

**1.**

STATEMENT  $S(i)$ : If  $1 \leq i < n$ , then

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

BASIS. The basis is  $i = 1$ , as given by the lower bound of the interval. Then we have the inductive definition of the recurrence hence the basis holds.

INDUCTION. If  $i \geq n$  then there is nothing to prove. Suppose that  $i+1 < n$ . From the inductive hypothesis we expand  $T(n-i)$  to get

$$\begin{aligned} T(n) &= T(n-i-1) \sum_{j=0}^{i-1} g(n-j) + g(n-i) \\ &= T(n-i-1) + \sum_{j=0}^i g(n-j). \end{aligned}$$

This is the statement  $S(i+1)$  and we have proved the inductive step. We conclude  $S(i)$  is true for  $1 \leq i < n$ . ♦

**2.**

a) We have

$$\begin{aligned} T(n) &= T(n-1) + g(n) \\ &= T(n-2) + g(n-1) + g(n) \\ &= T(n-3) + g(n-2) + g(n-1) + g(n). \end{aligned}$$

Thus we have

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

But we know this already by Exercise 3.11.1.

Let  $i = n-1$ . Then we have the recurrence in terms of the basis, which is

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j). \tag{1}$$

But  $g(n) = n^2$ . This is a summation of squares. By Exercise 2.3.1(b) we know that

$$\sum_{j=1}^n j^2 = n(n+1)(2n+1)/6.$$

We adjust the bounds to get the equivalent summation

$$\sum_{j=1}^n j^2 = \sum_{j=0}^{n-1} (n-j)^2.$$

But this has the extra term for when  $j = n - 1$ , that is  $g(1)$ . We subtract 1 and get

$$T(n) = a + \sum_{j=0}^{n-1} (n-j)^2 - 1,$$

the same as Equation (1). We could have just taken the summation and decomposed it with the  $(n-2)$ th index then wrote the remaining summation as the summation of squares. But now that we have the last term separate, we rewrite the summation in terms of the identity

$$T(n) = a + n(n+1)(2n+1)/6 - 1.$$

Clearly  $T(n)$  is cubic, hence  $T(n)$  is  $O(n^3)$  if  $g(n) = n^2$ .

b) We have

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j)$$

where  $g(n) = n^2 + 3n$ . Then we have two summations, one of the  $n^2$  terms and the other of  $3n$  terms. Write

$$T(n) = a + \sum_{j=0}^{n-2} (n-j)^2 + \sum_{j=0}^{n-2} 3(n-j).$$

The left summation is  $n(n+1)(2n+1)/6 - 1$  as given by (a), and the right summation is

$$\sum_{j=0}^{n-2} 3(n-j) = 3 \sum_{j=0}^{n-2} (n-j) = 3(n-1)(n+2)/2.$$

Thus

$$T(n) = a + n(n+1)(2n+1)/6 - 1 + 3(n-1)(n+2)/2.$$

We see that  $T(n)$  is cubic. We conclude that  $T(n)$  is  $O(n^3)$ .

c) Again the solution to  $T(n)$  is  $T(n) = a + \sum_{j=0}^{n-2} g(n-j)$ . Here  $g(n) = n^{3/2}$ . In (a) we found that there are  $n-1$  terms of  $n^2$  and the same follows in (b). We can treat the summation as  $n-1$  terms of  $n\sqrt{n}$  with lower terms. Clearly  $(n-1)n\sqrt{n}$  is  $O(n^{5/2})$ . We conclude that  $T(n)$  is  $O(n^{5/2})$ .

d) Following (c), there are  $n-1$  terms of  $n \log n$  plus lower terms. Hence  $(n-1)(n \log n)$  is  $O(n^2 \log n)$ . We conclude that  $T(n)$  is  $O(n^2 \log n)$ .

e) Following (d), there are  $n-1$  terms of  $2^n$  plus lower terms. Therefore  $T(n)$  is  $O(2^n n)$ .

For this recurrence we can assume that for any  $g(n)$  that  $T(n)$  is  $O(ng(n))$ . But that is just a guess.

**3.** Let us first solve this recurrence. We write some terms in the sequence and get

$$\begin{aligned} T(n) &= T(n/2) + g(n) \\ &= T(n/4) + g(n/2) + g(n) \\ &= T(n/8) + g(n/4) + g(n/2) + g(n) \\ &= T(n/16) + g(n/8) + g(n/4) + g(n/2) + g(n). \end{aligned}$$

The pattern to find is  $T(n) = T(n/2^i) + g(n/2^{i-1}) + g(n/2^{i-2}) + \cdots + g(n/2^0)$ . That is

$$T(n) = T(n/2^i) + \sum_{j=0}^{i-1} g(n/2^j).$$

Now to express this in terms of the basis. Let  $i = \log_2 n$ . Then we have

$$T(n) = a + \sum_{j=0}^{\log_2 n - 1} g(n/2^j).$$

a) Suppose that  $g(n) = n^2$ . This is

$$T(n) = a + \sum_{j=0}^{\log_2 n - 1} (n/2^j)^2.$$

The exponent changes with each index. This is a geometric series  $a, ar, ar^2, \dots, ar^{n-1}$ . Let  $a = n^2$  and  $r = 1/4$ . We know by Exercise 2.3.9 that this is

$$\begin{aligned} \sum_{j=0}^{\log_2 n - 1} (n/2^j)^2 &= \frac{n^2(1/4)^{\log_2 n} - n^2}{1/4 - 1} \\ &= n^2 - \frac{(1/4)^{\log_2 n} - 1}{3/4}. \end{aligned}$$

Regardless of what the right term is, we have  $n^2$  minus some smaller term. Therefore  $T(n)$  is  $O(n^2)$ .

b) We have

$$T(n) = a + \sum_{j=0}^{\log_2 n - 1} 2(n/2^j).$$

Let  $a = 2n$  and  $r = 1/2$ . Each number in the series gets halved. The first number is  $2n$ , the greatest. Hence  $T(n)$  is  $O(n)$ .

c) We have

$$T(n) = a + \sum_{j=0}^{\log_2 n - 1} 10.$$

This is  $\log_2 n$  terms of 10. The number of terms is dependent on the upper bound. Hence  $T(n)$  is  $O(\log n)$ .

4. We shall solve this recurrence first. We begin with

$$\begin{aligned} T(n) &= 2T(n/2) + bn \\ &= 4T(n/4) + 2bn \\ &= 8T(n/8) + 3bn \\ &= 16T(n/16) + 4bn. \end{aligned}$$

The pattern to observe is  $T(n) = 2^i(n/2^i) + ibn$ . Let  $i = \log_2 n$ . Know that  $2^{\log_2 n} = n$  and  $n/2^{\log_2 n} = 1$ . Then we have

$$T(n) = 2^{\log_2 n} T(1) + bn \log_2 n = an + bn \log_2 n.$$

a) We shall guess that  $cn \log_2 n + dn + e$  is the solution to  $T(n)$ . Note that we know the solution which is  $an + bn \log_2 n$ .

STATEMENT  $S(n)$ : If  $n$  is a power of 2 and  $n \geq 1$ , then  $T(n) \leq cn \log_2 n + dn + e$ .

BASIS. If  $n = 1$  then  $a \leq c \log_2 1 + d = d + e$ .

INDUCTION. Assume  $S(i)$  for all  $i < n$ . We shall prove  $S(n)$  for some  $n > 1$ . We may assume  $S(n/2)$ . That is,  $T(n/2) \leq c(n/2) \log_2(n/2) + d(n/2) + e$ . Substituting for  $T(n/2)$  in the definition of  $T$  we have

$$\begin{aligned} T(n) &\leq 2(c(n/2) \log_2(n/2) + d(n/2) + e + b(n/2)) + bn \\ &= cn(\log_2 n - 1) + dn + 2e + 2bn \\ &= cn \log_2 n + (2b - c)n + dn + 2e. \end{aligned}$$

We must show that  $T(n) \leq cn \log_2 n + dn + e$  by showing the constraints on the excess. That is,  $(2b - c)n + e$  must be at most 0.

b) We guess that  $cn + d$  is the solution to  $T(n)$ .

STATEMENT  $S(n)$ : If  $n$  is a power of 2 and  $n \geq 1$ , then  $T(n) \leq cn + d$ .

BASIS. If  $n = 1$ , then  $a \leq c + d$ .

INDUCTION. Assume  $S(i)$  for all  $i < n$ . We shall prove  $S(n)$  for some  $n > 1$ . We may assume  $S(n/2)$ . We substitute to get

$$\begin{aligned} T(n) &\leq 2(cn/2 + d + bn/2) + bn \\ &= cn + 2d + 2bn. \end{aligned}$$

We must show that the excess  $d + 2bn \leq 0$ .