The Hidden Beauty of the Double Pendulum

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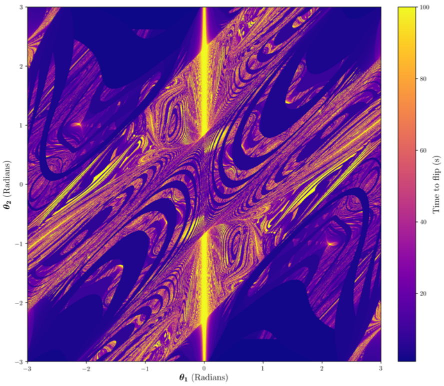

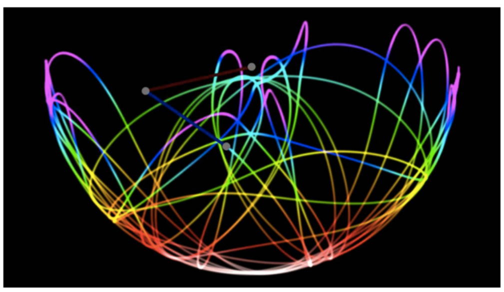

X and y coordinates of each pendulum based on their angle and lengths

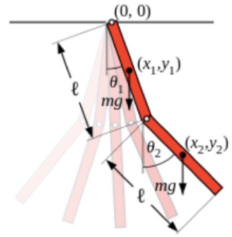

Figure 2: Diagram of angles on the double pendulum

$$(x_1, y_1) = \left(\frac{l}{2}\sin\theta_1, -\frac{l}{2}\cos\theta_1\right)$$

$$(x_2, y_2) = \left(l\left(\sin\theta_1 + \tfrac{1}{2}\sin\theta_2\right), -l\left(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right)\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

The Euler-Lagrange equation used to find the characteristic equations from the Lagrangian by taking partial derivatives with respect to the core variables.

Defining and simplifying the Lagrangian

$$L = T - V$$

$$T = \text{Linear kinetic energy} + \text{Rotational kinetic energy}$$

$$= \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}I(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}\left(\frac{1}{12}ml^2\right)(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$V = \text{Gravitational potential energy} = mg(y_1 + y_2)$$

$$L = \frac{1}{2}m\left(\frac{1}{4}l^2\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 + 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{12}ml^2\right)(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg\left(-\frac{1}{2}\cos\theta_1 - l(\cos\theta_1 + \frac{1}{2}\cos\theta_2)\right)$$

$$= \frac{1}{6}ml^2\left(4\dot{\theta}_1^2 + \dot{\theta}_2^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + \frac{1}{2}mgl(3\cos\theta_1 + \cos\theta_2)$$

Characteristic Equations

$$\dot{\theta}_1 = \frac{6}{ml^2}\frac{2p_{\theta_1} - 3p_{\theta_2}\cos(\theta_1 - \theta_2)}{16 - 9\cos^2(\theta_1 - \theta_2)}$$

$$\dot{\theta}_2 = \frac{6}{ml^2}\frac{8p_{\theta_2} - 3p_{\theta_1}\cos(\theta_1 - \theta_2)}{16 - 9\cos^2(\theta_1 - \theta_2)}$$

$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial \theta_1} = -\frac{1}{2}ml^2\left(3\frac{g}{l}\sin\theta_1 + \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\right)$$

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial \theta_2} = -\frac{1}{2}ml^2\left(3\frac{g}{l}\sin\theta_2 - \dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\right)$$