Application of Derivatives Part One

by: Joshua Bautista

Condition	Event
s<0	Object to the left (below) of the origin
s=0	Object at the origin
s>0	Object to the right (above) of the origin
v<0	Moving to the left (downward)
v=0	At rest
v>0	Moving to the right (upward)
a<0	s-t graph concave down; slow down ; speed up Acceleration directed to the left (downward)
a=0	Constant velocity
a>0	s-t graph concave up; slow down ; speed up Acceleration directed to the right (upward)

Condition	Event
$s \cdot v < 0$	Object moving toward the origin
$s \cdot v > 0$	Object moving away from the origin
$s \cdot a < 0$	Acceleration is directed toward the origin
$s \cdot a > 0$	Acceleration is directed away from the origin
$v \cdot a < 0$	Object is slowing down
$v \cdot a > 0$	Object is speeding up

1. Application of Derivatives Part One

1.1 Sketching

It is important to sketch out the function in the question to reveal all of its characteristics (increasing/decreasing intervals, concave up/down, inflection points, etc). There is an algorithm to help sketch a graph given its function/equation:

1. From the original graph:
   - You must first **factor** to check if any **holes** are in the graph.
   - State **VA's** and **Domain**.
   - Find the **x and y intercepts**.
   - Find the **end behaviour**.
   - Look at the behaviour near **zeros** (x-intercepts) and **VA's**. (Remember - do this by looking at multiplicities of zeros)
   - *CAN BE SKIPPED* - Find **positive and negative intervals** between zeros and VA's.

2. From the first derivative:
   - Find **critical points**.
   - *CAN BE SKIPPED* - Find **increasing/decreasing intervals**.

3. From the second derivative:
   - Find possible **inflection points**.
   - Find **concave up/down intervals**.
   - Decide if the possible inflection points found are actual inflection points and classify the critical points using the 2nd derivative test.

Example

Sketch and label all intercepts, asymtotes, crticial points and inflection points. Show all justifying steps.

$$y = (x^{\frac{1}{3}})(x-4)$$

1. Cannot be factored any further. $\therefore$ **no holes, or VA's. Domain** $x \in \Re$

1. Find x and y intercepts:

x-intercept

$$0 = (x^{\frac{1}{3}})(x-4)$$

by property of zeros:

$\therefore (0,0)$ **and** $(4,0)$

y-intercept

$$y = ((0)^{\frac{1}{3}})((0)-4)$$
$$y = (0)(-4)$$
$$y = 0$$
$$\therefore (0,0)$$

1. Find end behaviour: The function is not rational/exponential. $\therefore$ no HA/OA.

2. Find critical points: Find y' (first derivative) and find when it is equal to 0 or DNE.

$$\begin{aligned}
y &= (x^{\frac{1}{3}})(x-4)\\
y' &= (x^{\frac{1}{3}})(1) + (x-4)(\tfrac{1}{3})(x^{\frac{-2}{3}})\\
&= (x^{\frac{-2}{3}})[x + \tfrac{1}{3}(x-4)]\\
&= (x^{\frac{-2}{3}})[x + \tfrac{(x-4)}{3}]\\
&= (x^{\frac{-2}{3}})(\tfrac{3x+x-4}{3})\\
&= \frac{4(x-1)}{3x^{\frac{2}{3}}}
\end{aligned}$$

$$0 = \frac{4(x-1)}{3x^{\frac{2}{3}}}$$
$$0 = 4(x-1)$$
$$\therefore \text{critical pt. } x = 1$$

DNE when denominator = 0

$$0 = 3x^{\frac{2}{3}}$$
$$\therefore \text{critical pt. } x = 0$$

3. Find possible inflection points: Find y'' and find when it is equal to 0 or DNE.

$$\begin{aligned}
y' &= x^{\frac{1}{3}} + \tfrac{1}{3}(x)^{\frac{-2}{3}}(x-4)\\
y'' &= \tfrac{1}{3}(x)^{\frac{-2}{3}} + [\tfrac{-2}{9}(x)^{\frac{-5}{3}}(x-4) + (\tfrac{1}{3}(x)^{\frac{-2}{3}})(1)]\\
&= \frac{1}{3x^{\frac{2}{3}}} + (\frac{-2(x-4)}{9x^{\frac{5}{3}}} + \frac{1}{3x^{\frac{2}{3}}})\\
&= \frac{3x}{9x^{\frac{5}{3}}} + (\frac{-2x+8+3x}{9x^{\frac{5}{3}}})\\
&= \frac{4(x+2)}{9x^{\frac{5}{3}}}
\end{aligned}$$

$$0 = \frac{4(x+2)}{9x^{\frac{5}{3}}}$$
$$0 = 4(x+2)$$
$$\therefore \text{possible inflection pt. } x = -2$$

DNE when denominator = 0

$$0 = 9x^{\frac{5}{3}}$$
$$\therefore \text{possible inflection pt. } x = 0$$

3. Find concave up/down intervals: Make a y'' sign chart.

	$x < -2$	$-2 < x < 0$	$x > 0$
$4(x+2)$	$-$	$+$	$+$
$9x^{\frac{5}{3}}$	$-$	$-$	$+$
y''	$+$	$-$	$+$
y	CU	CD	CU

When looking at a sign chart like this, you should look at where the function goes from **concave up** to **concave down**. Remember, a **positive interval** in the second derivative means it is concave up. A **negative interval** means it is concave down. There is a CU/CD switch at $x = -2$ and $x = 0$. These are your **inflection points.**

3. Classify your critical points using the 2nd derivative test.

$\underline{x = -2}$

$y''(-2) = 0$

$\therefore (-2, 7.56)$ **inflection pt**.

$\underline{x = 0}$

$y''(0) =$ DNE, inflection pt.

$\therefore (0,0)$ is a **V.T**

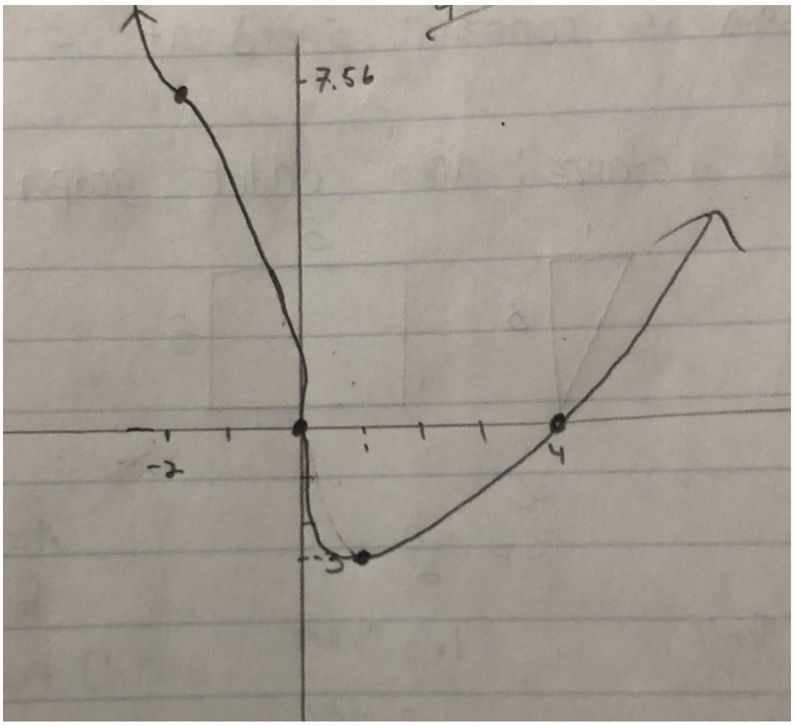
$\underline{x = 1}$

$y''(1) = \frac{4}{3}$

CU and $y'(1) = 0$

$\therefore (1, -3)$ **local min T.P**

Keep track of all of the classified points (intercepts, critical points, etc) and sketch!



1.2 Velocity and Acceleration

Before we start solving problems involving velocity and acceleration, we must make sure we know these key definitions:

Displacement

Displacement is the **change in position** of an object. It is concerned with the initial position of an object to its final position.

$$Displacement = s(t)$$

Velocity

Velocity is concerned with how fast or slow an objects moves as time changes. It is the derivative of $s(t)$.

$$Velocity = v(t) = \frac{ds}{dt} = \frac{\Delta s}{\Delta t}$$

Acceleration

Acceleration describes how an object is speeding up or speeding down as time changes. It is the derivative of $v(t)$. (also 2nd derivative of $s(t)$)

negative acceleration $\rightarrow$ decreasing velocity.
position acceleration $\rightarrow$ increasing velocity.

$$Acceleration = a(t) = \frac{d^2v}{dt^2}$$

Jerk/Turbulence

Jerk/Turbulence is the derivative of $a(t)$.

$$Jerk/Turbulence = j(t) = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The relationship between displacement, velocity, and acceleration:

$$s''(t) = v'(t) = a(t)$$

Graphs

When looking at displacement, velocity, acceleration, and jerk graphs, there are important things to note.

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s<0	Object to the left (below) of the origin
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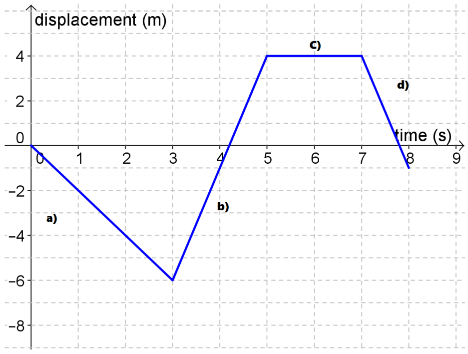
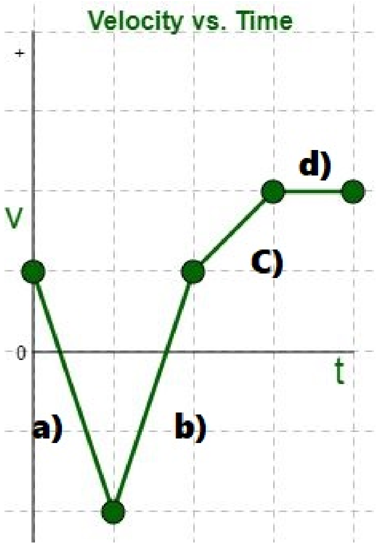


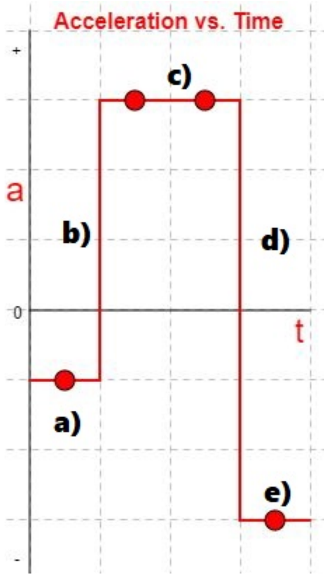
When looking at a displacement-time graph, it is important to note that the y-values represent the displacement from a **reference point**. The reference point is when $time = 0$. Positive displacement means the object is north or right of the reference point.

To look at velocity, you must take the **slope/derivative** ($\frac{m}{s}$).



When looking at velocity-time graphs, the y-values strictly represent the object's speed. A 0 does **not** mean the object is at the origin. A positive velocity means the object is moving north, and vice-versa. Whenever the graph crosses the x-axis, this means it has changed directions.

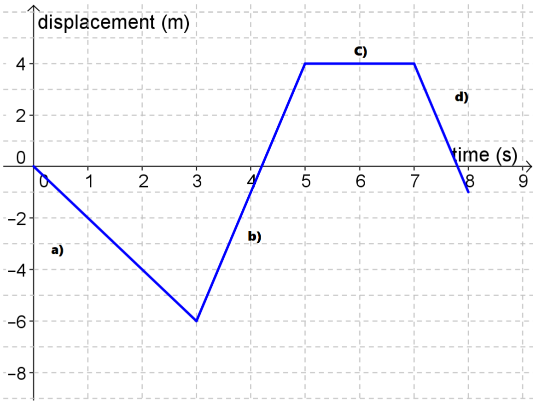
To look at acceleration, simply take the **slope/derivative** ($\frac{m}{s^2}$).



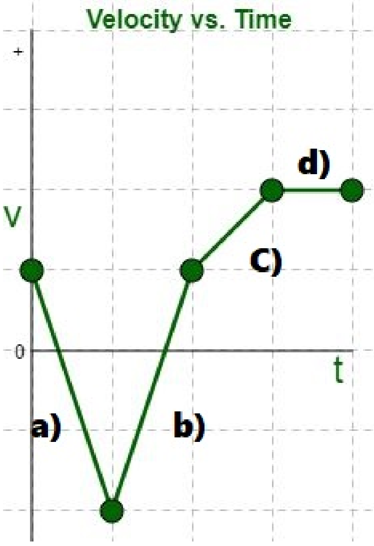
When looking at acceleration-time graphs, it is key to remember that it is the 'rate of change of the rate of change of displacement'. So you can visualize it as how is the velocity changing as time changes.

Jerk/Turbulence is **slope/derivative** of acceleration.

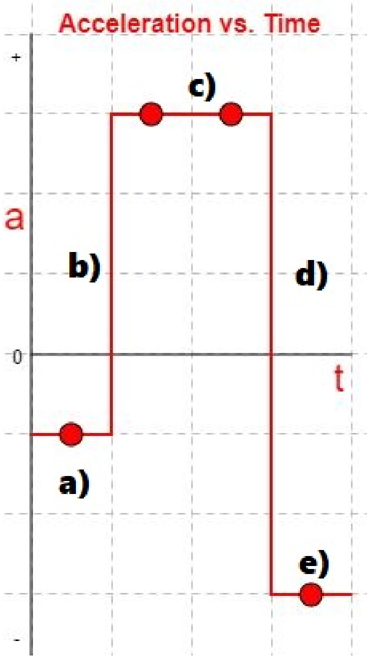
Examples Given Graph: 'What is happening?'



(a) The object is moving **south/left** (away) from the origin (it is becoming negative).
(b) The object is moving back towards the origin (reference point). It goes back to the origin (it crosses the x-axis). It then moves **north/right** from the origin (it is becoming positive).
(c) The object stays **still/not moving** (constant at +4m).
(d) The object moves back towards the origin and is at the origin again (crosses x-axis).

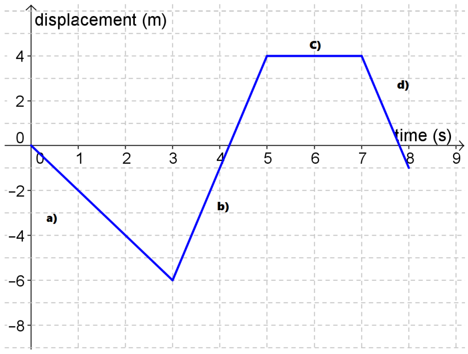


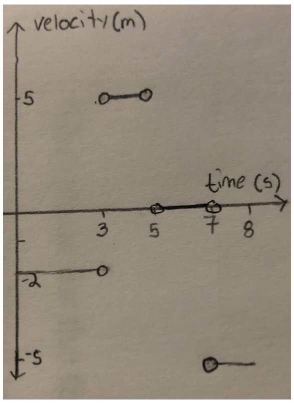
(a) The object is slowing down going right until it switches direction (crosses x-axis). After switching, the object speeds up going left.
(b) The object starts slowing down going left, until it switches direction again. After switching, it speeds up going right.
(c) The object speeds up going right.
(d) The object stays at a constant speed (zero slope) going right.



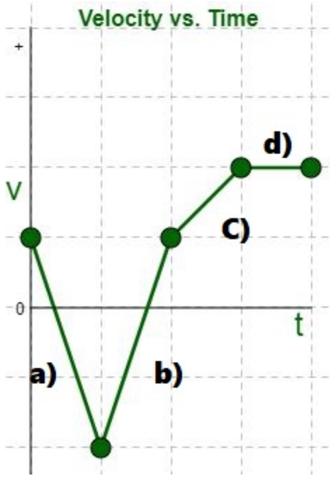
(a) The object's velocity is constantly decreasing.
(b) The object's velocity **instantly** goes from decreasing to increasing by a large number constantly.
(c) The object's velocity is increasing at a constant rate.
(d) The object's velocity **instantly** goes from increasing to decreasing by a large number constantly.
(e) The object's velocity is decreasing at a constant rate.

Sketching the Derivative/Anti-Derivative Graphs

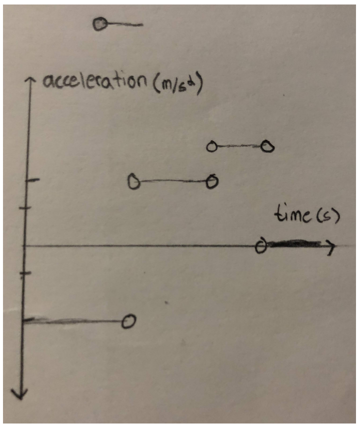




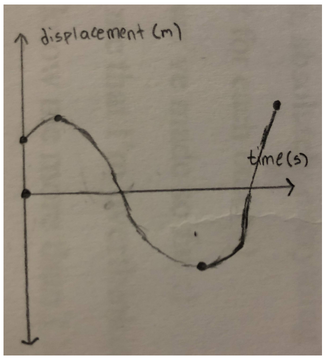
(a) Constant velocity of $-2\frac{m}{s}$.Object going left.
(b) Constant velocity of $5\frac{m}{s}$.Object going right.
(c) Constant velocity of $0\frac{m}{s}$.Object staying still.
(d) Constant velocity of $-5\frac{m}{s}$.Object going left.



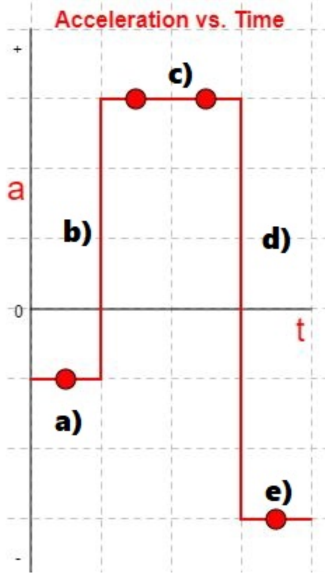
(zeros become t.p and t.p become inflection pt)

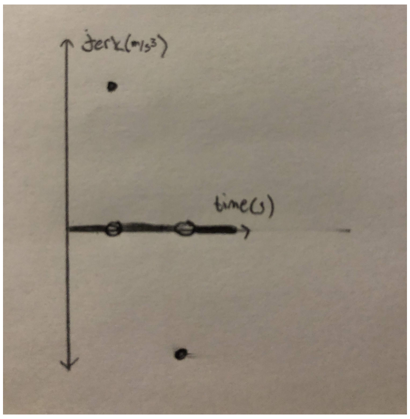


(a) Constant negative acceleration directed to the left (decreasing velocity).
(b) Constant positive acceleration directed to the right (increasing velocity).
(c) Larger constant positive acceleration directed to the right.
(d) 0 acceleration (constant velocity).

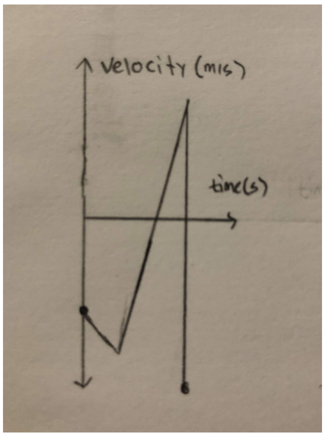


(a) NOTE: We do not know where the object starts. The object first moves away (right) from the origin, then switches direction and moves toward (left) the origin speeding up.
(b) The object slows down, still moving away from the origin, until it switches direction once again.
(c) Object speeds up while moving toward (right) the origin.
(d) Object is moving at a constant rate away (right) from the origin.





The derivative or 'rate of change' of acceleration in this case are all constant 0 (no slope in original acceleration graph), except the points where the acceleration instantly increases/decreases.



(a) Object is speeding up at a constant acceleration going away (left) from the origin (negative acceleration).
(b) Object speeds up instantly, now at a positive acceleration.
(c) Object is speeding up at a constant acceleration going toward the origin (positive acceleration). It passes origin (in this case) and continues will constant velocity away (right) from the origin.
(d) Objects slows down instantly, now at a negative acceleration.
(e) Object is speeding up at a constant acceleration going away (left) from the origin (negative acceleration).

1.3 Other Applications

If a ball is thrown into the air with a velocity of 10 m/s, its height (in meters) after t seconds is given by $y = 10t - 4.9t^2$.

a) Find the velocity after 2 seconds.
b) When will the ball reach its highest point?
c) When will the ball be back on the ground?

a)
We are given a function that gives the height of the ball after t seconds. To get velocity, we must get the **slope/derivative** at 2 seconds. Let's get the derivative of the given function:

$$y = f(t) = -4.9t^2 + 10t$$
$$f'(t) = -9.8t + 10$$

Plug-in 2 seconds into the derivative.

$$\begin{aligned} f'(2) &= -9.8(2) + 10 \\ &= -19.6 + 10 \\ &= -9.6 \end{aligned}$$
$$\therefore \text{velocity } -9.6\tfrac{m}{s} \text{ at 2 seconds.}$$

b)
At its highest point, the derivative will be equal to 0. We can set the derivative function equal to 0 and find at what second it is 0.

$$\begin{aligned} -9.8t + 10 &= 0 \\ -9.8t &= -10 \\ t &= \tfrac{10}{9.8} \\ t &= 1.020408163 \end{aligned}$$
$$\therefore \text{at 1.02 seconds the ball is at highest point}$$

c)
When the ball is on the ground, its height will be equal to 0. We can set the original function equal to 0 and solve for t.

$$-4.9t^2 + 10t = 0$$
$$t = \frac{-10 \pm \sqrt{10^2 - 4(-4.9)(0)}}{2(-4.9)}$$
$$t = \frac{-10 \pm \sqrt{100}}{-9.8}$$

$$t = \frac{-20}{-9.8} \qquad t = \frac{0}{-9.8}$$
$$t = 2.04081 \qquad t = 0$$

$\therefore$ The ball will be back on the ground at 2.04 seconds.

A company's productivity, P in terms of items completed, is a function of the number of people, n, working on a job. This is: P = f (n).

(a) What are the units of f ' (n)?

(b) In practical terms, what does dp/dn mean in this case?

(c) What can you say about the sign of dp/dn?

(d) Given that dp/dn = 300 when n = 6, what can you say about the effect of increasing the number of people from 6 to 7?

a)

The derivative is also the slope, the units you get when calculating the slope is $\frac{P}{n}$.

$$\frac{\Delta P}{\Delta n} = \frac{itemscompleted}{person}$$

b)

$\frac{\Delta P}{\Delta n}$ refers to the number of items completed as the number of people working on the job changes.

c)

Using logic, we can say that more items should be completed as the number of people working on the job increase. $\therefore \frac{\Delta P}{\Delta n} > 0$

d)

Given $\frac{\Delta P}{\Delta n} = 300$ when $n = 6$, we can say that adding one more person working on the job when there are already 6 people working, will increase the number of items completed by 300.