

PH4044 Tutorial 3

March 2023

1 Evaluating the Fermi-Dirac Distribution and Boltzmann Approximation

The concentration of electrons in the conduction band of a solid-state structure, n , can be modelled using Fermi-Dirac statistics.

$$n = \frac{2}{\sqrt{\pi}} N_c F_{1/2}(\eta_f) \quad (1)$$

Where n (m^{-3}) is the electron concentration, N_c (m^{-3}) is the effective concentration of electrons in the conduction band, $F_{1/2}$ is the Fermi Integral, and η_f is a dimensionless variable related to the difference in energy between the conduction band and Fermi surface.

$$\eta_f = \frac{E_f - E_c}{kT} \quad (2)$$

When $E_f \ll E_c$, we can approximate electron concentration with a simpler function for n known as the Boltzmann approximation.

$$n = N_c \exp(\eta_f) = N_c \exp\left(-\frac{E_c - E_f}{kT}\right) \quad (3)$$

Using Python, we can define functions for (1) and (2) to graphically compare the Fermi-Dirac distribution function and the Boltzmann approximation over a range of η_f values (Fig 1).

Fig 1 shows the nature of the Boltzmann approximation - it is very good for values of $E_f < E_c$ but rapidly becomes unreliable for $E_f > E_c$.

To further demonstrate this behaviour, we can use the same Python to calculate explicit values of n by defining a function that defines (2).

In Table 1, we see that, for $E_f = E_C - 100\text{meV}$, the two functions predict almost exactly the same values of n . However, for the case $E_f = E_C + 50\text{meV}$, they differ by an order of magnitude. This once again shows the Boltzmann approximation is only suitable for $E_f < E_c$.

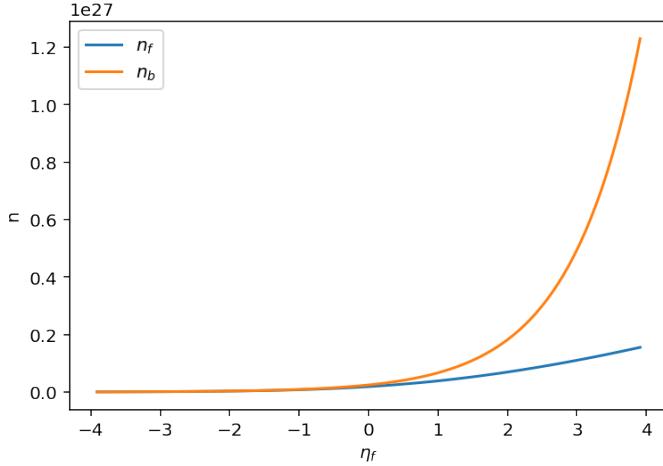


Figure 1: Carrier concentration, n , as a function of η_f for both the Fermi-Dirac distribution function (1) and the Boltzmann approximation (3) for a semiconductor at $T = 23C$ with an effective mass $m^* = m_e$, (where m_e is the rest mass of an electron). The values of η_f correspond to $-0.1 \text{ eV} \leq E_f - E_c \leq 0.1 \text{ eV}$, with positive values of η_f resulting from a Fermi energy greater than the conduction band energy.

E_f	n_f	n_b
$E_f = E_C - 100\text{meV}$	$4.88e^{23}$	$4.92e^{23}$
$E_f = E_C + 50\text{meV}$	$6.79e^{25}$	$1.74e^{26}$

Table 1: Electron concentration predicted by the Fermi-Dirac function and Boltzmann approximation for specific values of E_f .

2 Spatio-Temporal Dynamics of Excess Minority Hole Carriers, in an n-Doped Semiconductor

The dynamics of excess minority carriers in an n-doped semiconductor can be described with the following equation

$$D_p \frac{\partial^2(\delta p)}{\partial x^2} - \mu_p E_0 \frac{\partial(\delta p)}{\partial x} - \frac{\delta p}{\tau_p} = \frac{\partial(\delta p)}{\partial t} \quad (4)$$

Where $\delta p (\text{m}^{-3})$ is the excess hole concentration, $D_p (\text{m}^2\text{s}^{-1})$ is the diffusion constant, $\mu_p (\text{m}^2\text{V}^{-1}\text{s}^{-1})$ is the carrier mobility, $E_0 (\text{NC}^{-1})$ is the electric field in the semiconductor, and $\tau_p (\text{s}^{-1})$ is the recombination rate. D_p can be expressed in terms of temperature (5), which will prove useful when examining the physical behaviour of the semiconductor under variable conditions

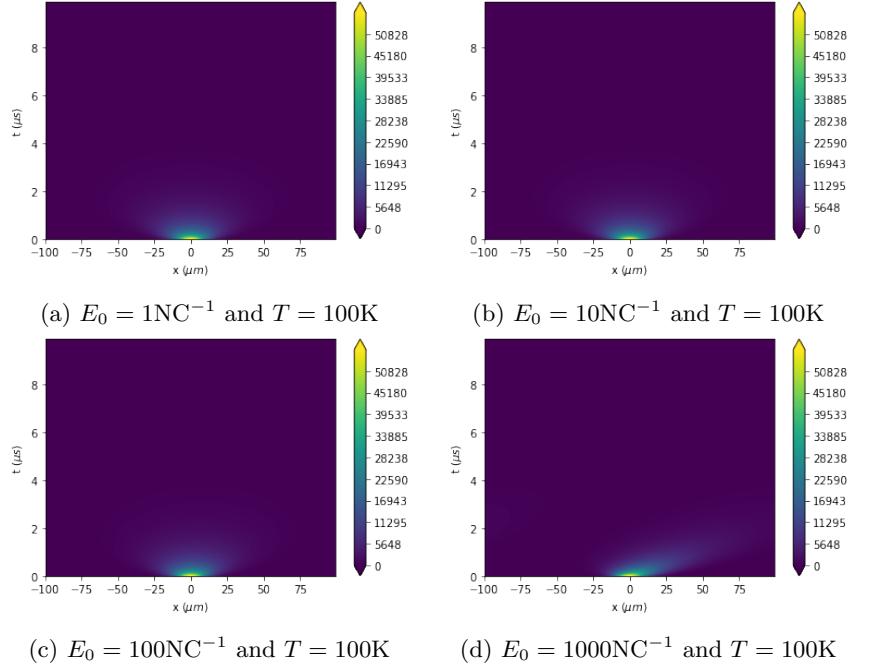


Figure 2: Dispersion of excess carriers δp over space and time for varying E_0 . The initial state δp_0 is of the form seen in (6) with an alpha value of $1e6$.

$$D_p = \frac{\mu_p kT}{e} \quad (5)$$

Eq 4, at its core, is simply a conservation equation based on the current in the semiconductor due to an applied and internal electric field and electron diffusion. Using a method proposed by Dr Steve Brunton on his YouTube channel, we can solve this partial differential equation using the FFT algorithm [1] for an initial normalised state δp_0 described by 6

$$\delta p_0 = \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha x^2) \quad (6)$$

were α is a parameter that allows us to tune the width of the initial distribution. The solutions for varying E_0 can be seen in Fig 2.

In Fig 2, we do that the carriers naturally spread out over time, which is what we would expect of like charges. We notice that the time dependence of the hole dispersion is unaffected by the increasing electric field until it is an order of magnitude greater than the temperature, suggesting thermal fluctuations dictate behaviour at higher temperatures. We see this behaviour displayed more completely in Fig 3, shown by a slant to the right in carrier concentration.

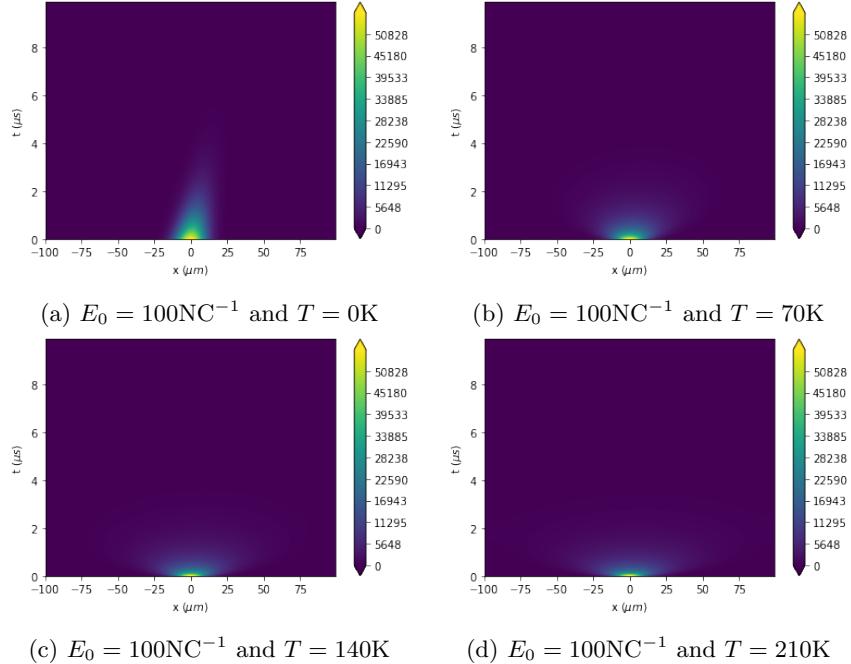


Figure 3: Dispersion of excess carriers δp over space and time for varying T . The initial state δp_0 is of the form seen in (6) with an alpha value of $1e6$.

Given we are applying an electric field in the positive x direction, this is the sort of behaviour we would expect.

In Fig 3a, we see dispersion is dominated by the applied electric field, once again in a case where E_0 is orders of magnitude greater than the temperature. As soon as T is raised to a magnitude on par with E_0 , we see thermal effects dominate. Both Fig 2 and Fig 3 suggest thermal effects dictate dispersion except at low temperatures or when the applied field is orders of magnitude greater than the temperature. We can compare the numeric results displayed in Fig 2 and Fig 3 to the analytical solution of (4) (see (7)).

$$\delta p(x, t) = \frac{e^{-t/\tau_p}}{(4\pi D_p t)^2} \exp \left[\frac{-(x - \mu_p E_0 t)^2}{4D_p t} \right] \quad (7)$$

We see this looks familiar to Fig 3a, which is unsurprising given their similar temperatures and applied fields. There is a dense concentration of carriers around $x = t = 0$, that gently spreads out over time. We again see the dominant effect is the electric field, causing the slight rightward inclination of the distribution through time. The analytic solution does appear to be broader than the numeric solution in Fig 3a, something that could be modified with our choice of the initial distribution δp_0 .

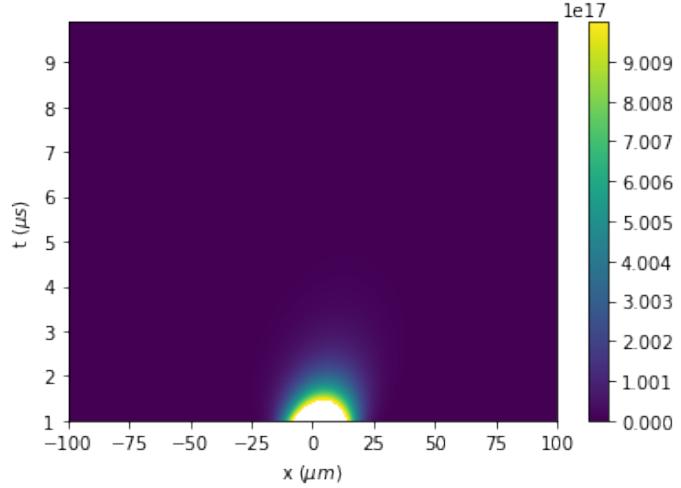


Figure 4: The analytic solution to the continuity equation (7), assuming a finite initial concentration $P_0 = 1$ at $t = x = 0$. E_0 and T are set to 100NC^{-1} and 10K respectively (4). The t axis starts at 1 to avoid destroying the detail in the graph that arises if the singularity at $t = 0$ is included in the plot.

Overall, we see that the dispersion of excess carriers is largely due to thermal fluctuations unless the semiconductor is cooled to near 0K or E_0 is greater than T by several orders of magnitude. In these limits, the electric field is seen to have a noticeable effect on the carriers characterised by a 'slant' in the positive x direction.

References

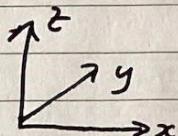
- [1] Steve Brunton (YouTube: @Eigensteve). Title of Video: Solving PDEs with the FFT [Python]. (Apr 18, 2020). Accessed: Mar 13, 2023. [Online Video]. Available: <https://www.youtube.com/watch?v=hDeARtZdq-U>

Question 2

a) Force experienced due to magnetic field by holes:

$$\underline{F}_B = q \underline{v} \times \underline{B}$$

$$= e(v_x \hat{x} \times B_z \hat{z})$$

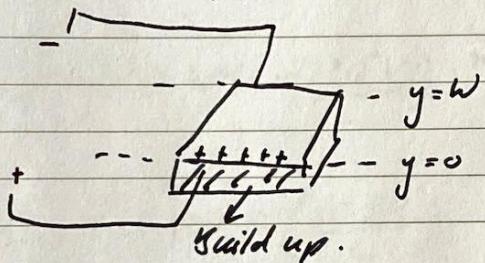


$$= e \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & 0 & 0 \\ 0 & 0 & B_z \end{vmatrix}$$

$$= -e \hat{y} (v_x - B_z)$$

$$= -F_y \hat{y}$$

- Holes build up on the southern β semi-conductor face



- Which sets up an E -field in the \hat{y} direction. Thus the potential would increase in the $-\hat{y}$ direction.
- Given the V_H potential is also measured in the \hat{y} direction

$$\Phi = - \int \underline{E} \cdot d\underline{s} = - \int_0^w E_y \hat{y} dy = - E_y w$$

$$E_y = v_x B_{xz}$$

$$\therefore \Phi = - E_y w v_x B_{xz} \quad v_x = \frac{I}{\rho A} \quad \rho \text{ is hole concentration.}$$

$$\therefore \Phi = - \frac{W J B_{xz}}{\rho e} \quad (\text{For holes})$$

Thus, there must be a build-up of holes, as this is the only configuration that creates a p.d. opposing V_H giving a negative value.

b) From part a:

$$\underline{\Phi} = - \frac{W \underline{J} \underline{B}_z}{\rho e}$$

$$\underline{J} = \frac{\underline{I}}{A} = \frac{\underline{I}}{wd}$$

$$\therefore \underline{\Phi} = - \frac{\underline{I} \underline{B}_z}{d \rho e}$$

$$\Rightarrow - \frac{\underline{\Phi} de}{\underline{I} \underline{B}_z} = p$$

$$\therefore p = - \frac{(-4 \times 10^{-3} V)(10^{-5} m)(1.6 \times 10^{-19} C)}{(1.0 \times 10^{-3} A)(5 \times 10^{-2} T)}$$

$$= 1.28 \times 10^{-22} \frac{V m C}{A \text{ kg } s^{-2} A^{-1}}$$

$$= 1.28 \times 10^{-22} \frac{J \text{ C } m \text{ s}}{\text{kg } s^{-2}}$$

$$= 1.28 \times 10^{-22} \frac{Nm}{\text{kg } s^{-2}}$$

$$= 1.28 \times 10^{-22} \frac{\text{kg } m \text{ s}^{-2} m^{-2}}{\text{kg } s^{-2}}$$

$$= 1.28 \times 10^{-22} m^3$$

↪ Need m^{-3}

$$\Rightarrow p = - \frac{\underline{I} \underline{B}_z}{d \underline{\Phi} e}$$

$$\Rightarrow p = - \frac{(1 \times 10^{-3} A)(5 \times 10^{-2} T)}{(-4 \times 10^{-3} V)(10^{-5} m)(1.6 \times 10^{-19} C)}$$

$$= 7.81 \times 10^{26} m^{-3}$$



c) In the equilibrium case, we know the magnetic force on current is matched by an electrostatic repulsion. $E(x)$

Assuming diffusion effects are negligible (particularly as p has a constant value):

$$J_p(x) = e \mu_p p(x) E_x(x)$$

$$\begin{aligned} \underline{E_x(x)} &= 12.5 \text{ V} = - \int \underline{E} \cdot d\underline{s} \xrightarrow{\substack{\text{direction of increasing } V \\ (-dx) \text{ in this case}}} \\ &= - \int_{40}^{90} E_x \cdot dx \\ &= E_x L \end{aligned}$$

$$\therefore E_x = \frac{12.5 \text{ V}}{L}$$

$$\therefore J_p(x) = e \mu_p p \left(\frac{12.5 \text{ V}}{L} \right)$$

$$\Rightarrow \frac{I_x}{A} = e \mu_p p \left(\frac{12.5 \text{ V}}{L} \right)$$

$$\Rightarrow \mu_p = \frac{I_x L}{A e p (12.5 \text{ V})}$$

$$\begin{aligned} &= (1.0 \times 10^{-3} \text{ A}) (10^{-3} \text{ m}) \\ &\quad \overbrace{(10^{-54} \text{ m})}^w \overbrace{(10^{-5} \text{ m})}^d \overbrace{(1.6 \times 10^{-19} \text{ C})}^e \overbrace{(7.81 \times 10^{-26.21} \text{ m}^{-3})}^p \overbrace{(12.5 \text{ V})}^V \\ &= 0.6064 \text{ u} \end{aligned}$$

$$\text{Where } u = \frac{A m}{m^2 C m^{-3} V} = \frac{k g^{-1}}{e V} \frac{A m}{m^2 C m^{-3} V} = \cancel{A} m^2 s^{-1} \cancel{C} V^{-1} = m^2 s^{-1} V^{-1}$$

$$\therefore \mu_p = 0.064 \text{ m}^2 \text{s}^{-1} \text{V}^{-1}$$

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Q8^{4.a)} Diffusion length is given by:

$$L_p = \sqrt{D_p \tau_p}$$

If we assume room temperature to be $\sim 290K$:

$$D_p = \frac{2 \mu_p k T}{e}$$

$$\Rightarrow L_p = \sqrt{\frac{\mu_p k T}{e} \tau_p}$$

$$= \sqrt{\frac{(480 \text{ cm}^2 \text{ V}^{-1} \text{ s}^{-1})(1.38 \times 10^{-23} \text{ kg s}^{-2} \text{ K}^{-1})(290 \text{ K}) (10 \mu\text{s})}{1.6 \times 10^{-19} \text{ C}}}$$

$$= 0.12 \text{ mm} = 1.1 \times 10^{-4} \text{ m}$$

b) Check

$$\frac{(10^{-4})(10^{-23})(2 \times 10^{-2})(10 \times 10^{-6})}{(-10^{-19})}$$

b) Forward current: 10 mA

Injection ratio: ~ 1

$$J = e \left[\frac{D_n n_{p0}}{L_n} + \frac{D_p p_{n0}}{L_p} \right] \left[\exp\left(\frac{eV}{kT}\right) - 1 \right]$$

$$\text{Hole injection ratio} = \frac{D_p p_{n0} / L_p}{D_n n_{p0} / L_n + D_p p_{n0} / L_p} \approx 1$$

$$\Rightarrow J = e \left[\frac{D_p p_{n0}}{L_p} \right] \left[\exp\left(\frac{eV}{kT}\right) - 1 \right]$$

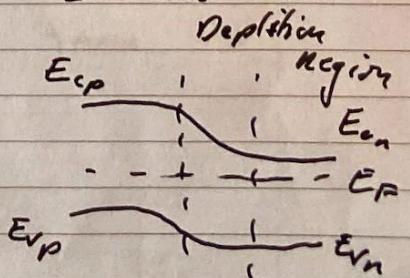
Want J just in terms of p_{n0} .

Zero bias case.

From the Boltzmann approximation:

$$n_n = N_c \exp\left\{-\left(\frac{E_{cn} - E_F}{kT}\right)\right\} \quad \text{where}$$

$$n_p = N_c \exp\left\{-\left(\frac{E_{cp} - E_F}{kT}\right)\right\}$$



$$n_p = n_n \frac{\exp\left(-\frac{(E_{pc} - E_F)}{kT}\right)}{\exp\left(-\left\{\frac{E_{nc} - E_{EF}}{kT}\right\}\right)}$$

$$\begin{aligned} n_p &= n_n \exp\left(\frac{-E_{pc} + E_F}{E_{nc} - E_F}\right) \frac{-E_{pc} + E_F + E_{nc} - E_F}{kT} \\ &= n_n \exp\left(-\left\{\frac{E_{pc} - E_{nc}}{kT}\right\}\right) \end{aligned}$$

If we want the carrier concentration in the n -region:

$$n_{nn} = n_p \exp\left(-\left\{\frac{E_{cn} - E_{cp}}{kT}\right\}\right)$$

If we define V_{bi} such that

$$eV_{bi} = E_{cp} - E_{cn}$$

$$\Rightarrow n_n = n_p \exp\left\{\frac{eV_{bi}}{kT}\right\}$$

If we apply a forward bias

$$E_{cp} - E_{cn} = e(V_{bi} - V_a)$$

$$\therefore n_n = n_p \exp\left\{\frac{eV_{bi} - eV_a}{kT}\right\}$$

$$\therefore n_n = n_{n0} \exp\left\{-\frac{eV_a}{kT}\right\}$$

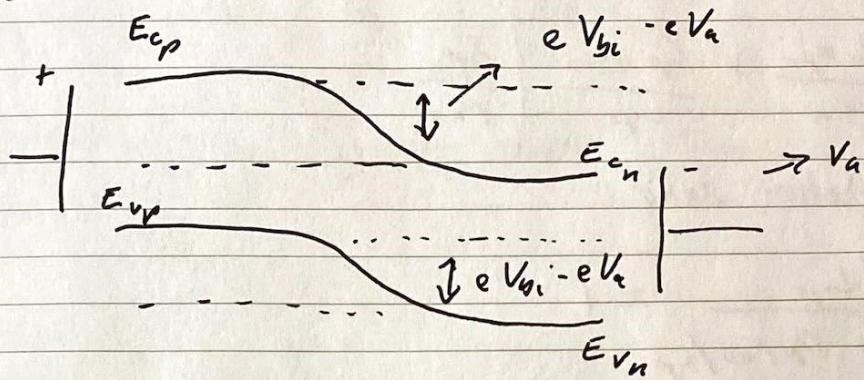
Replace V with V_a :

$$n_n = n_{n0} \exp\left\{-\frac{eV}{kT}\right\}$$

We want $\Delta n_n = n_n - n_{n0}$ $\xrightarrow{\text{? get this extra - sign}}$
 $= n_{n0} \left\{ \exp\left\{-\frac{eV}{kT}\right\} - 1 \right\}$ in more. Should this be here?

Attaching my calculation is

Having just reviewed my work, I think we have an issue. I've just realised that I've calculated everything for ~~today's~~ today's ~~holes~~ holes for electrons.



From the Boltzmann approximation:

$$p = N_v \exp\left(\frac{E_v - E_F}{kT}\right)$$

Thus, in the zero-bias case:

$$p_n = N_v \exp\left(\frac{E_{vn} - E_F}{kT}\right)$$

$$p_p = N_v \exp\left(\frac{E_{vp} - E_F}{kT}\right)$$

$$\Rightarrow p_n = p_p \exp\left(\frac{E_{vn} - E_F - E_{vp} + E_F}{kT}\right)$$

$$\therefore p_n = p_p \exp\left(\frac{E_{vn} - E_{vp}}{kT}\right)$$

$$= p_p \exp\left(-\frac{eV_{bi}}{kT}\right) = \frac{p_{n0}}{p_{p0}} \quad V_{bi} = E_{cp} - E_{vn}$$

In the forward bias case; the zero bias case above is the density at $E = 0$; before the set-up of the internal field. At some later time:

$$p_n = p_p \exp\left(-\frac{eV_{bi}}{kT} + \frac{eV_a}{kT}\right)$$

$$\Rightarrow \Delta p_n = p_n - p_{n0} = p_{n0} \left\{ \exp\left(\frac{eV_a}{kT}\right) - 1 \right\}.$$

The current density is given by :

$$J = e \left[\frac{D_n n_{po}}{L_n} + \frac{D_p p_{no}}{L_p} \right] \left[\exp\left(\frac{eV}{kT}\right) - 1 \right]$$

$$= e \left[\frac{D_n n_{po}}{L_n} + \frac{D_p p_{no}}{L_p} \right] \frac{\delta p_n}{p_{no}}$$

Using the injection ratio :

$$\frac{D_p p_{no} / L_p}{D_n n_{po} / L_n + D_p p_{no} / L_p} \approx 1$$

$$\Rightarrow D_n n_{po} / L_n = 0$$

Thus

$$J = e \left[\frac{D_p p_{no}}{L_p} \right] \frac{\delta p_n}{p_{no}}$$

$$= e \frac{\delta p_n D_p}{L_p}$$

Given $J = I/A$:

$$\delta p_n = \frac{I L_p}{A e D_p} = \frac{I \tau_p}{A e L_p} = \frac{(10 \times 10^{-3})(10 \times 10^{-6})}{(1 \times 10^{-6})(1.6 \times 10^{-19})(1.1 \times 10^{-4})}$$

$$= 5.68 \times 10^{21} \text{ m}^{-3}$$

$$\Rightarrow \delta p_n = \frac{(10 \times 10^{-3})(0.12 \times 10^{-3} \text{ m})}{(1 \times 10^{-6} \text{ m}^2)(1.6 \times 10^{-19})(1.1 \times 10^{-4})} = 5.68 \times 10^{15} \text{ cm}^{-3}$$

// δp_n is the no. holes in the n region due to the applied field.

Check units :

$$\frac{I \tau_p}{A e L_p} = \frac{A \tau_p}{m^2 C m} = \frac{C s^{-1} A}{m^3 C} = \frac{1}{m^3}$$

c) // When we say $n_p p_p = n_i^2$, is this simply an assertion?

Saturation current is given by

$$I_o = A e \left[\frac{O_n n_{po}}{L_n} + \frac{O_p p_{no}}{L_p} \right]$$

Assuming an injection ratio of one:

$$\frac{O_n n_{po}}{L_n} = 0$$

$$\therefore I_o = A e \frac{O_p p_{no}}{L_p} \quad (1)$$

We can relate p_{no} to δp_n :

$$\delta p_n = p_n - p_{no}$$

$$\Rightarrow p_{no} = p_n - \delta p_n \quad (2)$$

~~$p_n = N_D$~~ for the tip region of the diode

~~$\Rightarrow \overline{p_{no}} = \overline{p_n} =$~~
we know

$$p_n n_n = n_i^2 = p_n N_D$$

$$\Rightarrow p_n = \frac{n_i^2}{N_D} \quad (3)$$

Sub (3) \Rightarrow (2) :

$$p_{no} = \frac{n_i^2}{N_D} - \delta p_n \quad (4)$$

Sub (4) \Rightarrow (1) :

$$p_n I_o = A e \frac{O_p p_{no}}{L_p} \left(\frac{n_i^2}{N_D} - \delta p_n \right)$$

$$= A e \frac{L_p}{\tau_p} \left(\frac{n_i^2}{N_D} - \delta p_n \right)$$

$$\therefore I_o = \frac{(1 \times 10^{-6} \text{ m}^2)(1.6 \times 10^{-19} \text{ C})(1.1 \times 10^{-4} \text{ m})}{(10 \times 10^{-6} \text{ s})} \times \frac{(1.5 \times 10^{10} \times 10^{-6} \text{ m}^3)}{10^{17} \times 10^{-6} \text{ m}^3} = 6.9 \times 10^{-18}$$

Need to conduct some unit analysis!

$$\begin{aligned} I_o &= \frac{A e L_p}{\tau_p} \left(\frac{n_i^2}{N_0} - p_{n_0} \right) \\ &= \frac{m^2 C \text{ m}}{\text{s}} \left(\text{cm}^{-3} - \text{m}^{-3} \right) \\ &= \text{C/s} \\ &= \underline{\underline{A}} \end{aligned}$$

Unit analysis suggests this is of the correct form:

$$\begin{aligned} \therefore I_o &= \frac{(1 \times 10^{-6} \text{ cm}^2)(1.6 \times 10^{-19} \text{ C})(1.1 \times 10^{-4} \text{ cm})}{(10 \times 10^{-6} \text{ s})} \left(\frac{(1.5 \times 10^{10} \text{ cm}^{-3})}{(10^{15} \text{ cm}^{-3})} - 5.68 \times 10^{15} \text{ cm}^{-3} \right) \\ &= - \approx 1.0 \times 10^{-3} \text{ A} \\ &= - 1.0 \text{ mA} \end{aligned}$$

[^] This negative doesn't make a whole lot of sense. In review, I've found on the lecture slides $n_{n_0} p_{n_0} = n_i^2$:

$$\therefore n_{n_0} p_{n_0} = \frac{n_i^2}{n_{n_0}} = \frac{n_i^2}{N_0}$$

$$\begin{aligned} \therefore I_o &= \frac{A e L_p}{\tau_p} \frac{n_i^2}{N_0} = \frac{(1 \times 10^{-6} \text{ cm}^2)(1.6 \times 10^{-19} \text{ C})(1.1 \times 10^{-4} \text{ cm})}{(10 \times 10^{-6} \text{ s})} \left(\frac{(1.5 \times 10^{10} \text{ cm}^{-3})}{10^{15} \text{ cm}^{-3}} \right) \\ &= 3.96 \times 10^{-13} \text{ A} \end{aligned}$$

Question: When can we use $p_{n_0} n_0 = n_i^2$? Is it equivalent to $p_{n_0} n_{n_0} = n_i^2$? What do N_A and N_D actually represent? Is p_{n_0} the hole concentration in the n-region before diffusion? What is n_i ? Does I_o have to be positive?

1) We can use the reverse saturation current:

$$\begin{aligned} I &= I_0 \left[\exp\left(\frac{eV}{kT}\right) - 1 \right] \\ &= I_0 \left[\exp\left(\frac{(1.6 \times 10^{-19} C)(0.5 V)}{(1.38 \times 10^{-23} J)(290 K)}\right) - 1 \right] \\ &\quad m^2 \text{A} \text{s}^{-1} \text{K}^{-1} \\ &= (3.96 \times 10^{-13}) (\dots) \\ &= 1.9021 \times 10^{-4} \text{ A} \end{aligned}$$

c) Using $V_{oc} = \frac{kT}{e} \ln \left[1 + \frac{I_L}{I_0} \right]$

$$\Rightarrow 1 + \frac{I_L}{I_0} = \exp\left(\frac{V_{oc}}{kT}\right)$$

$$\Rightarrow I_L = I_0 \left[\exp\left(\frac{eV_{oc}}{kT}\right) - 1 \right]$$

~~⇒~~ If I_L is current from photoconductive emission, then $I_L = \Phi e$ where Φ is no. of photons per second

$$\Rightarrow \Phi = \frac{I_0}{e} \left[\exp\left(\frac{eV_{oc}}{kT}\right) - 1 \right]$$

$$= \frac{(3.96 \times 10^{-13} \text{ A})}{(1.6 \times 10^{-19} \text{ C})} \left[\exp\left(\frac{(1.6 \times 10^{-19} \text{ C})(0.5 \text{ V})}{(290)(1.38 \times 10^{-23})}\right) - 1 \right]$$

$$= \underline{\underline{1.19 \times 10^{15} \text{ photons per second.}}}$$