1.

a) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

BASIS. The basis is $S(1)$. That is $\sum_{i=1}^{1} i = 1(1+1)/2 = 1$. This is indeed true and thus the basis of $S(n)$ holds.

INDUCTION. Let $n \geq 1$. We must prove that $S(n)$ implies $S(n+1)$. To prove $S(n+1)$, write

$$\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}. \tag{1}$$

The left side of Equation (1) is defined in terms of the inductive hypothesis $S(n)$. That is, we have

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1. \tag{2}$$

By the inductive hypothesis, the right side of Equation (2) is $n(n+1)/2 + n + 1$, which is equal to the right side of (1). We have thus proved Equation (1), which is $S(n+1)$, in terms of $S(n)$. Therefore $S(n)$ is true for $n \geq 1$. ✦

b) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

BASIS. The basis is $S(1)$. We substitute $n = 1$ and find

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}. \tag{3}$$

The summation on the left side of Equation (3) is equal to 1, and the right side of (3) is also 1. Thus we have proved the basis of $S(n)$.

INDUCTION. We must prove that $S(n)$ implies $S(n+1)$. Let $n \geq 1$ and write

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \tag{4}$$

Since $S(n+1)$ is defined in terms of $S(n)$, then we can write

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2, \tag{5}$$

where the term $(n+1)^2$ is added to the summation. By the inductive hypothesis the right side of Equation (5) is equal to the right side of (4). Thus we have proven that $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

c) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

BASIS. The basis is $S(1)$. That is

$$\sum_{i=1}^{1} i^3 = 1.$$

The summation has value 1 and is equal to the right side of the basis. Hence the basis of $S(n)$ is proven.

INDUCTION. We must prove the statement $S(n+1)$. Let $n \geq 1$. Write

$$\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4}. \tag{6}$$

By the inductive hypothesis, we decompose the summation on the left side of Equation (6) and get

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3. \tag{7}$$

The right side of Equation (7) is equal to the right side of Equation (6). Hence $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

d) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{(n+1)}$$

BASIS. The basis is $S(1)$. Write

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{(1+1)}. \tag{8}$$

This summation has value 1/2, which satisfies the basis $S(1)$.

INDUCTION. We must show that $S(n)$ implies $S(n+1)$. That is, if $S(n)$ is true and $S(n+1)$ is derived from $S(n)$, then $S(n+1)$ is true. Let $n \geq 1$ and write $S(n+1)$:

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{((n+1)+1)}. \tag{9}$$

The left side of Equation (9) is composed of the summation in $S(n)$ and one other term, namely $1/(n+1)(n+2)$. We expand this and get

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)}. \tag{10}$$

By $S(n)$, the summation on the right side of Equation (10) is equal to $n/(n+1)$. Thus the right side of (10) has value $(n+1)/(n+2)$, which is equal to the right side of Equation (9). Hence $S(n)$ implies $S(n+1)$ meaning $S(n)$ is true for all $n \geq 1$. ✦

We change the style of the proofs from hereon.

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2. We prove the following statement $S(n)$ by induction on n, for $n \geq 1$.

STATEMENT $S(n)$:

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6$$

That is, the summation of triangular numbers, or a sum of sums.

BASIS. We take the basis to be $S(1)$ because $j = 1$. We have

$$\sum_{j=1}^{1} t_j = 1.$$

There is only one term in the summation, namely $1(1+1)/2 = 1$. Thus the basis of $S(n)$ is true.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$, which is

$$\sum_{j=1}^{n+1} t_j = (n+1)(n+2)(n+3)/6.$$

We decompose the summation in terms of $S(n)$ to get

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6 + t_{n+1}.$$

We know that the term $t_{n+1} = (n+1)(n+2)/2$. Hence the right side is

$$\begin{aligned} &n(n+1)(n+2)/6 + (n+1)(n+2)/2 \\ &= n(n+1)(n+2)/6 + 3(n+1)(n+2)/6 \\ &= (n+1)(n+2)(n+3)/6. \end{aligned}$$

This expression is equal to the right side of $S(n+1)$. Thus we have shown that $S(n+1)$ is true. Therefore $S(n)$ is true for all $n \geq 1$. ✦

3.

a) 01101 has three 1's, so its parity is odd.
b) 111000111 has six 1's, so its parity is even.
c) 010101 has three 1's, so its parity is odd.

4. We shall prove the following statement $S(n)$:

STATEMENT $S(n)$: If C is any error-detecting set of strings of length n using the digits $0, 1$, and 2, then C cannot have more than 3^{n-1} strings for any $n \geq 1$.

BASIS. We take the basis to be $S(1)$. Then C is an error-detecting set of strings of length 1. There are three possible sets C, those are $0, 1, 2$. If C has more than one element then C would not be error-detecting, thus the three sets above are the only ones possible. Therefore C cannot have more than $3^{1-1} = 1$ string. This proves the basis.

INDUCTION. Assume that $n \geq 1$ and that $S(n)$ is true. We must prove $S(n+1)$. That is, an error-detecting set C of strings of length $n+1$ cannot have more than 3^n strings.

Let C_1 be the set such that to each element c in C, one digit is concatenated with c. Thus C_1 has strings of length $n+1$.