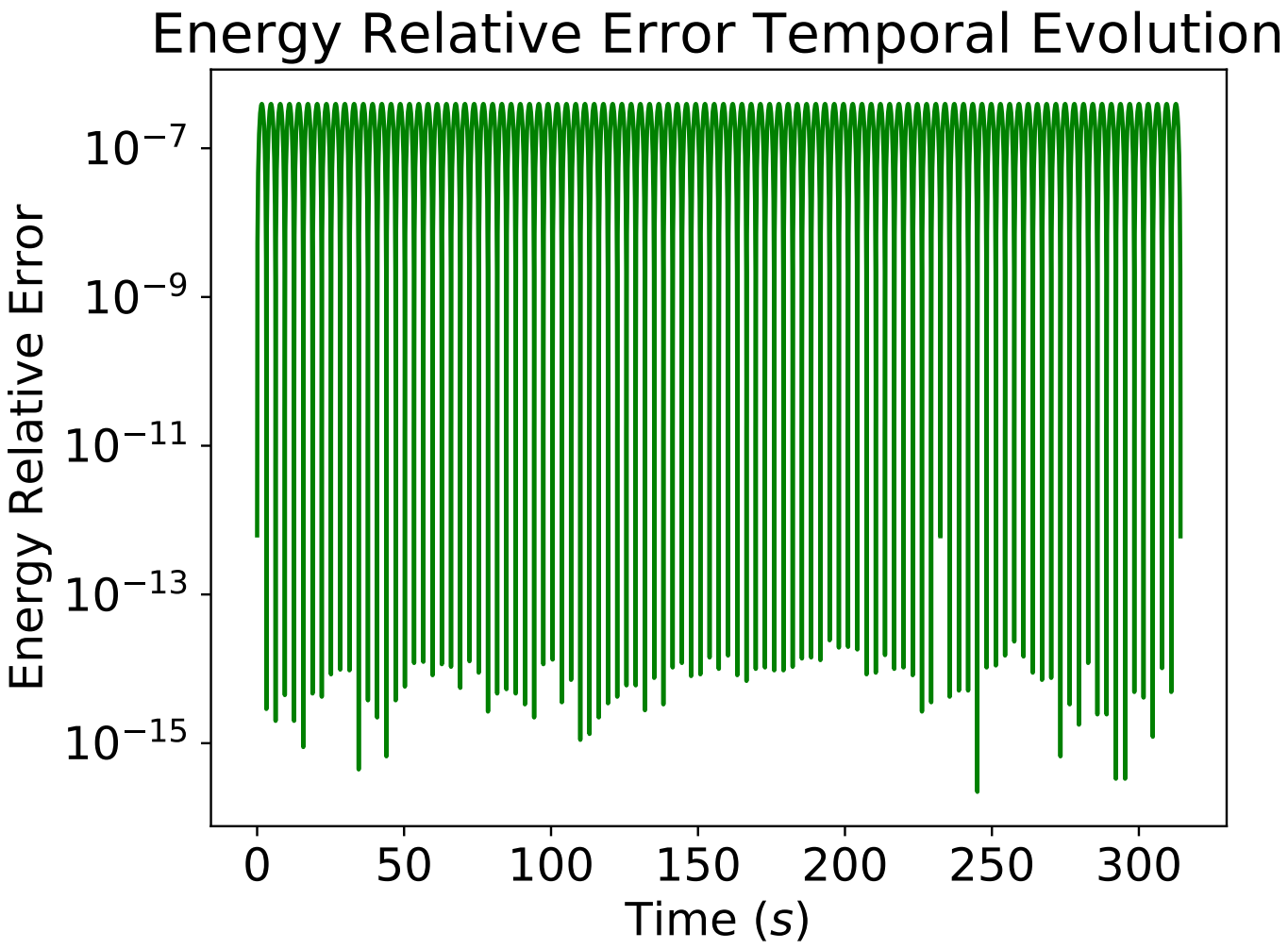

Figure 1. Energy relative error for Leapfrog integration for 50 cycles, $T_0 = 2\pi$ (Scaled to $\log_{10}$).

checked by altering the masses. The second set shows the solution where each of the bodies immediately fly off to infinity with only the red and green bodies interating with each other in the slightest.

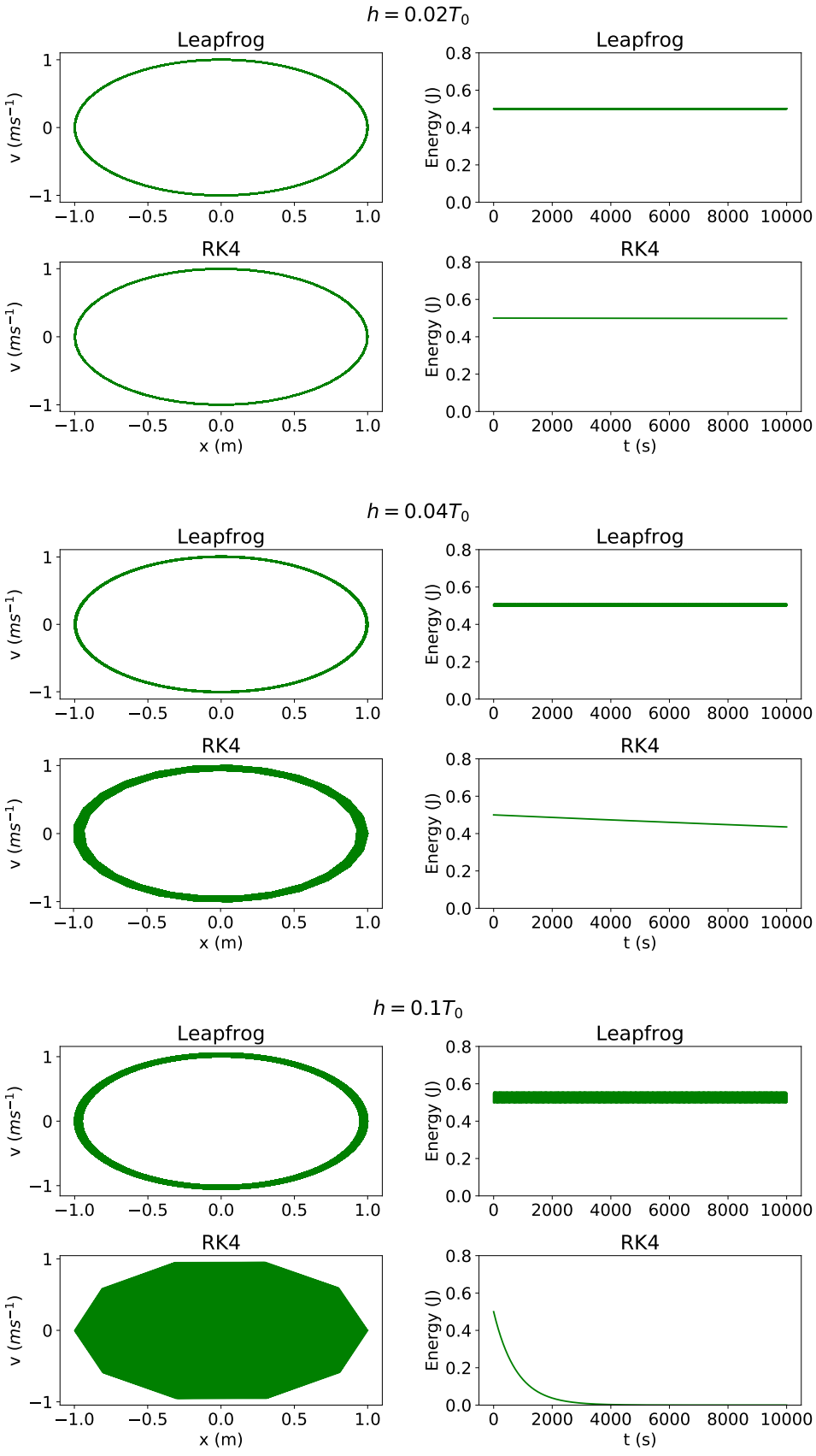

Figure 2. Phase Space diagrams ($x - v$) and total energy of the system for 800 periods of Leapfrog and Runge Kutta 4 integration for varying step sizes ($T_0 = 2\pi$).

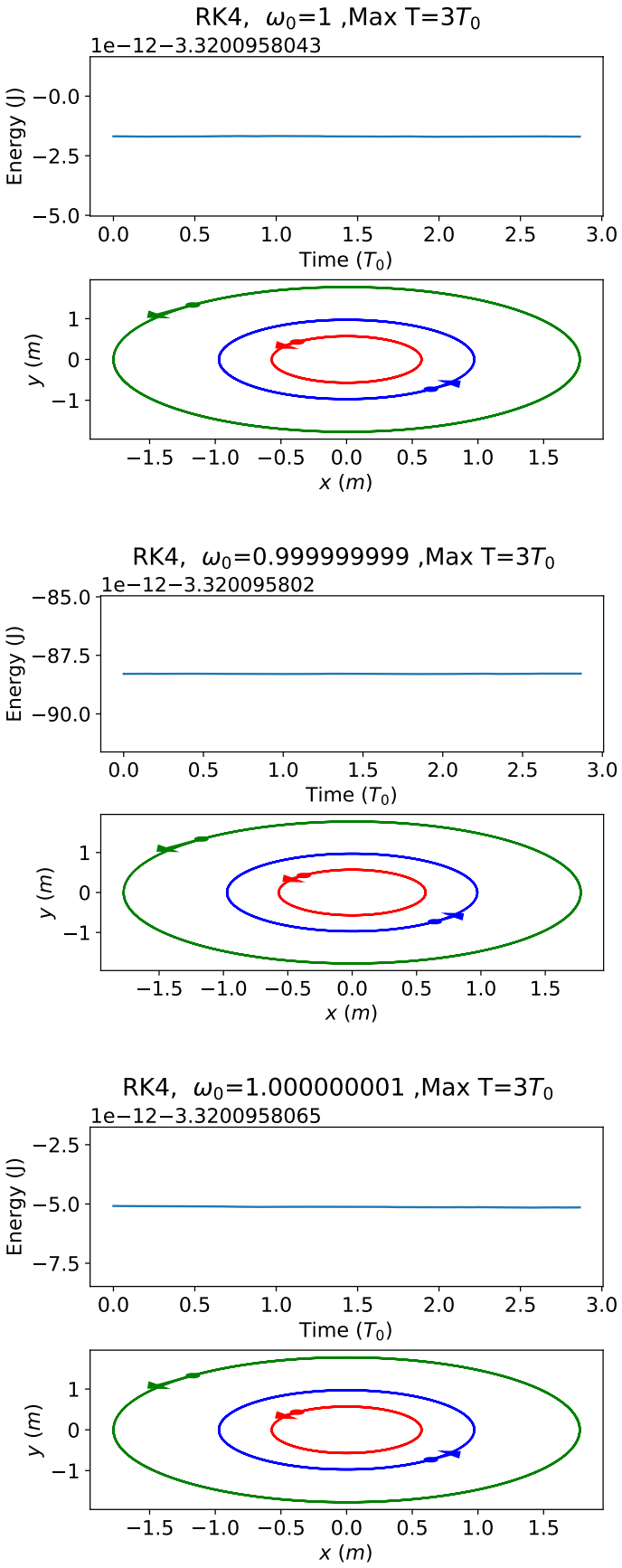

Figure 3. RK4 graphs showing the energy evolution and contour plots for 3 time periods.

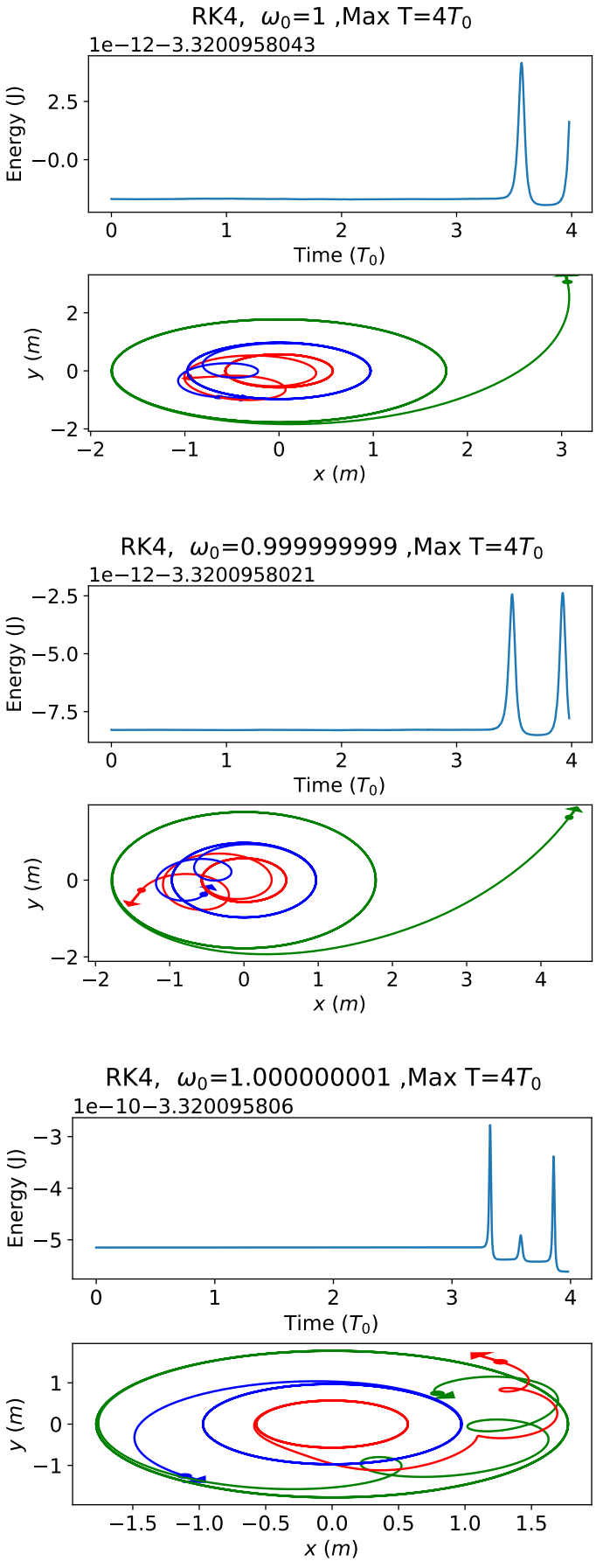

Figure 4. RK4 graphs showing the energy evolution and contour plots for slightly increased maximum time.

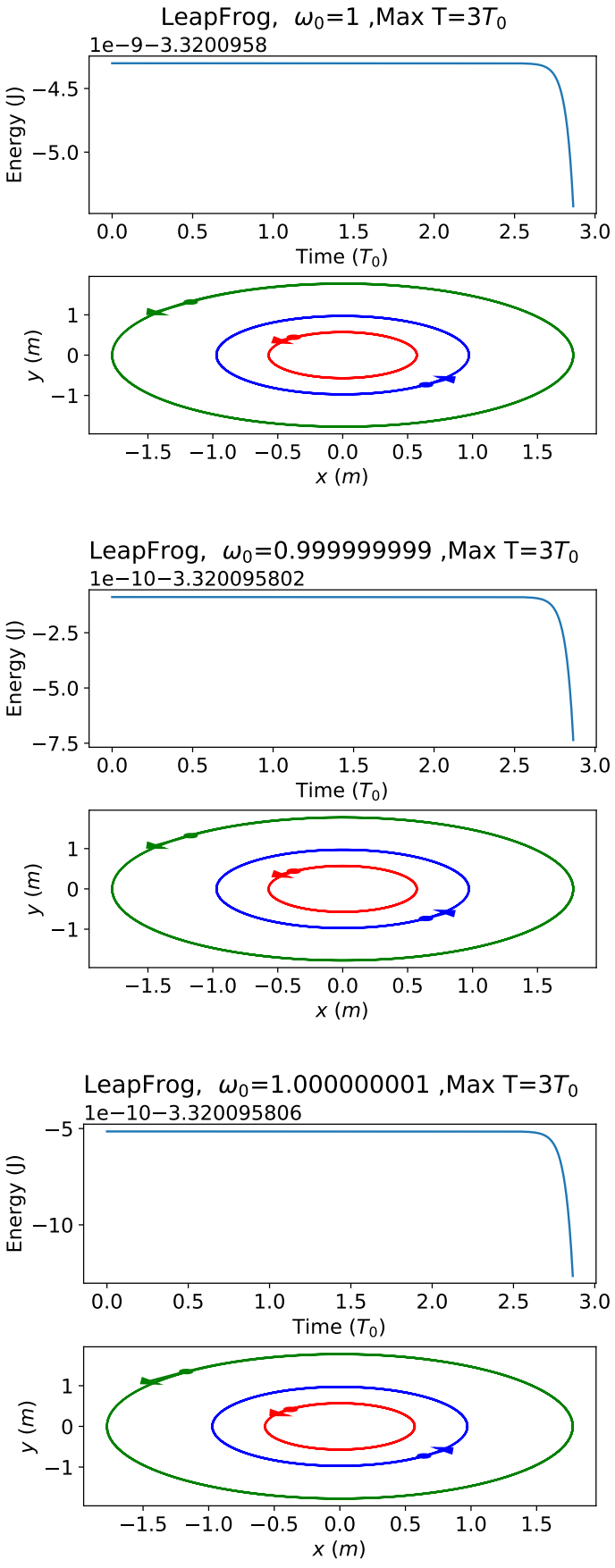

Figure 5. Leapfrog graphs showing the energy evolution and contour plots for 3 time periods.

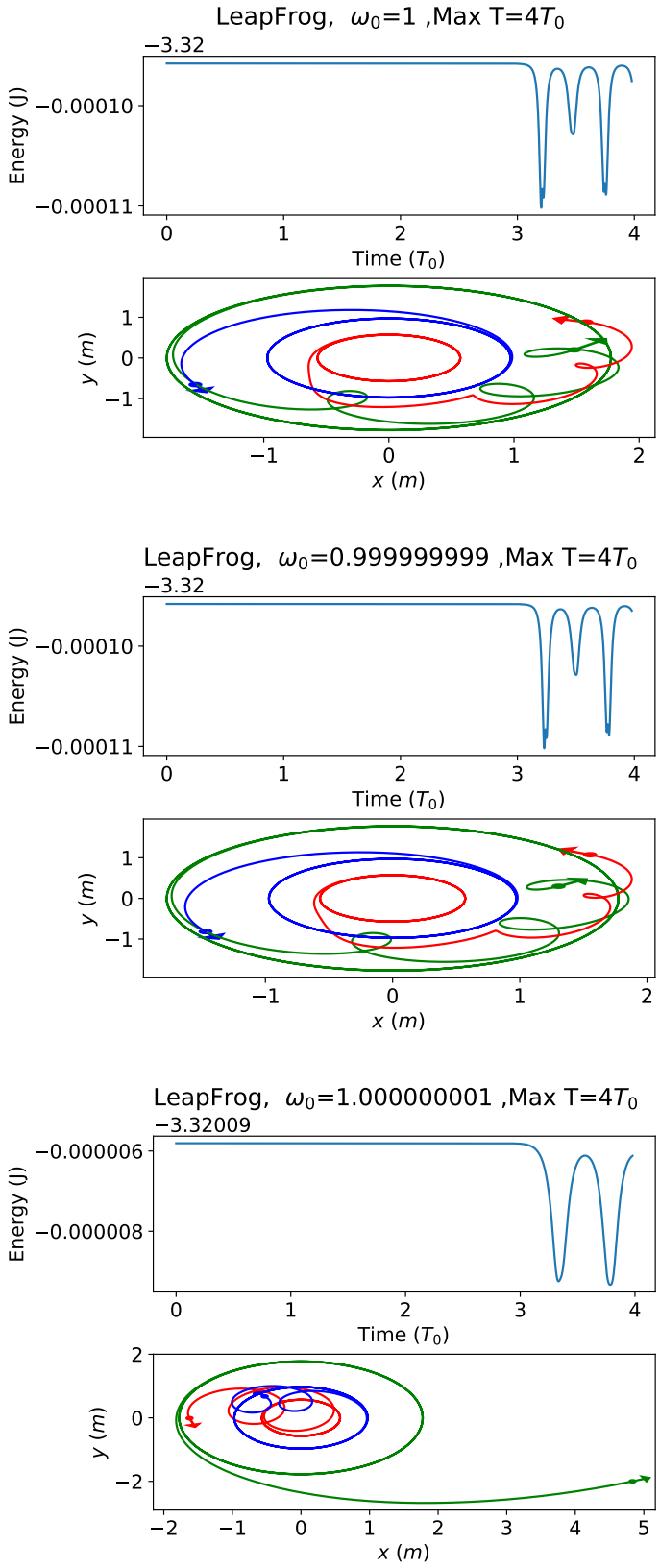

Figure 6. Leapfrog Graphs showing the Energy evolution of the system and contour plots for slightly increased maximum time.

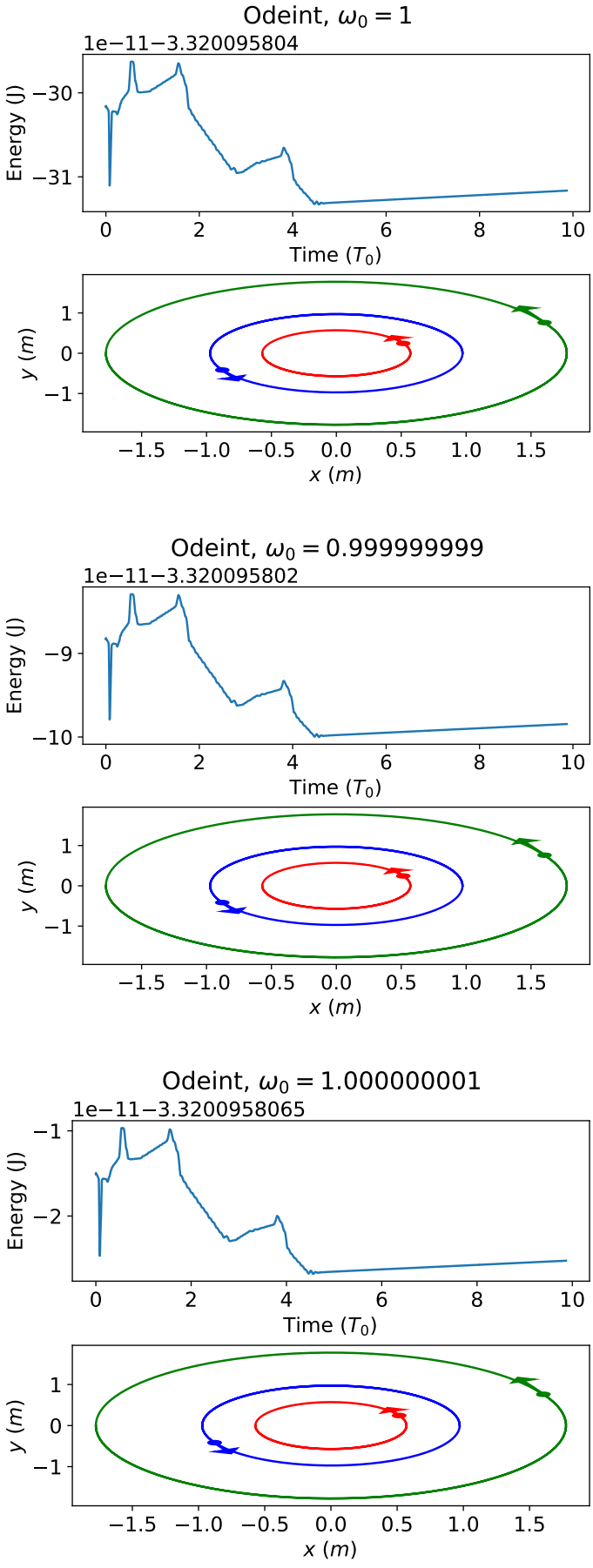

Figure 7. Python package , scipy, inbuilt integration function performed on the ordinary differential equations.

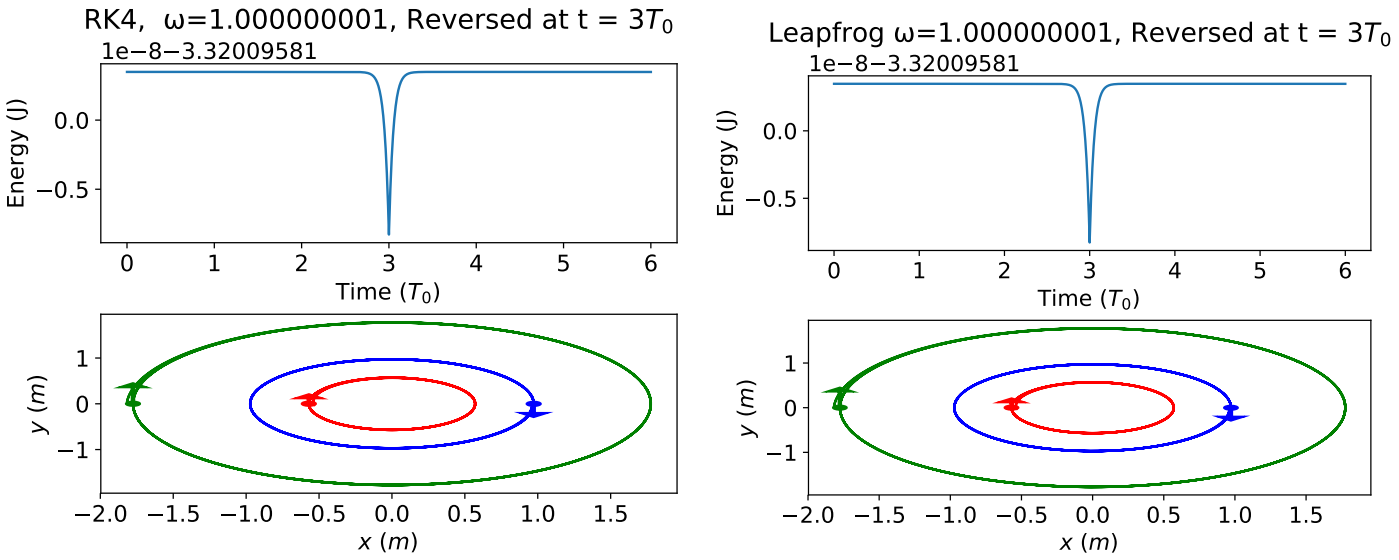

Figure 8. Graphs showing the contour paths for each integration method with the reverse occurring halfway through.

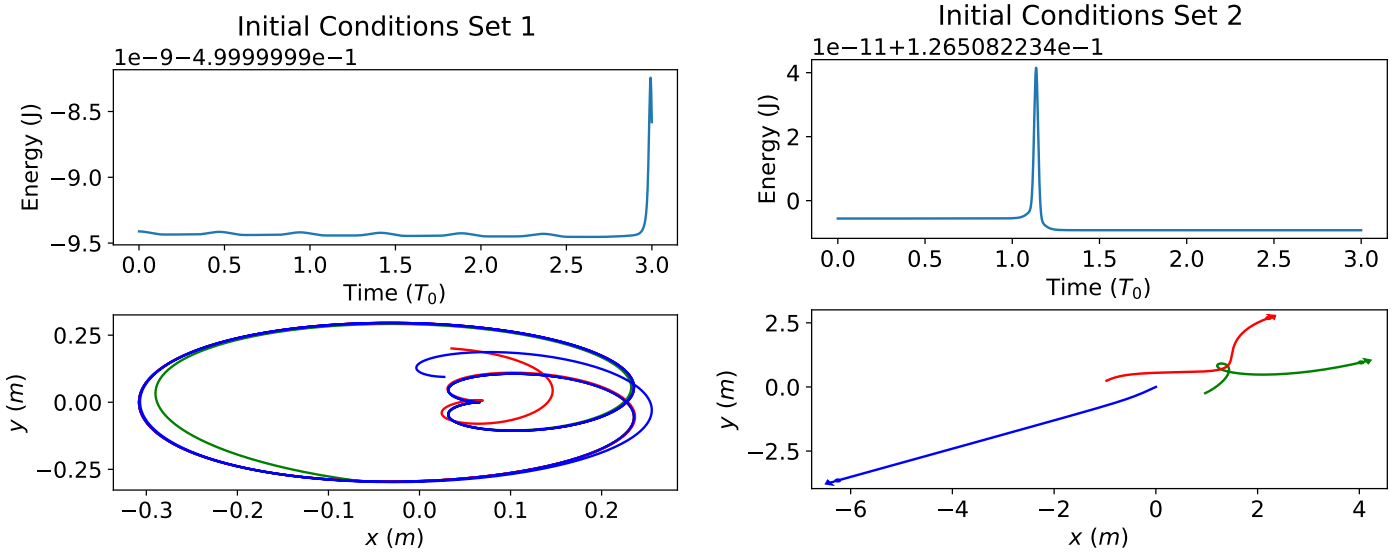

Figure 9. Suggested initial conditions, Set 1 left , Set 2 right.