*Date: 05-03-19

*From here, After mid-term exam.

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Homogeneous Linear differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_{n-1}\frac{dy}{dx} + a_n y = 0$$

Solve the following equation.

(i) $\frac{d^2y}{dx^2} + 4y = 0$, (ii) $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2 y = 0$

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Soln:

(i) $\frac{d^2y}{dx^2} + 4y = 0$ ——— (1)

let, $y = e^{mx}$ be a solution of (1)

Now form (1) we have

$\frac{d^2}{dx^2} e^{mx} + 4e^{mx} = 0$

$\Rightarrow m^2 e^{mx} + 4e^{mx} = 0$

$\Rightarrow e^{mx}(m^2+4) = 0$

Since, $e^{mx} \neq 0$

$\therefore m^2 + 4 = 0$

$\Rightarrow m^2 = -4 = (-1)\cdot 4 = i^2 2^2 = (2i)^2$

$\Rightarrow m^2 = (2i)^2$

$\therefore m = \pm 2i$

$m = a \pm ib$

$\therefore y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$\therefore y = c_1 \cos 2x + c_2 \sin 2x$ is a solution

(11) $\frac{d^2y}{dx^2} - 2K\frac{dy}{dx} + K^2y = 0$

Soln:

$$\frac{d^2y}{dx^2} - 2K\frac{dy}{dx} + K^2y = 0 \quad \text{---(1)}$$

Let, $y = e^{mx}$ be a solution of (1).

Now form (1) we have,

$\frac{d^2}{dx^2}e^{mx} - 2K\frac{d}{dx}e^{mx} + K^2e^{mx} = 0$

$\Rightarrow m^2e^{mx} - 2Kme^{mx} + K^2e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 - 2Km + K^2) = 0$

Since, $e^{mx} \neq 0$

$\therefore m^2 - 2Km + K^2 = 0$

$\Rightarrow m^2 - Km - Km + K^2 = 0$

$\Rightarrow m(m-K) - K(m-K) = 0$

$\Rightarrow (m-K)(m-K) = 0$

$\therefore m = K, K$

$y = (c_1 + c_2x)e^{kx}$ be a solution.

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Soln:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \text{ ——— (1)}$$

Let, $y = e^{mx}$ be a solution of (1)

Now, form (1) we have

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

$\frac{d^2}{dx^2} e^{mx} + \frac{d}{dx} e^{mx} - 2e^{mx} = 0$

$\Rightarrow m^2 e^{mx} + m e^{mx} - 2e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 + m - 2) = 0$

Since, $e^{mx} \neq 0$

$\therefore m^2 + m - 2$

$\Rightarrow m^2 + 2m - m - 2$

$\Rightarrow m(m+2) - 1(m+2) = 0$

$\Rightarrow (m+2)(m-1) = 0$

$m+2 \neq 0$

$\therefore m = -2, 1$

$y = (c_1 e^{-2x} + c_2 e^{x})$ is a solution

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$D^2 = \frac{d^2}{dx^2}$

$D^3 = \frac{d}{d3}$

* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ ⟶ (Full form) } of Homogeneous Linear diff. equ

$\Rightarrow D^2y - 3Dy + 2y = 0$ ⟶ (Short form)

So, the auxaliory equation is —

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m = 2, 1$

So, the solution is

$y = (c_1 e^{2x} + c_2 e^{x})$

$D^3y - 5D^2y + 7Dy - 3y = 0$

Soln:

The auxiliary equation is –

$m^3 - 5m^2 + 7m - 3 = 0$

$\Rightarrow m^3 - 3m^2 - 2m^2 + 6m + m - 3 = 0$

$\Rightarrow m^2(m-3) - 2m(m-3) + 1(m-3) = 0$

$\Rightarrow (m-3)(m^2 - 2m + 1) = 0$

$m - 3 = 0$

$\therefore m = 3$

and, $m^2 - 2m + 1 = 0$

~~$\Rightarrow m(m$~~

$\Rightarrow m^2 - m - m + 1 = 0$

$\Rightarrow m(m-1) - 1(m-1) = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\therefore m = 1, 1$

$y = c_1 e^{3x} + (c_2 + c_3)e^x$ be a solution.

Theory of Vector Analysis

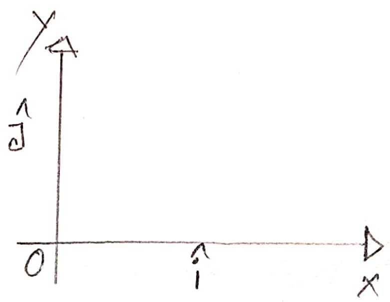
Vector: A vector is a quantity having both magnitude and direction such as displacement, velocity, force, and acceleration.

Scalar: A scalar is a quantity having magnitude but no direction. eg: mass, lenth, time, temparature etc.

$$\vec{A} = x\hat{i} + y\hat{j}$$

$$\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{A}| = \sqrt{x^2 + y^2 + z^2}$$



Q: Given $r_1 = 3\hat{i} - 2\hat{j} + \hat{k}$, $r_2 = 2\hat{i} - 4\hat{j} - 3\hat{k}$, $r_3 = -\hat{i} + 2\hat{j} + 2\hat{k}$. Find the magnitude of $2r_1 - 3r_2 - 5r_3$

Soln:

$$2r_1 = 2(3\hat{i} - 2\hat{j} + \hat{k})$$
$$= 6\hat{i} - 4\hat{j} + 2\hat{k}$$

$$3r_2 = 3(2\hat{i} - 4\hat{j} - 3\hat{k})$$
$$= 6\hat{i} - 12\hat{j} - 9\hat{k}$$

$$5r_3 = 5(-\hat{i} + 2\hat{j} + 2\hat{k})$$
$$= -5\hat{i} + 10\hat{j} + 10\hat{k}$$

Now, $2r_1 - 3r_2 - 5r_3 = 6\hat{i} - 4\hat{j} + 2K - 6\hat{i} - 12\hat{j} - 9\hat{k} + 5\hat{i} - 10\hat{j} - 10\hat{k}$

$\Rightarrow 6\hat{i} - 4\hat{j} + 2\hat{k} - 6\hat{i} + 12\hat{j} + 9\hat{k} + 5\hat{i} - 10\hat{j} - 10\hat{k}$

$\Rightarrow 5\hat{i} - 2\hat{j} + \hat{k}$

So, $|2r_1 - 3r_2 - 5r_3| = \sqrt{(5)^2 + (-2)^2 + (1)^2}$
$= \sqrt{30}$ Ans

Q: Find the projection of the vector $\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$ on the vector $\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Soln:

We know,

the projection of the vector $\vec{A}$ on $\vec{B} = |\vec{A}|\cos\theta$

$\therefore \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$\Rightarrow \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = |\vec{A}|\cos\theta$

$\Rightarrow |\vec{A}|\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|}$ —— (1)

$\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

$\vec{A}\cdot\vec{B} = 4 + 8 + 7 = 19$

$\therefore |\vec{B}| = \sqrt{(4)^2 + (-4)^2 + (7)^2}$

$= 9$

Now from (1) we have —

$|\vec{A}|\cos\theta = \dfrac{19}{9}$

So, the projection of $\vec{A}$ on $\vec{B}$ is $= \dfrac{19}{9}$

Rules:

Dot product of $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

Cross " of $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{\eta}$

(1) $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{B} = 5\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{A}\cdot\vec{B} = 10 + 6 + 12$

$= 28$

** Gradient, Divergence & Curl:

The vector differential operator del, written ∇ is defined by, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

*Gradient: let, $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space. (ie: ϕ defines as differentiable scalar field) Then the gradient of ϕ, written grad ϕ, is defined $\nabla\phi = \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)$

$$= \phi \Rightarrow \frac{\partial \phi \hat{i}}{\partial x} + \frac{\partial \phi \hat{j}}{\partial y} + \frac{\partial \phi \hat{k}}{\partial z}$$

Q: if $\phi(x,y,z) = 3x^2y - y^3z^2$. Find $\nabla\phi$ (or grad ϕ) at the point $(1,-2,-1)$

Soln: we know,

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= \left\{3y\frac{\partial}{\partial x}x^2 - 0\right\}\hat{i} + \hat{j}\left\{3x^2\frac{\partial y}{\partial y} - z^2\frac{\partial}{\partial y}y^3\right\} + \hat{k}\left\{\frac{\partial}{\partial z}3x^2y - y^3\frac{\partial}{\partial z}z^2\right.$$

$\Rightarrow 3y \cdot 2x \cdot \hat{i} + \hat{j}\left(3x^2 - z^2 3y^2\right) + \hat{k}\left(0 - y^3 2z\right)$

$\Rightarrow 6xy\hat{i} + \hat{i} + \hat{j}\left(3x^2 - z^2 3y^2\right) + \hat{k}\left(0 - y^3 2z\right)$

$\Rightarrow$ Now at the point $(1, -2, -1)$

$\therefore \text{grad}\,\phi = 6 \cdot 1(-2)\hat{i} + \left(3 \cdot \hat{i} - 3 \cdot 4 \cdot 1\right)\hat{j} - 2(-2)^3 \cdot (-1)\hat{k}$

$= -12\hat{i} + (3 - 12)\hat{j} - 2 \cdot 8 \cdot \hat{k}$

$= -12\hat{i} - 9\hat{j} - 16\hat{k}$

* **Curl:** If $\vec{V}(x,y,z)$ is a differentiable vector field then the curl or rotation of V written, $\vec{\nabla}\times\vec{\nabla}$, curl $\vec{V}$ or rot $\vec{V}$ is defined as

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\right)\times\left(V_1\hat{i}+V_2\hat{j}+V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} + K\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}V_3 - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial}{\partial x}V^2 - \frac{\partial V_1}{\partial y}\right)$$

Q: if $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$. Find curl $\vec{A}$ at the point $(1,-1,1)$

Soln: We know,

$$\text{curl } \vec{A} = \vec{\nabla}\times\vec{A}$$

$$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times\left(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$\Rightarrow \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x^2yz & 2yz^4 \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz^3 & 2yz^4 \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz^3 & -2x^2yz \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(2z^4\frac{\partial y}{\partial y} + 2x^2y\frac{\partial z}{\partial z}\right) - \hat{j}\left(0 - x\frac{\partial}{\partial z}z^3\right) + \hat{k}\left(-2yz\frac{\partial}{\partial x}x^2 - 0\right)$$

$$\Rightarrow \hat{i}\left(2z^4 + 2x^2y\right) + \hat{j}\cdot x\cdot 3z^2 - \hat{k}\cdot 2yz\cdot 2x$$

$$\Rightarrow \left(2z^4 + 2x^2y\right)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

Now at point $(1, -1, 1)$

$$\therefore \text{Curl } \vec{A} = (2-2)\hat{i} + 3\hat{j} + 4\hat{k}$$

$$= 3\hat{j} + 4\hat{k}$$ Ans.

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*Divergence: Let, $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain point/region of space. Then the divergence of $\vec{V}$, written $\vec{\nabla}\cdot\vec{V}$ or div $\vec{V}$ is defined,
$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$
$$= \frac{\partial}{\partial x}V_1 + \frac{\partial}{\partial y}V_2 + \frac{\partial}{\partial z}V_3$$

Q: A particle moves along a curve whose parametric equation are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where 't' is the time.

(a) Determine it's velocity and acceleration at any time

(b) Find the magnitudes of the velocity and acceleration at $t=0$

Soln: The position vector $\vec{r}$ of the particle is
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

$\therefore \frac{d}{dt}\left(e^{-t} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}\right)$

$= \hat{i}\frac{d}{dt}e^{-t} + 2\hat{j}\frac{d}{dt}\cos 3t + 2\hat{k}\frac{d}{dt}\sin 3t$

$= -\hat{i}\,e^{-t} - \hat{j}\,2\cdot 3\sin 3t + 2\hat{k}\cdot 3\cos 3t$

$= -e^{-t} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$

accelaration $\vec{a} = \frac{d\vec{v}}{dt}$ -1

$= \frac{d}{dt}\left(-e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}\right)$

$\Rightarrow -\frac{d}{dt}e^{-t}\hat{i} - 6\frac{d}{dt}\sin 3t\,\hat{j} + 6\frac{d}{dt}\cos 3t\,\hat{k}$

$\Rightarrow e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$

$\Rightarrow e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$

Now at point $= 0$

velocity $\vec{v} = -\hat{i} + 0 + 6\hat{k} = -\hat{i} + 6\hat{k}$

megnetude $\vec{v} = \sqrt{(-1)^2 + (6)^2} = \sqrt{1+36} = \sqrt{37}$

accelaration $= \hat{i} - 18\hat{j} + 0 = \hat{i} - 18\hat{j}$

megnetude $= \sqrt{(1)^2 + (-18)^2}$

$= \sqrt{1+324}$

$= \sqrt{325}$

Homogeneous Linear differential equation:

Home work:

① $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Soln: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ ——— ①

let, $y = e^{mx}$ be a solution of ①

Now, From ①, we have,

$\Rightarrow \frac{d^2}{dx^2} e^{mx} - 8\frac{d}{dx} e^{mx} + 16 e^{mx} = 0$

$\Rightarrow m^2 e^{mx} - 8m e^{mx} + 16 e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 - 8m + 16) = 0$

Since, $e^{mx} \neq 0$

$\therefore m^2 - 8m + 16 = 0$

$\Rightarrow m^2 - 4m - 4m + 16 = 0$

$\Rightarrow m(m-4) - 4(m-4) = 0$

$\Rightarrow (m-4)(m-4)$

$\therefore m = 4, 4$

$\therefore y = (c_1 + c_2 x) e^{4x}$ be a solution (A)

(11) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$

Soln: $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0$ ——— (1)

let, $y = e^{mx}$ be a solution of (1)

Now form (1) we have,

$\frac{d^2}{dx^2} e^{mx} + 6\frac{d}{dx} e^{mx} + 25e^{mx} = 0$

$\Rightarrow m^2 e^{mx} + 6e^{mx} + 25e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 + 6 + 25) = 0$

Since, $e^{mx} \neq 0$

$m^2 + 6 + 25 = 0$

$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 25}}{2 \cdot 1}$

$= \frac{-6 \pm \sqrt{36 - 100}}{2}$

$= \frac{-6 \pm 8i}{2}$

$= \frac{-6-8i}{2}, \frac{-6+8i}{2}$

$= (-3-4i), (-3+4i)$

$= (-3 \pm 4i)$

$\therefore y = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$

(iii) $\frac{d^2y}{dx^2} + 9y = 0$

Soln: $\frac{d^2y}{dx^2} + 9y = 0$ ——— (1)

let $y = e^{mx}$ be a solution of (1).

Now from (1) we have

$\frac{d^2}{dx^2} e^{mx} + 9e^{mx} = 0$

$\Rightarrow m^2 e^{mx} + 9e^{mx} = 0$

$\Rightarrow e^{mx}(m^2+9) = 0$

Since, $e^{mx} \neq 0$

$m^2 + 9 = 0$

$\Rightarrow m^2 = -9 = (-1).9 = i^2 3^2 = (i3)^2$

$\therefore m = \pm 3i$

$y = (c_1 \cos 3x + c_2 \sin 3x)$ be a solution. Ans:

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