1.

STATEMENT $S(i)$: If $1 \leq i < n$, then

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

BASIS. The basis is $i = 1$, as given by the lower bound of the interval. Then we have the inductive definition of the recurrence hence the basis holds.

INDUCTION. If $i \geq n$ then there is nothing to prove. Suppose that $i + 1 < n$. From the inductive hypothesis we expand $T(n-i)$ to get

$$\begin{aligned} T(n) &= T(n-i-1) \sum_{j=0}^{i-1} g(n-j) + g(n-i) \\ &= T(n-i-1) + \sum_{j=0}^{i} g(n-j). \end{aligned}$$

This is the statement $S(i+1)$ and we have proved the inductive step. We conclude $S(i)$ is true for $1 \leq i < n$. ✦

2.

a) We have

$$\begin{aligned} T(n) &= T(n-1) + g(n) \\ &= T(n-2) + g(n-1) + g(n) \\ &= T(n-3) + g(n-2) + g(n-1) + g(n). \end{aligned}$$

Thus we have

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

But we know this already by Exercise 3.11.1.

Let $i = n - 1$. Then we have the recurrence in terms of the basis, which is

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j). \tag{1}$$

But $g(n) = n^2$. This is a summation of squares. By Exercise 2.3.1(b) we know that

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6.$$

We adjust the bounds to get the equivalent summation

$$\sum_{j=1}^{n} j^2 = \sum_{j=0}^{n-1} (n-j)^2.$$

But this has the extra term for when $j = n-1$, that is $g(1)$. We subtract 1 and get

$$T(n) = a + \sum_{j=0}^{n-1} (n-j)^2 - 1,$$

the same as Equation (1). We could have just taken the summation and decomposed it with the $(n-2)$th index then wrote the remaining summation as the summation of squares. But now that we have the last term separate, we rewrite the summation in terms of the identity

$$T(n) = a + n(n+1)(2n+1)/6 - 1.$$

Clearly $T(n)$ is cubic, hence $T(n)$ is $O(n^3)$ if $g(n) = n^2$.

b) We have

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j)$$

where $g(n) = n^2 + 3n$. Then we have two summations, one of the n^2 terms and the other of $3n$ terms. Write

$$T(n) = a + \sum_{j=0}^{n-2} (n-j)^2 + \sum_{j=0}^{n-2} 3(n-j).$$

The left summation is $n(n+1)(2n+1)/6-1$ as given by (a), and the right summation is

$$\sum_{j=0}^{n-2} 3(n-j) = 3\sum_{j=0}^{n-2} (n-j) = 3(n-1)(n+2)/2.$$

Thus

$$T(n) = a + n(n+1)(2n+1)/6 - 1 + 3(n-1)(n+2)/2.$$

We see that $T(n)$ is cubic. We conclude that $T(n)$ is $O(n^3)$.

c) Again the solution to $T(n)$ is $T(n) = a + \sum_{j=0}^{n-2} g(n-j)$. Here $g(n) = n^{3/2}$. In (a) we found that there are $n-1$ terms of n^2 and the same follows in (b). We can treat the summation as $n-1$ terms of $n\sqrt{n}$ with lower terms. Clearly $(n-1)n\sqrt{n}$ is $O(n^{5/2})$. We conclude that $T(n)$ is $O(n^{5/2})$.

d) Following (c), there are $n-1$ terms of $n \log n$ plus lower terms. Hence $(n-1)(n \log n)$ is $O(n^2 \log n)$. We conclude that $T(n)$ is $O(n^2 \log n)$.

e) Following (d), there are $n-1$ terms of 2^n plus lower terms. Therefore $T(n)$ is $O(2^n n)$.

For this recurrence we can assume that for any $g(n)$ that $T(n)$ is $O(ng(n))$. But that is just a guess.

3. Let us first solve this recurrence. We write some terms in the sequence and get

$$\begin{aligned}
T(n) &= T(n/2) + g(n) \\
&= T(n/4) + g(n/2) + g(n) \\
&= T(n/8) + g(n/4) + g(n/2) + g(n) \\
&= T(n/16) + g(n/8) + g(n/4) + g(n/2) + g(n).
\end{aligned}$$

The pattern to find is $T(n) = T(n/2^i) + g(n/2^{i-1}) + g(n/2^{i-2}) + \cdots + g(n/2^0)$. That is

$$T(n) = T(n/2^i) + \sum_{j=0}^{i-1} g(n/2^j).$$

Now to express this in terms of the basis. Let $i = \log_2 n$. Then we have

$$T(n) = a + \sum_{j=0}^{\log_2 n-1} g(n/2^j).$$

a) Suppose that $g(n) = n^2$. This is

$$T(n) = a + \sum_{j=0}^{\log_2 n-1} (n/2^j)^2.$$

The exponent changes with each index. This is a geometric series $a, ar, ar^2, \ldots, ar^{n-1}$. Let $a = n^2$ and $r = 1/4$. We know by Exercise 2.3.9 that this is

$$\begin{aligned}
\sum_{j=0}^{\log_2 n-1} (n/2^j)^2 &= \frac{n^2(1/4)^{\log_2 n} - n^2}{1/4 - 1} \\
&= n^2 - \frac{(1/4)^{\log_2 n} - 1}{3/4}.
\end{aligned}$$

Regardless of what the right term is, we have n^2 minus some smaller term. Therefore $T(n)$ is $O(n^2)$.

b) We have

$$T(n) = a + \sum_{j=0}^{\log_2 n-1} 2(n/2^j).$$

Let $a = 2n$ and $r = 1/2$. Each number in the series gets halved. The first number is $2n$, the greatest. Hence $T(n)$ is $O(n)$.

c) We have

$$T(n) = a + \sum_{j=0}^{\log_2 n-1} 10.$$

This is $\log_2 n$ terms of 10. The number of terms is dependent on the upper bound. Hence $T(n)$ is $O(\log n)$.

4. We shall solve this recurrence first. We begin with

$$\begin{aligned} T(n) &= 2T(n/2) + bn \\ &= 4T(n/4) + 2bn \\ &= 8T(n/8) + 3bn \\ &= 16T(n/16) + 4bn. \end{aligned}$$

The pattern to observe is $T(n) = 2^i(n/2^i) + ibn$. Let $i = \log_2 n$. Know that $2^{\log_2 n} = n$ and $n/2^{\log_2 n} = 1$. Then we have

$$T(n) = 2^{\log_2 n}T(1) + bn\log_2 n = an + bn\log_2 n.$$

a) We shall guess that $cn\log_2 n + dn + e$ is the solution to $T(n)$. Note that we know the solution which is $an + bn\log_2 n$.

STATEMENT $S(n)$: If n is a power of 2 and $n \geq 1$, then $T(n) \leq cn\log_2 n + dn + e$.

BASIS. If $n = 1$ then $a \leq c\log_2 1 + d = d + e$.

INDUCTION. Assume $S(i)$ for all $i < n$. We shall prove $S(n)$ for some $n > 1$. We may assume $S(n/2)$. That is, $T(n/2) \leq c(n/2)\log_2(n/2) + d(n/2) + e$. Substituting for $T(n/2)$ in the definition of T we have

$$\begin{aligned} T(n) &\leq 2(c(n/2)\log_2(n/2) + d(n/2) + e + b(n/2)) + bn \\ &= cn(\log_2 n - 1) + dn + 2e + 2bn \\ &= cn\log_2 n + (2b - c)n + dn + 2e. \end{aligned}$$

We must show that $T(n) \leq cn\log_2 n + dn + e$ by showing the constraints on the excess. That is, $(2b - c)n + e$ must be at most 0. Thus $(2b - c)n \leq -e$.