1. We "pull out" each of the operands of $E = (u+v)+((w+(x+y))+z)$. We perform this arbitrarily from left to right.

By the associative law, E can be transformed into $u+(v+((w+(x+y))+z))$. Thus we have $E = u+E_1$ where $E_1 = v+((w+(x+y))+z)$. We trivially pull out v from E_1 to get an expression of the form $v+E_2$ where $E_2 = (w+(x+y))+z$. With the associative law we transform E_2 into an expression of the form $w+E_3$ where $E_3 = (x+y)+z$. Similarly we transform E_3 into an expression of the form $x+E_4$ where $E_4 = y+z$. We transform E_4 into an expression of the form $y+E_5$ where $E_5 = z$. The sequence of transformations is

$$\begin{aligned}
&(u+v)+((w+(x+y))+z)\\
&u+(v+((w+(x+y))+z))\\
&u+(v+(w+((x+y)+z)))\\
&u+(v+(w+(x+(y+z)))).
\end{aligned}$$

2.

a) We transform $E = w+(x+(y+z))$ into $F = ((w+x)+y)+z$. We do so by "pulling out" one operand a from both expressions, which are equivalent, and then repeating with the next operand until none are left.

We first choose to "pull out" w from both expressions first. This is already so for E. For F we follow the sequence

$$((w+x)+y)+z \to (w+(x+y))+z \to w+((x+y)+z). \tag{1}$$

Now to transform $E_1 = x+(y+z)$ into $F_1 = (x+y)+z$. We "pull out" x which is already accomplished for E_1. For F_1 we perform the transformation

$$(x+y)+z \to x+(y+z). \tag{2}$$

We "pull out" y next from $E_2 = y+z$ and $F_2 = y+z$. This is done so trivially.

We now transform what is left of the expressions E_2 and F_2 without y. Consider the expressions $E_3 = z$ and $F_3 = z$. E_3 naturally transforms into F_3. Furthermore, $E_2 = y+E_3$ can transform into $F_2 = y+F_3$, and $E_1 = x+E_2$ can transform into $F_1 = x+F_2$. Finally, $E = w+E_1$ can transform into $F = w+F_1$, and we are done. The sequence of transformations is

$w+(x+(y+z))$	Expression E
$w+((x+y)+z)$	(2) in reverse
$(w+(x+y))+z$	Middle of (1) in reverse
$((w+x)+y)+z$	Expression F, beginning of (1) in reverse

b) We transform $E = (v+w)+((x+y)+z)$ into $F = ((y+w)+(v+z))+x$.

We "pull out" v first from both expressions. The sequences of transformations for E and F respectively are

$$(v+w)+((x+y)+z) \to v+(w+((x+y)+z)) \tag{3}$$

and

$$\begin{aligned}
&((y+w)+(v+z))+x \to (((y+w)+v)+z)+x \\
&\to ((v+(y+w))+z)+x \to (v+((y+w)+z))+x \\
&\to v+(((y+w)+z)+x).
\end{aligned} \tag{4}$$

We "pull out" w from the subexpressions $w+((x+y)+z)$ and $((y+w)+z)+x$:

$$((y+w)+z)+x \to ((w+y)+z)+x \to (w+(y+z))+x \to w+((y+z)+x). \tag{5}$$

We shall "pull out" x from the subexpressions $(x+y)+z$ and $(y+z)+x$:

$$(x+y)+z \to x+(y+z) \tag{6}$$

and

$$(y+z)+x \to x+(y+z). \tag{7}$$

We then "pull out" y from the subexpressions $y+z$ and $y+z$. We are then left with the operand z in both expressions, which means we can transform one expression into the other. Thus $y+A_1$ can transform into $y+B_1$ if we consider $A_1 = z = B_1$. By successively letting the subexpressions of E and F (starting with z) being added to y, x, w, v in order, we transform E into F. The sequence of transformations is

$(v+w)+((x+y)+z)$	Expression E
$v+(w+((x+y)+z))$	(3)
$v+(w+(x+(y+z)))$	(6)
$v+(w+((y+z)+x))$	(7) in reverse
$v+((w+(y+z))+x)$	Middle-right of (5) in reverse
$v+(((w+y)+z)+x)$	Middle-left of (5) in reverse
$v+(((y+w)+z)+x)$	Beginning of (5) in reverse
$(v+((y+w)+z))+x$	Middle-right of (4) in reverse
$((v+(y+w))+z)+x$	Middle of (4) in reverse
$(((y+w)+v)+z)+x$	Middle-left of (4) in reverse
$((y+w)+(v+z))+x$	Expression F, beginning of (4) in reverse

3. We shall prove the following statement by complete induction on n, the number of occurrences of operators in an expression.

STATEMENT $S(n)$: Let E be an expression with operators $+, -, *$, and $/$. If E has n operator occurrences, then E has $n+1$ operands.

We choose zero as the basis because it is the least nonnegative number. By induction, the intuitive basis of one would be proved as well.

BASIS. Let $n = 0$. Then E has 1 operand, hence $S(0)$ is true.

INDUCTION. Assume $n \geq 0$ and $S(0), S(1), \ldots, S(n)$ are true. We shall prove $S(n+1)$. We assume that E has at least one operator, therefore E has at least two operands. Let the operands of E be the expressions E_1 and E_2. Since E has exactly $n+1$ operators, then either E_1 or E_2 has at most n operators, but not both. We apply the inductive hypothesis to E_2, meaning it has $n+1$ operands. Thus E_1 has only one operand, because E_1 has no operators. Together, E has $n+2$ operands. This proves the inductive step, and we conclude that $S(n)$ for all $n \geq 0$. ✦

We should have written that E_1 has n_1 operator occurrences and E_2 has n_2 operator occurrences and together there are $n_1 + n_2 = n$ operator occurrences. We also could have used a symbol to represent the operator in E, like θ.

6. We prove by complete induction the following statement on n, the length of the expression E.

STATEMENT $S(n)$: An expression E of length n having all binary operators has an odd length.

BASIS. Let $n = 1$. The expression E is only an operand, hence $S(1)$ is true.

INDUCTION. Assume $n \geq 1$ and $S(i)$ for $i = 1, 2, \ldots, n$. We shall prove $S(n+1)$. Let E be an expression of length $n+1$ having binary operators that can be written in the form $E_1 \theta E_2$, where E_1 and E_2 are expressions and θ is a binary operator. Let the length of E_1 be n_1 and the length of E_2 be n_2, and $n_1 + n_2 = n$. By the inductive hypothesis, n_1 and n_2 must be odd. The length of $E = E_1 \theta E_2$ is $n + 1 = n_1 + 1 + n_2$, which must be odd. Hence the inductive step is proven, and therefore $S(n)$ for $n \geq 1$. ✦

7. We prove the following statement by complete induction on n.

STATEMENT $S(n)$: Given a positive integer n, the integer $-n$ can be written in the form $2a + 3b$ for some integers a and b.

BASIS. Let $n = 1$. Select $a = 1$ and $b = -1$. Then $-n = -1 = 2 \cdot 1 + 3 \cdot -1$.

INDUCTION. Assume $n \geq 1$ and $S(1), S(2), \ldots, S(n)$ are true. We shall prove $S(n+1)$. That is, given a positive integer $n+1$, the integer $-(n+1)$ can be written in the form $2a + 3b$ for some integers a and b.

By the inductive hypothesis, we have $-n = 2a' + 3b'$ for some integers a' and b'. We subtract 1 from both sides to get $-n - 1 = 2a' + 3b' - 1$. The left side is $-(n+1)$ and we can express the right side as $2a' + 3b' + 2 - 3$. Hence we have

$$-(n+1) = 2(a'+1) + 3(b'-1).$$

If we let $a = a' + 1$ and $b = b' - 1$, then we have $S(n+1)$. Thus we have proved the induction. Therefore $S(n)$ for $n \geq 1$. ✦

As for the intuition of what to do after subtracting 1 from both sides, the basis tells us that -1 can be expressed in terms of 2 and 3. We cannot invoke S with -1, because S takes only positive integers, but we do not need S to yield the fact that $-1 = 2 - 3$.

8. We prove the following statement by complete induction on n.

STATEMENT $S(n)$: Every nonzero integer n can be written in the form $5a + 7b$ for some integers a and b.

We prove this statement for both positive and negative n sequentially.

BASIS.

i) Basis for positive n. Let $n = 1$. Select $a = 10$ and $b = -7$. Then $1 = 5 \times 10 + 7 \times -7$.

ii) Basis for negative n. Let $n = -1$. Select $a = -10$ and $b = 7$. Then $-1 = 5 \times -10 + 7 \times 7$.

INDUCTION. We first prove the inductive step for positive n. Assume $n \geq 1$ and $S(1), \ldots, S(n)$ are true. We must prove $S(n+1)$. By the inductive hypothesis $n = 5a' + 7b'$ for some integers a' and b'. We add 1 to both sides to get $n + 1 = 5a' + 7b' + 1$. We know by $S(1)$ that 1 can be expressed as $5 \times 10 + 7 \times -7$. We can thus rewrite the equation as

$$n + 1 = 5(a' + 10) + 7(b' - 7).$$

If we let $a = a' + 10$ and $b = b' - 7$, then we have $S(n+1)$. Therefore $S(n)$ is true for all $n \geq 1$.

Now we prove the inductive step for negative n. Assume $n \leq -1$ and $S(j)$ for $j = -1, -2, \ldots, n$ is true. We must prove $S(n-1)$. By the inductive hypothesis $n = 5a' + 7b'$ for some integers a' and b'. We follow the steps analogous to the inductive step for positive n, and end up with

$$n - 1 = 5(a' - 10) + 7(b' + 7).$$

If we let $a = a' - 10$ and $b = b' + 7$, then we have $S(n-1)$. Therefore $S(n)$ is true for all $n \leq -1$.

With both inductive steps proven, together they imply that $S(n)$ is true for all integers $n \neq 0$. ✦