1.

a) We select 3 letters from 26. Thus there are $\Pi(26, 3) = 26!/(26-3)! = 26 \times 25 \times 24 = 15{,}600$ ways to form a sequence of 3 letters out of the 26 letters without replacement.

b) We select 5 letters from 26. Thus there are $\Pi(26, 5) = 26!/(26-5)! = 26 \times 25 \times 24 \times 23 \times 22 = 7{,}893{,}600$ ways to form a sequence of 5 letters out of the 26 letters without replacement.

2. We select 4 students from 200. That is $\Pi(200, 4) = 200!/(200-4)! = 200 \times 199 \times 198 \times 197 = 1{,}552{,}438{,}800$ ways of selecting these four officers.

3.

a) $100!/97! = 100 \times 99 \times 98 = 970{,}200$.

b) $200!/195! = 200 \times 199 \times 198 \times 197 \times 196 = 304{,}278{,}004{,}800$.

4.

a) A code is a sequence of four pegs each having one of six colors and colors can be repeated. There are $6^4 = 1296$ different codes.

b) The number of codes with pegs having non-distinct colors is the solution to (a). The number of codes with pegs having distinct colors is $\Pi(6, 4) = 6\times5\times4\times3 = 360$. The number of codes having at least two or more pegs of the same color, which is the same as saying the codes with pegs not having distinct colors, is the difference. That being $1296 - 360 = 936$.

c) Remove red from the pool of colors. By (a) we have $5^4 = 625$ different codes.

d) The number of codes with pegs having distinct colors without red is $\Pi(5, 4) = 5 \times 4 \times 3 \times 2 = 120$. We subtract the number of codes with pegs having distinct colors without red from the number of codes with pegs having non-distinct colors without red. That is $625 - 120 = 505$.

5. We prove the following statement by induction on n.

STATEMENT $S(n)$: For any m such that $1 \le m \le n$, $\Pi(n, m) = n!/(n-m)!$.

BASIS. Let $n = 1$. Then m must be 1. We find that $\Pi(1, 1) = 1(1-1+1) = 1 = 1!/(1-1)!$. This proves the basis.

INDUCTION. Assume $S(n)$ is true and consider $S(n+1)$. We have that

$$\frac{(n+1)!}{(n+1-m)!} = \frac{n+1}{n+1-m}\frac{n!}{(n-m)!}.$$

By the inductive hypothesis, we must have

$$\frac{(n+1)!}{(n+1-m)!} = \frac{n+1}{n+1-m}\Pi(n, m). \tag{1}$$

Now we must show that the right side is $\Pi(n+1, m)$. We know that

$$\Pi(n, m) = n(n-1)\cdots(n-m+1)$$

and

$$\Pi(n+1,m) = (n+1)(n)(n-1)\cdots(n-m+2).$$

Therefore the right side of (1) is

$$\begin{aligned}\frac{(n+1)\Pi(n,m)}{n+1-m} &= \frac{(n+1)(n)(n-1)\cdots(n-m+2)(n-m+1)}{n+1-m} \\ &= (n+1)(n)(n-1)\cdots(n-m+2) \\ &= \Pi(n+1,m).\end{aligned}$$

Observe that we pulled out $n-m+2$ from the sequence. This proves the inductive step. We conclude that $S(n)$ is true for all $1 \leq m \leq n$. ✦

6. We prove the following statement by induction on $m = a - b$.

STATEMENT $S(m)$: $a!/b! = a(a-1)(a-2)\cdots(b+1)$.

BASIS. Let $m = 1$. Then $a = b+1$. Hence $(b+1)!/b! = b+1$. This proves the basis.

INDUCTION. Assume $S(m)$ is true and $m > 1$. We shall prove $S(m+1)$. Observe that as m increases, the difference of a and b increases. The consequence is that there will be more factors in the equality.

We assume $m+1 = a-b$. That is, $a = b+m+1$. Write

$$(b+m+1)!/b! = (b+m+1)(b+m)!/b!.$$

By the inductive hypothesis, where $a = b+m$, we must have

$$(b+m+1)(a)(a-1)(a-2)\cdots(b+1) = (a+1)(a)(a-1)(a-2)\cdots(b+1).$$

This is the statement $S(m+1)$. This proves the inductive step and we conclude that $S(n)$ is true for all $m \geq 1$. ✦