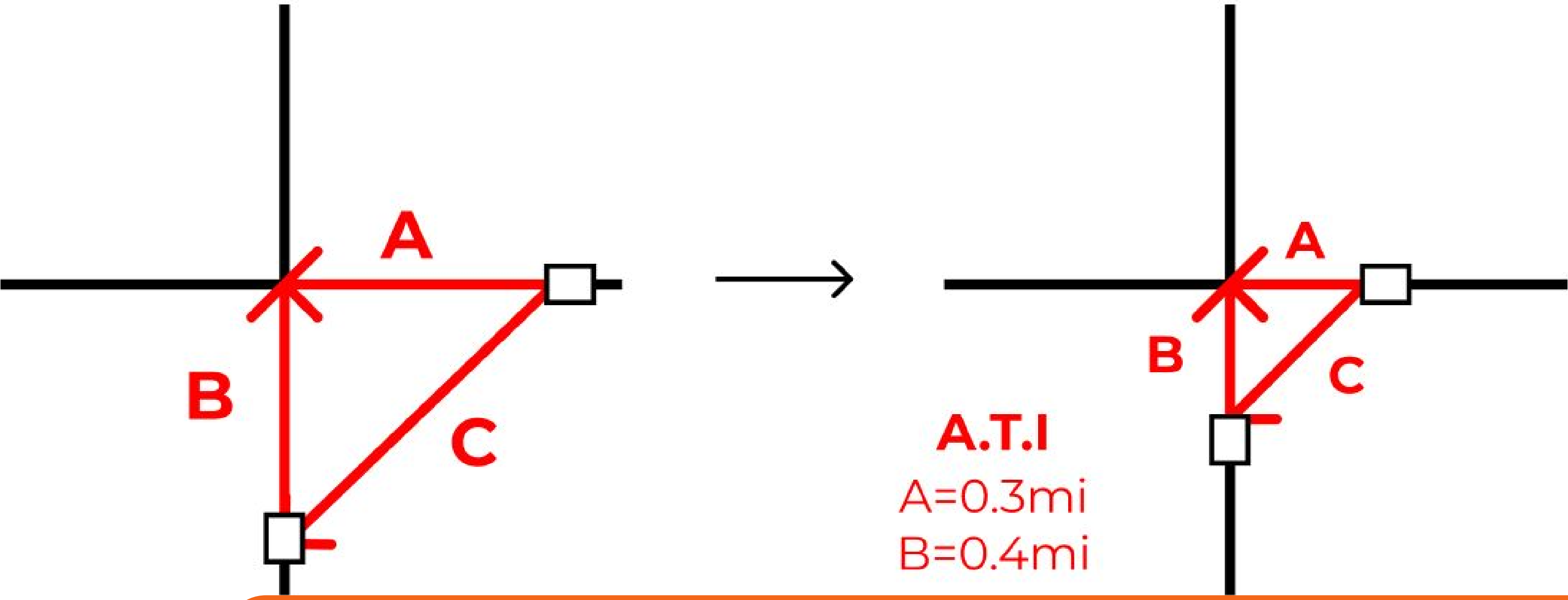
Application of Derivatives Part Two by: Joshua Bautista



1. Application of Derivatives Part Two

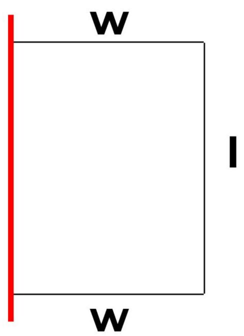
1.1 Optimization

For optimization problems, you must clearly identify what the question is asking for. The wording of the question is the biggest thing to look out for. Read the problem carefully, then identify the quantity that is to be optimized and the constraints.

Example 1: Area

Sarah has a new puppy and she wants to maximize her outdoor time, so she builds her a fenced-in play area. She has 40 feet of fencing, and she wants to fence off a rectangular area next to her house. The house will be one side of the play area, so that side needs no fencing. In order for the puppy to have adequate space, the area needs to be at least 5 feet long and 5 feet wide. What is the largest area she could have? Is there a smallest area she could have if she wants to use all 40 feet of fencing?

1. **Find out what you are trying to optimize.**
   In this example, we are trying to maximize and minimize the area enclosed.
2. **Draw a sketch.**



3. **Write what you are trying to optimize as a function of the unknown variables idenitied in the picture.**

$$A = lw$$

We cannot take the derivative of this function and find the max/min as we must first get the function in terms of only one variable. Let's go back to the question and see what constraints are given. We are told that she has 40 feet of fencing, let's make an equation out of that.

$$2w + l = 40$$
$$w = \frac{40-l}{2}$$

We have isolated w, now we can plug it back into the original function we made to make it in terms of only one variable.

$$A = lw = (l)(\frac{40-l}{2})$$

4. **Find the critical points.**

$$A(l) = 10l - \frac{1}{2}l^2$$
$$A'(l) = 10 - l$$
$$20 - l = 0$$
$$l = 20$$

We must make sure that this critical point is not an extraneous solution. We need to make sure this value for l lies in the domain of l. We are told the area must be at least 5 feet long and 5 feet wide. Plugging in $w = 5$ in $2w + l = 40$ gives us $l = 30$

$$\text{domain of } l : [5, 30]$$

$l = 20$ is within the domain. Now, we must see if this is a local max, local min, or neither. We can check max with the 2nd derivative test.

5. **2nd derivative test.**

$$A''(l) = -1$$
$$A''(20) = -1$$

Since the function is concave down (negative 2nd derivative) at any point l, $l = 20$ must be a local maximum.

6. **Find minimum using EVT (plug-in endpoints of domain).**

At $l = 5$:
$$A(5) = 20(5) - \frac{1}{2}(5)^2$$
$$= 87.5ft^2$$

At $l = 30$: $A(30) = 20(30) - \frac{1}{2}(30)^2$
$$= 600 - \frac{900}{2}$$
$$= 150ft^2$$

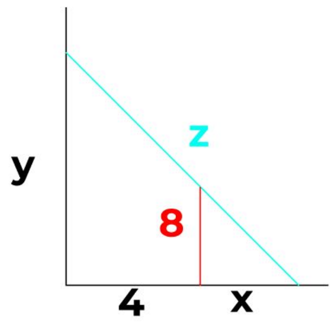
$$2w + 5 = 40$$
$$2w = 35$$
$$w = 17.5$$

$$2w + 20 = 40$$
$$2w = 20$$
$$w = 10$$

$\therefore$ Maximum enclosed area at $l = 20$ feet long and $w = 10$ feet wide. Minimum enclosed area at $l = 5$ feet long and $w = 17.5$ feet wide.

Example 2: Distance

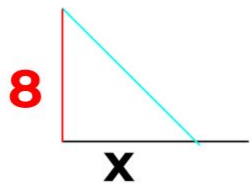
A fence 8ft tall runs parellel to a tall building at a distance of 4ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



Once we draw/sketch the problem out, we must create an equation that will relate all of the variables. Note that we are minimizing z (shortest ladder). We have a right angled triangle, so we can use the **pythagorean theorem**.

$$z^2 = y^2 + (4+x)^2$$

This equation does not help as of right now as it contains two variables, we need it in terms of one variable. Let's use what else we are given - the smaller triangle.



These two triangles are **similar**, so we can use proportions to help us solve for y in terms of x.

$$\frac{y}{8} = \frac{4+x}{x}$$
$$yx = 8(4+x)$$
$$y = [\frac{8(4+x)}{x}]$$

Now, we can use this equation in the original equation we found at the start to get it in terms of one variable. It is very important to note that if **z is minimized, z^2 is also minimized**. This is important because instead of isolating for z^2 and getting a square root on the other side (which will be annoying to take the derivative of), we can simply take the derivative of z^2 and find the critical points.

$$z^2 = (\frac{8(4+x)}{x})^2 + (4+x)^2$$
$$(z^2)' = 2(\frac{8(4+x)}{x})(\frac{-32}{x^2}) + 2(4+x)$$
$$0 = 2(\frac{8(4+x)}{x})(\frac{-32}{x^2}) + 2(4+x)$$
$$-2(4+x) = 2(\frac{8(4+x)}{x})(\frac{-32}{x^2})$$
$$2 = 2(\frac{8(4+x)}{x})(\frac{32}{x^2})$$
$$4+x = (\frac{8(4+x)}{x})(\frac{32}{x^2})$$
$$1 = (\frac{8}{x})(\frac{32}{x^2})$$
$$1 = (\frac{256}{x^3})$$
$$x = \sqrt[3]{256}$$

First we need to make sure this x-value falls within the domain: x **cannot** be negative and has to be at least smaller than z (we confirm this up ahead). Next, we need to make sure that this critical point is actually the minimum value. Let's do the 1st derivative test - plug in two numbers into the derivative: one lower (4) and one higher (8) than $\sqrt[3]{256}$.

$$\begin{gathered}\underline{x=4}\\ =2(\tfrac{8(6)}{4})(\tfrac{-32}{16})+2(8)\\ =\text{negative}\end{gathered} \qquad \begin{gathered}\underline{x=8}\\ =2(\tfrac{8(12)}{8})(\tfrac{-32}{64})+2(12)\\ =\text{positive}\end{gathered}$$

By drawing what we found from the 1st derivative test, we can assume that at $x=\sqrt[3]{256}$, it is a minimum. Now let's find out the y value.

$$\begin{gathered}z^2=(\tfrac{8((\sqrt[3]{256})+4)}{\sqrt[3]{256}})^2+((\sqrt[3]{256})+4)^2\\ z^2=277.15\\ z=\sqrt{277.15}\\ z=16.65\end{gathered}$$

$\therefore$ The length of the shortest ladder that will reach will be 16.65ft long.

Example 3: Revenue

The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

Revenue = units sold x price per unit

$$R(x)=x\cdot p(x)$$

We will need to find an expression for $p(x)$. Note that for every \$10 increase in price, there will be a 1 unit decrease. This is a contant/linear relationship: -10 slope.

$$p(x)=-10x+b$$

We are also given that the price of the units will be \$800 when all 100 units are being used.

$$p(100)=800$$

Bringing these two equations together we get:

$$\begin{gathered}800=-10(100)+b\\ 800=-1000+b\\ b=1800\\ \therefore p(x)=-10x+1800\end{gathered}$$

We can then use $p(x)$ in our original equation:

$$\begin{gathered}R(x)=x\cdot p(x)\\ R(x)=x\cdot(-10x+1800)\\ R(x)=-10x^2+1800x\\ R'(x)=-20x+1800\end{gathered}$$

We can, as always, find the critical points by finding the x-values that make this equal to zero.

$$0 = -20x + 1800$$
$$-1800 = -20x$$
$$x = 90$$

To verify that this is the maximum, we need to plug in the endpoints of the domain as well (EVT): $0 \leq x \leq 100$.

$$R(0) = -10(0)^2 + 1800(0)$$
$$R(0) = \$0$$

$$R(90) = -10(90)^2 + 1800(90)$$
$$R(90) = -81000 + 162000$$
$$R(90) = \$81000$$

$$R(100) =$$
$$-10(100)^2 + 1800(100)$$
$$R(100) = -100000 + 180000$$
$$R(100) = \$80000$$

The maximum revenue will be \$81000 when 90 units are sold, so to find price per unit we need to divide revenue by number of units.

Revenue = units sold · price per unit
Price per unit = revenue/units sold
$$= \frac{81000}{90}$$
$$= 900$$

$\therefore$ The manager should charge a **\$900** rent per month to maximize revenue to a value of \$81000.

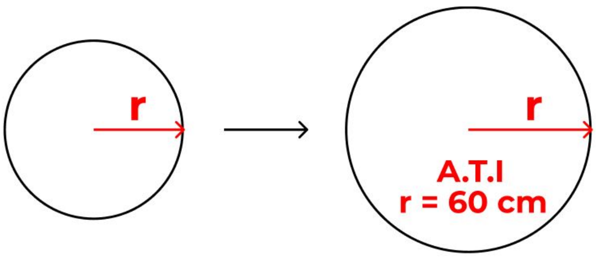
1.2 Related Rates

We will be putting our differentiation skills to the test in this related rates section. These are real applications of derivatives - exactly what we are learning!

Example 1: Area of Circle

When a circular metal plate is heated in an over, its radius increases at the rate of 0.02 cm/min. At what rate is the plate's area increasing when the radius is 60cm?

As always, let's first draw/sketch it out so we can have an easier time understanding the problem and what it is asking for: what rate the plate's area is increasing **at the instant** (A.T.I) the radius is 60cm.



Notice in the question it says that the radius **increases at a rate of 0.02 cm/min**. This basically gives us the derivative; whenever you see 'rate', it refers to the derivative. We are also given the radius.

<u>given:</u>

$\frac{dr}{dt} = 0.02\frac{cm}{min}$ $r = 60cm$

The question asks us **what rate is the plate's area increasing** when the radius is 60cm? Again, it is asking for the 'rate', so they are asking for the derivative:

want:
$\frac{dA}{dt} = ?$
A.T.I when r = 60 cm

Next, we need to find an equation that relates the area and the radius. Since we are working with circles, we already know this formula.

$$A = \pi r^2$$

Since we want $\frac{dA}{dt}$, we can simply take the derivative of this formula with respect to t and after, plug in what we are given. **NOTE:** in the last line, remember that we are taking the derivative with respect to t, so we cannot simply use the power rule.

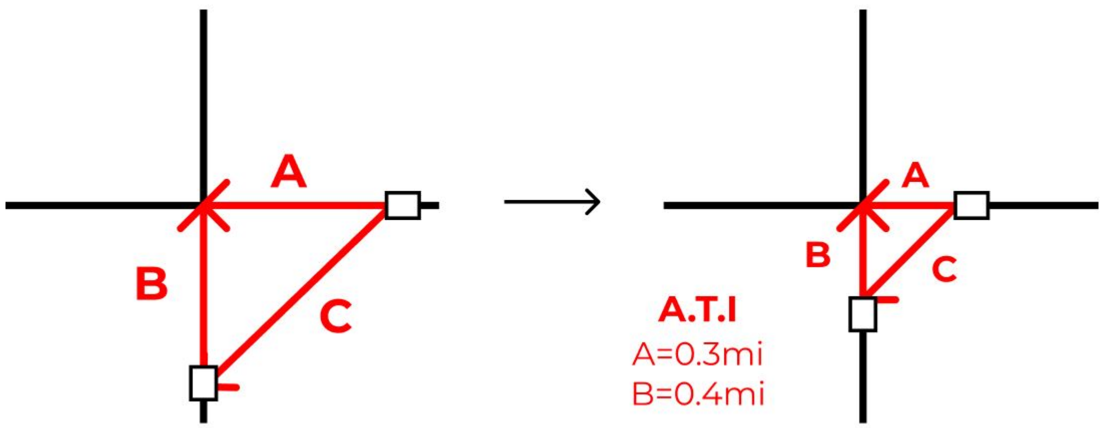
$$\frac{d}{dt}(A = \pi r^2)$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi(60cm)(0.02\frac{cm}{min})$$
$$\frac{dA}{dt} = 2.4\pi \frac{cm^2}{min}$$

$\therefore$ At the instant when the radius is 60cm, the rate of which the plate's area is increasing by is $2.4\pi \frac{cm^2}{min}$.

Example 2: Pythagorean

Car A is travelling west at 50 mi/h and Car B is travelling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when Car A is 0.3 mi and Car B is 0.4 mi from the intersection?

Let's start by drawing a diagram to have a better understanding of what's happening. We will indicate the parts that are changing with red arrows and label all the information given at the instant (A.T.I) the question is asking for.



Reading the question again it tells us the rate of which the cars are travelling with respect to time. **NOTE:** the length of A and B are actually decreasing, so the derivative should be negative as the distance is getting shorter - easy to see from the diagram.

given:

$\frac{dA}{dt} = -50\frac{mi}{h}$ $\frac{dB}{dt} = -60\frac{mi}{h}$ $A = 0.3mi$ $B = 0.4mi$

The question also asks, 'at what rate are the cars approaching each other?':

want:
$\frac{dc}{dt} =?$

Now, we need an equation that relates A, B, and C. Since we are working with triangles, it should be easy to notice that we can use pythagorean's theorem. Everything that we are given and need ($\frac{dA}{dt}$, $\frac{dB}{dt}$, etc) are the derivatives of these variables with respect to time. So let's take the derivative of both sides.

$$A^2 + B^2 = C^2$$
$$\frac{d}{dt}(A^2 + B^2 = C^2)$$
$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

To reiterate, it is very important to double-check if the rates are positive or negative. If Car A was going in the opposite direction, the rate would be positive as the length/distance of A would be increasing. In this case, both Car A and Car B are going towards the intersection; the length/distance of A and B are getting shorter, hence why $\frac{dA}{dt}$ and $\frac{dB}{dt}$ are negative. Let's plug in what we know now.

$$2(0.3mi)(-50\frac{mi}{h}) + 2(0.4\frac{mi}{h})(-60mi) = 2C\frac{dC}{dt}$$

You should notice that we do not have the value of C yet. Here we can use the equation we got from pythagorean's theorem.

$$A^2 + B^2 = C^2$$
$$(0.3mi)^2 + (0.4mi)^2 = C^2$$
$$0.25mi^2 = C^2$$
$$C = 0.5mi$$

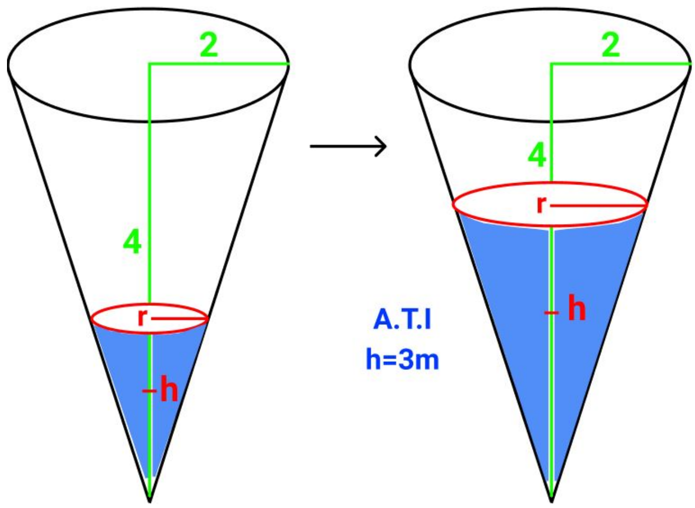
We can continue where we left off and plug in this C value.

$$2(0.3mi)(-50\frac{mi}{h}) + 2(0.4mi)(-60\frac{mi}{h}) = 2(0.5mi)\frac{dC}{dt}$$
$$(0.6mi)(-50\frac{mi}{h}) + (0.8mi)(-60\frac{mi}{h}) = (1mi)\frac{dC}{dt}$$
$$-78\frac{mi^2}{h} = (mi)\frac{dC}{dt}$$
$$-78\frac{mi}{h} = \frac{dC}{dt}$$

$\therefore$ The cars the approaching each other at a rate of 78 $\frac{mi}{h}$.

Example 3: Volume of Cone

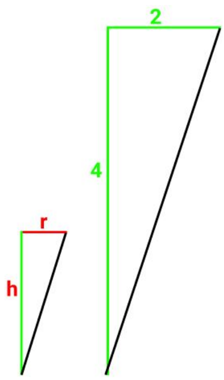
A water tank has the shape of an inverted (meaning up-side down) circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is 3m deep.



In the diagram, whatever is changing as time passes by is indicated in red (the height - h and radius - r). We are given a rate: 'water is being pumped into the tank at a rate of 2 $\frac{m^3}{min}$. This is our $\frac{dV}{dt}$ as it is the rate of which the volume is changing as time passes by. We want to find 'the rate at which the water level is rising', which is $\frac{dh}{dt}$; the rate at which the height is changing as time passes by.

given:	want:
$\frac{dV}{dt} = 2\frac{m^3}{min}$	$\frac{dA}{dt} = ?$
	A.T.I when h = 3m

Like the previous examples on related rates, we need to find an equation that can relate all of the variables together (V, r, h). We can use the volume of a cone formula: $V = \frac{1}{3}\pi r^2 h$. Like before, we need this in terms of only one variable. In the diagram there are two *similar* triangles that can be used to set up a **proportion** - this is why it is important to draw a diagram.



$$\frac{2}{4} = \frac{r}{h}$$
$$\frac{1}{2} = \frac{r}{h}$$
$$r = \frac{1}{2}h$$

We can use this in the formula to get it in terms of one variable. After that, we can take the derivative of both sides, plug in what we are given, and solve for $\frac{dh}{dt}$.

$$V_{cone} = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi(\frac{1}{2}h)^2 h$$
$$V = \frac{1}{3}\pi\frac{1}{4}h^2 h$$
$$\frac{d}{dt}(V = \frac{1}{12}\pi h^3)$$
$$\frac{dV}{dt} = \frac{3}{12}\pi h^2 \frac{dh}{dt}$$
$$2 = \frac{1}{4}\pi(3)^2 \frac{dh}{dt}$$
$$2 = \frac{9\pi}{4}\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{8}{9\pi}\frac{m}{min}$$

$\therefore$ The water level is rising at a rate of $\frac{8}{9\pi}\frac{m}{min}$ when the water is 3m deep.

1.3 L'Hopital's Rule (AP)

L'Hospital's Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). If you get the indeterminate forms of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ if the limit on the right side exists or even if it approaches ∞ or $-\infty$

L'Hopital's rule provides us a technique to **calculate the limit of indeterminate forms.** ($\frac{\infty}{\infty}$, $\frac{0}{0}$) When you have an indeterminate form, you can simply take the limit of the derivative of the numerator divided by the derivative of the denominator. We can use L'Hopital's rule on the following forms:

$$\frac{\infty}{\infty} \qquad\qquad \frac{0}{0}$$

If you have other indeterminate forms, you can manipulate it to a form where you *can* apply L'Hopital's rule.

Example 1: Indeterminate Quotients

$$\lim_{x\to\infty}\frac{x}{\ln(1+2e^x)}$$

If we substitute ∞ for x, we get an indeterminate form of $\frac{\infty}{\infty}$. Now that we know L'Hopital's rule, we can simply divide the derivative of the numerator by the derivative of the denominator to solve this. Notice we label L'H above the equal sign to signify we are using L'Hopital's rule.

$$\overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{1}{\frac{1}{(1+2e^x)}(2e^x)}$$

$$= \lim_{x\to\infty} \frac{1+2e^x}{2e^x} = \frac{\infty}{\infty}$$

After applying L'Hopital's rule,it still turns out to be an indeterminate form of $\frac{\infty}{\infty}$, but we can apply L'Hopital's rule **as many times as we want**, as long as it is applicable.

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2e^x}{2e^x} = 1$$

Example 2: Indeterminate Products

The indeterminate form $0 \cdot \pm\infty$ is a form that we can manipulate into an indeterminate quotient/fraction, and then apply L'Hopital's rule.

$$\lim_{x\to 0^+} \sqrt{x} \cdot \ln x$$

Like always, we try to substitute the x-value we are approaching into the equation first. This time, it actually gives us an indeterminate form of $(0 \cdot -\infty)$. In order to manipulate this into an indeterminate quotient, we'll need to bring one of the terms into the denominator. We need to make sure we are not changing the overall expression. Note that the following step is equivalent to the one before (negative power in denom.) We can then apply L'Hopital's rule; we have an indeterminate form of $\frac{0}{0}$.

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{\frac{-1}{2}}}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2}x^{\frac{-3}{2}}} = -2 \lim_{x\to 0^+} \frac{x^{\frac{3}{2}}}{x} = \lim_{x\to 0^+} -2\sqrt{x} = -2\sqrt{0} = 0$$

Example 3: Indeterminate Differences

The indeterminate form $\pm\infty \pm \infty$ is an indeterminate difference form. With these forms, we will try to manipulate it to one of the previous forms to solve it.

$$\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1}$$

Plugging in $x = 1$ will give us $(\infty - \infty)$. Since we have two fractions, we want to manipulate it so it is one (Indeterminate Quotient). Our first step will be to get common denominators, then apply L'Hopital's rule wherever applicable. **NOTE:** always try to simplify after applying L'Hopital's rule to make it cleaner.

$$= \lim_{x\to 1} \frac{x-1-\ln x}{(\ln x)(x-1)}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} \cdot \frac{x}{x}$$

$$= \lim_{x\to 1} \frac{x-1}{x-1+x\ln x}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 1} \frac{1}{1+1\cdot \ln x + x\frac{1}{x}}$$

$$= \frac{1}{1+0+1\cdot\frac{1}{1}} = \frac{1}{2}$$

Example 4: Indeterminate Powers

∞^0, 0^0, and 1^∞ are all examples of an indeterminate power. With the case of indeterminate powers, we will be carefully manipulating the expression with $e^{\ln x}$. Note that this **cancels out to x / is equal to x**.

$$\lim_{x\to\infty} x^{\frac{1}{x}}$$

$$= \lim_{x\to\infty} (e^{\ln x})^{\frac{1}{x}}$$

$$= \lim_{x\to\infty} e^{\frac{\ln x}{x}}$$

Since e^x is continuous, we can bring the limit in.

$$= e^{\lim_{x\to\infty} \frac{\ln x}{x}}$$

Now we apply L'Hopital's rule since it is applicable.

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{(\frac{1}{x})}{1}}$$

The limit in the power is 0: $\lim_{x\to\infty}(x^{\frac{1}{x}}) = 1$

$$= e^0 = 1$$

1.4 Approximations (AP)

Linearization

Draw any curve and a tangent at one of the points.

The **linear approximation** or **tangent line approximation** of $f(x)$ at the point $x = a$ is

$f(x) \approx f(a) + f'(a)(x-a)$. $\longrightarrow y \approx y_1 + m(x - x_1)$

curve ~ line near x = a

The linear function whose graph is this tangent line, i.e.,

$L = f(a) + f'(a)(x-a)$

is called the **linearization** of $f(x)$ at the point $x = a$.

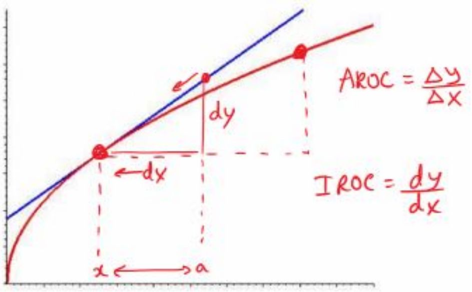
Linear approximation is a process that is used to find a good estimate about the value of a function we **don't know**, by using information about a value of a function we **already know**. Basically, you find the tangent line of a point you **do know** that is close to the point you are trying to estimate - by using the point-slope formula and finding the slope/derivative at that point. Then, you plug in the x-value you are trying to find in the tangent line and solve for its y-value. Since the tangent line should be relatively close to the point you are trying to estimate, the y-value should be relatively close as well.

Differentials

Instead of linearization one can use differentials to find approximations. Recall that $\boxed{f'(x) = \frac{dy}{dx}}$

We call dx and dy **differentials,** and they are related through $\boxed{dy = f'(x)dx}$.

For differentials, we utilize the fact that $f'(x) = \frac{dy}{dx}$. We isolate for dy and get $dy = f'(x)dx$. This is saying that the change in y is equivalent to the slope/derivative at x, times the change in x.



This is helpful as when trying to estimate a number such as $\sqrt{16.4}$, we can 'split' it up into $\sqrt{16} + \sqrt{0.4}$. Then we can say that the change in y (from $\sqrt{16}$ to $\sqrt{16.4}$) is the derivative at 16 $\cdot$ the change in x - which is 0.4, and add Δy to $\sqrt{16}$. Note this is an **approximation**.