Chaotic Dynamics in Nonlinear Dynamical systems

G14PJS

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I have read and understood the School and University guidelines on plagiarism. I confirm that this work is my own, apart from the acknowledged references.

Abstract

In this project I will look into the abstract theory of Chaos in Mathematics. In particular I will be looking at three models; The Tent map, the Smale horseshoe, and the F(x,y) Model.

To help explain my findings I have produced graphs in Matlab that will support my arguments, and help illustrate certain Chaotic features.

The objective of this project is to communicate to the reader how each of these systems exhibit chaos, and why chaos arises in the system. I will use both analytical and numerical methods to explain how small changes of parameters can have huge implications on the outcome of the system, and use this to highlight how such systems display high sensitivity to initial conditions-a feature of Chaos.

The project concludes stating that we can successfully prove that all three systems exhibit chaotic behaviour. Key features of chaos discussed are initial condition sensitivity, how the system contains multiple periodic orbits of arbitrarily long lengths, and how the system contains dense orbits.

Contents

1 Introduction

Chaos Theory is concerned with the behaviour of nonlinear dynamical systems and their highly sensitive dependence on initial conditions. Under certain conditions, these systems can produce completely unpredictable and wildly chaotic behaviour over time. To measure the effects of Chaos in this report, we will consider a systems sensitivity to initial conditions; A high sensitivity to initial conditions is a hallmark of Chaos. We will also consider the density of orbits and the existence of periodic orbits of multiple periods for each system; all of which indicate the presence of chaos within a system.

I will be looking at Discrete Dynamical systems, where we can reduce the study of the flow of a differential equation to that of an iterated function. An advantage of using this method is that we can look at maps on a lower dimensional space, therefore making visualisation easier; we simply iterate the function over and over to determine the behaviour of the orbit over time. These simplifications make it much easier to understand the chaotic behaviour that arises for systems of differential equations. We will aim to answer the following questions; Does the system converge to a certain value? what is Chaos and when is our dynamical system chaotic?

This project will focus on 3 models; the Tent map, the Smale horseshoe and the F(x,y) model. My results are derived from intense numerical and analytically study of these models, using plots such as cobweb diagrams and bifurcation diagrams to support my arguments and build to the following overall conclusion; that each model exhibits chaotic behaviour.

The project can be seen as a review of existing chaotic dynamical systems, and will support existing results highlighting the chaotic tendencies of such systems.

The report has been structured as follows. After mentioning some key definitions I will present the first model; the Tent map. I will mention some key features of the tent map and present specific graphical examples that highlight chaotic behaviour, I will finalise this section by confirming Chaotic behaviour for certain parameter values. I will then move on to analysing the Smale horseshoe with a brief mention of the Baker's map, analysis of the horseshoe will be carried out numerically, I will then analyse a map with very similar dynamics, the Baker's map, in much greater analytical depth. In the final section I will discuss the F(x,y) model, and produce bifurcation diagrams to display chaotic behaviour for certain parameter values. I will then reflect on my results in the conclusion, and make recommendations for future work.

All figures in this report were made using MATLAB Version 7.10.0 [1] unless otherwise stated. Codes are provided in Section 8.

2 Key definitions

All definitions are taken exactly as shown in [2] spanning pages 355-459 and [3] pages 329-381 (unless stated otherwise).

Definition 2.1 ***An iterative Map*** *Define an iterative map by*

$$x_{n+1} = f(x_n) \tag{2.1}$$

where f is a smooth function $f : \mathbb{R} \to \mathbb{R}$.

Definition 2.2 ***Orbit*** *The sequence $x_0, x_1, x_2, ...$ starting from an initial value x_0. i.e. the orbit of x_0 is the sequence $x_0, x_1 = f(x_0), x_2 = f^2(x_0), ..., x_n = f^n(x_0)$*

Definition 2.3 ***The nth iterate of f*** *We denote f^n the* n*th iterate of f. Given $x_0 \in \mathbb{R}$*

Definition 2.4 ***Fixed Point*** *Suppose x^* satisfies $f(x^*) = x^*$, then x^* is a fixed point. Then $x_{n+1} = f(x_n) = f(x^*) = x^*$; The orbit will remain at x^* for all future iterations.*

Definition 2.5 ***Stability****(Informal) In this report we will determine the stability of x^* by considering a nearby point $x_n = x^* + \mu_n$ and calculate whether the orbit is attracted to or repelled from x^*.*

Definition 2.6 ***Chaos*** ”Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on IC's. *3 main features;*
1)Aperiodic long term behaviour; there exist trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as time tends to infinity.
2)Deterministic; the system has no random or noisy inputs of parameters. Irregular behaviour arises from the systems' nonlinearity rather than from a noisy driving force.
3)Sensitive dependence on Initial conditions; nearby trajectories separate exponentially fast, i.e. the system has a positive Liapunov exponent.” [2] page 331

Definition 2.7 ***Attractor*** "A set to which all neighbouring trajectories converge, examples are stable limit cycles and fixed points. *More precisely, an attractor is a closed set A, with the properties;*
1)A is an invariant set; any trajectory x(t) that starts in A stays in A for all time.
2)A attracts an open set of initial conditions; there is an open set U containing A such that if x(0) is an element of U, then the distance from x(t) to A tends to 0 as t tends to infinity. This means A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.
3)A is minimal; there is no proper subset of A that satisfies conditions 1) and 2)." [2] page 332.

Definition 2.8 ***Strange attractor****(Informal) An attractor that exhibits sensitive dependence on Initial conditions (Chaos).*

Definition 2.9 ***Period of an orbit*** *"If $f^n(x) = f^{n+m}(x)$ for some integer m, the orbit is called a periodic orbit. The smallest such value of m for a given* x *is called the* ***period of the orbit****. The point* x *itself is called a periodic point. For example for a fixed point* x*, the period of the orbit of* x *is 1." [5]*

Definition 2.10 ***Sink/Attracting Fixedpoint*** *Suppose that x_0 is a fixed point for f. We can say x_0 is a sink/attracting fixed point for f if there is a neighbourhood U of x_0 in $\mathbb{R}$ having the property that, if $y_0 \in U$, then $f^n(y_0) \in U$ for all n, and, moreover $f^n(y_0) \to x_0$ as $n \to \infty$.*

Definition 2.11 ***Source/Repelling Fixedpoint*** *Similarly, x_0 is a source/repelling fixedpoint if all orbits (except x_0) leave U under iteration of f.*

Definition 2.12 ***Neutral/Indifferent Fixedpoint*** *A fixed point is called neutral/indifferent if it is neither attracting nor repelling.*

Definition 2.13 ***Fixed point period N*** *"A fixed point period N is a point at which $x_{n+N} = f^N(x_n)$ for all n." [4] page 45."*

Definition 2.14 ***Bifurcation (simple definition)*** *"In a dynamical system, a bifurcation is a period doubling, quadrupling, etc, that accompanies the onset of chaos. It represents the sudden appearance of a qualitatively different solution for a nonlinear system as some parameter is varied." [13]*

Definition 2.15 ***Fractal*** *"Fractals are complex geometric shapes with fine structure at arbitrarily small scales, usually having some degree of self similarity i.e. if we magnify a tiny part of the fractal, we will see features reminiscent of the whole (in some cases the similarity is exact)."[2] page 445.*

3 The Tent Map

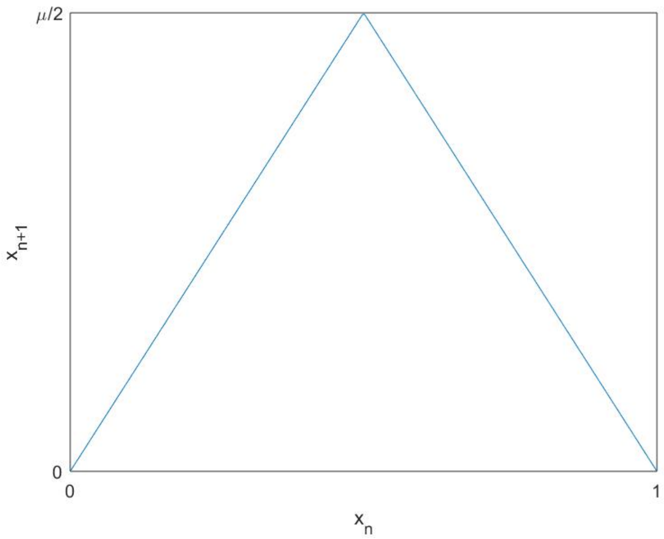
This section will reference [4].

We use the Tent map to introduce one-dimensional nonlinear discrete dynamical systems. We present the tent map, T:[0,1]⟶[0,1], defined by

$$T(x) = \begin{cases} \mu(x) & 0 \leq x < 1/2 \\ \mu(1-x) & 1/2 \leq x \leq 1 \end{cases}$$

where $0 \leq \mu \leq 2$

Figure 1: A plot of the Tent map as defined using 'Simple Tent function plot' from Section 8



"The name "tent map" comes from the shape of the graph T(x) on [0,1]." [6]

As before in Section 2, we can define an iterative map by

$$x_{n+1} = T(x_n) \tag{3.1}$$

where $x_n \in [0,1]$. We notice that complex behaviour and in some cases chaotic phenomena can be displayed for specific parameter values. Correspondingly, for certain parameter values that we will discuss, the mapping can display high sensitivity to initial conditions and display periodicity, leading to chaotic orbits.

Before advancing, we recall the definition of a periodic orbit

Definition 3.1 ***Period of an orbit*** *If $f^n(x) = f^{n+m}(x)$ for some integer m, the orbit is called a periodic orbit. The smallest such value of m for a given* x *is called the* ***period of the orbit****. The point* x *itself is called a periodic point. For example for a fixed point* x*, the period of the orbit of* x *is 1. [5]*

3.1 Cobweb diagrams and Time Series plots

This section will reference and illustrate some examples from [4] pages 265-266.

This section will use the codes 'simple tent function plot', 'tent map cobweb diagram', and 'Tent map x_n vs n graph' from section 8.

We will start our analysis by plotting the tent map for various parameter μ values, and initial conditions x_0.

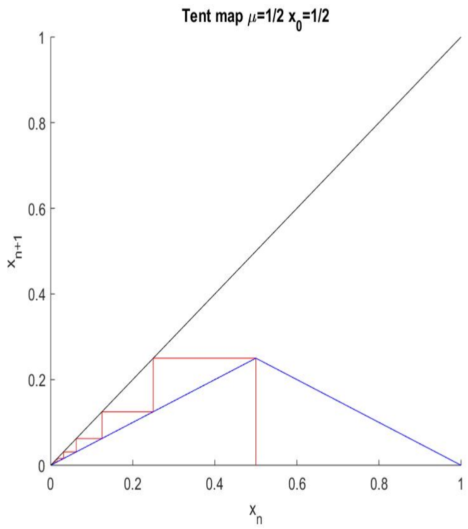
The procedure for plotting a cobweb diagram is as follows; firstly we prescribe an initial condition x_0. from this point on the x axis we draw a vertical line until we hit the function. We then draw a horizontal line to the leading diagonal. We then repeat this two step algorithm.

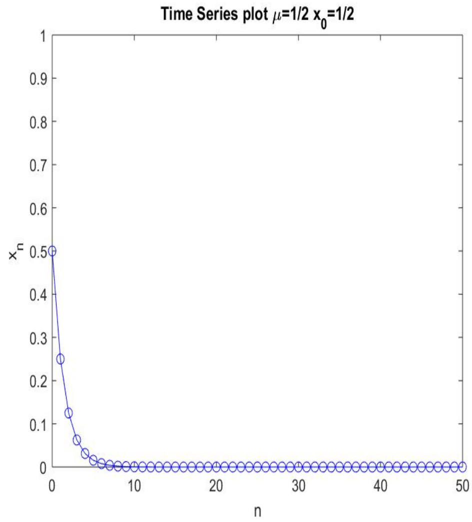
The successive points along the x axis generated by this algorithm correspond to the orbit of points $x_0, x_1, x_2, x_3, ..., x_n, ...$.

We can further analyse the periodicity of solutions by looking at time series plots of x_n vs n. These graphs make it clear what kind of period the orbit follows as iterations are carried out.

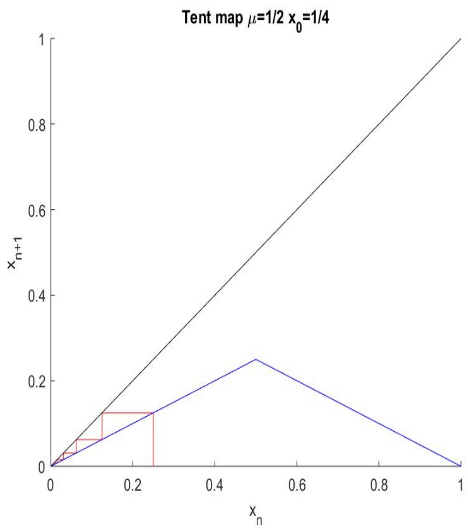
We begin by setting $\mu = 1/2$, $x_0 = 1/2$, followed by $x_0 = 1/4$.

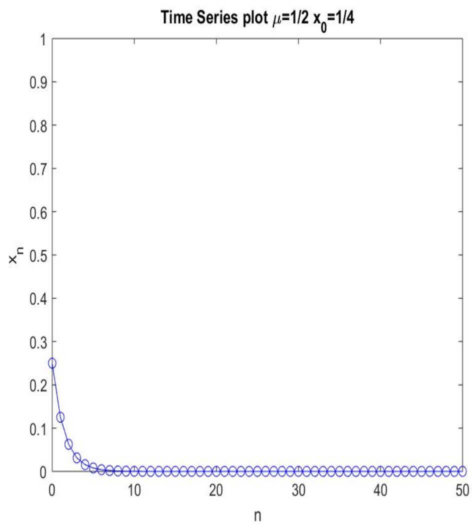
(a) Tent map plot $\mu = 1/2$, $x = 1/2$





(a) Tent map plot $\mu = 1/2$, $x = 1/4$





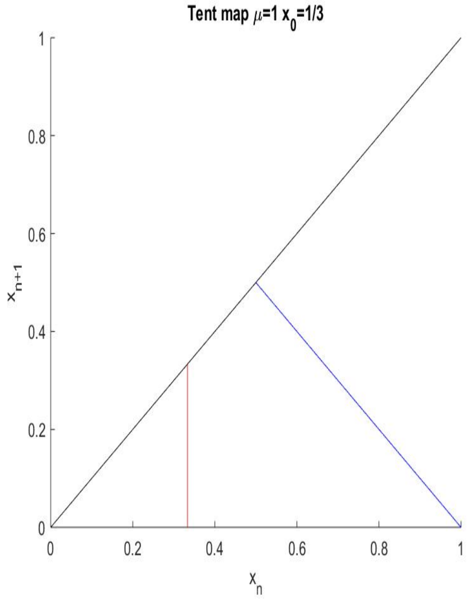
In both cases, $x_n \to 0 \quad as \quad n \to \infty$.

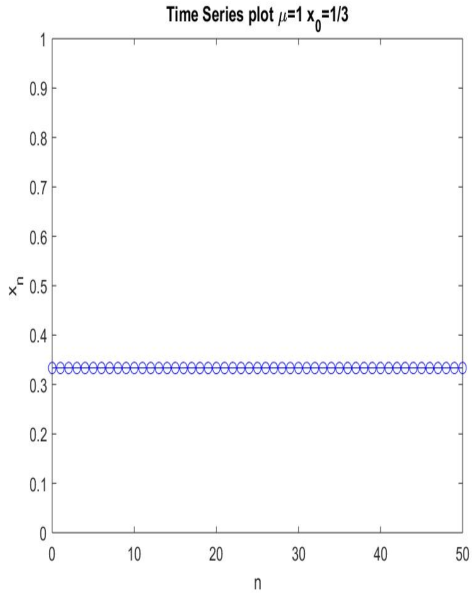
The orbit for 2a looks like $x_n = 1/2, 1/4, ...\frac{1}{2^n+1}, ...$

The orbit for 3a looks like $x_n = 1/4, 1/8, ...\frac{1}{4\times 2^n}, ...$ We will see in section 3.3 that if $0 < \mu < 1$ we see the emergence of a single fixedpoint at x=0, which supports the findings of our plots.

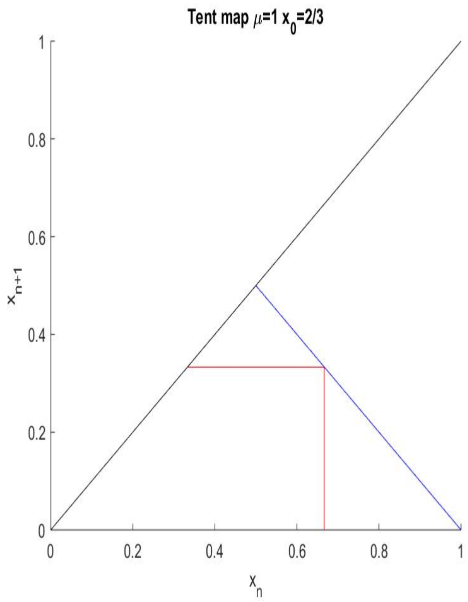
We will now consider the case $\mu = 1, x_0 = 1/3, 2/3$.

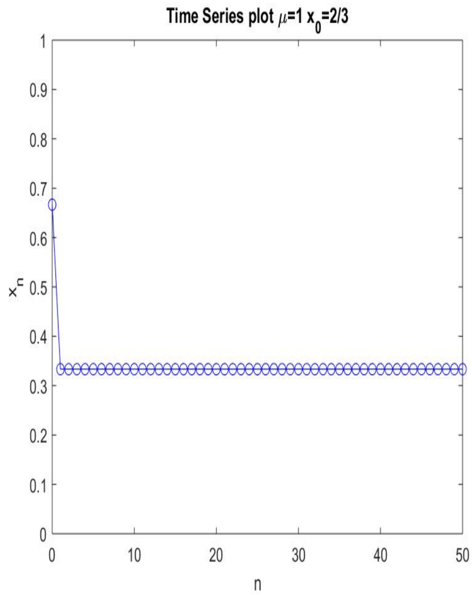
(a) Tent map plot $\mu = 1$, $x = 1/3$





(a) Tent map plot $\mu = 1$, $x = 2/3$





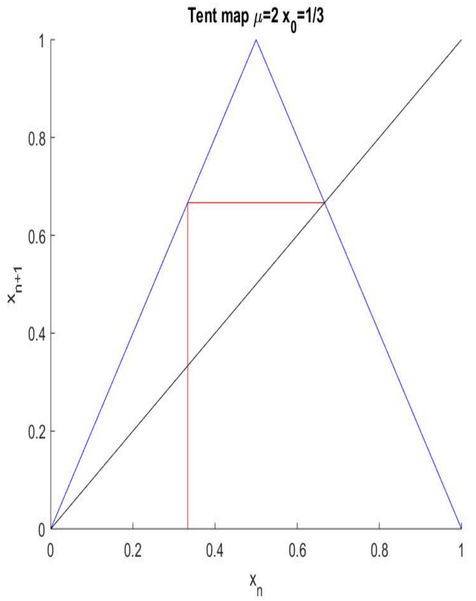
We obtain a sequence of iterations $x_n = 1/3, 1/3, 1/3...$ and $x_n = 2/3, 1/3, 1/3...$ Both of these orbits tends to fixedpoints in [0,1/2].

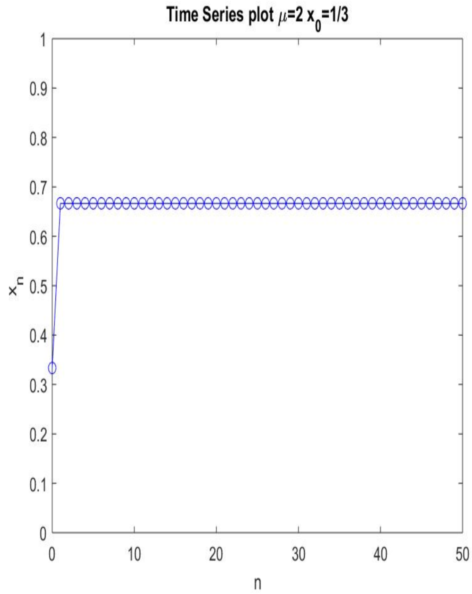
Again, we will see later in section 3.3 that when $\mu = 1$ the branch μx of $T(x)$ runs along the diagonal and all points lying in the interval $0 \leq x \leq 1/2$ are fixed points.

We also note that as we increase the parameter μ, the height of the graph T increases due to the factor μ in the formula for T(x).

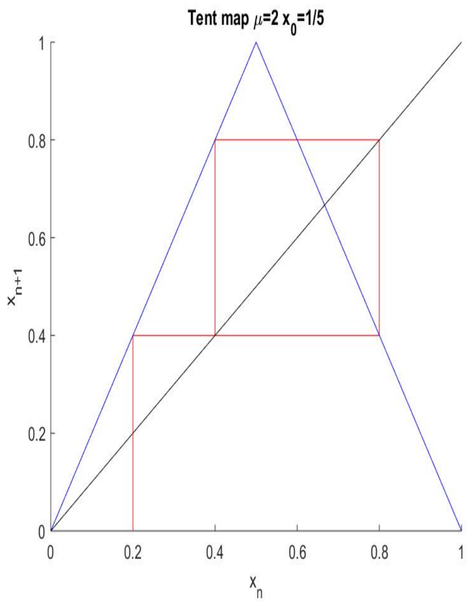
We now consider $\mu = 2$, $x_0 = 1/3, 1/5, 1/7, 1/11$.

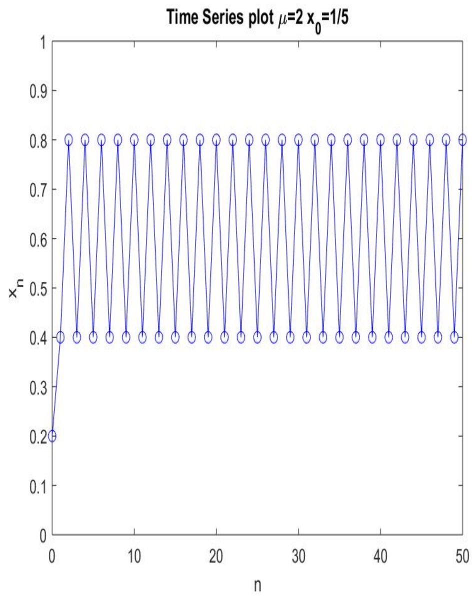
(a) Tent map plot $\mu = 2$, $x = 1/3$



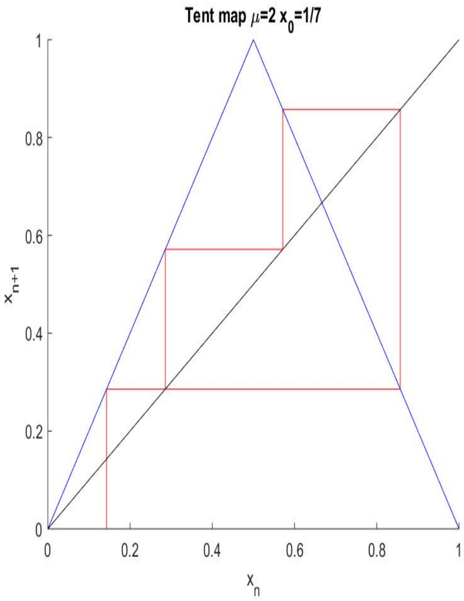


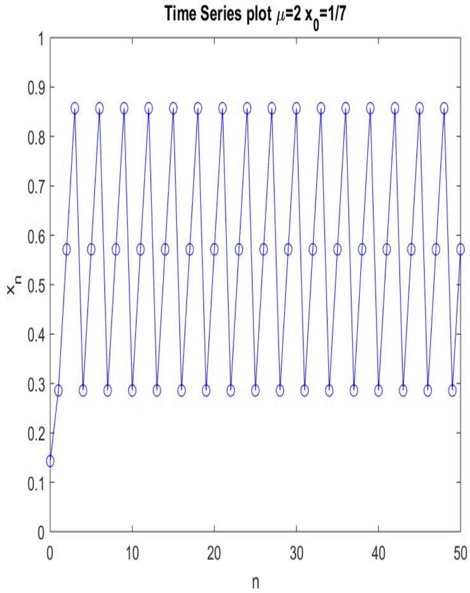
(a) Tent map plot $\mu = 2$, $x = 1/5$



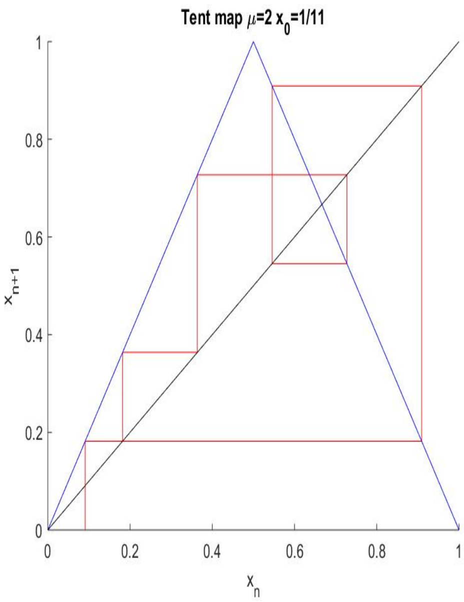


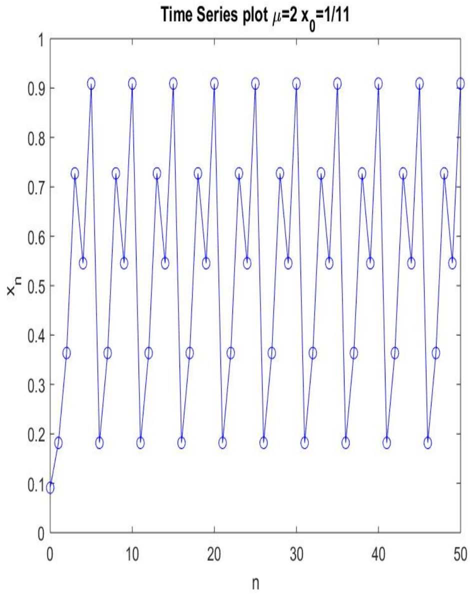
(a) Tent map plot $\mu = 2$, $x = 1/7$





(a) Tent map plot $\mu = 2$, $x = 1/11$





Initial condition $x_0 = 1/3$ shows period one behaviour;

$x_n = (1/3, 2/3, ..., 2/3, ...)$

Initial condition $x_0 = 1/5$ shows period two behaviour;

$x_n = (1/5, 2/5, 4/5, 2/5, 4/5, ..., 2/5, 4/5, ...)$

Initial condition $x_0 = 1/7$ shows period three behaviour;

$x_n = (1/7, 2/7, 4/7, 6/7, 2/7, 4/7, 6/7..., 2/7, 4/7, 6/7, ...)$

Initial condition $x_0 = 1/11$ shows period five behaviour;

$x_n = (1/11, 2/11, 4/11, 8/11, 6/11, 10/11, 2/11, ..., 2/11, 4/11, 8/11, 6/11, 10/11, ...)$.

Again, this supports a result we mention in section 3.3, that when $\mu = 2$ there exist points of all period, as well as chaotic and non-periodic points. This result is supported by the finding of Li and Yorke, which are discussed in section 3.4.

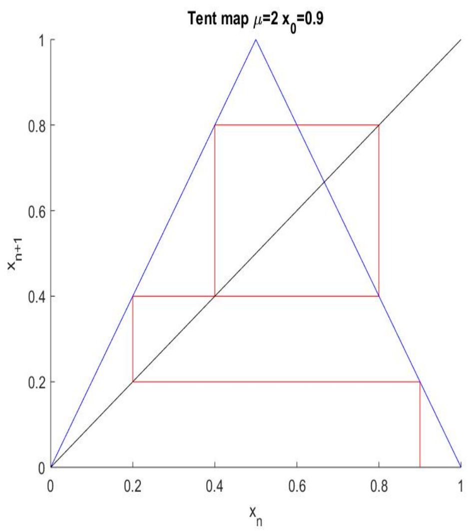
3.2 Some Chaotic examples

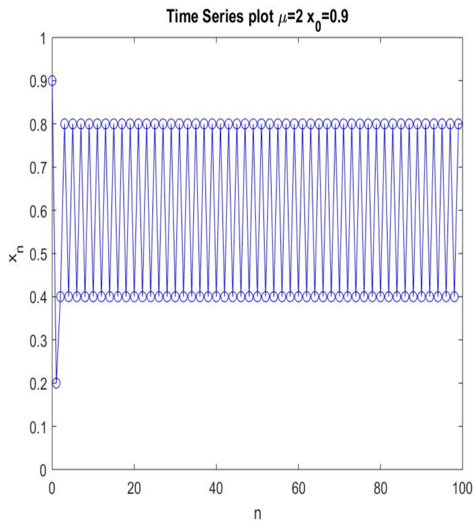
This section will reference [4] pages 266-267.

We will now turn our attention some examples that show extremely sensitive dependence on initial conditions, a condition for a chaotic system.

This subsection will illustrate some examples of Chaos in the tent map, all Graphs were produced using Matlab using the same codes as above as referenced in 8.

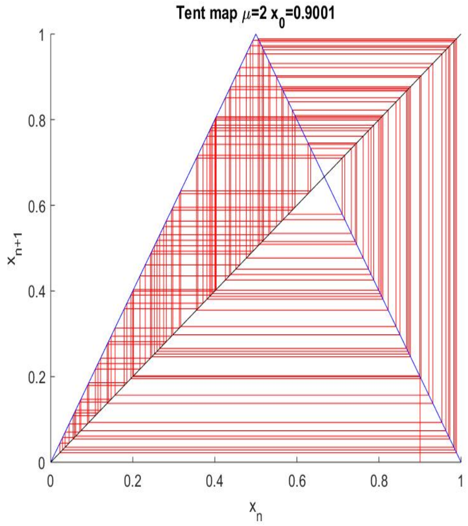
(a) Tent map plot $\mu = 2$, $x = 0.9$

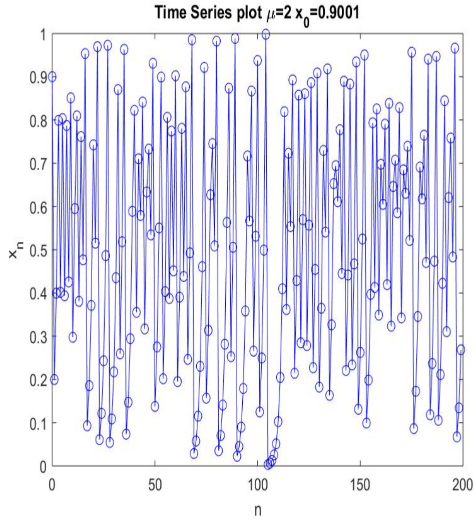




The orbit in this case is $x_n = 9/10, 1/5, 2/5, 4/5...2/5, 4/5$ i.e. the sequence shows period two behaviour.

(a) Tent map plot $\mu = 2$, $x = 0.9001$

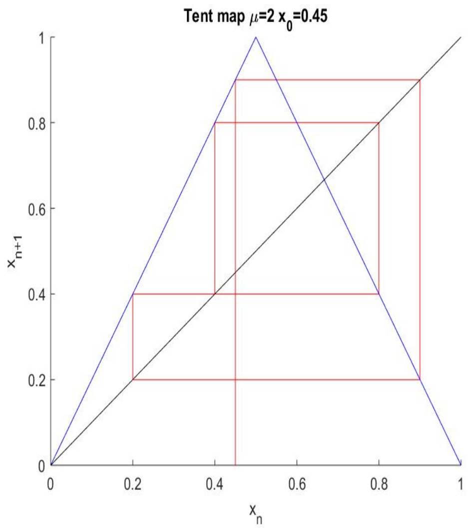


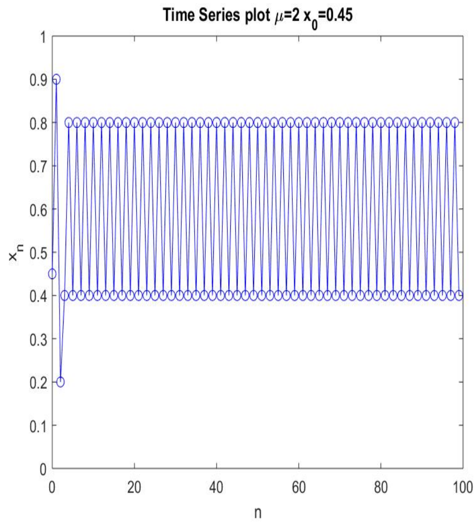


The first few values in the orbit for $x_0 = 0.9001$ are

$x_n = 9001/10000, 999/5000, 999/2500, 999/1250, 251/625, 502/625....$ By calculating the first 100 iterations in Matlab there is no obvious periodic behaviour. We notice that the difference between the initial values is $\epsilon = 0.001$, yet the orbits differ greatly, even after a small number of iterations. This high sensitivity to initial conditions is a key feature of chaos.

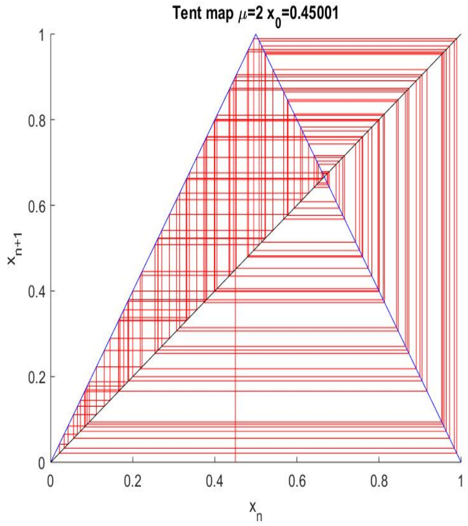
(a) Tent map plot $\mu = 2$, $x = 0.45$

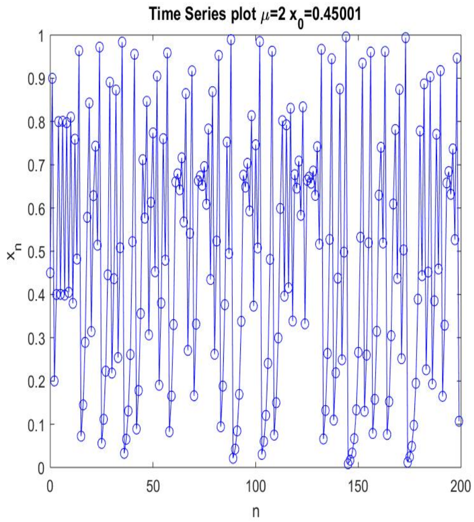




When $x_0 = 0.45$ the orbit is $x_n = 9/20, 9/10, 1/5, 2/5, 4/5...2/5, 4/5...$ and we see that this eventually settles down to period two behaviour.

(a) Tent map plot $\mu = 2$, $x = 0.45001$





When $x_0 = 0.45001$ the orbit is

$x_n = 45001/100000, 45001/50000, 4999/25000, 4999/12500, 4999/6250, 1251/3125$ and again we see this is Chaotic by our definition. In this case the difference between our initial values was even smaller, only 0.00001! yet its clear from our figures 12a, 13a that the orbits vary wildly! this supports our argument for the tent map showing chaotic properties for certain parameter values and initial conditions!

By assigning different parameter values and initial conditions to the Matlab code used, it is easy to come across orbits of various periods, as well as chaotic orbits.

3.3 Fixed points and periodic orbits

In this section I will reference [8] and [7].

We will now turn our attention to fixed points and periodic orbits of the tent map. We consider the general map (3.1) , and note the definition of a **fixed point**;

Definition 3.2 ***Fixed Point*** *Suppose x^* satisfies $f(x^*) = x^*$, then x^* is a fixed point. Then $x_{n+1} = f(x_n) = f(x^*) = x^*$; The orbit will remain at x^* for all future iterations.*

For the tent map, we see that for fixed point $T(x_n) = x_n$ then fixed points where the map $T(x)$ intersects $y = x$.

Using this definition, it is possible to determine the fixed points of period one for the tent map (3.1).

Case 1 If $0 < \mu < 1$ then there exists only one fixed point at x=0. It follows from the way we have defined T(x), that if $0 < \mu < 1$, then if $0 \leq x < 1/2$, we have

$$0 \leq T(x) = \mu x < x \tag{3.2}$$

and if $1/2 \leq x \leq 1$ we have

$$0 \leq T(x) = \mu(1-x) < 1 - x \leq 1/2 \leq x \tag{3.3}$$

"Therefore for any $x \in [0,1]$, the sequence T(x) is bounded and decreasing by the continuity of T(x), the sequence converges to the fixed point at 0. Therefore, **0 is an attracting fixed point whose basin of attraction is [0,1].** The fixed point is stable and we call the origin the *trivial fixed point*." [7]

Case 2 When $\mu = 1$ the branch μx of $T(x)$ runs along the $y = x$, therefore all points in the interval $0 \leq x \leq 1/2$ are fixed points. We note that when $T(x)$ passes through $y = x$, the origin becomes unstable as the gradient of the tent map now exceeds one. We will see more on this result in Section 3.5.

Case 3 When $1 < \mu \leq 2$, we obtain two fixedpoints period one; $x_{1,1} = 0$ and $x_{1,2} = \frac{\mu}{1+\mu}$. This second fixedpoint is identified by the simple calculation;

$$T(x) = x$$
$$\mu(1-x) = x$$
$$x(\tfrac{1}{\mu} + 1) = 1$$
$$x = \tfrac{\mu}{1+\mu}$$

Notation used; the periodic point denoted $x_{i,j}$ indicates the jth point of period i. Each of these fixedpoints will be unstable.

Before proceeding, we will mention the definition of a dense orbit;

Definition 3.3 ***Dense orbit*** *We say the orbit of x_0 is dense if every open sub-interval of [0,1] contains an iterate of x_0. [6]*

i.e. we say the orbit is 'dense' in [0,1], we mean that the path will visit every part of the interval [0,1].

When $\mu = 2$ the system maps the interval [0,1] onto itself. We now have a system in which there are periodic points of all lengths in this interval, along with non-periodic points. The periodic points are dense in [0,1], and therefore we expect our map to show chaotic features. Our examples from section 3.2 are good examples of dense orbits, this is clear from the cobweb diagrams produced. With more iterations, these diagrams would become more and more dense.

The Sequences we obtain from these chaotic starting values x_0 shows behaviour that is very different to periodic behaviour, yet still remains in the interval [0,1]; we can say that the sequence passes arbitrarily close to every point in the interval.

We will now mention an important result that supports our argument for parameter values $\mu = 2$ being chaotic.

3.4 Li and Yorke

This section will reference [11].

A very famous paper was published by Li and Yorke (1975) titled **"Period Three implies Chaos"**[11]. In this paper it was proved that any one-dimensional continuous map $F : W \to W$ that has a period three orbit must have the following two properties;

(1) For each P, where P is a member of the positive integers, there exists a point in W that will return to its starting position after P iterations of the map. It does not return to this starting position before these P iterations.

This implies that there are infinitely many periodic points which may be stable or unstable, with different set of points for each P.

We introduce a new definition...

Definition 3.4 ***Scrambled Points and Sets*** *"let a and b be a pair of points, then we call these points 'scrambled' if as the mapping is applied iteratively to the pair, they begin to get drawn together, before moving apart, before drawing together again, and repeating this behaviour i.e. they get arbitrarily close together, but do not stay close together permanently.*

We then say that a set S is scrambled if every pair of distinct points a and b in S is scrambled." [11].

This allows us to define our second condition...

(2) There exists an uncountably infinite set S that is scrambled.

As the title of their paper suggests, their results imply that if a system shows behaviour of period-three, then the system can rightfully show periodic behaviour of any period, and even chaotic phenomena.

We relate this back to our own findings, and state that in the case of example 8a, we have $\mu = 2$ and $x_0 = \frac{1}{7}$, produces a period three orbit, which implies that when $\mu = 2$ the tent map is chaotic because it has a period three sequence!

3.5 Attracting and Repelling Fixedpoints of the Tent Map

This section will reference [4] pages 40-45.

As stated in 3.3 we can find the fixed points of the tent map graphically by simply observing where $T(x)$ intersects $y = x$.

Therefore the type of fixed point (attracting, repelling or indifferent as defined in section 2) can be determined by the gradient of the function $T(x)$. For straight line segments defined by the equation $y = mx + c$ where m defines the gradient of the line, it is trivial to show that if

(1) $m < -1$, iterations repel (spiral away from the fixed pointx)

(2) $-1 < m < 0$, iterations are attracted to the fixed point and will spiral towards it.

(3) $0 < m < 1$, iterations attract, and move towards the fixedpoint.

(4) $m > 1$, iterations repel, and move away from the fixed point.

(5) $|m| = 1$ indicates the fixed point is neither attracting nor repelling.

(6) m=0 indicates the trivial case.

Alternatively, we can identify the stability of fixed points using the following;

We suppose that a mapping $g(x)$ has a fixed point x^*. We say the fixed point is stable (sink) if

$$|\frac{d}{dx}g(x^*)| < 1 \tag{3.4}$$

and unstable (source) if

$$|\frac{d}{dx}g(x^*)| > 1 \tag{3.5}$$

our results are inconclusive in the case

$$|\frac{d}{dx}g(x^*)| = \pm 1 \tag{3.6}$$

[2] pages 356-357.

3.6 Fixed Points of Higher Periods

This section uses codes from [7], and therefore are not my own, for that reason I have not included these codes in Section 8. This section will reference results from [12] pages 273-276.

We recall the definition of a fixed point period N from Section 2...

Definition 3.5 ***Fixed point period N*** *"A fixed point period N is a point at which $x_{n+N} = f^N(x_n)$ for all n."*

We will now look at determining fixed points period two for the map $T(x)$, similar to what we have seen so far, these will be given by where $T^2(x)$ intersects $y = x$.

We will use the parameter value $\mu = 2$ and define $T(T(x)) = T^2(x)$ by...

$$T(x) = \begin{cases} 2(x) & 0 \leq x < 1/2 \\ 2(1-x) & 1/2 \leq x \leq 1 \end{cases}$$

Giving

$$T^2(x) = \begin{cases} 2T(x) & 0 \leq x < 1/2 \\ 2(1-T(x)) & 1/2 \leq x \leq 1 \end{cases}$$

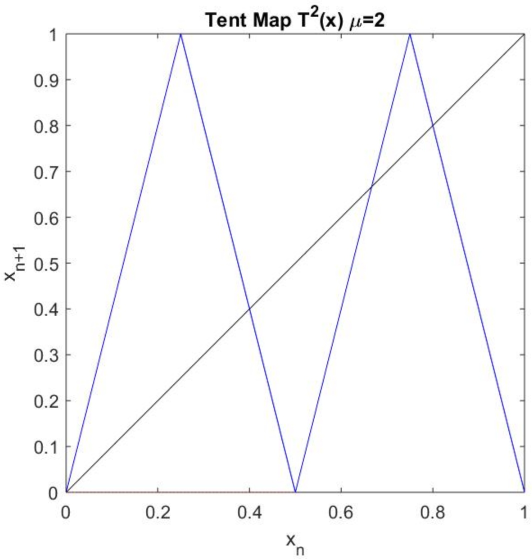
Here the interval $0 \leq T(x) < \frac{1}{2}$ along the y axis now corresponds to two intervals; $0 \leq x < T^-1(\frac{1}{2})$ and $T^-1(\frac{1}{2}) \leq x \leq 1$ on the x axis. We can calculate $T^-1(x)$ as

$$T^{-1}(x) = \begin{cases} \frac{1}{2}x & 0 \leq x < 1/2 \\ 1 - \frac{1}{2}x & 1/2 \leq x \leq 1 \end{cases}$$

Therefore $T^{-1}(\frac{1}{2}) = \frac{1}{4}$ or $\frac{3}{4}$. We can repeat this for T(x) in the interval $[\frac{1}{2}, 1]$, giving

$$T^2(x) = \begin{cases} 4x & 0 \leq x < 1/4 \\ 2-4x & 1/4 \leq x < 1/2 \\ 4x-2 & 1/2 \leq x < 3/4 \\ 4-4x & 3/4 \leq x \leq 1 \end{cases}$$

Figure 14: Tent map plot T^2 $\mu = 2$



[7]

We find that this function intersects $y = x$ at **4 points;** $x = 0, \frac{2}{5}, \frac{2}{3}, \frac{4}{5}$. Stability is easy to determine; the gradient of $|T^2(x)| > 1$ (m=-4) at $x = \frac{2}{5}, \frac{4}{5}$, therefore these points are unstable, we also note that both these fixedpoints are of period 2. By the same argument, fixed points at $x = 0, \frac{2}{3}$ are period 1 fixedpoints.

Similarly we can find fixed points period 3 using the same method of identifying intersections of $T^3(x)$ with $y = x$. Again we set our parameter $\mu = 2$.

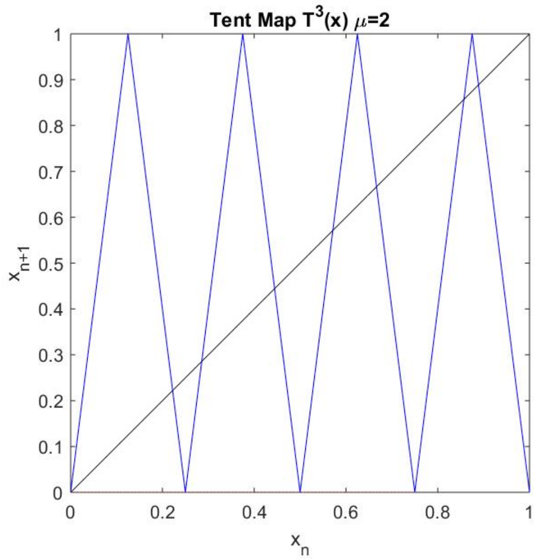
This time we replace x with $T(x)$ in the mapping for $T^2(x)$. This gives;

$$T^3(x) = \begin{cases} 4T(x) & 0 \leq x < 1/4 \\ 2 - 4T(x) & 1/4 \leq x < 1/2 \\ 4T(x) - 2 & 1/2 \leq x < 3/4 \\ 4 - 4T(x) & 3/4 \leq x \leq 1 \end{cases}$$

We find the interval $0 \leq T(x) < \frac{1}{4}$ on the y axis corresponds to two intervals on the x axis; $0 \leq x < T^{-1}(\frac{1}{4})$ and $T^{-1}(\frac{1}{4} \leq x \leq 1)$. By the same method above we find $T^{-1}(\frac{1}{4}) = \frac{1}{8}$ or $\frac{7}{8}$. This gives $T^3(x)$;

$$T^3(x) = \begin{cases} 8x & 0 \leq x < 1/8 \\ 2-8x & 1/8 \leq x < 1/4 \\ 8x-2 & 1/4 \leq x < 3/8 \\ 4-8x & 3/8 \leq x \leq 1/2 \\ 8x-4 & 1/2 \leq x < 5/8 \\ 6-8x & 5/8 \leq x < 3/4 \\ 8x-6 & 3/4 \leq x < 7/8 \\ 8-8x & 7/8 \leq x \leq 1 \end{cases}$$

Figure 15: Tent map plot T^3 $\mu = 2$



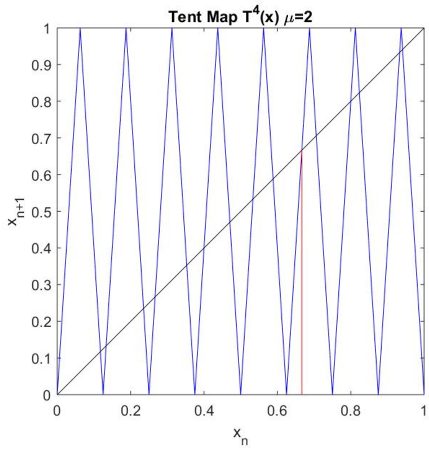
[7]

Here we have 8 fixed points; $x = 0, \frac{2}{9}, \frac{2}{7}, \frac{4}{9}, \frac{4}{7}, \frac{2}{3}, \frac{6}{7}, \frac{8}{9}$. We find the gradient of $|T^3(x)| > 1$ (m=8) at $x = \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}$ therefore these 6 points are unstable. We can note that these points are all period 3, which implies that an initial condition starting arbitrarily close to one of these points will move away from the fixedpoint with more iterations. As before, fixedpoints at $x = 0, \frac{2}{3}$ are period 1 fixedpoints.

This method can be used to identify fixedpoints of any period for the mapping $T(x)$. As discussed in Section 3.4 the mapping $T(x)$ has periodic points of all periods, as well as aperiodic orbits where $T(x)$ is sensitive to initial conditions.

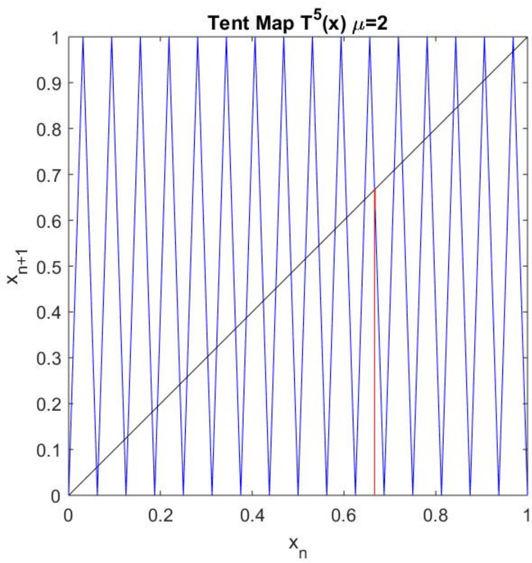
For example...

Figure 16: Tent map plot T^4 $\mu = 2$



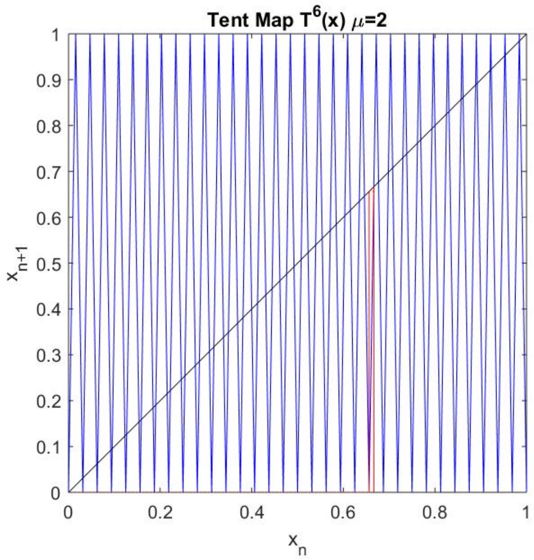
[7] Has 16 fixed points, 14 of which are period 4. These can be found by a similar method to that shown above.

Figure 17: Tent map plot T^5 $\mu = 2$



[7] Has 32 fixed points, 30 of which are period 5.

Figure 18: Tent map plot T^6 $\mu = 2$



[7] Has 64 fixed points, 62 of which are period 6.

These calculations were made in the case that $\mu = 2$, and show that as we continue to iterate $T(x)$ we obtain fixed points of higher and higher periods and these fixedpoints will always be unstable. This supports our argument that the Tent map contains fixed points of all periods, and arbitrarily long period lengths.

We highlight the relevance of these results, by this method we are able to find fixed points of arbitrarily long periods, which again is a feature that indicates chaos within a system.

3.7 The Liapunov Exponent

This section will reference [2] pages 373-374.

We have seen that the Tent map exhibits aperiodic orbits for specific parameter values, but we have not yet proved rigorously that this is really Chaos.

As defined in section 2 high sensitivity to initial conditions is a signature of chaos, meaning orbits that start arbitrarilly close to an initial value x_0 separate exponentially fast. To check this sensitivity we can introduce the Liapunov exponent for 1-dimensional maps.

We introduce the definition of a Liapunov exponent;

Definition 3.6 ***Liapunov Exponent*** *"Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$, where the initial separation δ_0 is extremely small. Let δ_n be the separation after n iterates. If $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then λ is called the Liapunov exponent. A positive Liapunov exponent is a signature of chaos." [2] page 373.*

We can derive a more precise and more computationally useful formula for the Liapunov exponent. We note that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0$, and take logarithms;

$$\delta \approx \frac{1}{n} ln|\frac{\delta_n}{\delta_0}|$$
$$= \frac{1}{n} ln|\frac{f^n(x_0+\delta_0)-f^n(x_0)}{\delta_0}$$
$$= \frac{1}{n} ln|(f^n)'(x_0)|$$

note we have taken $\delta_0 \to 0$. We expand the term inside the natural logarithm using the chain rule as follows;

$$(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$$

Hence we obtain

$$\lambda \approx \frac{1}{n} ln|\prod_{i=0}^{n-1} f'(x_i)|$$
$$= \frac{1}{n} \sum_{i=0}^{n-1} ln|f'(x_i)|$$

And we define the limit of this expression as $n \to \infty$ to be the **Liapunov Exponent** for the orbit starting at x_0.

$$\lambda = \lim_{n\to\infty} (\frac{1}{n} \sum_{i=0}^{n-1} ln|f'(x_i|)) \tag{3.7}$$

We note that the Liapunov Exponent changes depending on the initial value x_0, however, λ will remain the same $\forall\ x_0$ in the basin of attraction of a given attractor. We can find the value of λ analytically. Stable fixedpoints and cycles are indicated by $\lambda < 0$, whereas chaotic attractors are indicated by $\lambda > 0$. [2].

We can compute the liapunov exponent for the Tent map as follows;

Firstly we note that $f'(x) = \pm\mu, \forall x$ we find

$$\lambda = \lim_{n\to\infty} (\frac{1}{n}) \sum_{i=0}^{n-1ln|f'(x_i|)} \tag{3.8}$$

$$= \lim_{n\to\infty} (\frac{ln\mu}{n} \sum_{i=0}^{n-1} 1) \tag{3.9}$$

$$= ln\mu \tag{3.10}$$

Which suggests that $T(x)$ contains Chaotic solutions $\forall \mu > 1$, since $\lambda = ln\mu > 0$. We will see in the next section the implications this has graphically...

3.8 Bifurcation Diagram of the Tent Map

This section will reference [7]. Returning to our original set of equations for $T(x)$. We recall our definition of a bifurcation;

Definition 3.7 ***Bifurcation (simple definition)*** *"In a dynamical system, a bifurcation is a period doubling, quadrupling, etc, that accompanies the onset of chaos. It represents the sudden appearance of a qualitatively different solution for a nonlinear system as some parameter is varied." [13]*

We have seen that for different parameter values μ the tent map can exhibit wildly varying long term solutions. For example we can see a behaviour change from a fixed point to a period 2 orbit. Such changes are called bifurcations, we can model the bifurcations that occur within the tent map system, as well as the different types of orbits in a **Bifurcation Diagram**.

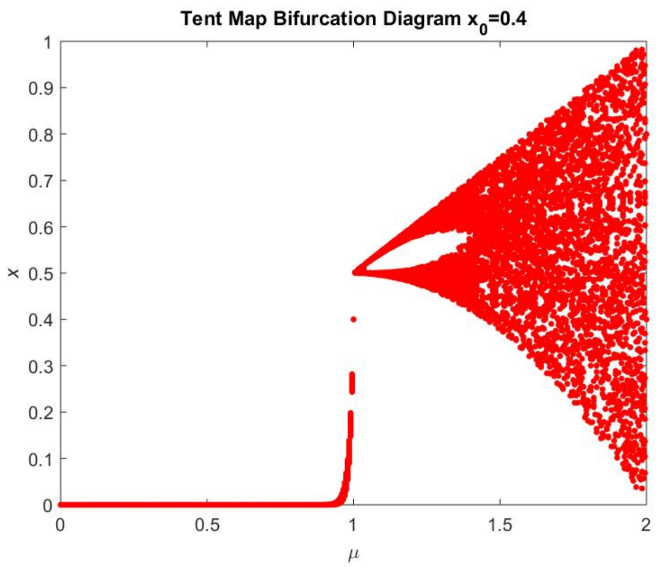
To construct a bifurcation diagram we plot the parameter μ along the x axis, and all values of x that are visited after a certain number of iterations plotted as points along the y axis. A change from a fixed point to a period 2 orbit will look like a change from a single curve splitting into two curves.

Chaotic orbits will be visually obvious, these are cases where the orbit visits an infinite number of points i.e. not periodic. These will appear as bands of continua of points.

From previous analysis we already know there is a stable equilibrium for $\mu \leq 1$ whereas for $\mu > 1$ we have seen that there exist two unstable equilibria, some unstable periodic solutions, and for certain parameter values and initial conditions, chaotic solutions.

In our plot we will vary the parameter $0 \leq \mu \leq 2$, we will perform 100 iterations for each value, using the 'Tent Map Bifurcation Code' from section 8.

Figure 19: Tent Map Bifurcation Diagram $x_0 = 0.4$



This plot requires a large amount of data, therefore Matlab takes a very long time to run the code. For this reason I will include an image of the full bifurcation diagram for comparison:

Figure 20: Tent Map Bifurcation Diagram Image $x_0 = 0.4$

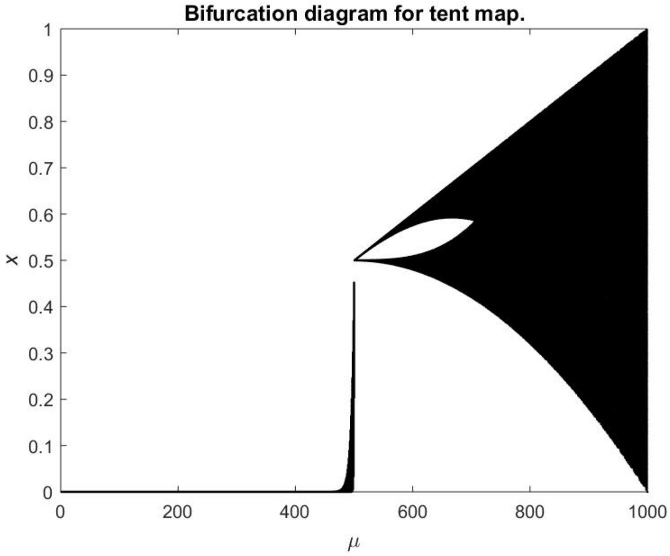


Image taken from [7].

It is easy to see that for $0 < \mu < 1$ the only fixed point is at 0, when $\mu = 1$ all points lying in the interval $0 \leq x \leq \frac{1}{2}$ are fixedpoints. It is clear to see that at $\mu = 1$ the period one cycle changes to a period two cycle, therefore we say there is a bifurcation point at $\mu = 1$, and we call this type of bifurcation a period-doubling bifurcation. When $1 < \mu \leq 2$, as we proved in section 3.7, for values $\mu > 1$ the tent map is Chaotic. This is clear from the diagram as there are a thick band of points for these values, and we can say that the orbits are dense for these parameter values.

Therefore we have shown both numerically and analytically that for certain parameter values and initial conditions, the tent map exhibits chaotic solutions! We have shown this through highlighting the high sensitivity to initial conditions of the tent map, as well as the presence of dense orbits, particularly for parameter value $\mu = 2$. Additionally, using conclusions from the work of Li and Yorke, we have shown that, because the tent map has an orbit period 3, it has periodic orbits of all periods! therefore we can expect chaotic solutions from the tent map, which we have shown to be true. The results we have found are particularly interesting because although the system looks like a simple set of differential equations, the dynamics behind them are quite fascinating!

4 The Smale Horseshoe

4.1 A Simple Introduction to the Smale Horseshoe

The Smale horseshoe is another simple idea attempting to reduce chaos to its most elementary expression. The Smale horseshoe mapping consists of a sequence of operations on the unit square, namely scaling, contracting and folding. We can then inverse this transformation in the opposite direction and iterate this procedure. Chaos once again arises since there is heavy dependence on initial conditions, i.e. two points that begin arbitrarily close together can diverge and follow wildly different orbits as the function is iterated.

4.2 The Smale Horseshoe Map

This section will reference material from [3] Chapter 16 'Homoclinic Phenomena' page 368-375 and [12] Chapter 8 pages 190-192.

We will begin by introducing a key definition;

Definition 4.1 ***Stable Manifold*** *The stable manifold consists of points that approach the fixed point in the limit* $t \to \infty$*. [2] pg 130.*

Definition 4.2 ***Unstable Manifold*** *The unstable manifold consists of points that approach the fixed point in the limit* $t \to -\infty$ *[2] pg 130.*

Definition 4.3 ***Cantor Set*** *"We start with a closed interval* $S_0 = [0, 1]$ *and remove its open middle third, i.e. delete the interval (1/3,2/3), leaving the endpoints behind. This leaves the pair of closed intervals* $S_1 = (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$*. We then remove the open middle thirds of both these open intervals to produce* S_2*, and so on. The limiting set* $C = S_\infty$ *we call the Cantor Set." [2] pages 408-409.*

We begin by considering a square containing our initial points that we will call S. We suppose the map is given by $H : S \to \mathbb{R}^2$ and that H contracts S in the horizontal direction, before expanding S in the vertical direction and finally folding the rectangle back onto itself to form the shape of a horseshoe (hence the name!) Similarly, H^{-1} on S carries out the same deformation but in the opposite direction, resulting in the horseshoe rotated 90 degrees. Instead of studying the trajectory in space we are simply looking at the sequence of returns on the square.

Figure 21: Smale Horseshoe Iterative Map visualised; the mapping H and H^{-1}

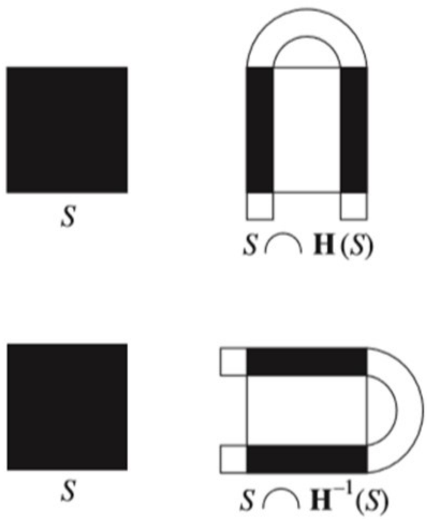
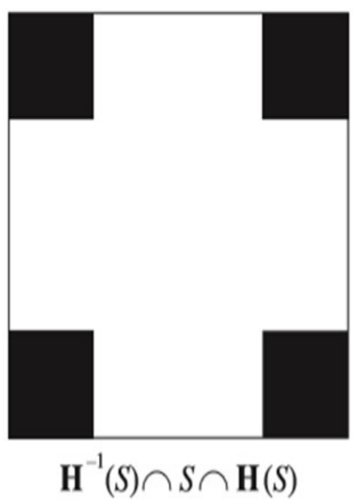


Figure 22: Dark squares are shown where iterates of the square intersect



Images taken from [12] Page 191-192.

If we continue to iterate this process, we would see that "points fall into the area contained by the original square in smaller and smaller subareas. We call the result an Invariant Cantor set that contains a countable set of periodic orbits and an uncountable set of bounded nonperiodic orbits"[4] i.e. the cross section of the final structure corresponds to a Cantor set. Every member of the Cantor set is a limit point.

It is clear to see that this system would hold high sensitivity to initial conditions. To demonstrate this we imagine injecting a small drop of food colouring somewhere on S, which represents a set of nearby initial conditions. After many iterations of stretching, contracting and folding the colouring would spread throughout the square.

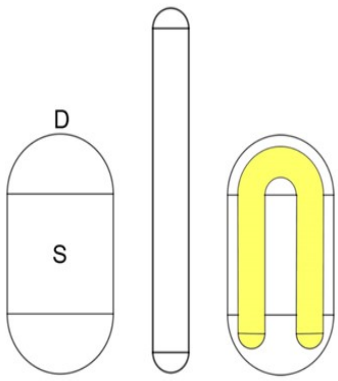
4.3 The Horseshoe as an example of Chaos

This section will reference [15].

In particular we are interested in the invariant set of points which will stay forever inside the iterated square.

We consider a region D, made from a square S with rounded off ends attached to the top and bottom of S. Again we imagine squeezing the square in the horizontal direction and stretching it in the vertical direction, before folding it round into a horseshoe shape. We then place this folded horseshoe back onto the region D.

Figure 23: The Horseshoe function



Our focus then turns to what happens when we iterate this process on D again. In this case, we would retrieve a 'four stranded' horseshoe sitting within the simple horseshoe. Iterating this procedure we would then retrieve an eight-stranded horseshoe, sixteen-stranded horseshoe, and so on as more folds are added, with 'snakier' horseshoes of more and more strands. We see that under more iterations the structure of D gets increasingly complex.

4.3.1 Fixpoints of the Horseshoe

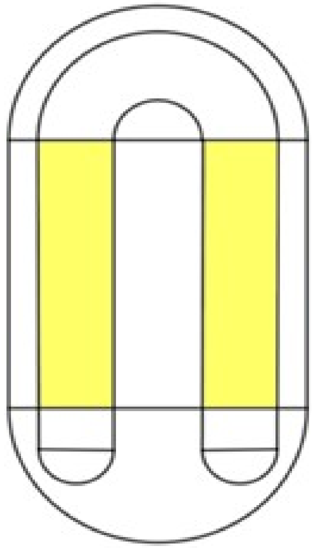
This section will reference [15] and [16] pages 30-33.

The operations described above give a dynamical system; as we apply repeated operations, points within the region D are moved around it. We consider an initial point x_0 in D that is mapped by the operation we will call f, to another point in the horseshoe x_1. This point is then mapped to x_2 and so on, giving the sequence of points $x_0, x_1, x_2, x_3, ...$ We call this sequence the *forward orbit* of x_0.

In the opposite direction, we question whether there exists a point P that maps to x_0 under f? if this point exists we call it x_{-1}, if the following points exist, we say x_{-2} is the point that maps to x_{-1} under f, x_{-3} maps to x_{-2} under f, and so on. The sequence obtained x_{-1}, x_{-2}, x_{-3} we call the *backward orbit* of x_0.

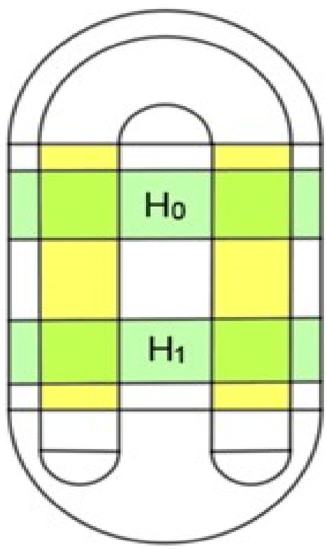
We note that there is no guarantee that a point x_0 in S will map to an x_1 also in S, in fact most points are lost under iteration due to them being part of the overhang of the square. In the case that x_1 does lie in S, then it must lie in one of the two vertical strips in which the horseshoe intersects the square as shown in 24.

Figure 24: x_1 must lie in one of the two vertical strips



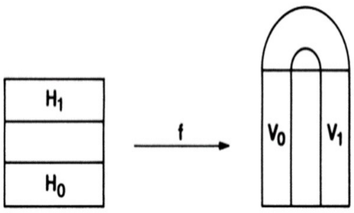
Considering the backward orbit as well as the forward orbit, we see that x_0 must in fact lie in either of two horizontal strips H_0 and H_1 as shown in 25. Points in S not in these strips will get mapped to the overhang under iteration.

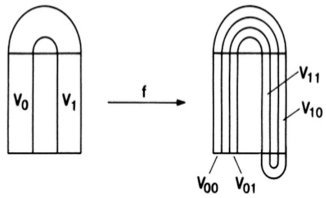
Figure 25: x_0 must lie in one of the two horizontal strips



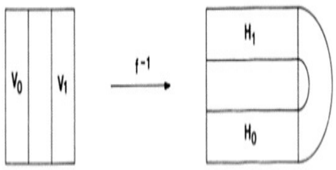
Here we can denote the set of invariant points that will never leave S under iteration of f; $\Lambda = \cap_{n=-\infty}^{\infty} f^n(S)$ i.e. the intersection of all images (forward orbits) and preimages (backward orbits) of S. We have that $f^{-1}(S) \cap S = H_1 \cup H_2$ and $S \cap f(S)$ equal to the union of the vertical strips, which we will call V_1 (Left) and V_2 (Right). We therefore see that $f^n(S)$ is the union of four rectangles. In a similar pattern we see that $\cap_{n=-2}^{2} f^n(S)$ is the union of 16 rectangles and so on, we therefore find that $\cap_{n=-k}^{k} f^n(S)$ is the union of 2^k rectangles. We can relate this back to our statement before that Λ turns out to be a **Cantor set**.

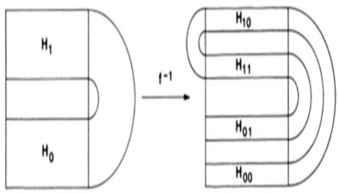
(a) First and Second iterations of the Horseshoe map in the forward direction





(a) First and Second iterations of the horseshoe map in the backward direction





26a and 27a were both taken from [18] page 559-563.

We will call the set of points whos forward and backward orbits dont leave S under iteration 'L'. Given a point x_0 in L we consider any other point x_n in the orbit of x_0 (this can be in either the forward or backward orbit). by the above argument we know x_n lies in one of the horizontal trips H_0 or H_1, and x_{n+1} lies in one of the vertical strips, therefore as we move with the orbit, points will bounce between H_0 and H_1.

We can then begin to write down a sequence for each x_0 in L. If x_0 is in H_0 we will write 0, if it is in H_1 we will write 1. We then do the same for x_1, x_2, x_3 and so on, giving an infinite sequence of 0s and 1s. We do the same for the backward orbit of x_0, where the sequence from left to right looks like $...x_{-1}.x_0x_1, ...$, where the dot separates the two sequences. Such as sequence from $-\infty$ to $+\infty$ we call bi-infinite.

It is trivial to prove that "for every point in L there exists a unique bi-infinite sequence with a dot, and for every bi-infinite sequence with a dot there exists a unique point in L."[15] The closer the starting values, the more places in the sequence will agree. We also note that applying f to a point in S corresponds to shifting the dot one place to the right, whereas applying the inverse of f shifts the dot one place to the left.

We can see from the above dynamics of f that we are immediately presented with two fixedpoints that stay in the same position as f is applied, these are the points ...111.111... and ...000.000... . It is clear to see that the entire backward and forward orbits of these points like in H_1 and H_2 respectively.

We can also see there are two points of period two given by ...0101.0101... and ...1010.1010...

In fact, following this same pattern, we can find fixed points for any period n; given any bi-infinite sequence with a dot, we can always construct a periodic sequence that agrees with it.

To see the chaotic features of the horseshoe we consider two points that may begin arbitrarily close together i.e. ...1111.1111... and ...1111.1110..., we see that after applying f 3 times we end up with ...1111111.1... and ...1111111.0..., the first of which lies in H_1 and the second in H_0, i.e. two points that are far apart! therefore our initial slight imprecision has grown into a large difference.

We have therefore shown that the invariant set Λ of the horseshoe contains;

(1)periodic orbits of all periods, including orbits of arbitrarily long periods.
(2) nonperiodic orbits.
(3) A dense orbit as defined in section 3.3.

We have also illustrated how a pair of points who start arbitrarily close together can follow wildly different orbits, and this sensitivity to initial conditions is considered hallmarks of chaos, therefore we conclude by making the statement that although the horseshoe looks simple on the surface, if we study it in greater depth we see that it exhibits chaotic dynamics.

4.4 Conley-Moser Conditions:Horseshoe type Map

This section will reference [20]

Our invariant set of all iterations $\Lambda = \cap_{n=-\infty}^{\infty} f^n(S)$ can be constructed in a standard way satisfying Conley-Moser conditions.

These conditions are a combination of geometric and analytical conditions, and are sufficient for proving the existence of a Chaotic invariant set for nonautonomous maps such as the horseshoe we have described above.

"(1) Stip condition; f maps horizontal strips H_0, H_1 to vertical strips V_0, V_1, mapping horizontal boundaries to horizontal boundaries and vertical boundaries to vertical boundaries.

(2) Hyperbolicity condition; f has uniform contraction in the horizontal direction and expansion in the vertical direction."[20]

We have illustrated condition (1) by considering the forward and backward orbits of iterations, and considering where these maps intersect.

The second condition is inherent in the Horseshoe itself; we have discussed how a horizontal contraction and vertical stretch are applied to our surface S, before a fold into the horseshoe shape.

It is clear to see from our above analysis that these conditions are satisfied. For a much more in depth description of the Conley-Moser conditions see [18] Chapter 25 "The Conley-Moser conditions, or 'How to prove that a dynamical system is Chaotic'."

5 The Bakers Map

This section will reference [2] pages 431-432.

In this section we will look at the Bakers map, which has the same dynamics as the horseshoe in that it iterates the unit square.

We define the Baker's map B of the square $0 \leq x \leq 1, 0 \leq y \leq 1$ to itself by...

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \leq x_n \leq 1/2 \\ (2x_n - 1, ay_n + \frac{1}{2}) & 1/2 \leq x \leq 1 \end{cases}$$

We define a as a parameter ranging $0 < a \leq \frac{1}{2}$.

The mapping of the Baker's map is similar to that of the horseshoe; firstly the square is stretched into a rectangle length $2a$, this rectangle is then cut in half, producing two rectangles, each length a. The right rectangle is stacked on top of the left, in such as way that its base sits at $y = 1/2$.

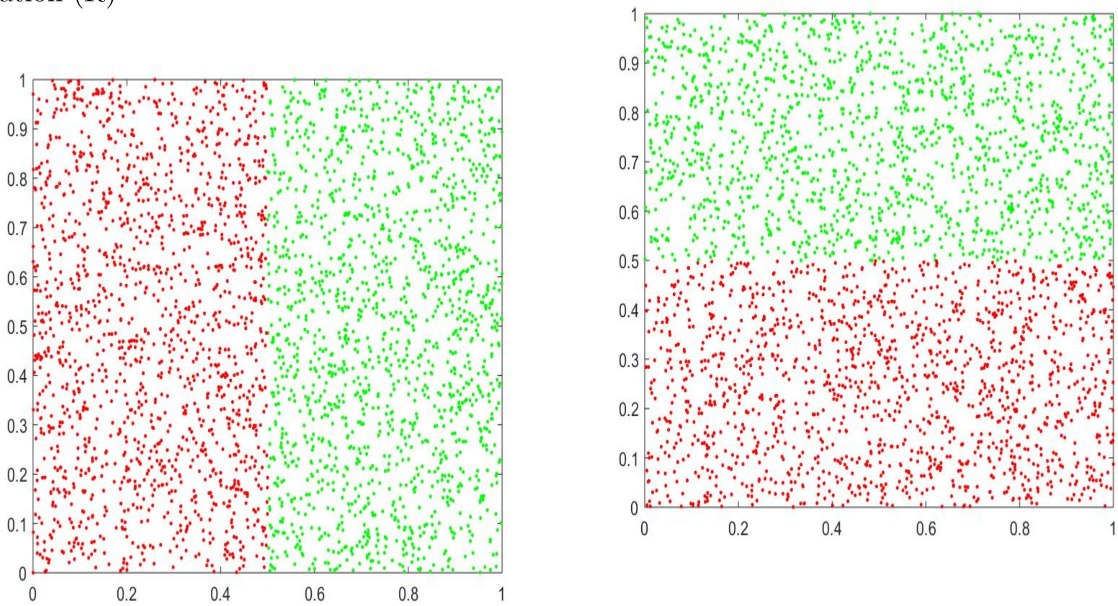
Similar to the Horseshoe, the bakers map also contains a fractal attractor that attracts all orbits i.e. for any initial points x_0, y_0, the distance from the iterations of x_0, y_0, to A converge as the number of iterations increases.

We will illustrate the chaotic tendencies of the Baker's map using Matlab. This section has used 'Bakers map initial plot' and 'Bakers map iterations' from section 8.

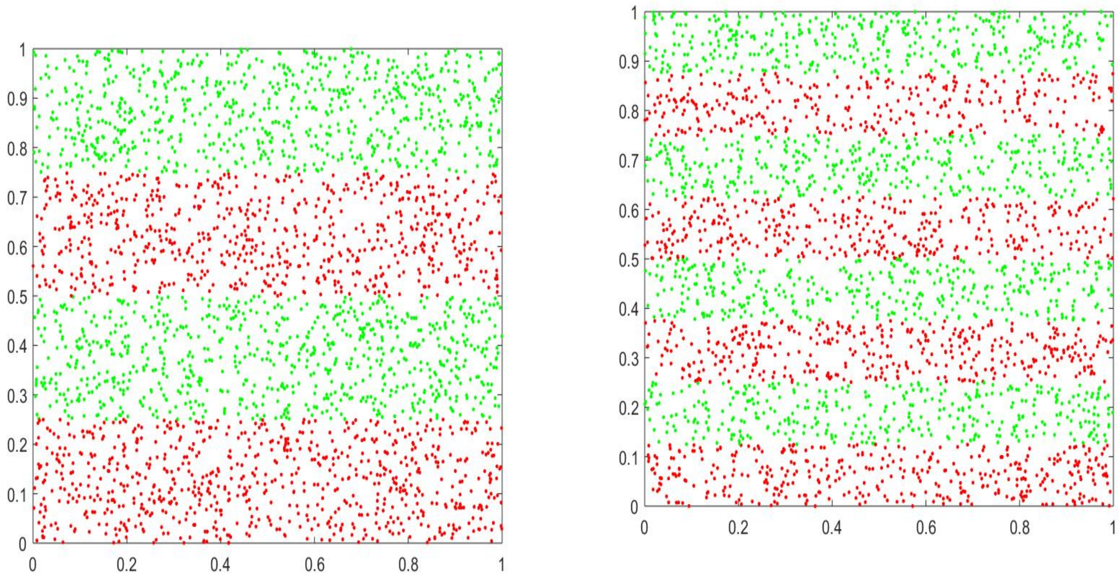
For these plots we have assigned the parameter a the value $a = 1/2$. We begin by creating a vector s of random points within the unit square. We then class these points in a way that if $x \leq 1/2$ we make that point red. If $x > 0.5$ it is assigned the colour green. The initial state is shown in 28a.

We then run these points through the Baker's Map function as defined above. As this set of points is iterated they keep the colours they were originally assigned, this makes it easier for us to observe the effects of the mapping after each iteration.

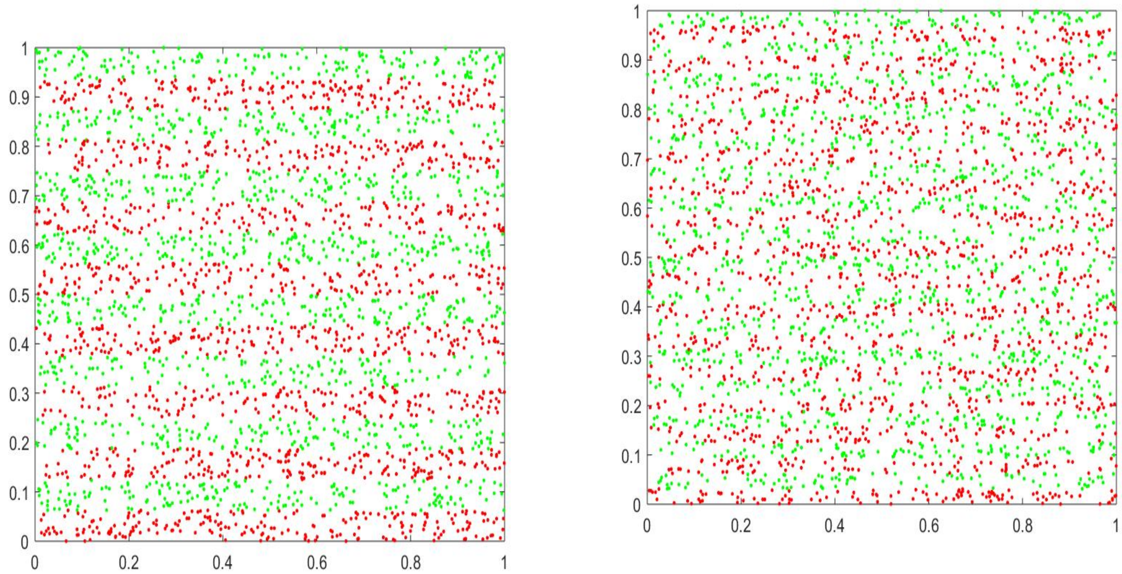
(a) Bakers map; starting values (L), and first iteration (R)



(a) Bakers map; second and third iteration

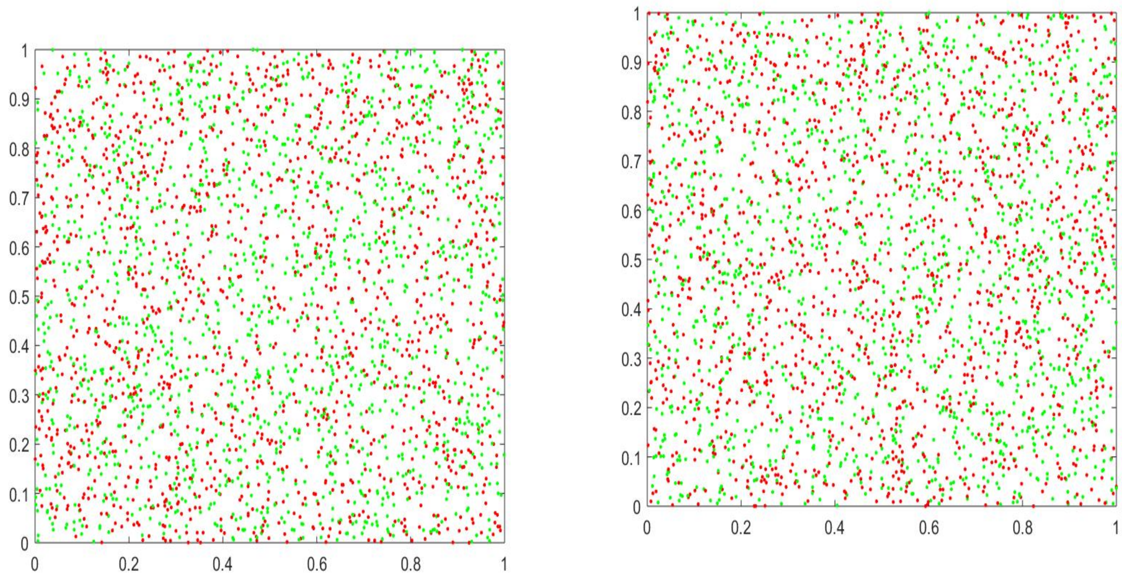


(a) Bakers map; fourth and fifth iteration



After 5 iterations, we observe from 30a that the plot is still layered, and there is not yet complete mixing. However, figure 31a shows us that as more iterations are applied, the the two colours become well mixed.

(a) Bakers map; ninth and tenth iteration



To see the chaotic features of the Baker's map, we consider the orbits of two points who's initial positions are arbitrarily close together. Chaos is not immediately obvious after the first few iterations, however it is clear that as more iterations are applied, these points can end up in completely different places. This illustrates the high sensitivity to initial conditions of the Baker's map, which as we have seen from prior analysis, is a hallmark of chaos.

The action of expansion in the horizontal direction and contraction in the vertical direction is familiar from our analysis of the horseshoe. Under iteration of this process we would expect, similar to the horseshoe, that there are infinitely many chaotic and periodic orbits in the unit square.

6 The F(x,y) Map

This section will heavily reference [19]. Our focus will be directed only to key results, all rigorous proofs and mathematical arguments can be found in [19].

Definition 6.1 ***Topological dynamical system*** *A Topological dynamical system (x,f) is the action of a continuous map f on a compact metric space X, by repeated composition. An example is the horseshoe map as seen in section 4.*

Definition 6.2 ***Topological Entropy*** *(informal definition) Topological Entropy is a quantity that measures the exponential growth rate of the number of distinguishable orbits when the rate increases. Topological Entropy is denoted h(f), and a positive value indicates the number of distinguishable trajectories increases exponentially with time, which in turn indicates a high level of complexity.[19].*

This is important because a positive topological entropy indicates sensitive dependence on initial conditions, a feature that is a hallmark of chaos.

A familiar example is the horseshoe map we analysed before. We showed that the horseshoe map has chaotic features, which agrees with its topological entropy; the horseshoe map has topological entropy $log(2)$.

In this section we will consider a planar map acting on the square $D = I^2$ where I=[-1,1]. This map takes the form...

$$F(x, y) = \frac{2b}{x + 2b + 1}(sgn(y), 1 - 2|y|) \tag{6.1}$$

where sgn has the meaning $\pm$.

We can make some observations about this map...

We see that the inverse of F is given by...

$$F^{-1}(x, y) = (\frac{2b}{|x|} - 2b - 1, \frac{sgn(x)}{2}(1 - \frac{y}{|x|})) \tag{6.2}$$

As before, when dealing with horseshoes we are interested in the dynamics of the invariant set only. In this case the invariant set is defined...

$$A^{\infty} = \cap_{n \in \mathbb{Z}} F^n(A) \tag{6.3}$$

where A is the surface on which we are applying the horseshoe mapping.

It is complicated to prove that this set has positive entropy, and therefore shows chaotic behaviour. We can also prove that the invariant set A^{∞} is structurally stable. Detailed proofs are left to [19].

In this section we are more interested in proving chaotic properties of this map analytically.

To do this we will be using the familiar tool of bifurcation diagrams as seen in section 3.8. The same method has been used to construct the bifurcation diagrams. We first consider the diagrams for x and y seperately.

All diagrams in this section we constructed in matlab using 'F(x,y) Bifurcation diagram for y vs b and x vs b (code is the same for both)' and 'F(x,y) norm and norm square Bifurcation Diagram (same code for both)' from 8.

Important notes;

(1) Since bifurcation diagrams require so much data, they are extremely

time-consuming to plot in matlab on a regular computer, therefore these plots may have less iterations than what is desirable, however they are sufficient to illustrate the key features.

(2) Birfurcation diagrams in this section may be incorrect! nonetheless I will explain them as if they are. The graphs shown are my attempts at the bifurcation diagram.

Figure 32: F(x,y) Bifurcation diagram x vs b

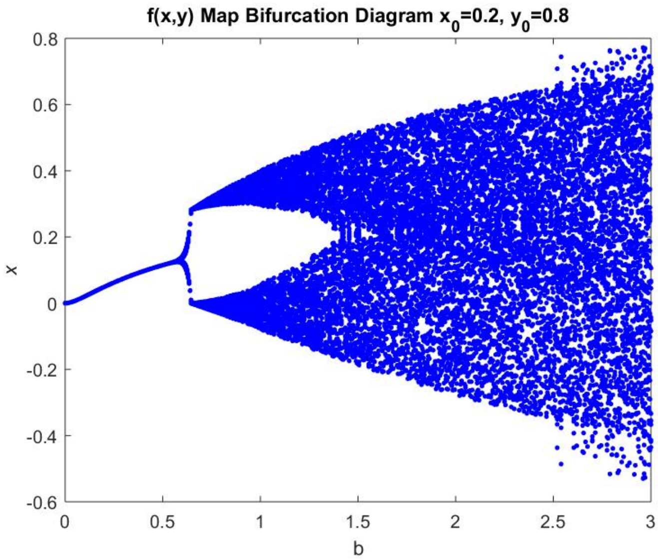
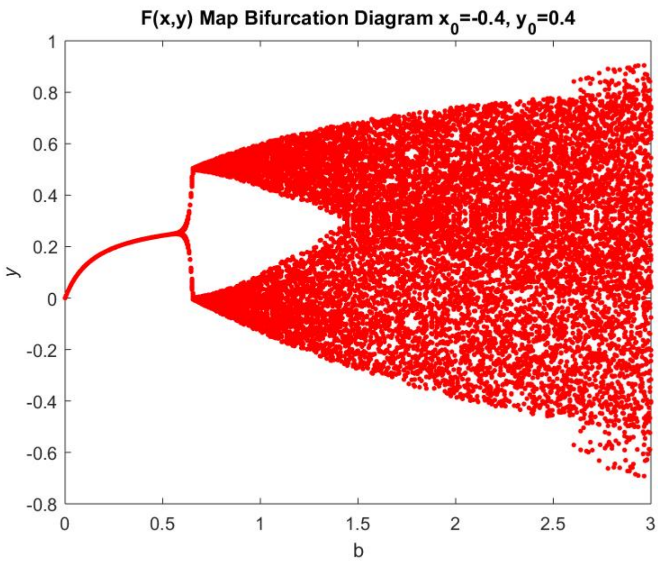
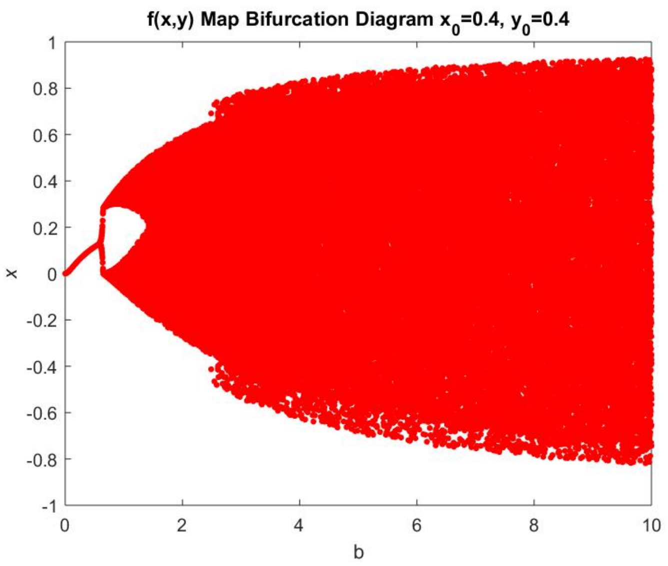


Figure 33: F(x,y) Bifurcation diagram y vs b



Our first two plots 32 and 33 show similar behaviour. For small values of b a single fixed point is shown. At just over $b = 1/2$ we see that a period doubling bifurcation occurs, and the period one orbit is split into a period two orbit. As before, period doubling bifurcations hint at the onset of chaos, and in this example this is confirmed. Both our plots show highly chaotic behaviour for $b > 1.5$ which is indicated by the dense plot of points, indicating dense orbits in this range.

Figure 34: F(x,y) Bifurcation diagram x vs b



34 Further argues that the mapping is chaotic; this figure shows a thick cloud of points for parameter values $b > 1/2$.

Figure 35: F(x,y) Bifurcation diagram norm vs b

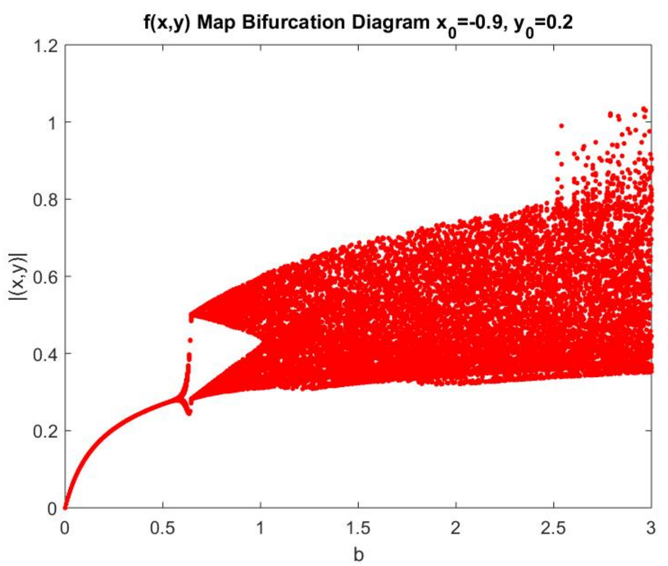


Figure 36: F(x,y) Bifurcation diagram norm vs b

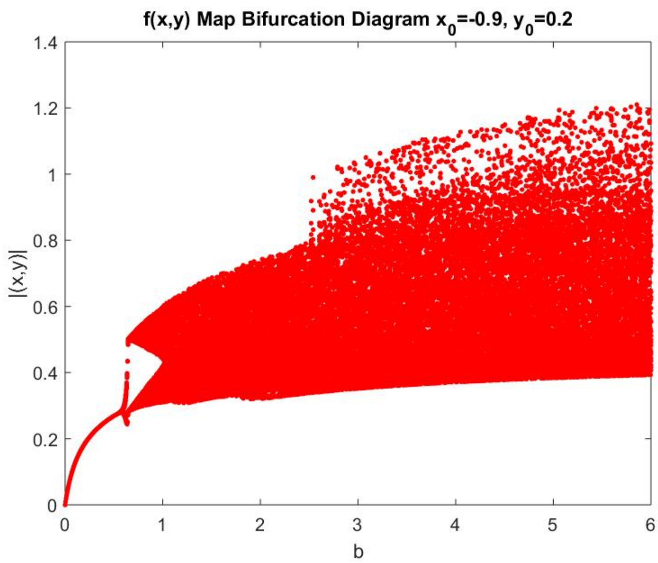


Figure 37: F(x,y) Bifurcation diagram norm squared vs b

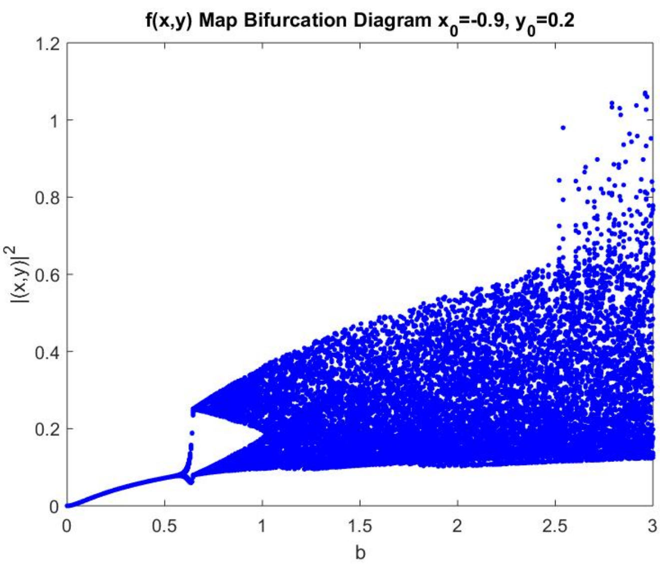
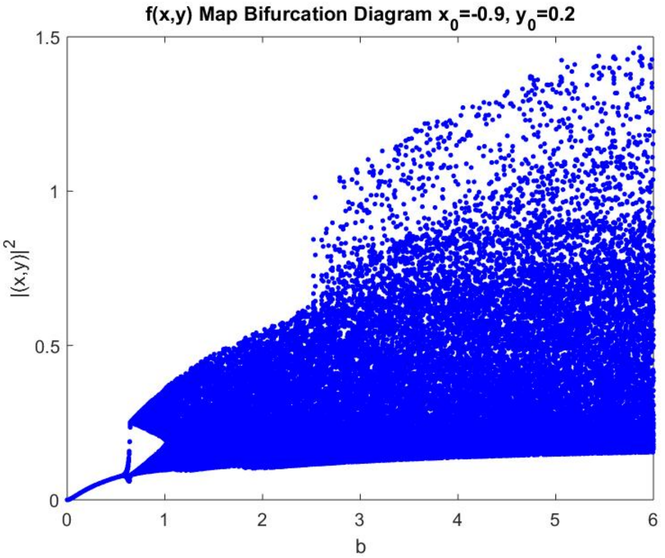


Figure 38: F(x,y) Bifurcation diagram norm squared vs b



The figures 35, 36, 37, 38 provide an additional viewpoint on the Chaos that our system exhibits. Again we see that for small parameter values $b < 1/2$ the system has one fixed point before undergoing a period doubling bifurcation, which commences the route to chaotic solutions.

Therefore, we have shown analytically that this system is chaotic, as made obvious by the bifurcation diagrams. This is backed up by proofs in [19] that show that the system has positive topological entropy, and therefore a high sensitivity to initial conditions.

7 Conclusion

In conclusion I have satisfied my objectives in proving that all three systems exhibit Chaotic behaviour.

I have shown that the Tent map exhibits high sensitivity to initial conditions for certain parameter values. This was made immediately obvious by cobweb plots, where the difference between initial values was small, yet under iteration each value followed a completely different path. This result was further supported by the Liapunov exponent for the tent map which proved that the Tent map had chaotic behaviour for $\mu > 1$. My claims were backed up by findings such as Li and Yorke, that period 3 behaviour implies chaos, and bifurcation diagrams that showed period doubling behaviour and chaotic dynamics for $\mu > 1$.

I have shown that the Smale horseshoe contains periodic orbits of all periods. This was done by reducing points in the square to a number in binary and considering shifting dynamics. I also illustrated how the Horseshoe map showed high sensitivity to initial conditions, and that through the iteration of forward and backward orbits, two points that are initially arbitrarily close together, can end up following completely different paths under iteration of the map. I analysed the Baker's map using matlab, and proved that, similar to the dynamics of the Horseshoe, there was a high sensitivity to initial conditions.

With the aid of bifurcation diagrams I showed how the F(x,y) model showed period one behaviour for small values of the parameter b, however after a period doubling bifurcation occured the system soon revealed itself as chaotic for values $b > 1/2$. Numerical analysis of the F(x,y) model would reveal a lot more about the complicated dynamics behind it, however unfortunately I did not find the time to look into this.

The findings suggest that all three maps show Chaotic features, and the conditions for chaotic behaviour were specified in each case.

Plans for future work

Throughout the project, the reader has come to understand analytical and theoretical arguments for Chaotic behaviour. The methods used in this project were a great introduction Chaos, and what Chaotic behaviour looks like in discrete dynamcial systems, however, the next step would be to prove my results more rigorously using proofs, and discover Chaotic arguments for these systems from a Pure mathematics perspective.

The F(x,y) model could also be looked into further to discover more about the biology that it represents, for example gene networks and how Chaos ties into them.

It would also be advantageous to take some of the ideas from this report and look into the real world application of Chaos theory, for example its uses in cryptography and private communications.

List of Figures

8 Matlab codes

Simple tent function plot:

```
mu = 2;  %parameter value
n = 300;    %number of iterations
x = linspace(0,1,n);     %creates a vector

for k=1:n
if x(k) < 0.5              %run the tent function
y(k)=mu*x(k);
elseif x(k) >= 0.5
y(k)=mu*(1-x(k));
end
end

plot(x,y)
xlabel('x_n')
ylabel('x_{n+1}')
set(gca,'XTick',0:1)
set(gca,'YTick',0:1,'YTickLabel',{'0','\mu/2'})  %specific labels on y axis
```

Tent Map cobweb diagram:

```
maxiter=200;
x=sym(zeros(1,maxiter));x1=sym(zeros(1,maxiter));x2=sym(zeros(1,maxiter));
x(1)=0.8;                  %IC x0=0.8 varies
mu=2;    %value of parameter varies
mid=maxiter/2;
axis([0 1 0 1]);
```

```
for k=2:maxiter
if (double(x(k-1)))>=0 && (double(x(k-1)))<=1/2
x(k)=sym(mu*x(k-1));
else
if (double(x(k-1)))<=1
x(k)=sym(mu*(1-x(k-1)));
end
end
end
for k=1:mid
x1(2*k-1)=x(k);
x1(2*k)=x(k);
end
x2(1)=0;x2(2)=double(x(2));
for k=2:mid
x2(2*k-1)=double(x(k));
x2(2*k)=double(x(k+1));
end
hold on
plot(double(x1),double(x2),'r');
a=[0 0.5 1];b=[0 mu/2 0];
plot(a,b,'b');
a=[0 1];b=[0 1];
plot(a,b,'k');
title('Tent map \mu=2 x_0=0.8')     %title varies
set(gca,'XTick',0:0.2:1)
set(gca,'YTick',0:0.2:1)
xlabel('x_n')
ylabel('x_{n+1}')
hold off
```

Tent map x_n vs n graph:

```
maxiter=200;   %max iterations
x=sym(zeros(1,maxiter));
x(1)=0.5;            %varying I.C
mu=2;    %varying parameter mu
nvec=[0:199];
for k=2:maxiter
if (double(x(k-1)))>=0 && (double(x(k-1)))<=1/2
x(k)=sym(mu*x(k-1));
else
if (double(x(k-1)))<=1
x(k)=sym(mu*(1-x(k-1)));
end
end
end
plot(nvec,x,'b--o')
xlabel('n')
ylabel('x_n')
xlim([0 200]) %can vary for more iterations
ylim([0 1])
title('Time Series plot \mu=2 x_0=0.5')  %%
hold on
```

Tent map Bifurcation Diagram

```
maxiter=100;   %varies
m=sym(zeros(1,maxiter));
lastit=30;
last=maxiter-(lastit -1);
for mu=0:0.005:2
```

```
m=0.4;mo=m;
for k=2:maxiter
if (double(m(k-1)))>=0 && (double(m(k-1)))<=1/2
m(k)=sym(mu*m(k-1));
else
if (double(m(k-1)))<=1
m(k)=sym(mu*(1-m(k-1)));
end
end
mo=m(k);
end
plot(mu*ones(lastit),m(last:maxiter),'r.','MarkerSize',10)
hold on
end
title('Tent Map Bifurcation Diagram x_{0}=0.4')
xlabel('{\mu}')
ylabel('\itx')
hold off
```

Bakers Map initial plot

```
nmax=3000;
s=rand(nmax,2);
c=zeros(nmax,1);
axis([0 1 0 1])

for i=1:nmax
if s(i,1)<=0.5
plot(s(i,1),s(i,2),'.r')
else
plot(s(i,1),s(i,2),'.g')
```

```
end
hold on
end
```

Bakers Map Iterations

```
nmax=5000;
s=rand(nmax,2);
c=zeros(nmax,1);
axis([0 1 0 1])

for i=1:nmax
if s(i,1)<=0.5
c(i)=0;
else
c(i)=1;
end
end

for k=1:4   %the span of k can be changed, giving different graphs
for i=1:nmax
if s(i,1) <=0.5
s(i,1) = 2*s(i,1);
s(i,2) = s(i,2)/2;
else
s(i,1) = 2*s(i,1)-1;
s(i,2) = 0.5+s(i,2)/2;
end
end
end
```

```
for i=1:nmax
if c(i)==0
plot(s(i,1),s(i,2),'.r')
hold on
else
plot(s(i,1),s(i,2),'.g')
end
hold on
end
```

F(x,y) Bifurcation diagram for y vs b and x vs b (code is the same for both)

```
maxiter=100;
x=(zeros(1,maxiter));
y=(zeros(1,maxiter));
x(1)=-0.4;
y(1)=0.4;
lastit=30;
last=maxiter-(lastit-1);
for b=0:0.005:3
xo=x; yo=y;
for n=2:maxiter
r=2*b;
s=x(n-1)+2*b+1;
t=r/s;
g=abs(y(n-1));
h=2*g;
p=1-h;
x(n)=t*y(n-1);
```

```
y(n)=t.*p;
xo=x(n);
yo=y(n);

end
plot(b*ones(lastit),x(last:maxiter),'r.','MarkerSize',10)  %x,y interchange
hold on
end
title('F(x,y) Map Bifurcation Diagram x_{0}=-0.4, y_{0}=0.4')
xlabel('b')
ylabel('\itx')   %ylabel changeable dependent on diagram
hold off
```

F(x,y) norm and norm square Bifurcation Diagram (same code for both)

```
maxiter=100;
x=(zeros(1,maxiter));
y=(zeros(1,maxiter));
d=(zeros(1,maxiter));
j=(zeros(1,maxiter));
x(1)=-0.9;
y(1)=0.2;
d(1)=sym(sqrt(85)/10);   %changes dependent on initial conditions
j(1)=(d(1))^2;
lastit=30;
last=maxiter-(lastit-1);
for b=0:0.005:3
xo=x; yo=y;
for n=2:maxiter
r=2*b;
s=x(n-1)+2*b+1;
```

```
t=r/s;
g=abs(y(n-1));
h=2*g;
p=1-h;
x(n)=t*y(n-1);
y(n)=t.*p;
xo=x(n);
yo=y(n);
z=[x(n) y(n)];
d(n)=norm(z);
j(n)=(d(n))^2;
end
plot(b*ones(lastit),j(last:maxiter),'b.','MarkerSize',8) %interchangeable
hold on
end
title('f(x,y) Map Bifurcation Diagram x_{0}=-0.9, y_{0}=0.2')
xlabel('b')
ylabel('|(x,y)|^{2}') %label varies dependent on diagram
hold off
```

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