Inductive Specification

Exercise 1.1

1. $\{3n+2 \mid n \in N\}$

Definition (top-down) *A natural number n is in S if and only if*

1. $n = 2$, or

2. $n - 3 \in S$.

Definition (bottom-up) *Define the set S to be the smallest set contained in N and satisfying the following two properties:*

1. $2 \in S$, and

2. if $n \in S$, then $n + 3 \in S$.

Definition (rules of inference)

$$2 \in S$$

$$\frac{n \in S}{n+3 \in S}$$

We generate some elements of S.

1. $2 \in S$.

2.

$$\frac{2 \in S}{(2+3) \in S}$$

3.

$$\frac{5 \in S}{(5+3) \in S}$$

2. $\{2n+3m+1 \mid n, m \in N\}$

Definition (top-down) *A natural number n is in S if and only if*

1. $n = 1$, or

2. $n - 2 \in S$, or

3. $n - 3 \in S$.

Definition (bottom-up) *Define the set S to be the smallest set contained in N and satisfying the following three properties:*

1. $1 \in S$, and

2. if $n \in S$, then $n + 2 \in S$, and

3. if $n \in S$, then $n + 3 \in S$.

Definition (rules of inference)

$$1 \in S$$

$$\frac{n \in S}{(n+2) \in S}$$

$$\frac{n \in S}{(n+3) \in S}$$

We generate some elements of S.

1. $1 \in S$.

2.

$$\frac{1 \in S}{(1+2) \in S}$$

3.

$$\frac{3 \in S}{(3+2) \in S}$$

4.

$$\frac{5 \in S}{(5+2) \in S}$$

We could have derived $7 \in S$ differently using only property 3.

$$\cfrac{\cfrac{1 \in S}{(1+3) \in S}}{(4+3) \in S}$$

3. $\{(n, 2n+1) | n \in N\}$

Definition (top-down) *A pair of natural numbers (n, m) is in the set S if and only if*

1. the pair is $(0, 1)$, or

2. $(n-1, 2(n-1)+1) \in S$.

Definition (bottom-up) *We define S to be the smallest set of pairs whose elements are contained in N and satisfying the following properties:*

1. $(0, 1) \in S$, and

2. if $(n, m) \in S$, then $(n+1, 2(n+1)+1) \in S$.

Definition (rules of inference)

$$(0, 1) \in S$$

$$\frac{(n, m) \in S}{(n+1, 2(n+1)+1) \in S}$$

We generate some elements of S.

1. $(0, 1) \in S$.

2.

$$\frac{(0,1) \in S}{(0+1, 2(0+1)+1) \in S}$$

3.

$$\frac{(1,3) \in S}{(1+1, 2(1+1)+1) \in S}$$

4. $\{(n, n^2) | n \in N\}$

Definition (top-down) *A pair of natural numbers (n, m) is in the set S if and only if*

1. the pair is $(0, 0)$, or

2. $(n-1, n(n-2)+1) \in S$.

Definition (bottom-up) *We define S to be the smallest set of pairs of natural numbers and satisfying the following properties:*

1. $(0, 0) \in S$, and

2. if $(n, m) \in S$, then $(n+1, n(n+2)+1) \in S$.

Definition (rules of inference)

$$(0,0) \in S$$

$$\frac{(n,m) \in S}{(n+1, n(n+2)+1) \in S}$$

We generate some elements of S.

1. $(0, 0) \in S$.

2.

$$\frac{(0,0) \in S}{(1,1) \in S}$$

3.

$$\frac{(1,1) \in S}{(2,4) \in S}$$

Exercise 1.2

1.

$$\cfrac{\cfrac{\cfrac{(0,1) \in S}{(1,8) \in S}}{(2,15) \in S}}{(3,22) \in S}$$

We find a pattern and determine the set is $\{(n, 7n+1) | n \in N\}$.

2.

$$\cfrac{\cfrac{\cfrac{(0,1) \in S}{(1,2) \in S}}{(2,4) \in S}}{(3,8) \in S}$$

We find a pattern and determine the set is $\{(n, 2^n) | n \in N\}$.

3.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(0,0,1) \in S}{(1,1,1) \in S}}{(2,1,2) \in S}}{(3,2,3) \in S}}{(4,3,5) \in S}}{(5,5,8) \in S}$$

This is a reasonable number of derivations to find the Fibonacci pattern. We determine the set is $\{(n, fib(n), fib(n+1)) | n \in N\}$.

4.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(0,1,0) \in S}{(1,3,1) \in S}}{(2,5,4) \in S}}{(3,7,9) \in S}}{(4,9,16) \in S}}{(5,11,25) \in S}$$

It is clear at this point what the pattern is. We determine the set is $\{(n, 2(n+1)+1, n^2) | n \in N\}$.

Exercise 1.3

Let $T = N$, the set of natural numbers. Then T clearly satisfies the properties of S and $T \neq S$.

Defining Sets Using Grammars

Exercise 1.4

$$
\begin{aligned}
&\quad List-of-Int \\
&\Rightarrow (Int\ .\ List-of-Int) \\
&\Rightarrow (-7\ .\ List-of-Int) \\
&\Rightarrow (-7\ .\ (Int\ .\ List-of-Int)) \\
&\Rightarrow (-7\ .\ (3\ .\ List-of-Int)) \\
&\Rightarrow (-7\ .\ (3\ .\ (Int\ .\ List-of-Int))) \\
&\Rightarrow (-7\ .\ (3\ .\ (14\ .\ List-of-Int))) \\
&\Rightarrow (-7\ .\ (3\ .\ (14\ .\ ())))
\end{aligned}
$$

Induction

Exercise 1.5

Theorem *If $e \in LcExp$, then there are the same number of left and right parentheses in e.*

Proof: We prove by induction on k, the number of times a production is used to compose e. The induction hypothesis, $IH(k)$, is that any member of $LcExp$ constructed by $\leq k$ production applications has the same number of left and right parentheses.

Under no production applications, there is no such $e \in LcExp$, so $IH(0)$ holds.

Let k be an integer such that $IH(k)$ holds. We need to show that $IH(k+1)$ holds. Consider that e is formed by $\leq k+1$ production applications. Then there are three possible forms of e:

e could be of the form id, where $id \in Identifier$. There are 0 left parentheses and 0 right parentheses.

e could be of the form `(lambda (`$Identifier$`)` e_1`)`, where e_1 is a member of $LcExp$. Here, e_1 must have been constructed by k production applications. Thus $IH(k)$ holds for e_1. Therefore e, which we can count has 2 left parentheses and 2 right parentheses, together with e_1, satisfy the inductive hypothesis.

e could be of the form `(`e_1 e_2`)`, where e_1 and e_2 are members of $LcExp$. Now e_1 and e_2 must be composed by $\leq k$ production applications. Thus $IH(k)$ applies to them. Lastly, e has 1 left parenthesis and 1 right parenthesis, and together with e_1 and e_2, satisfy the inductive hypothesis.

This completes the proof of the claim that $IH(k+1)$ holds and therefore completes the induction. □