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A First Course in Calculus

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Notes for Self Study

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Powers

Theorem 4.1. *Let n be an integer ≥ 1 and let $f(x) = x^n$. Then*

$$\frac{df}{dx} = nx^{n-1}.$$

Remarks on the proof. When we have some number $(x + h)^n$, writing each factor yields

$$(x + h)(x + h) \cdots (x + h).$$

If we were to distribute we would get many terms that we do not need to think about. We are able to select which terms from each factor we wish to distribute to find a particular number. There exists n number of x and we multiply them by each other, giving us x^n.

If we choose x from all but one factor, then the remaining factor has h and we get hx^{n-1}. But we do this for each factor. The idea is that it is not the h from one particular factor, but it could be the h from any factor. Since the terms are added when we distribute $(x + h)^n$, then we add the n instances of hx, and get nhx^{n-1}.

Now we have the term x^n and the only term nhx^{n-1} having a factor of h^1. We conclude that every other term must choose h from at least two factors. Hence we have

$$(x + h)^n = x^n + nhx^{n-1} + h^2 g(x, h),$$

where $g(x, h)$ is some expression involving powers of x and h with numerical coefficients. Of course h^2 is factored from the expression.

The rest of the proof follows very naturally using the Newton quotient.

Theorem 4.2. *Let a be any number and let $f(x) = x^a$ (defined for $x > 0$). Then $f(x)$ has a derivative, which is*

$$f'(x) = ax^{a-1}.$$

We do not prove this until we have more techniques available.

Sums, Products, and Quotients

Definition. A function is said to be **continuous at a point** x if and only if

$$\lim_{h\to 0} f(x+h) = f(x).$$

A function is said to be **continuous** if it is continuous at every point of its domain of definition.

Let f be a function having a derivative $f'(x)$ at x. Then f is continuous at x.

Remarks on the proof. We note that if a function $f(x)$ is continuous at x, then it is continuous at every point of its domain of definition. The proposition statement states that f has a derivative $f'(x)$ at x, this is equivalent to saying that f is differentiable. So what we wish to prove is:

Let f be a function that is differentiable. Then f is continuous.

We set the Newton quotient of f equal to itself then multiply by h and get

$$h\frac{f(x+h)-f(x)}{h} = f(x+h) - f(x).$$

As h approaches 0, the left term approaches $0f'$. Thus we have

$$\lim_{h\to 0} f(x+h) - f(x) = 0f'(x) = 0.$$

This is another way of stating that

$$\lim_{h\to 0} f(x+h) = f(x).$$

By definition, f is continuous.

We now show some computational rules.

Constant times a function. *The derivative of cf is then given by the formula*

$$(cf)'(x) = c \cdot f'(x).$$

In the other notation, this reads

$$\frac{d(cf)}{dx} = c\frac{df}{dx}.$$

Sum. *Let $f(x)$ and $g(x)$ be two functions which have derivatives $f'(x)$ and $g'(x)$, respectively. Then the sum $f(x) + g(x)$ has a derivative, and*

$$(f + g)'(x) = f'(x) + g'(x).$$

In the other notation, this reads

$$\frac{d(f + g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}.$$

Product. *Let $f(x)$ and $g(x)$ be two functions having derivatives $f'(x)$ and $g'(x)$. Then the product function $f(x)g(x)$ has a derivative, which is given by the formula*

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Special case with quotients. *Let $g(x)$ be a function having a derivative $g'(x)$, and such that $g(x) \neq 0$. Then the derivative of the quotient $1/g(x)$ exists, and is equal to*

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-1}{g(x)^2}g'(x).$$

Quotient. *Let $f(x)$ and $g(x)$ be two functions having derivatives $f'(x)$ and $g'(x)$ respectively, and such that $g(x) \neq 0$. Then the derivatie of the quotient $f(x)/g(x)$ exists, and is equal to*

$$\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Chain Rule

Chain rule. *Let f and g be two functions having derivatives, and such that f is defined at all numbers which are values of g. Then the composite function $f \circ g$ has a derivative, given by the formula*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Remarks on the proof. We distinguish two kinds of numbers h. Let H_1 be the set of h such that $g(x+h) - g(x) \neq 0$, and H_2 be the set of h such that $g(x+h) - g(x) = 0$.

For h in H_1, we must show that the limit of the Newton quotient of $f \circ g$ is $f'(u)g'(x)$. By definition, we have

$$\frac{f(g(x+h)) - f(g(x))}{h}.$$

Put $u = g(x)$, as we have practiced before in the examples, and let $k = g(x+h) - g(x)$. Then we have

$$\frac{f(g(x) + g(x+h) - g(x)) - f(u)}{h} = \frac{f(u+k) - f(u)}{h}.$$

We have essentially added 0 to the input of f. Since k is expressed in h, we say that k depends on h and tends to 0 as h approaches 0. Since we are dealing with h in H_1, then k is unequal to 0 for all small values of h. Then we can multiply and divide this quotient by k, and obtain

$$\frac{f(u+k) - f(u)}{k}\frac{k}{h} = \frac{f(u+k) - f(u)}{k}\frac{g(x+h) - g(x)}{h}.$$

Note that we multiply the Newton quotient by k/k. As h approaches 0, then our Newton quotient approaches

$$f'(u)g'(x).$$

For h in H_2, we show that the limit of the Newton quotient of $f \circ g$ is 0, and that 0 is equivalent to writing the formula for the chain rule anyway. We assume that we have $g(x+h) - g(x) = 0$ for arbitrarily small values of h. Then

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = 0,$$

because $g(x+h) - g(x) = 0$, so $g(x+h) = g(x)$ therefore

$$f(g(x+h)) - f(g(x)) = f(g(x)) - f(g(x)) = 0.$$

Since the limit approaches 0 as h approaches 0, we can choose any number equal to 0 to represent this limit. We choose $f'(g(x))g'(x)$ to keep the formula constant whether h is in H_1 or H_2.