The Hidden Beauty of the Double Pendulum

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Derivation of the Characteristic Equations

To derive the equations of motion for the double pendulum, start with two pendulums with a distributed mass both with length l and mass m and with angles θ_1 and θ_2. As shown in Figure 2 the following equations for the coordinates (x_1, y_1) and (x_2, y_2) follow by trigonometry:

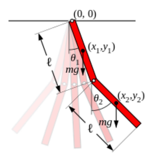


Figure 2: Diagram of angles on the double pendulum

$$
\begin{aligned}
(x_1, y_1) &= \left(\frac{l}{2}\sin\theta_1, -\frac{l}{2}\cos\theta_1\right) \\
(x_2, y_2) &= \left(l\left(\sin\theta_1 + \tfrac{1}{2}\sin\theta_2\right), -l\left(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right)\right)
\end{aligned} \tag{2.1}
$$

The Lagrangian is now used which is defined as $L = T - V$ where T is the kinetic energy and V is the potential energy of a system[5] and by taking the moment of inertia of a rod attached by the end to be $\frac{1}{12}ml^2$:

$$
\begin{aligned}
L &= T - V \\
T &= \text{Linear kinetic energy} + \text{Rotational kinetic energy} \\
&= \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}I(\dot\theta_1^2 + \dot\theta_2^2) \\
&= \frac{1}{2}m(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2) + \frac{1}{2}\left(\frac{1}{12}ml^2\right)(\dot\theta_1^2 + \dot\theta_2^2) \\
V &= \text{Gravitational potential energy} = mg(y_1 + y_2)
\end{aligned} \tag{2.2}
$$

To simplify the equation for kinetic energy we can just take the first bracket $(\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2)$ and observe that for example $\dot{x_1} = \frac{dx_1}{dt} = \frac{d\theta_1}{dt}\frac{dx_1}{d\theta_1} = \dot\theta_1 \frac{dx}{d\theta_1}$ by the chain rule and therefore using the coordinates defined in Eq. (2.1):

$$
\begin{aligned}
&\dot x_1^2 + \dot y_1^2 + \dot x_2^2 + \dot y_2^2 \\
&= \left(\dot\theta_1 \frac{l}{2}\cos\theta_1\right)^2 + \left(\dot\theta_1\frac{l}{2}\sin\theta_1\right)^2 + \left(\dot\theta_1 l\cos\theta_1 + \dot\theta_2\frac{l}{2}\cos\theta_2\right)^2 + \left(\dot\theta_1 l\sin\theta_1 + \dot\theta_2\frac{l}{2}\sin\theta_2\right)^2 \\
&= \frac{1}{4}l^2\left(5\dot\theta_1^2 + \dot\theta_2^2 + 4\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2)\right)
\end{aligned} \tag{2.3}
$$

And we can redefine the Lagrangian from Eq. (2.2) just in terms of angles by substituting in Eq. (2.3):

$$
\begin{aligned}
L &= \frac{1}{2}m\left(\frac{1}{4}l^2\left(5\dot\theta_1^2 + \dot\theta_2^2 + 4\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2)\right)\right) \\
&+ \frac{1}{2}\left(\frac{1}{12}ml^2\right)(\dot\theta_1^2 + \dot\theta_2^2) - mg\left(-\frac{1}{2}\cos\theta_1 - l(\cos\theta_1 + \frac{1}{2}\cos\theta_2)\right) \\
&= \frac{1}{6}ml^2\left(4\dot\theta_1^2 + \dot\theta_2^2 + 3\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2)\right) + \frac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right)
\end{aligned} \tag{2.4}
$$

Now as we know that in the linear case $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot x^2$ it can be seen that $\frac{dT}{d\dot x} = m\dot x = mv = p$ and since potential energy (V) is independent of velocity, $p = \frac{\partial L}{\partial \dot x}$ or in the case of the double pendulum $p_{\theta_1} = \frac{\partial L}{\partial \dot\theta_1}$ and $p_{\theta_2} = \frac{\partial L}{\partial \dot\theta_2}$ so we just take the partial derivatives of Eq. (2.4) in terms of $\dot\theta_1$ and $\dot\theta_2$:

$$
\begin{aligned}
p_{\theta_1} &= \frac{\partial L}{\partial \dot\theta_1} = \frac{1}{6}ml^2\left(8\dot\theta_1 + 3\dot\theta_2\cos(\theta_1 - \theta_2)\right) \\
p_{\theta_2} &= \frac{\partial L}{\partial \dot\theta_2} = \frac{1}{6}ml^2\left(2\dot\theta_2 + 3\dot\theta_1\cos(\theta_1 - \theta_2)\right)
\end{aligned} \tag{2.5}
$$

Eq. (2.5) can then be re-arranged for $\dot\theta_1$ and $\dot\theta_2$:

$$
\begin{aligned}
\dot\theta_2 &= \frac{1}{2}\left(\frac{6}{ml^2}p_{\theta_2} - 3\dot\theta_1\cos(\theta_1 - \theta_2)\right) \\
\dot\theta_1 &= \frac{1}{8}\left(\frac{6}{ml^2}p_{\theta_1} - 3\dot\theta_2\cos(\theta_1 - \theta_2)\right)
\end{aligned} \tag{2.6}
$$

By substituting the opposite expressions for p_{θ_1} and p_{θ_2} from Eq. (2.5) into Eq. (2.6) , the characteristic expressions for the rate of change of the angles are formed:

$$
\begin{aligned}
\dot\theta_1 &= \frac{6}{ml^2}\frac{2p_{\theta_1} - 3p_{\theta_2}\cos(\theta_1 - \theta_2)}{16 - 9\cos^2(\theta_1 - \theta_2)} \\
\dot\theta_2 &= \frac{6}{ml^2}\frac{8p_{\theta_2} - 3p_{\theta_1}\cos(\theta_1 - \theta_2)}{16 - 9\cos^2(\theta_1 - \theta_2)}
\end{aligned} \tag{2.7}
$$

The final step is to solve for the rate of change of momentum to get the 4 characteristic equations of this system. This can be done by differentiating the Lagrangian now with respect to θ_1 and θ_2 rather than $\dot\theta_1$ and $\dot\theta_2$ as $\dot p_{\theta_1} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = \frac{\partial}{\partial t}\left(\partial L \frac{\partial t}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1}$ [6] and from (2.4):

$$
\begin{aligned}
\dot p_{\theta_1} &= \frac{\partial L}{\partial \theta_1} = -\frac{1}{2}ml^2\left(3\frac{g}{l}\sin\theta_1 + \dot\theta_1\dot\theta_2\sin(\theta_1 - \theta_2)\right) \\
\dot p_{\theta_2} &= \frac{\partial L}{\partial \theta_2} = -\frac{1}{2}ml^2\left(3\frac{g}{l}\sin\theta_2 - \dot\theta_1\dot\theta_2\sin(\theta_1 - \theta_2)\right)
\end{aligned} \tag{2.8}
$$

Eq. (2.7) and Eq. (2.8) are the *characteristic equations* of the double pendulum system which define their motion[7] and can be integrated numerically over time using methods such as the *Euler method* or the *Runge Kutte method.*[8]

Numerical Integration Code in Python

```python
@jit
def doublePend(y, t, l, m, g):
    # Define the differential equations
    theta1, theta2, p1, p2 = y

    dtheta1 = (6 * (2 * p1 - 3 * p2 * math.cos(theta1 - theta2))) / (m * l**2 * (16 - 9 * (math.cos(theta1 - theta2)**2)))
    dtheta2 = (6 * (2 * p2 - 3 * p1 * math.cos(theta1 - theta2))) / (m * l**2 * (16 - 9 * (math.cos(theta1 - theta2)**2)))

    dp1 = - 0.5 * m * l**2 * (dtheta1 * dtheta2 * math.sin(theta1 - theta2) + 3 * (g / l) * math.sin(theta1))
    dp2 = - 0.5 * m * l**2 * (-dtheta1 * dtheta2 * math.sin(theta1 - theta2) + 3 * (g / l) * math.sin(theta1))

    dydx = [dtheta1, dtheta2, dp1, dp2]
    return dydx

@np.vectorize
def doublePendTestSegments(theta1, theta2):
    y0 = [theta1, theta2, 0.0, 0.0]

    for t in range(Tmax):
        sol = odeint(doublePend, y0, [0,1], args=(l, m, g), mxstep=5000000)

        prevTheta2 = y0[1]  % (2 * pi)
        Theta2 = sol[1][1] % (2 * pi)

        if prevTheta2 < pi and prevTheta2 > pi - 0.52 and Theta2 > pi:
            return t + 1

        if prevTheta2 > pi and prevTheta2 < pi + 0.52 and Theta2 < pi:
            return t + 1

        y0 = sol[1]
        nextTheta2 = y0[1]

    return Tmax + 1
```

ODE system which is solved numerically using the scipy.odeint package

```python
# Define mesh space for the colour map
x = np.linspace(-3, 3, gridSize)
y = np.linspace(-3, 3, gridSize)
X,Y = np.meshgrid(x,y)
Z = doublePendTestSegments(X, Y)

# Set up 2D plot using matplotlib
fig = plt.figure(figsize = (12,10))
fig.subplots_adjust(wspace=0.5)

# Set up latex usage and font to match final write up
plt.rc('text', usetex=True)
plt.rcParams['text.latex.preamble']=[r"\usepackage{amsmath}"]
plt.rc('font', family='cooper')

# Create colour map
plt.pcolormesh(X, Y, Z, cmap=plt.cm.get_cmap('plasma'), vmin = 0, vmax = 100)
cbar = plt.colorbar(fraction=0.046, pad=0.04)
cbar.set_label('Time to flip (s)', fontsize = 14)

# Label the axes and fix the aspect ratio to make the colour map a square
plt.xlabel(r'$\boldsymbol{\theta_{1}} $ (Radians)', fontsize = 14)
plt.ylabel(r'$\boldsymbol{\theta_{2}} $ (Radians)', fontsize = 14)
plt.axis([-3, 3, -3, 3])
plt.axes().set_aspect('equal')

# Save the colour map as a PNG with the parsed arguments in the name
name = "length" + str(l)  + "_mass" + str(m) + "_res" + str(gridSize) + ".png"
plt.savefig(name, bbox_inches='tight', dpi = 500)
```

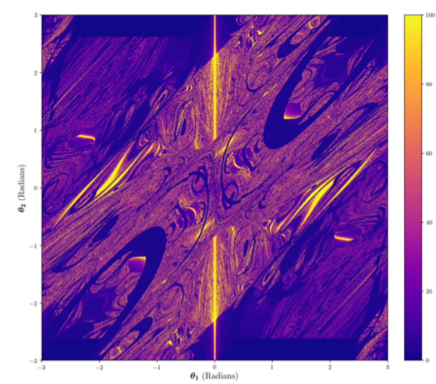
Colormap formatting using matplotlib in Python

Overall aims of the investigation:

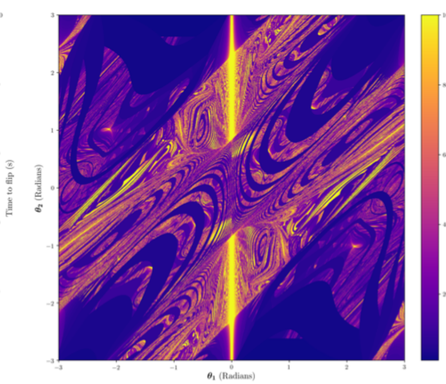
- Investigate the chaotic properties of the double pendulum system.
- Look at the “flip time” of the outer pendulum depending on different starting conditions.
- Generate colour maps of the flip time based on the starting angles of both pendulums.

NB All work here is original research carried out by the author

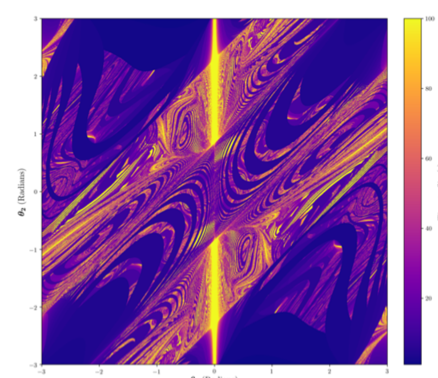
Colour Maps



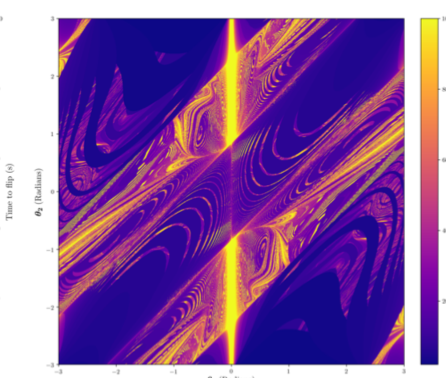
l = 3, m = 1



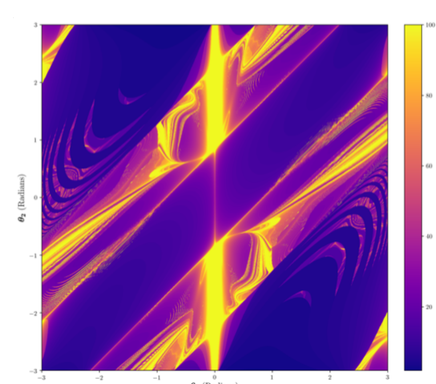
l = 30, m = 1



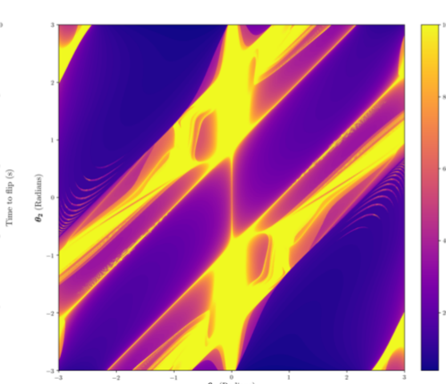
l = 50, m = 1



l = 100, m = 1



l = 300, m = 1



l = 1000, m = 1