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A First Course in Calculus

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Notes for Self Study

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Limits

We assume the properties of limits.

Property 1. *Suppose that we have two functions F and G defined for small values of h, and assume that the limits*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist. Then

$$\lim_{h\to 0}(F(h) + G(h))$$

exists and

$$\lim_{h\to 0}(F + G)(h) = \lim_{h\to 0} F(h) + \lim_{h\to 0} G(h).$$

Property 2. *Let F, G be two functions for small values of h, and assume that*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist. Then the limit of the product exists and we have

$$\begin{aligned}\lim_{h\to 0}(FG)(h) &= \lim_{h\to 0}(F(h)G(h)) \\ &= \lim_{h\to 0} F(h) \cdot \lim_{h\to 0} G(h).\end{aligned}$$

Property 3. *Assume that the limits*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist, and that

$$\lim_{h\to 0} G(h) \neq 0.$$

Then the limit of the quotient exists and we have

$$\lim_{h\to 0} \frac{F(h)}{G(h)} = \frac{\lim F(H)}{\lim G(H)}.$$

We discuss two further properties at a later time.

Powers

Theorem 4.1. *Let n be an integer ≥ 1 and let $f(x) = x^n$. Then*

$$\frac{df}{dx} = nx^{n-1}.$$

Remarks on the proof. When we have some number $(x+h)^n$, writing each factor yields

$$(x+h)(x+h)\cdots(x+h).$$

If we were to distribute we would get many terms that we do not need to think about. We are able to select which terms from each factor we wish to distribute to find a particular number. There exists n number of x and we multiply them by each other, giving us x^n.

If we choose x from all but one factor, then the remaining factor has h and we get hx^{n-1}. But we do this for each factor. The idea is that it is not the h from one particular factor, but it could be the h from any factor. Since the terms are added when we distribute $(x+h)^n$, then we add the n instances of hx, and get nhx^{n-1}.

Now we have the term x^n and the only term nhx^{n-1} having a factor of h^1. We conclude that every other term must choose h from at least two factors. Hence we have

$$(x+h)^n = x^n + nhx^{n-1} + h^2 g(x,h),$$

where $g(x,h)$ is some expression involving powers of x and h with numerical coefficients. Of course h^2 is factored from the expression.

The rest of the proof follows very naturally using the Newton quotient.

Theorem 4.2. *Let a be any number and let $f(x) = x^a$ (defined for $x > 0$). Then $f(x)$ has a derivative, which is*

$$f'(x) = ax^{a-1}.$$

We do not prove this until we have more techniques available.

Sums, Products, and Quotients

Definition. A function is said to be **continuous at a point** x if and only if

$$\lim_{h \to 0} f(x+h) = f(x).$$

A function is said to be **continuous** if it is continuous at every point of its domain of definition.

Let f be a function having a derivative $f'(x)$ at x. Then f is continuous at x.

Remarks on the proof. We note that if a function $f(x)$ is continuous at x, then it is continuous at every point of its domain of definition. The proposition statement states that f has a derivative $f'(x)$ at x, this is equivalent to saying that f is differentiable. So what we wish to prove is:

Let f be a function that is differentiable. Then f is continuous.

We set the Newton quotient of f equal to itself then multiply by h and get

$$h\frac{f(x+h) - f(x)}{h} = f(x+h) - f(x).$$

As h approaches 0, the left term approaches $0f'$. Thus we have

$$\lim_{h \to 0} f(x+h) - f(x) = 0f'(x) = 0.$$

This is another way of stating that

$$\lim_{h \to 0} f(x+h) = f(x).$$

By definition, f is continuous.

We now show some computational rules.

Constant times a function. *The derivative of cf is then given by the formula*

$$(cf)'(x) = c \cdot f'(x).$$

In the other notation, this reads

$$\frac{d(cf)}{dx} = c\frac{df}{dx}.$$

Sum. *Let $f(x)$ and $g(x)$ be two functions which have derivatives $f'(x)$ and $g'(x)$, respectively. Then the sum $f(x)+g(x)$ has a derivative, and*

$$(f+g)'(x) = f'(x) + g'(x).$$

In the other notation, this reads

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}.$$

Product. *Let $f(x)$ and $g(x)$ be two functions having derivatives $f'(x)$ and $g'(x)$. Then the product function $f(x)g(x)$ has a derivative, which is given by the formula*

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Special case with quotients. *Let $g(x)$ be a function having a derivative $g'(x)$, and such that $g(x) \neq 0$. Then the derivative of the quotient $1/g(x)$ exists, and is equal to*

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-1}{g(x)^2}g'(x).$$

Quotient. *Let $f(x)$ and $g(x)$ be two functions having derivatives $f'(x)$ and $g'(x)$ respectively, and such that $g(x) \neq 0$. Then the derivative of the quotient $f(x)/g(x)$ exists, and is equal to*

$$\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Chain Rule

Chain rule. *Let f and g be two functions having derivatives, and such that f is defined at all numbers which are values of g. Then the composite function $f \circ g$ has a derivative, given by the formula*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Remarks on the proof. We distinguish two kinds of numbers h. Let H_1 be the set of h such that $g(x+h) - g(x) \neq 0$, and H_2 be the set of h such that $g(x+h) - g(x) = 0$.

For h in H_1, we must show that the limit of the Newton quotient of $f \circ g$ is $f'(u)g'(x)$. By definition, we have

$$\frac{f(g(x+h)) - f(g(x))}{h}.$$

Put $u = g(x)$, as we have practiced before in the examples, and let $k = g(x+h) - g(x)$. Then we have

$$\frac{f(g(x) + g(x+h) - g(x)) - f(u)}{h} = \frac{f(u+k) - f(u)}{h}.$$

We have essentially added 0 to the input of f. Since k is expressed in h, we say that k depends on h and tends to 0 as h approaches 0. Since we are dealing with h in H_1, then k is unequal to 0 for all small values of h. Then we can multiply and divide this quotient by k, and obtain

$$\frac{f(u+k) - f(u)}{k}\frac{k}{h} = \frac{f(u+k) - f(u)}{k}\frac{g(x+h) - g(x)}{h}.$$

Note that we multiply the Newton quotient by k/k. As h approaches 0, then our Newton quotient approaches

$$f'(u)g'(x).$$

For h in H_2, we show that the limit of the Newton quotient of $f \circ g$ is 0, and that 0 is equivalent to writing the formula for the chain rule anyway. We assume that we have $g(x+h) - g(x) = 0$ for arbitrarily small values of h. Then

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = 0,$$

because $g(x+h) - g(x) = 0$, so $g(x+h) = g(x)$ therefore

$$f(g(x+h)) - f(g(x)) = f(g(x)) - f(g(x)) = 0.$$

Since the limit approaches 0 as h approaches 0, we can choose any number equal to 0 to represent this limit. We choose $f'(g(x))g'(x)$ to keep the formula constant whether h is in H_1 or H_2.

Higher Derivatives

Given a differentiable function f defined on an interval, its derivative f' is also a function on this interval. If it turns out to be also differentiable, then its derivative is called the **second derivative** of f and is denoted by $f''(x)$. We write

$$f^{(n)}$$

to denote the n-th derivative of f. Thus f'' is also written $f^{(2)}$. To refer to the variable x, we also write

$$f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

Implicit Differentiation

Suppose that a curve is defined by an equation

$$F(x, y) = 0.$$

Assuming that $y = f(x)$ is a differentiable function, we can find an expression for the derivative.

Example. Find the derivative dy/dx in terms of x and y if $x^2 + xy = 2$.

We differentiate both sides of the equation. The right-hand side is 0 and the left-hand side is

$$\frac{d}{dx}(x^2 + xy) = \frac{d(x^2)}{dx} + \frac{d(xy)}{dx}.$$

The left term is $2x$ and by the product rule we have

$$2x + x\frac{dy}{dx} + y\frac{dx}{dx} = 2x + x\frac{dy}{dx} + y.$$

Now we find dy/dx in terms of x and y, which is

$$\frac{dy}{dx} = \frac{-y - 2x}{x},$$

and we are done.

Rate of Change

Suppose that a particle moves along some straight line a certain distance depending on time t. Then the distance s is a function of t, which we write $s = f(t)$.

For two values of the time t_1 and t_2, the quotient

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

can be regarded as a sort of average speed of the particle. We of course see this pattern as the slope of a line in analytic geometry. We note that distance divided by time should intuitively yield speed. We regard the limit

$$\lim t \to t_0 \frac{f(t) - f(t_0)}{t - t_0}$$

as the rate of change of s with respect to t. This is the derivative $f'(t)$, which is called the **speed**.

Let us denote the speed by $v(t)$, that is, speed is a function of time. Then the speed is given by the derivative of the distance with respect to time, which is

$$v(t) = \frac{ds}{dt}.$$

The rate of change of the speed is called the **acceleration**. Thus

$$\frac{dv}{dt} = \frac{d^2 s}{dt^2},$$

which is the second derivative of the distance with respect to time. We essentially found the rate of change of the rate of change of distance relative to time is acceleration.

In general, given a function $y = f(x)$, the derivative $f'(x)$ is interpreted as the **rate of change of** y **with respect to** x.