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A First Course in Calculus

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Notes for Self Study

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Powers

Theorem 4.1. *Let n be an integer ≥ 1 and let $f(x) = x^n$. Then*

$$\frac{df}{dx} = nx^{n-1}.$$

Remarks on the proof. When we have some number $(x+h)^n$, writing each factor yields

$$(x+h)(x+h)\cdots(x+h).$$

If we were to distribute we would get many terms that we do not need to think about. We are able to select which terms from each factor we wish to distribute to find a particular number. There exists n number of x and we multiply them by each other, giving us x^n.

If we choose x from all but one factor, then the remaining factor has h and we get hx^{n-1}. But we do this for each factor. The idea is that it is not the h from one particular factor, but it could be the h from any factor. Since the terms are added when we distribute $(x+h)^n$, then we add the n instances of hx, and get nhx^{n-1}.

Now we have the term x^n and the only term nhx^{n-1} having a factor of h^1. We conclude that every other term must choose h from at least two factors. Hence we have

$$(x+h)^n = x^n + nhx^{n-1} + h^2 g(x,h),$$

where $g(x,h)$ is some expression involving powers of x and h with numerical coefficients. Of course h^2 is factored from the expression.

The rest of the proof follows very naturally using the Newton quotient.

Sums, Products, and Quotients

Definition. A function is said to be **continuous at a point** x if and only if

$$\lim_{h \to 0} f(x+h) = f(x).$$

A function is said to be **continuous** if it is continuous at every point of its domain of definition.

Let f be a function having a derivative $f'(x)$ at x. Then f is continuous at x.

Remarks on the proof. We note that if a function $f(x)$ is continuous at x, then it is continuous at every point of its domain of definition. The proposition statement states that f has a derivative $f'(x)$ at x, this is equivalent to saying that f is differentiable. So what we wish to prove is:

Let f be a function that is differentiable. Then f is continuous.

We set the Newton quotient of f equal to itself then multiply by h and get

$$h\frac{f(x+h) - f(x)}{h} = f(x+h) - f(x).$$

As h approaches 0, the left term approaches $0f'$. Thus we have

$$\lim_{h \to 0} f(x+h) - f(x) = 0f'(x) = 0.$$

This is another way of stating that

$$\lim_{h \to 0} f(x+h) = f(x).$$

By definition, f is continuous.

The Chain Rule

Chain rule. *Let f and g be two functions having derivatives, and such that f is defined at all numbers which are values of g. Then the composite function $f \circ g$ has a derivative, given by the formula*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$