STATE SPACE MODEL OF 2 DOF BEARING SYSTEM

Equation of motion for Bearing Dynamics

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Contents

List of Figures:

Problem Formulation:

Bearing dynamics is well known to be associated with nonlinearity due to **Hertzian force /deformation relationship,** varying stiffness resulting from load transmission, etc. whereas the application of filters requires the system to be linear.

The study by *White (1979)* has mentioned that linearized rolling bearing stiffness coefficient may be assumed, if the dynamic forces cause a small variation from the equilibrium. It has also been concluded in the same study that the fluctuation of stiffness due to changing number and position of rolling elements in the load zone are less than half percent of the total value for a given load.

Moreover, for the given set up, the first natural frequency of the system is 492 Hz, which is much higher than the maximum rotational speed of the shaft (25 Hz). Therefore, though non-linear, displacement between inner race and outer race is small at the operating speed and hence the effect of non-linearity does not show up much.

"Therefore, the system has been modeled as a simple linear 2 DOF system for the direct applicability of Kalman and H_∞ filters"

Test Set up:

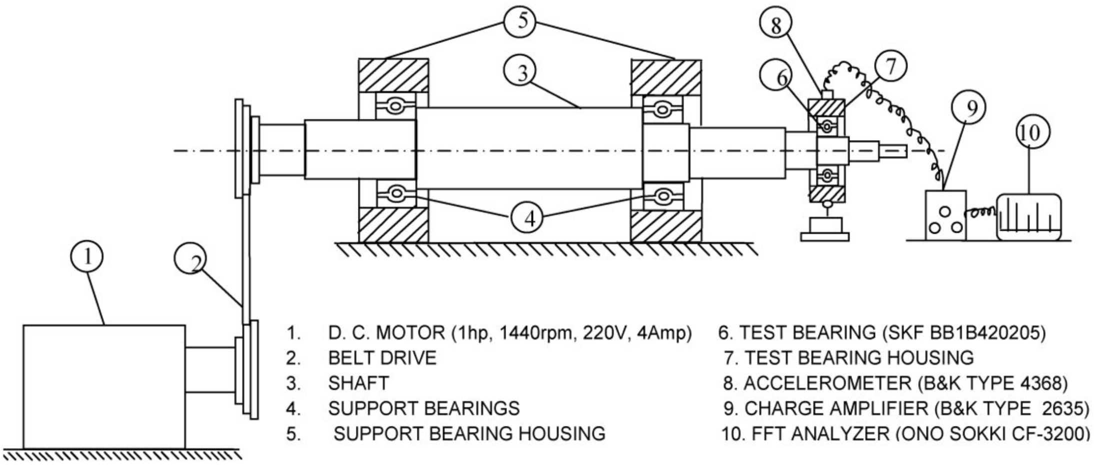

Figure 1: Test Set Up [1]

2 DOF Model of Test Set up:

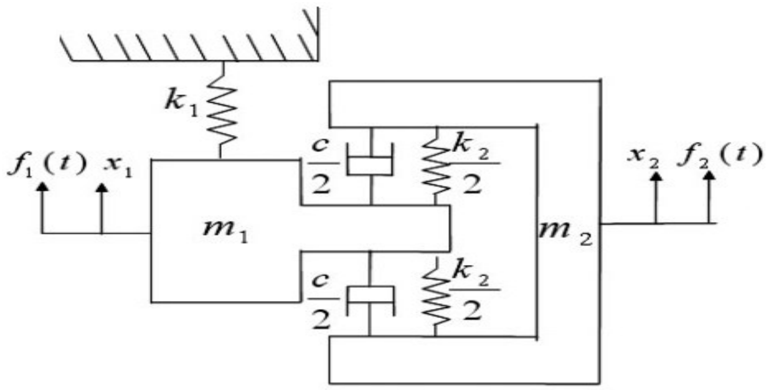

Figure 2: 2-DOF Model of Test Set Up [1]

Equation of Motion:

The bearing dynamics has been modeled as that of a linear system, subject to simultaneous actions of a deterministic $\{f(t)\}^d$ as well as a stochastic (random) $\{f(t)\}^s$ force.

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f(t)\}^d + \{f(t)\}^s$$

Where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices respectively. The forcing function has got deterministic part, $\{f(t)\}^d$, generated by the impact when a rolling element negotiates a defect for which satisfactory model exists *(Choudhury and Tandon (2006))*. And a stochastic part, $\{f(t)\}^s$, generated by a number of unmodelled effects, viz. non-linearity due to the Hertzian force/ deformation relationship varying stiffness resulting from load transmission via a finite number of rolling elements, the presence of clearance between the rolling elements, the presence of clearance between the rolling elements and the bearing races, and the effect of lubricant film.

Free Body Diagrams:

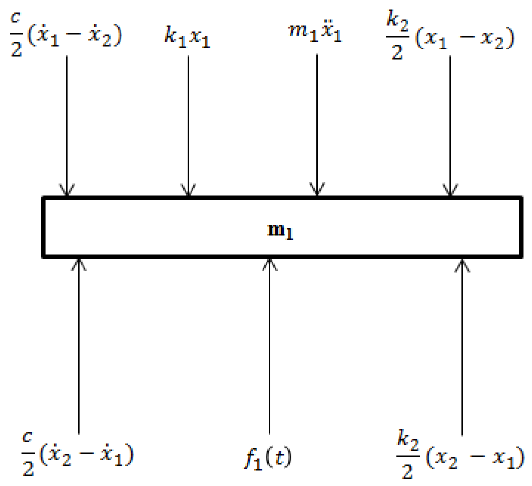

Figure 3: Free body diagram of mass 1

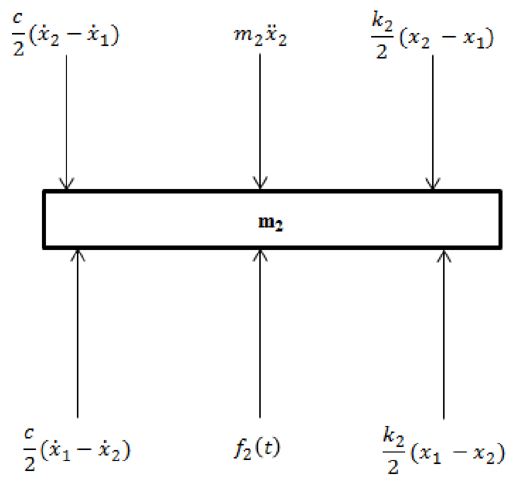

Figure 4: Free body diagram of mass 2

m_1 = mass of extended portion of the shaft and inner race, m_2 = mass of race and housing, c = damping coefficient of balls, due to oil film that builds up during rotation, k_1 = stiffness coefficient of shaft (on which bearing under investigation is mounted), k_2 = linearized bearing stiffness coefficient. The vectors $\{f_i(t)\}^d$ $(i = 1, 2)$ denote excitation force vector due to localized defect acting on mass m_i at time t.

Respective Equation of Motion:

For mass (m_1):

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + k_1x_1 + k_2x_1 - k_2x_2 = \{f_1(t)\}^d + \{f_1(t)\}^s$$

For mass (m_2):

$$m_2\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - k_2x_1 + k_2x_2 = \{f_2(t)\}^d + \{f_2(t)\}^s$$

Matrix formulation:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}^d + \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}^s$$

Determining constants (k, c, m) and Input excitations ($f_i(t)$):

Determining mass:

1. m_1 = mass of extended portion of the shaft and inner race, the mass of extended portion of shaft can be evaluate by the idea of density, $\rho = \frac{m}{V}$ where m is mass of the shaft(extended portion), V is volume of the shaft, ρ is density of the material of the shaft(which is standard value).
2. m_2 = mass of race and housing, this can be evaluated by just weighing the bearing.

Determining Stiffness:

k_1 = stiffness coefficient of shaft (on which bearing under investigation is mounted). The shaft is assumed to be fixed at right support bearing (fig. 1) and the extended portion is assumed to act as cantilever. Therefore stiffness of shaft is stiffness of cantilever beam with variable cross section. Also here stiffness is equivalent stiffness for element in series.

$$k_{cantilever} = \frac{3.E.I}{l^3}$$

Where E = Youngs Modulus (standard material property), I = moment of inertia$\left(\frac{\pi}{64} \times d^4\right)$, where d is diameter of shaft, l = length of the shaft. As the beam is of variable cross section it is considered as element in series.

$$\frac{1}{k_1} = \frac{1}{k_{eq}} = \frac{1}{k_{c1}} + \frac{1}{k_{c2}} + \cdots + \frac{1}{k_{cn}}$$

Where k_{c1}, k_{c2}, k_{cn} are stiffness of cantilever for respective diameter and length.

k_2 = linearized bearing stiffness coefficient, here the model considers outer race to be rigidly mounted in the housing and the inner race to be rigidly mounted on the shaft, the torsional(or flexural) vibration of the races are neglected, on the basis of this assumption rotor bearing system has been modeled as spring mass damper system.

According to *Kramer [Dynamics of rotor and foundation]*, stiffness of rolling element bearing can be determined by force – deflection relationship of a rolling body. The force/contact force F_i for the i^{th} rolling body to give radial displacement δ_i is.

$$F_i = \left(\frac{\delta_i}{C}\right)^n$$

$$for\ ball\ bearing \begin{cases} n = \dfrac{3}{2} \\ C = 4.37 \times 10^{-4} \times d^{-1/3}\ mm\ N^{-2/3}\ (d\ is\ diameter\ of\ ball) \end{cases}$$

$$for\ roller\ bearing \begin{cases} n = \dfrac{1}{0.9} \\ C = 0.77 \times 10^{-4} \times l^{-1/3}\ mm\ N^{-2/3}\ \ (l\ is\ length\ of\ roller) \end{cases}$$

To obtain displacement δ_i, consider outer race is held rigidly and inner race is displaced by amounts of x and y or vice versa. Then

$$\delta_i = \mathrm{x} \cos\varphi_i + \mathrm{y} \sin\varphi_i - \gamma$$

Where $\varphi_i = i\ \varphi_1\ and\ \varphi_1 = 2\pi/z$, z is no. of rolling elements, γ is the radial play that exists between races.

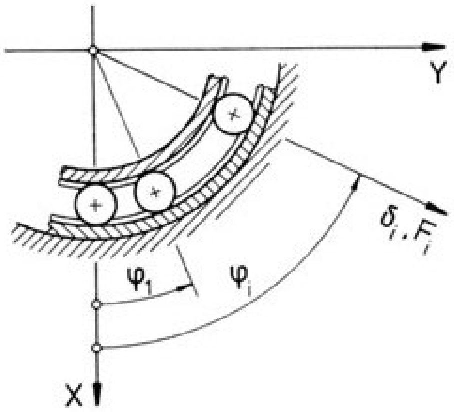

Figure 5: Coordinates and angles for rolling element bearing [5]

Contact force only arises only when δ_i is positive, (i.e. displacement in x direction) x > 0, y = 0. Also for bearing with no play ($\gamma = 0$)

$$\delta_i = \begin{cases} \mathrm{x} \cos\varphi_i & with \quad \dfrac{-\pi}{2} < \varphi_i < \dfrac{\pi}{2} \\ 0 & with \quad \dfrac{\pi}{2} < \varphi_i < \dfrac{3\pi}{2} \end{cases}$$

The components of force F_i in the x and y direction are,

$$F_{xi} = F_i \cos\varphi_i\ , F_{yi} = F_i \sin\varphi_i$$

When the displacement is considered in only x direction then the y component of force is neglected and therefore, **the sum of components of F_{xi} is the bearing force.**

$$F = \sum F_{xi} = \left(\frac{x}{C}\right)^n \left[1 \quad + \quad 2\sum \cos{\varphi_i}^{n+1}\right], where\ 0 < \varphi_i < \frac{\pi}{2}$$

The $[1 \quad + \quad 2\sum \cos \varphi_i{}^{n+1}]$ is approximated to Sz, where S is constant with value (0.23 for ball bearing and 0.24 for roller bearing) and z is number of rolling elements (i.e. balls or rollers). Therefore, bearing force is given as

$$\boldsymbol{F = Sz\left(\frac{x}{C}\right)^{n}}$$

Here the displacement of the bearing in x direction can be measured using sensor (amplitude measurement) at static condition and bearing force can be calculated accordingly.

The **stiffness** is evaluated by differentiating bearing force w.r.t displacement.

$$k = \frac{dF}{dx} = \frac{n}{C} \quad (Sz)^{\frac{1}{n}} \quad F^{\frac{n-1}{n}}$$

Therefore the stiffness k_2 after substituting the constants n, C, S respectively,

$$\boldsymbol{k_2 = \ 1.3 \quad z^{2/3} \quad d^{1/3} \quad F^{1/3}} \ (for\ ball\ bearing)$$

$$\boldsymbol{k_2 = \ 4 \quad z^{0.9} \quad l^{0.8} \quad F^{0.1}} \ (for\ roller\ bearing)$$

Determining damping coefficient:

c = damping coefficient of balls, due to oil film that builds up during rotation. According to *Kramer [Dynamics of rotor and foundation]*, In rolling element bearings the damping can be defined as

$$\boldsymbol{c = x\, k_2,} \qquad \forall\ (0.25\ \times 10^{-5}) < x < \ (2.5\ \times 10^{-5})$$

Determining Excitation Force:

The vectors $\{f_i(t)\}^d$ $(i = 1, 2)$ denote excitation force vector due to localized defect acting on mass m_i at time t. the excitation force $F_1(t)$ is the response of defect on the inner race or mass m_1 and $F_2(t)$ is response of defect on the outer race or mass m_2.

Localized Defects:

Localized defects are modeled according to *C Mishra et.al [2017]*, accordingly, constant angular width $(\Delta\emptyset_d)$ and depth (C_d) of the fault is assumed.

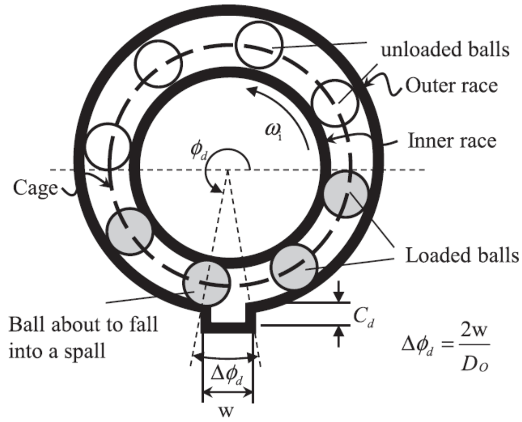

Figure 6: Simplified Outer race defect schematic [4]

Modeling of Outer Race Defect:

Consider a single defect on the outer race of a bearing at an angular position of $\emptyset_d$. The interaction of this defect with rolling elements causes the large amount of impulsive force $F_2(t)$ to be excited, whereas the force $F_1(t)$ is assumed to be zero, under this condition according to deformation equation as mentioned by *C Mishra et.al [2017].*

$$\delta_{0i} = max\big((x_d \cos\varphi_i + y_d \sin\varphi_i - \gamma - \delta_f),\quad 0\big)$$

Where $\delta_f = \begin{cases} C_d & \emptyset_d < \varphi_i < (\emptyset_d + \Delta\emptyset_d) \\ 0 & otherwise \end{cases}$, and ($x_d = x_2 - x_1$) and ($y_d = y_2 - y_1$) is the relative displacement between inner and outer races in vertical and horizontal direction. For bearing with no slippage (no radial play, $\gamma = 0$).

Now according to *N.Sawalhi et.al [2008]*, the exicitation force in vertical direction is

$$F_2(t) = K_d \sum_{i=1}^{Z} \varepsilon_i \quad {\delta_0}_i^n \quad \cos\varphi_i$$

Where deformation constant $K_d = \frac{34{,}300}{k^{0.35}} d^{0.5}\ (N/mm^{3/2})$, n value is same as that of the one use for stiffness calculation. Curvature ratio $k = \frac{r_o + r_i - d}{d}$, (r_o, r_i are groove radii of outer and inner race), d is ball diameter, δ_0 is substituted accordingly. ε_i is contact state of ${\delta_0}_i$ rolling element which is $\begin{cases} 1 & {\delta_0}_i > 0 \\ 0 & otherwise \end{cases}$.

The anglular position of rolling element φ_i is the function of time increment dt, and previous cage/ball position φ_o and the cage speed ω_c.

$$\varphi_i = \frac{2\pi\,(i-1)}{Z} + \omega_c dt + \varphi_o$$

Where $\omega_c = \frac{\omega_s}{2}\left(1 - \frac{d}{D}\cos\alpha\right)$, ω_s is shaft speed, d is ball diameter, D is Pitch diameter Z is no. of balls, α is contact angle (standard value).

Modeling Inner Race Defect:

Inner race fault is modeled in similar way to that of in outer race, but in case of inner race fault, the defect rotates with inner race and its position changes continually. Thus $\emptyset_d$ is dependent on the angle of shaft rotation $(\omega_s t)$, which can be expressed as $\emptyset_d = \omega_s t + \emptyset_o$ where $\emptyset_o$ is the initial location of the defect.

Therefore the contact deformation can be given as,

$$\delta_{ii} = max\big((x_d \cos\varphi_i + y_d \sin\varphi_i - \gamma - \delta_f),\quad 0\big)$$

Where $\delta_f = \begin{cases} C_d & \emptyset_d < \varphi_i < (\emptyset_d + \Delta\emptyset_d) \\ 0 & otherwise \end{cases}$

Interaction of this defect with rolling elements causes the large amount of impulsive force $F_1(t)$ to be excited, whereas the force $F_2(t)$ is assumed to be zero. For this condition,

$$F_1(t) = K_d \sum_{i=1}^{Z} \varepsilon_i \quad \delta_{i_i}^{\ n} \quad \cos\varphi_i$$

Modeling of Rolling Element Defect:

The defect in the ball rotates at the same speed as that of ball. At any instant, the angular position of the spall can be expressed as $\emptyset_d = \frac{\omega_s D}{2d}\left(1 - \left(\frac{d}{D}\cos\alpha\right)^2\right)t + \emptyset_0$. Where $\emptyset_o$ is the initial location of the defect. It is to be noted that ball rotates in opposite direction to that of shaft rotation.

Given below is representation of effect of ball defect.

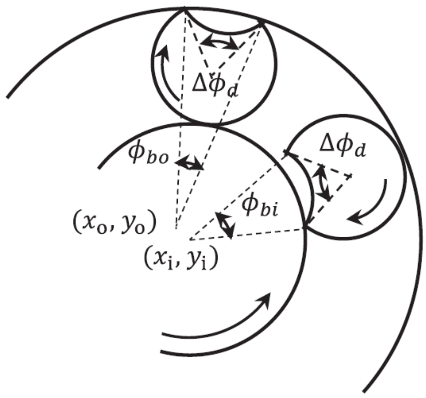

Figure 7: Effect of rolling element defect on inner and outer race [4]

The angular width of ball fault in terms of inner and outer races can be expressed as $\emptyset_{bi} = \frac{\Delta\emptyset_d d}{D_i}$, $\emptyset_{bo} = \frac{\Delta\emptyset_d d}{D_o}$, where D_i & D_o are the diameter of inner and outer races.

For each complete rotation of ball, loss of contact is detected twice, once contact with inner race is lost and once contact with outer race is lost. As the inner race and outer race have different curvatures and hence, the angular fault widths are different for two contact losses.

This difference in curvature also influences the depth of ball's entry into the defect pit on the inner and outer races.

Case 1: When the rolling element is in contact with the inner race, the maximum depth to which the inner race enters the defect pit can be expressed as.

$$C_{di} = \frac{1}{2\left(D_i - \sqrt{D_i^2 - 4x^2}\right)}$$

x = w/2, half defect width. When the defect is in contact with inner race, the inner race moves C_{di} downward while the rolling element moves upward by

$$C_{dr} = \frac{1}{2\left(d - \sqrt{d^2 - 4x^2}\right)}$$

The net contact loss is expressed as $C_d = C_{dr} + C_{di}$

Case 2: When the defect is in contact with outer race, the outer race moves towards center by C_{d0} while the rolling element moves outward by C_{dr}.

$$C_{d0} = \frac{1}{2\left(D_o - \sqrt{D_o^2 - 4x^2}\right)}$$

x = w/2, half defect width.

The net contact loss is expressed as $C_d = C_{dr} - C_{do}$

Therefore the overall contact deformation in case of rolling element fault, where j[th] rolling element is faulty can be expressed as.

$$\delta_{ji} = max\left(\left(x_d \cos\varphi_i + y_d \sin\varphi_i - \gamma - \delta_f\right),\ \ 0\right)$$

Where $\delta_f = \begin{cases} C_{dr} - C_{do} & 0 < \emptyset_d < \emptyset_{bo} \\ C_{dr} + C_{di} & \pi < \emptyset_d < (\pi + \emptyset_{bi}) \\ 0 & otherwise \end{cases}$

Case 1: Interaction of ball defect with inner race causes the large amount of impulsive force $F_1(t)$ to be excited,

$$F_1(t) = K_d \sum_{i=1}^{Z} \varepsilon_i \quad \delta_{j_i}^{\ n} \quad \cos\varphi_i$$

Here the δ_j is dependent on δ_f such that for $\pi < \emptyset_d < (\pi + \emptyset_{bi})$, $\delta_f = C_{dr} + C_{di}$. Or 'zero' otherwise

Case 2: Interaction of ball defect with outer race causes the large amount of impulsive force $F_2(t)$ to be excited,

$$F_2(t) = K_d \sum_{i=1}^{Z} \varepsilon_i \quad {\delta_j}_i^n \quad \cos\varphi_i$$

Here the δ_j is dependent on δ_f such that for $0 < \emptyset_d < \emptyset_{bo}$, $\delta_f = C_{dr} - C_{do}$. Or 'zero' otherwise.

Therefore for rolling element defect both $F_1(t)$ & $F_2(t)$ exists within the limits.

Equation of Motion with Influence of Defects:

For mass (m_1):

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + k_1x_1 + k_2x_1 - k_2x_2 = \{F_1\left((x_d), \omega_s, t\right)\}^d$$

For mass (m_2):

$$m_2\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - k_2x_1 + k_2x_2 = \{F_2\left((x_d), \omega_s, t\right)\}^d$$

State Space Model of 2-DOF Bearing System:

State variables: Minimum number of variables required to define state of system (i.e. Bearing Test Setup). These variables are defined by the energy storing components (i.e. mass, spring etc).

Here only four state variables are needed to completely define the state. $X_1 = x_1, X_2 = \dot{x}_1$, $X_3 = x_2, X_4 = \dot{x}_2$.

State Equation:

$$\dot{\boldsymbol{X}}_{\boldsymbol{1}} = \frac{\boldsymbol{dX_1}}{\boldsymbol{dt}} = \dot{\boldsymbol{x}}_{\boldsymbol{1}} = \boldsymbol{X_2}$$

$$\dot{X}_2 = \frac{dX_2}{dt} = \ddot{x}_1$$

Consider equation of motion for m_1, substitute state variables in respective places, to determine the equation of $\ddot{x}_1$ or $\dot{X}_2$, also only deterministic force is taken into account for linearizing the equation

$$m_1\dot{X}_2 + c\,X_2 - c\,X_4 + k_1X_1 + k_2X_1 - k_2X_3 = \{f_1(t)\}^d$$

$$\dot{X}_2 = \frac{1}{m_1}\left(-k_1X_1 - k_2X_1 - c\,X_2 + k_2X_3 + c\,X_4 + \{f_1(t)\}^d\right)$$

$$\dot{\boldsymbol{X}}_2 = \frac{-(\boldsymbol{k_1} + \boldsymbol{k_2})}{\boldsymbol{m_1}}\boldsymbol{X_1} - \frac{\boldsymbol{c}}{\boldsymbol{m_1}}\boldsymbol{X_2} + \frac{\boldsymbol{k_2}}{\boldsymbol{m_1}}\boldsymbol{X_3} + \frac{\boldsymbol{c}}{\boldsymbol{m_1}}\boldsymbol{X_4} + \frac{\{\boldsymbol{f_1(t)}\}^{\boldsymbol{d}}}{\boldsymbol{m_1}}$$

$$\dot{\boldsymbol{X}}_3 = \frac{\boldsymbol{dX_3}}{\boldsymbol{dt}} = \dot{\boldsymbol{x}}_2 = \boldsymbol{X_4}$$

$$\dot{X}_4 = \frac{dX_4}{dt} = \ddot{x}_2$$

Consider equation of motion for m_1, substitute state variables in respective places, to determine the equation of $\ddot{x}_2$ or $\dot{X}_4$, also only deterministic force is taken into account for linearizing the equation

$$m_2\dot{X}_4 - c\,X_2 + c\,X_4 - k_2X_1 + k_2X_3 = \{f_2(t)\}^d$$

$$\dot{X}_4 = \frac{1}{m_2}\left(k_2X_1 + c\,X_2 - k_2X_3 - c\,X_4 + \{f_2(t)\}^d\right)$$

$$\dot{\boldsymbol{X}}_4 = \frac{\boldsymbol{k_2}}{\boldsymbol{m_2}}\boldsymbol{X_1} + \frac{\boldsymbol{c}}{\boldsymbol{m_2}}\boldsymbol{X_2} - \frac{\boldsymbol{k_2}}{\boldsymbol{m_2}}\boldsymbol{X_3} - \frac{\boldsymbol{c}}{\boldsymbol{m_2}}\boldsymbol{X_4} + \frac{\{\boldsymbol{f_2(t)}\}^{\boldsymbol{d}}}{\boldsymbol{m_2}}$$

Therefore, the state equation for above system is given as,

$$\begin{aligned}
\dot{\boldsymbol{X}}_1 &= \boldsymbol{X_2}\\
\dot{\boldsymbol{X}}_2 &= \frac{-(\boldsymbol{k_1} + \boldsymbol{k_2})}{\boldsymbol{m_1}}\boldsymbol{X_1} - \frac{\boldsymbol{c}}{\boldsymbol{m_1}}\boldsymbol{X_2} + \frac{\boldsymbol{k_2}}{\boldsymbol{m_1}}\boldsymbol{X_3} + \frac{\boldsymbol{c}}{\boldsymbol{m_1}}\boldsymbol{X_4} + \frac{\{\boldsymbol{f_1(t)}\}^{\boldsymbol{d}}}{\boldsymbol{m_1}}\\
\dot{\boldsymbol{X}}_3 &= \boldsymbol{X_4}\\
\dot{\boldsymbol{X}}_4 &= \frac{\boldsymbol{k_2}}{\boldsymbol{m_2}}\boldsymbol{X_1} + \frac{\boldsymbol{c}}{\boldsymbol{m_2}}\boldsymbol{X_2} - \frac{\boldsymbol{k_2}}{\boldsymbol{m_2}}\boldsymbol{X_3} - \frac{\boldsymbol{c}}{\boldsymbol{m_2}}\boldsymbol{X_4} + \frac{\{\boldsymbol{f_2(t)}\}^{\boldsymbol{d}}}{\boldsymbol{m_2}}
\end{aligned}$$

Matrix formulation of State Equation:

$$\begin{Bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{-(k_1+k_2)}{m_1} & -\frac{c}{m_1} & \frac{k_2}{m_1} & \frac{c}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k_2}{m_2} & \frac{c}{m_2} & -\frac{k_2}{m_2} & -\frac{c}{m_2}\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\\ X_3\\ X_4\end{Bmatrix} + \begin{bmatrix}0 & 0\\ 1/m_1 & 0\\ 0 & 0\\ 0 & 1/m_2\end{bmatrix}\begin{Bmatrix}f_1(t)\\ f_2(t)\end{Bmatrix}^d + \begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{Bmatrix}$$

Here ω_i (i = 1, 2, 3, 4) represents the process noise corresponding to ith state.

Taking t_s as sampling time, and with scalar multiplication done on state equation.

$$\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{Bmatrix} = t_s \times \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & -\frac{c}{m_1} & \frac{k_2}{m_1} & \frac{c}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c}{m_2} & -\frac{k_2}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}^d \right) + \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{Bmatrix}$$

The above equation is of form of,

$$\dot{X} = AX + BU + noise$$

Where,

$\boldsymbol{A}$	$state\ matrix$	$\begin{bmatrix} 0 & t_s & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1}t_s & -\frac{c}{m_1}t_s & \frac{k_2}{m_1}t_s & \frac{c}{m_1}t_s \\ 0 & 0 & 0 & t_s \\ \frac{k_2}{m_2}t_s & \frac{c}{m_2}t_s & -\frac{k_2}{m_2}t_s & -\frac{c}{m_2}t_s \end{bmatrix}$
$\boldsymbol{X}$	$state\ variables$	$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix}$
$\boldsymbol{B}$	$input\ matrix$	$\begin{bmatrix} 0 & 0 \\ (1/m_1)t_s & 0 \\ 0 & 0 \\ 0 & (1/m_2)t_s \end{bmatrix}$
$\boldsymbol{U}$	$Input\ (i.e.\ force)\ vector$	$\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}^d$

Inorder to represent the state equation to satisfy the form of Kalman filter equation, Identity matrix is added to state matrix (i.e. A),

$$\bar{A} = I + A$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t_s & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1}t_s & -\frac{c}{m_1}t_s & \frac{k_2}{m_1}t_s & \frac{c}{m_1}t_s \\ 0 & 0 & 0 & t_s \\ \frac{k_2}{m_2}t_s & \frac{c}{m_2}t_s & -\frac{k_2}{m_2}t_s & -\frac{c}{m_2}t_s \end{bmatrix}$$

$$
\bar{A} = \begin{bmatrix} 1 & t_s & 0 & 0 \\ \frac{-(k_1 + k_2)}{m_1} t_s & 1 - \frac{c}{m_1} t_s & \frac{k_2}{m_1} t_s & \frac{c}{m_1} t_s \\ 0 & 0 & 1 & t_s \\ \frac{k_2}{m_2} t_s & \frac{c}{m_2} t_s & -\frac{k_2}{m_2} t_s & 1 - \frac{c}{m_2} t_s \end{bmatrix}
$$

The state equation after the above modification is given below,

$$
\begin{Bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{Bmatrix} = \begin{bmatrix} 1 & t_s & 0 & 0 \\ \frac{-(k_1 + k_2)}{m_1} t_s & 1 - \frac{c}{m_1} t_s & \frac{k_2}{m_1} t_s & \frac{c}{m_1} t_s \\ 0 & 0 & 1 & t_s \\ \frac{k_2}{m_2} t_s & \frac{c}{m_2} t_s & -\frac{k_2}{m_2} t_s & 1 - \frac{c}{m_2} t_s \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ \left(1/m_1\right) t_s & 0 \\ 0 & 0 \\ 0 & \left(1/m_2\right) t_s \end{bmatrix} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}^d + \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{Bmatrix}
$$

According to defect modeling solution, input vector can be given as.

Type of Defect	Input vector
Outer race defect	$\{0 \quad f_2(t)\}^T$
Inner race defect	$\{f_1(t) \quad 0\}^T$
Rolling element defect	$\{f_1(t) \quad f_2(t)\}^T$

Conclusion:

In-order to model the defects on the bearing, the required inputs is approximate dimensions of defect and its location on different elements of the bearing. It is to be noted that excitation force is function of x_1 and x_2 and the speed of the shaft.

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