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A First Course in Calculus

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Notes for Self Study

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Limits

We assume the properties of limits.

Property 1. *Suppose that we have two functions F and G defined for small values of h, and assume that the limits*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist. Then

$$\lim_{h\to 0}(F(h) + G(h))$$

exists and

$$\lim_{h\to 0}(F + G)(h) = \lim_{h\to 0} F(h) + \lim_{h\to 0} G(h).$$

Property 2. *Let F, G be two functions for small values of h, and assume that*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist. Then the limit of the product exists and we have

$$\begin{aligned}\lim_{h\to 0}(FG)(h) &= \lim_{h\to 0}(F(h)G(h)) \\ &= \lim_{h\to 0} F(h) \cdot \lim_{h\to 0} G(h).\end{aligned}$$

Property 3. *Assume that the limits*

$$\lim_{h\to 0} F(h) \quad \textit{and} \quad \lim_{h\to 0} G(h)$$

exist, and that

$$\lim_{h\to 0} G(h) \neq 0.$$

Then the limit of the quotient exists and we have

$$\lim_{h\to 0} \frac{F(h)}{G(h)} = \frac{\lim F(H)}{\lim G(H)}.$$

We discuss two further properties at a later time.

Powers

Theorem 4.1. *Let n be an integer* ≥ 1 *and let* $f(x) = x^n$. *Then*

$$\frac{df}{dx} = nx^{n-1}.$$

Remarks on the proof. When we have some number $(x + h)^n$, writing each factor yields

$$(x + h)(x + h) \cdots (x + h).$$

If we were to distribute we would get many terms that we do not need to think about. We are able to select which terms from each factor we wish to distribute to find a particular number. There exists n number of x and we multiply them by each other, giving us x^n.

If we choose x from all but one factor, then the remaining factor has h and we get hx^{n-1}. But we do this for each factor. The idea is that it is not the h from one particular factor, but it could be the h from any factor. Since the terms are added when we distribute $(x + h)^n$, then we add the n instances of hx, and get nhx^{n-1}.

Now we have the term x^n and the only term nhx^{n-1} having a factor of h^1. We conclude that every other term must choose h from at least two factors. Hence we have

$$(x + h)^n = x^n + nhx^{n-1} + h^2 g(x, h),$$

where $g(x, h)$ is some expression involving powers of x and h with numerical coefficients. Of course h^2 is factored from the expression.

The rest of the proof follows very naturally using the Newton quotient.

Theorem 4.2. *Let a be any number and let* $f(x) = x^a$ *(defined for* $x > 0$*). Then* $f(x)$ *has a derivative, which is*

$$f'(x) = ax^{a-1}.$$

We do not prove this until we have more techniques available.

Sums, Products, and Quotients

Definition. A function is said to be **continuous at a point** x if and only if

$$\lim_{h \to 0} f(x+h) = f(x).$$

A function is said to be **continuous** if it is continuous at every point of its domain of definition.

Let f be a function having a derivative $f'(x)$ at x. Then f is continuous at x.

Remarks on the proof. We note that if a function $f(x)$ is continuous at x, then it is continuous at every point of its domain of definition. The proposition statement states that f has a derivative $f'(x)$ at x, this is equivalent to saying that f is differentiable. So what we wish to prove is:

Let f be a function that is differentiable. Then f is continuous.

We set the Newton quotient of f equal to itself then multiply by h and get

$$h\frac{f(x+h)-f(x)}{h} = f(x+h) - f(x).$$

As h approaches 0, the left term approaches $0f'$. Thus we have

$$\lim_{h \to 0} f(x+h) - f(x) = 0f'(x) = 0.$$

This is another way of stating that

$$\lim_{h \to 0} f(x+h) = f(x).$$

By definition, f is continuous.

We now show some computational rules.

Constant times a function. *The derivative of cf is then given by the formula*

$$(cf)'(x) = c \cdot f'(x).$$

In the other notation, this reads

$$\frac{d(cf)}{dx} = c\frac{df}{dx}.$$

Sum. *Let* $f(x)$ *and* $g(x)$ *be two functions which have derivatives* $f'(x)$ *and* $g'(x)$*, respectively. Then the sum* $f(x)+g(x)$ *has a derivative, and*

$$(f+g)'(x) = f'(x) + g'(x).$$

In the other notation, this reads

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}.$$

Product. *Let* $f(x)$ *and* $g(x)$ *be two functions having derivatives* $f'(x)$ *and* $g'(x)$*. Then the product function* $f(x)g(x)$ *has a derivative, which is given by the formula*

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Special case with quotients. *Let* $g(x)$ *be a function having a derivative* $g'(x)$*, and such that* $g(x) \neq 0$*. Then the derivative of the quotient* $1/g(x)$ *exists, and is equal to*

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-1}{g(x)^2}g'(x).$$

Quotient. *Let* $f(x)$ *and* $g(x)$ *be two functions having derivatives* $f'(x)$ *and* $g'(x)$ *respectively, and such that* $g(x) \neq 0$*. Then the derivative of the quotient* $f(x)/g(x)$ *exists, and is equal to*

$$\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Chain Rule

Chain rule. *Let f and g be two functions having derivatives, and such that f is defined at all numbers which are values of g. Then the composite function $f \circ g$ has a derivative, given by the formula*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Remarks on the proof. We distinguish two kinds of numbers h. Let H_1 be the set of h such that $g(x+h) - g(x) \neq 0$, and H_2 be the set of h such that $g(x+h) - g(x) = 0$.

For h in H_1, we must show that the limit of the Newton quotient of $f \circ g$ is $f'(u)g'(x)$. By definition, we have

$$\frac{f(g(x+h)) - f(g(x))}{h}.$$

Put $u = g(x)$, as we have practiced before in the examples, and let $k = g(x+h) - g(x)$. Then we have

$$\frac{f(g(x) + g(x+h) - g(x)) - f(u)}{h} = \frac{f(u+k) - f(u)}{h}.$$

We have essentially added 0 to the input of f. Since k is expressed in h, we say that k depends on h and tends to 0 as h approaches 0. Since we are dealing with h in H_1, then k is unequal to 0 for all small values of h. Then we can multiply and divide this quotient by k, and obtain

$$\frac{f(u+k) - f(u)}{k}\frac{k}{h} = \frac{f(u+k) - f(u)}{k}\frac{g(x+h) - g(x)}{h}.$$

Note that we multiply the Newton quotient by k/k. As h approaches 0, then our Newton quotient approaches

$$f'(u)g'(x).$$

For h in H_2, we show that the limit of the Newton quotient of $f \circ g$ is 0, and that 0 is equivalent to writing the formula for the chain rule anyway. We assume that we have $g(x+h) - g(x) = 0$ for arbitrarily small values of h. Then

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = 0,$$

because $g(x+h) - g(x) = 0$, so $g(x+h) = g(x)$ therefore

$$f(g(x+h)) - f(g(x)) = f(g(x)) - f(g(x)) = 0.$$

Since the limit approaches 0 as h approaches 0, we can choose any number equal to 0 to represent this limit. We choose $f'(g(x))g'(x)$ to keep the formula constant whether h is in H_1 or H_2.

Higher Derivatives

Given a differentiable function f defined on an interval, its derivative f' is also a function on this interval. If it turns out to be also differentiable, then its derivative is called the **second derivative** of f and is denoted by $f''(x)$. We write

$$f^{(n)}$$

to denote the n-th derivative of f. Thus f'' is also written $f^{(2)}$. To refer to the variable x, we also write

$$f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

Implicit Differentiation

Suppose that a curve is defined by an equation

$$F(x, y) = 0.$$

Assuming that $y = f(x)$ is a differentiable function, we can find an expression for the derivative.

Example. Find the derivative dy/dx in terms of x and y if $x^2 + xy = 2$.

We differentiate both sides of the equation. The right-hand side is 0 and the left-hand side is

$$\frac{d}{dx}(x^2 + xy) = \frac{d(x^2)}{dx} + \frac{d(xy)}{dx}.$$

The left term is $2x$ and by the product rule we have

$$2x + x\frac{dy}{dx} + y\frac{dx}{dx} = 2x + x\frac{dy}{dx} + y.$$

Now we find dy/dx in terms of x and y, which is

$$\frac{dy}{dx} = \frac{-y - 2x}{x},$$

and we are done.

Sine and Cosine

On wholes and parts. Note that I made this up and the terminology is nonstandard.

We wish to determine the area of a sector S having θ radians in a disc D of radius r. To get a part P of some whole W, we note that all of a whole is simply $1 \cdot W$. As $0 \leq P \leq 1$, we determine portions of W by

$$P \cdot W.$$

The whole of the area of D is πr^2 so

$$A = \text{part} \cdot \pi r^2.$$

The whole of the radians is 2π. We take a portion of this, which is θ. Then there exists some number q such that

$$\theta = 2\pi q.$$

We see that $q = \theta/2\pi$ and we have our part. The area of S is

$$\frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}.$$

Determine the length L of an arc of θ radians on a circle of radius r. The whole of the length is $2\pi r$ and the whole of the radians is 2π. Again we take $\theta/2\pi$ and we get that

$$L = \frac{\theta}{2\pi} 2\pi r = r\theta.$$

Theorem 1 *The functions* $\sin x$ *and* $\cos x$ *have derivatives and*

$$\frac{d(\sin x)}{dx} = \cos x,$$
$$\frac{d(\cos x)}{dx} = -\sin x.$$

Proof The Newton quotient of $\sin x$ is

$$\frac{\sin(x+h) - \sin x}{h}.$$

Using the addition formula, the Newton quotient is then

$$\frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

We factorize and get

$$\frac{\cos x \sin h + \sin x(\cos h - 1)}{h}.$$

We separate our quotient:

$$\cos x \frac{\sin h}{h} + \sin x \frac{\cos h - 1}{h}.$$

For now, assume that as h approaches 0 the limits of $(\sin h)/h$ and $(\cos h - 1)/h)$ are 1 and 0 respectively. It follows that $\cos x$ remains, proving that

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x.$$

We can get the derivative of $\cos x$ in the same manner, but we do so in terms of $\sin x$ and the chain rule. We know that $\cos x = \sin(x + \pi/2)$. Let $u = x + \pi/2$. Taking the derivative of both sides with respect to x, we get

$$\frac{d(\cos x)}{dx} = \cos u = \cos\left(x + \frac{\pi}{2}\right) = -\sin x,$$

and our theorem is proven. □

We prove

$$\lim_{h\to 0} \frac{\sin h}{h} = 1.$$

Taking the figure given, we see that the area of the small triangle is lesser than the area of the sector, and that is lesser than the area of the big triangle. We see that

$$\frac{|AB|}{|OB|} = \sin h = s$$

and

$$\frac{|CD|}{|OC|} = \frac{|CD|}{1} = t$$

where we equivalently find that

$$t = \frac{\sin h}{\cos h} = \tan h.$$

We could also have found that $t = \sin h/\cos h$ by similar triangles. We do this now. Take $|DC|/|BA| = |OC|/|OA|$. This relation is the same as $t/s = 1/\cos h$, and multiplying by $s = \sin h$ we get $t = \sin h/\cos h$.

The area of the small triangle is $(\cos h \sin h)/2$, the area of the big triangle is $t/2 = (1/2)\sin h \cos h$, and the area of the sector is $h/2$. Hence

$$\frac{1}{2}\cos h \sin h < \frac{1}{2}h < \frac{1}{2}\frac{\sin h}{\cos h}.$$

Multiplying by 2, we get

$$\cos h \sin h < h < \frac{\sin h}{\cos h}.$$

Recall we assumed $h > 0$, and know that we take h to be small. Then it follows that $\sin h > 0$ so we multiply the inequalities by $1/\sin h$ to get

$$\cos h < \frac{h}{\sin h} < \frac{1}{\cos h}.$$

As h approaches 0, both $\cos h$ and $1/\cos h$ approach 1. Thus $h/\sin h$ is squeezed between two quantities which approach 1, and therefore $h/\sin h$ must approach 1 also. This is given by Property 5 of limits.

Our desired quotient $(\sin h)/h$ is equal to $1/(h/\sin h)$. Since

$$\lim_{h\to 0} \frac{1}{h/\sin h} = \frac{\lim 1}{\lim h/\sin h} = \frac{1}{1} = 1,$$

then

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$