

## Inductive Specification

Exercise 1.1

1.  $\{3n + 2 \mid n \in N\}$

**Definition (top-down)** *A natural number  $n$  is in  $S$  if and only if*

1.  $n = 2$ , or
2.  $n - 3 \in S$ .

**Definition (bottom-up)** *Define the set  $S$  to be the smallest set contained in  $N$  and satisfying the following two properties:*

1.  $2 \in S$ , and
2. if  $n \in S$ , then  $n + 3 \in S$ .

**Definition (rules of inference)**

$$2 \in S$$

$$\frac{n \in S}{n + 3 \in S}$$

We generate some elements of  $S$ .

1.  $2 \in S$ .
- 2.

$$\frac{2 \in S}{(2 + 3) \in S}$$

- 3.

$$\frac{5 \in S}{(5 + 3) \in S}$$

2.  $\{2n + 3m + 1 \mid n, m \in N\}$

**Definition (top-down)** *A natural number  $n$  is in  $S$  if and only if*

1.  $n = 1$ , or
2.  $n - 2 \in S$ , or
3.  $n - 3 \in S$ .

**Definition (bottom-up)** *Define the set  $S$  to be the smallest set contained in  $N$  and satisfying the following three properties:*

1.  $1 \in S$ , and
2. if  $n \in S$ , then  $n + 2 \in S$ , and
3. if  $n \in S$ , then  $n + 3 \in S$ .

**Definition (rules of inference)**

$$1 \in S$$

$$\frac{n \in S}{(n + 2) \in S}$$

$$\frac{n \in S}{(n + 3) \in S}$$

We generate some elements of  $S$ .

1.  $1 \in S$ .
- 2.

$$\frac{1 \in S}{(1 + 2) \in S}$$

- 3.

$$\frac{3 \in S}{(3 + 2) \in S}$$

4.

$$\frac{5 \in S}{(5 + 2) \in S}$$

We could have derived  $7 \in S$  differently using only property 3.

$$\frac{\frac{1 \in S}{(1 + 3) \in S}}{(4 + 3) \in S}$$

3.  $\{(n, 2n + 1) | n \in N\}$

**Definition (top-down)** *A pair of natural numbers  $(n, m)$  is in the set  $S$  if and only if*

1. *the pair is  $(0, 1)$ , or*
2.  *$(n - 1, 2(n - 1) + 1) \in S$ .*

**Definition (bottom-up)** *We define  $S$  to be the smallest set of pairs whose elements are contained in  $N$  and satisfying the following properties:*

1.  *$(0, 1) \in S$ , and*
2. *if  $(n, m) \in S$ , then  $(n + 1, 2(n + 1) + 1) \in S$ .*

**Definition (rules of inference)**

$$(0, 1) \in S$$

$$\frac{(n, m) \in S}{(n + 1, 2(n + 1) + 1) \in S}$$

We generate some elements of  $S$ .

1.  $(0, 1) \in S$ .

2.

$$\frac{(0,1) \in S}{(0+1, 2(0+1)+1) \in S}$$

3.

$$\frac{(1,3) \in S}{(1+1, 2(1+1)+1) \in S}$$

4.  $\{(n, n^2) | n \in N\}$

**Definition (top-down)** A pair of natural numbers  $(n, m)$  is in the set  $S$  if and only if

1. the pair is  $(0, 0)$ , or
2.  $(n-1, n(n-2)+1) \in S$ .

**Definition (bottom-up)** We define  $S$  to be the smallest set of pairs of natural numbers and satisfying the following properties:

1.  $(0, 0) \in S$ , and
2. if  $(n, m) \in S$ , then  $(n+1, n(n+2)+1) \in S$ .

**Definition (rules of inference)**

$$(0,0) \in S$$

$$\frac{(n,m) \in S}{(n+1, n(n+2)+1) \in S}$$

We generate some elements of  $S$ .

1.  $(0, 0) \in S$ .
- 2.

$$\frac{(0,0) \in S}{(1,1) \in S}$$

3.

$$\frac{(1,1) \in S}{(2,4) \in S}$$

Exercise 1.2

1.

$$\frac{(0,1) \in S}{\frac{(1,8) \in S}{\frac{(2,15) \in S}{(3,22) \in S}}}$$

We find a pattern and determine the set is  $\{(n, 7n + 1) | n \in N\}$ .

2.

$$\frac{(0,1) \in S}{\frac{(1,2) \in S}{\frac{(2,4) \in S}{(3,8) \in S}}}$$

We find a pattern and determine the set is  $\{(n, 2^n) | n \in N\}$ .

3.

$$\frac{(0,0,1) \in S}{\frac{(1,1,1) \in S}{\frac{(2,1,2) \in S}{\frac{(3,2,3) \in S}{\frac{(4,3,5) \in S}{(5,5,8) \in S}}}}}$$

This is a reasonable number of derivations to find the Fibonacci pattern. We

determine the set is  $\{(n, fib(n), fib(n + 1)) | n \in N\}$ .

4.

$$\begin{array}{c}
 (0, 1, 0) \in S \\
 \hline
 (1, 3, 1) \in S \\
 \hline
 (2, 5, 4) \in S \\
 \hline
 (3, 7, 9) \in S \\
 \hline
 (4, 9, 16) \in S \\
 \hline
 (5, 11, 25) \in S
 \end{array}$$

It is clear at this point what the pattern is. We determine the set is  $\{(n, 2(n + 1) + 1, n^2) | n \in N\}$ .

Exercise 1.3

Let  $T = N$ , the set of natural numbers. Then  $T$  clearly satisfies the properties of  $S$  and  $T \neq S$ .

## Defining Sets Using Grammars

Exercise 1.4

$$\begin{aligned}
 &List - of - Int \\
 \Rightarrow &(Int \ . \ List - of - Int) \\
 \Rightarrow &(-7 \ . \ List - of - Int) \\
 \Rightarrow &(-7 \ . \ (Int \ . \ List - of - Int) \ ) \\
 \Rightarrow &(-7 \ . \ (3 \ . \ List - of - Int) \ ) \\
 \Rightarrow &(-7 \ . \ (3 \ . \ (Int \ . \ List - of - Int) \ ) \ ) \\
 \Rightarrow &(-7 \ . \ (3 \ . \ (14 \ . \ List - of - Int) \ ) \ ) \\
 \Rightarrow &(-7 \ . \ (3 \ . \ (14 \ . \ ( \ ) \ ) \ ) \ ) \ )
 \end{aligned}$$