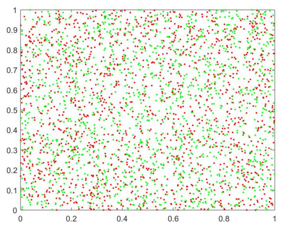

Chaotic Dynamics in Nonlinear Dynamical systems

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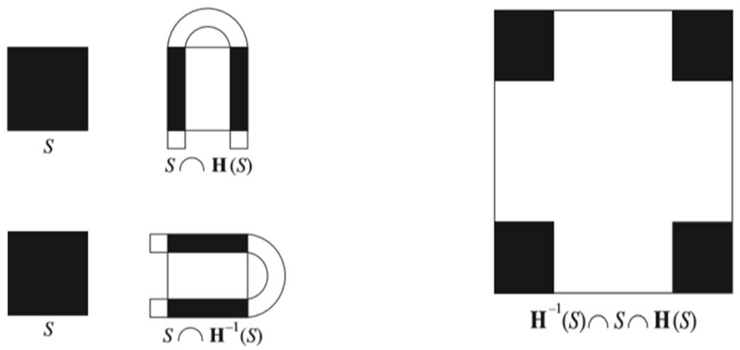
What is Chaos Theory?

Chaos Theory is concerned with the behaviour of nonlinear dynamical systems and their highly sensitive dependence on initial conditions. Under certain conditions, these systems can produce completely unpredictable and wildly chaotic behaviour over time. **Aim of this project**: to test for the presence of Chaos within a system; we consider a systems sensitivity to initial conditions, as well as the density of orbits and the existence of periodic orbits of multiple periods.

The Smale Horseshoe

Consider a square containing our initial points that we will call S.
We suppose the horseshoe map is given by $H : S \to \mathbb{R}^2$ and that H contracts S in the horizontal direction, before expanding S in the vertical direction and finally folding the rectangle back onto itself to form a horseshoe shape (hence the name!)
Similarly, H^{-1} on S carries out the same deformation but in the opposite direction, resulting in the horseshoe rotated 90 degrees. Instead of studying the trajectory in space we are simply looking at the sequence of returns on the square, and where these mappings intersect. We find that when we consider the forward orbit (H) and backward orbit (H^{-1}) under iteration and find where they intersect, we are left with a set of invariant points that never leave S under iteration of H, from here we can prove that the Horseshoe has Chaotic properties.
To see how this system would display high sensitivity to initial conditions, we imagine injecting a small drop of food colouring somewhere on S, which represents a set of nearby initial conditions. After many iterations of stretching, contracting and folding the colouring would spread throughout the square.



The Tent Map

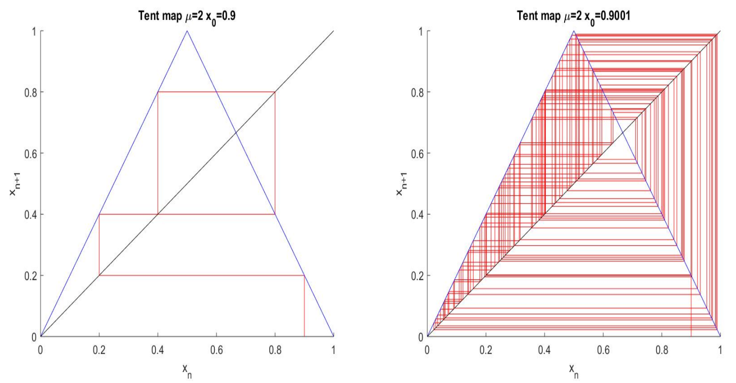
The Tent map is an example of a one-dimensional nonlinear discrete dynamical system. We present the tent map, T:[0,1]⟶[0,1], defined by

$$T(x) = \begin{cases} \mu(x) & 0 \le x < 1/2 \\ \mu(1-x) & 1/2 \le x \le 1 \end{cases}$$

where $0 \le \mu \le 2$

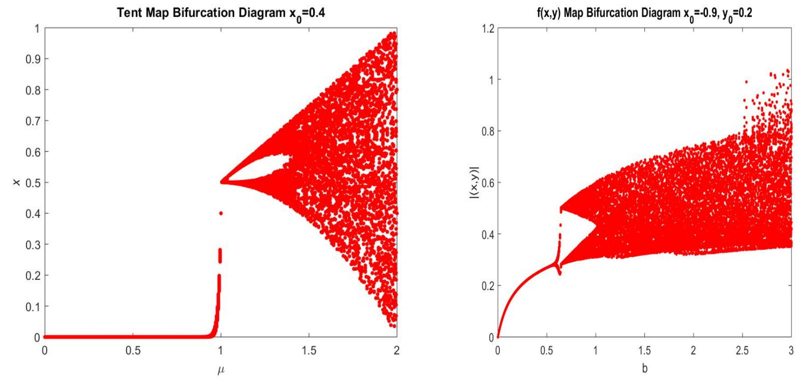
We notice that complex behaviour and in some cases chaotic phenomena can be displayed for specific parameter values. Correspondingly, for certain parameter values, the mapping undergoes stretching and folding transformations, and can display high sensitivity to initial conditions and periodicity.

Is the Tent Map Chaotic?



$x_0 = 0.9$ (L), $x_0 = 0.9001$ (R), Initial points are only $\epsilon = 0.001$ apart, however their orbits differ greatly even after a small number of iterations, this high sensitivity to initial conditions is a hallmark of Chaos. On further study of the tent map we see that Chaotic behaviour is present for all parameter values $\mu > 1$. This is proven by calculating the Liapunov exponent.

How can we formally define Chaos? *Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on IC's.*
3 Ingredients;
1)Aperiodic long term behaviour; there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as time tends to infinity.
2)Deterministic; the system has no random or noisy inputs of parameters. Irregular behaviour arises from the systems' nonlinearity rather than from a noisy driving force.
3)Sensitive dependence on Initial conditions; nearby trajectories separate exponentially fast.



Bifurcation Diagrams; Bifurcation Diagrams have been produced to show when Chaos is present in the system, this is when there is a strong band of points corresponding to parameter values on the x axis. Tent map (L) F(x,y) Model (R) It is clear that both of these maps exhibit chaos for certain parameter values, and that past a certain parameter value, the system becomes chaotic.

Bakers Map

The Baker's map has the same dynamics as the horseshoe in that it iterates the unit square.
We define the Baker's map B of the square $0 \le x \le 1, 0 \le y \le 1$ to itself by...

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \le x_n \le 1/2 \\ (2x_n - 1, ay_n + \frac{1}{2}) & 1/2 \le x \le 1 \end{cases}$$

We define a as a parameter ranging $0 < a \le \frac{1}{2}$. Using Matlab to produce plots, we can show that a set of initial values separated into green and red points, become well mixed under iteration of the Baker's map, and that sensitivity to initial conditions is embedded in the system. A plot of the mapping after the 10th iteration is shown in the top left of this poster.

Conclusions

We can show that the Tent map exhibits high sensitivity to initial conditions for certain parameter values. This result is supported by the Liapunov exponent for the tent map which proves the Tent map has chaotic behaviour for $\mu > 1$. Our claims are backed up by findings such as Li and Yorke, that period 3 behaviour implies chaos, and bifurcation diagrams that illustrate period doubling behaviour and chaotic dynamics for $\mu > 1$.
We show Smale horseshoe contains periodic orbits of all periods. This is done by reducing points to a number in binary and considering shifting dynamics. We also illustrate how the Horseshoe map shows high sensitivity to initial conditions, and that through the iteration of forward and backward orbits, two points that are initially arbitrarily close together, can end up following completely different paths under iteration of the map. Similar analysis is carried out for the F(x.y) modell leading to the same result.