1. We "pull out" each of the operands of $E = (u + v) + ((w + (x + y)) + z)$. We perform this arbitrarily from left to right.

By the associative law, E can be transformed into $u + (v + ((w + (x + y)) + z))$. Thus we have $E = u + E_1$ where $E_1 = v + ((w + (x + y)) + z)$. We trivially pull out v from E_1 to get an expression of the form $v + E_2$ where $E_2 = (w + (x + y)) + z$. With the associative law we transform E_2 into an expression of the form $w + E_3$ where $E_3 = (x + y) + z$. Similarly we transform E_3 into an expression of the form $x + E_4$ where $E_4 = y + z$. We transform E_4 into an expression of the form $y + E_5$ where $E_5 = z$. The sequence of transformations is

$$(u + v) + ((w + (x + y)) + z)$$
$$u + (v + ((w + (x + y)) + z))$$
$$u + (v + (w + ((x + y) + z)))$$
$$u + (v + (w + (x + (y + z)))).$$

2.

a) We transform $E = w + (x + (y + z))$ into $F = ((w + x) + y) + z$. We do so by "pulling out" one operand a from both expressions, which are equivalent, and then repeating with the next operand until none are left.

We first choose to "pull out" w from both expressions first. This is already so for E. For F we follow the sequence

$$((w + x) + y) + z \rightarrow (w + (x + y)) + z \rightarrow w + ((x + y) + z). \tag{1}$$

Now to transform $E_1 = x + (y + z)$ into $F_1 = (x + y) + z$. We "pull out" x which is already accomplished for E_1. For F_1 we perform the transformation

$$(x + y) + z \rightarrow x + (y + z). \tag{2}$$

We "pull out" y next from $E_2 = y + z$ and $F_2 = y + z$. This is done so trivially.

We now transform what is left of the expressions E_2 and F_2 without y. Consider the expressions $E_3 = z$ and $F_3 = z$. E_3 naturally transforms into F_3. Furthermore, $E_2 = y + E_3$ can transform into $F_2 = y + F_3$, and $E_1 = x + E_2$ can transform into $F_1 = x + F_2$. Finally, $E = w + E_1$ can transform into $F = w + F_1$, and we are done. The sequence of transformations is

$$w + (x + (y + z)) \qquad \text{Expression } E$$
$$w + ((x + y) + z) \qquad \text{(2) in reverse}$$
$$(w + (x + y)) + z \qquad \text{Middle of (1) in reverse}$$
$$((w + x) + y) + z \qquad \text{Expression F, beginning of (1) in reverse}$$

b) We transform $E = (v + w) + ((x + y) + z)$ into $F = ((y + w) + (v + z)) + x$.

We "pull out" v first from both expressions. The sequences of transformations for E and F respectively are

$$(v + w) + ((x + y) + z) \rightarrow v + (w + ((x + y) + z)) \tag{3}$$

and

$$((y + w) + (v + z)) + x \to (((y + w) + v) + z) + x$$
$$\to ((v + (y + w)) + z) + x \to (v + ((y + w) + z)) + x \quad (4)$$
$$\to v + (((y + w) + z) + x).$$

We "pull out" w from the subexpressions $w + ((x + y) + z)$ and $((y + w) + z) + x$:

$$((y + w) + z) + x \to ((w + y) + z) + x \to (w + (y + z)) + x \to w + ((y + z) + x). \quad (5)$$

We shall "pull out" x from the subexpressions $(x + y) + z$ and $(y + z) + x$:

$$(x + y) + z \to x + (y + z) \quad (6)$$

and

$$(y + z) + x \to x + (y + z). \quad (7)$$

We then "pull out" y from the subexpressions $y + z$ and $y + z$. We are then left with the operand z in both expressions, which means we can transform one expression into the other. Thus $y + A_1$ can transform into $y + B_1$ if we consider $A_1 = z = B_1$. By successively letting the subexpressions of E and F (starting with z) being added to y, x, w, v in order, we transform E into F. The sequence of transformations is

$(v + w) + ((x + y) + z)$	Expression E
$v + (w + ((x + y) + z))$	(3)
$v + (w + (x + (y + z)))$	(6)
$v + (w + ((y + z) + x))$	(7) in reverse
$v + ((w + (y + z)) + x)$	Middle-right of (5) in reverse
$v + (((w + y) + z) + x)$	Middle-left of (5) in reverse
$v + (((y + w) + z) + x)$	Beginning of (5) in reverse
$(v + ((y + w) + z)) + x$	Middle-right of (4) in reverse
$((v + (y + w)) + z) + x$	Middle of (4) in reverse
$(((y + w) + v) + z) + x$	Middle-left of (4) in reverse
$((y + w) + (v + z)) + x$	Expression F, beginning of (4) in reverse

3. We shall prove the following statement by complete induction on n, the number of occurrences of operators in an expression.

STATEMENT $S(n)$: Let E be an expression with operators $+, -, *$, and $/$. If E has n operator occurrences, then E has $n + 1$ operands.

We choose zero as the basis because it is the least nonnegative number. By induction, the intuitive basis of one would be proved as well.

BASIS. Let $n = 0$. Then E has 1 operand, hence $S(0)$ is true.

INDUCTION. Assume $n \geq 0$ and $S(0), S(1), \ldots, S(n)$ are true. We shall prove $S(n+1)$. We assume that E has at least one operator, therefore E has at least two operands. Let the operands of E be the expressions E_1 and E_2. Since E has exactly $n+1$ operators, then either E_1 or E_2 has at most n operators, but not both. We apply the inductive hypothesis to E_2, meaning it has $n+1$ operands. Thus E_1 has only one operand, because E_1 has no operators. Together, E has $n+2$ operands. This proves the inductive step, and we conclude that $S(n)$ for all $n \geq 0$. ✦

We should have written that E_1 has n_1 operator occurrences and E_2 has n_2 operator occurrences and together there are $n_1 + n_2 = n$ operator occurrences. We also could have used a symbol to represent the operator in E, like θ.

6. We prove by complete induction the following statement on n, the length of the expression E.

STATEMENT $S(n)$: An expression E of length n having all binary operators has an odd length.

BASIS. Let $n = 1$. The expression E is only an operand, hence $S(1)$ is true.

INDUCTION. Assume $n \geq 1$ and $S(i)$ for $i = 1, 2, \ldots, n$. We shall prove $S(n+1)$. Let E be an expression of length $n+1$ having binary operators that can be written in the form $E_1 \theta E_2$, where E_1 and E_2 are expressions and θ is a binary operator. Let the length of E_1 be n_1 and the length of E_2 be n_2, and $n_1 + n_2 = n$. By the inductive hypothesis, n_1 and n_2 must be odd. The length of $E = E_1 \theta E_2$ is $n+1 = n_1 + 1 + n_2$, which must be odd. Hence the inductive step is proven, and therefore $S(n)$ for $n \geq 1$. ✦

7. We prove the following statement by complete induction on n.

STATEMENT $S(n)$: Every negative integer n can be written in the form $2a + 3b$ for some integers a and b.

BASIS.