The Beauty of the Double Pendulum

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1. Literature Review

Pendulums have fascinated scientists since Galileo Galilei observed a swinging chandelier in Pisa Cathedral in 1582 and went on to discover the property that made them useful as timekeepers, called isochronism, where the time period of the pendulum is independent of both the mass and the amplitude. He also discovered that the period is proportional to the square root of the length which is fully described by the equation $T = 2\pi\sqrt{\frac{l}{g}}$ ADD IN GENERAL CASE HERE IN PAT 2007 where l is the length and g is the gravitational constant. These useful properties of the humble pendulum meant that it became essential for time keeping over the next centuries as the only variable that needed to be controlled precisely was the length to fix the time period.

Pendulums can be found now in almost every mechanics textbook now but a natural question to ask would be what if two pendulums were attached together end to end and allowed to swing? If this is set up in the real world and the motion is tracked, interesting curves emerge and as we change the starting conditions by a small amount, such as the starting angles or lengths of both pendulums, the outcome changes by a large amount which shows that it exhibits *chaotic motion*. Due to the complexity of this motion there is no “analytical solution” to the system which means there is no way to work out what the pendulum does after a given time but there is a method using Lagrangian mechanics to work out what happens at the next time interval.

Figure 1: *Long exposure of double pendulum tracked with an LED*

To derive the equations of motion for the double pendulum, start with two pendulums with a distributed mass both with length l and mass m and with angles θ_1 and θ_2 as shown in Figure 2 which yields the following equations for the coordinates (x_1, y_1) and (x_2, y_2)

$$\begin{gathered}(x_1, y_1) = \left(\frac{l}{2}\sin\theta_1, -\frac{l}{2}\cos\theta_1\right) \\ (x_2, y_2) = \left(l\left(\sin\theta_1 + \tfrac{1}{2}\sin\theta_2\right), -l\left(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right)\right)\end{gathered} \tag{1}$$

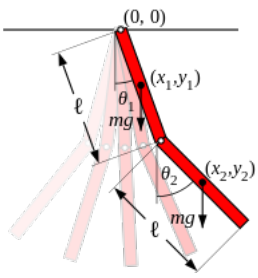

Figure 2: *Double pendulum with coordinates labelled*

The Lagrangian is now used which is defined as $L = T - V$ where T is the kinetic energy and V is the potential energy of a system and by taking the moment of inertia of a rod attached by the end to be $\frac{1}{12}ml^2$:

$$\begin{gathered}L = T - V \\ T = \text{Linear kinetic energy} + \text{Rotational kinetic energy} = \frac{1}{2}m\left(v_1^2 + v_2^2\right) + \frac{1}{2}I\left(\dot\theta_1^2 + \dot\theta_2^2\right) = \frac{1}{2}m\left(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2\right) + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\left(\dot\theta_1^2\right. \\ V = \text{Gravitational potential energy} = mg(y_1 + y_2)\end{gathered} \tag{2}$$

To simplify the equation for Kinetic energy we can just take the first bracket and observe that for example $\dot{x_1} = \frac{dx_1}{dt} = \frac{d\theta_1}{dt}\frac{dx_1}{d\theta_1} = \dot{\theta}_1\frac{dx}{d\theta_1}$ by the product rule and therefore using (1):

$$\begin{aligned}\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 &= \left(\dot{\theta}_1 \frac{l}{2}\cos\theta_1\right)^2 + \left(\dot{\theta}_1 \frac{l}{2}\sin\theta_1\right)^2 + \left(\dot{\theta}_1 l\cos\theta_1 + \dot{\theta}_2 \frac{l}{2}\cos\theta_2\right)^2 + \left(\dot{\theta}_1 l\sin\theta_1 + \dot{\theta}_2 \frac{l}{2}\sin\theta_2\right)^2 \\ &= \frac{1}{4}l^2\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 + 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)\end{aligned} \tag{3}$$

And we can redefine the Lagrangian just in terms of angles:

$$\begin{aligned}L &= \frac{1}{2}m\left(\frac{1}{4}l^2\left(5\dot{\theta}_1^2 + \dot{\theta}_2^2 + 4\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)\right) + \frac{1}{2}\left(\frac{1}{12}ml^2\right)(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg\left(-\frac{1}{2}\cos\theta_1 - l(\cos\theta_1 + \frac{1}{2}\cos\theta_2)\right) \\ &= \frac{1}{6}ml^2\left(4\dot{\theta}_1^2 + \dot{\theta}_2^2 + 3\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + \frac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right)\end{aligned} \tag{4}$$

Now as we know that in the linear case $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$ it can be seen that $\frac{dT}{d\dot{x}} = m\dot{x} = p$ and since potential energy (V) is independent of velocity, $p = \frac{\partial L}{\partial \dot{x}}$ or in the case of the double pendulum $p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1}$ and $p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2}$ so we just take the partial derivatives of (4) in terms of θ_1 and θ_2 :

$$\begin{aligned}p_{\theta_1} &= \frac{\partial L}{\partial \theta_1} = \frac{1}{6}ml^2\left(8\dot{\theta}_1 + 3\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) \\ p_{\theta_2} &= \frac{\partial L}{\partial \theta_2} = \frac{1}{6}ml^2\left(2\dot{\theta}_2 + 3\dot{\theta}_1\cos(\theta_1 - \theta_2)\right)\end{aligned} \tag{5}$$

And finally we can re-arrange