1.

a) 1.

b) 8, 9, 13, 6, 11, 15, 12.

c) 1, 2, 4, 3, 5, 10, 14, 7.

d) 5, 7.

e) 5, 10, 13, 14, 15.

f) 1, 3, 5, 10.

g) 10, 13, 14, 15.

h) 2, 4, 8, 9.

i) 6, 7, 11, 12

j) 1, 3, 5, 10, 14, 15.

k) 4.

l) 4.

m) 5.

2. A leaf is its own descendant. A leaf cannot have any proper descendants.

3. Let A and B be distinct leaves. If A is an ancestor of B, then A has at least one descendant. Thus A is an interior node. But A is a leaf. We have reached a contradiction. We conclude that a leaf cannot be an ancestor of another leaf. The same follows if we swap the names.

4. We prove the following statement by induction on n, the number of nodes in the tree.

STATEMENT $S(n)$: A tree having n nodes under the nonrecursive definition is a tree under the recursive definition.

BASIS. The basis is $n = 1$. The tree has one node, the root. This is a one-node tree under the recursive definition. This proves the basis.

INDUCTION. Assume $S(n)$ for $n \geq 1$. We shall prove $S(n+1)$. Let T_n be a tree having n nodes and root c_n satisfying the nonrecursive definition. By the inductive hypothesis, T_n is a tree under the recursive definition. Let T_{n+1} be a tree with $n+1$ nodes formed from T_n having root c_{n+1} such that c_{n+1} is the parent of c_n. Since properties (1), (2), and (3) of the nonrecursive definition apply to T_{n+1}, then T_{n+1} is a tree under the nonrecursive definition.

If we let $r = c_{n+1}$ and $T_1 = T_n$ and $c_1 = c_n$ within the recursive definition, then the construction of T_{n+1} with the nonrecursive definition satisfies the recursive definition of trees. This proves the inductive step. We conclude $S(n)$ is true for $n \geq 1$. ✦

We prove the following statement by induction on n, the number of rounds used in the recursive definition.

STATEMENT $S(n)$: A tree constructed under n applications of the recursive rule is a tree under the nonrecursive definition.

BASIS. Let $n = 1$. There is no node other than the root so properties (2) and (3) do not apply. Property (1) is satisfied, thus the tree having only one node is a tree under the nonrecursive definition. This proves the basis.

INDUCTION. Assume $S(n)$ for $n \geq 1$. We shall prove $S(n+1)$. Let T_n be a tree constructed under n applications of the recursive rule having root c_n. By the inductive hypothesis, T_n is a tree under the nonrecursive definition. Let c_{n+1} be a new node being the root of T_{n+1} having child c_n.

We have that c_{n+1} is the root of T_{n+1}, thus property (1) is satisfied. The nodes other than c_{n+1} must be connected by an edge because those nodes make up T_n, thus property (2) is satisfied. We need only consider that the parent of c_n is c_{n+1}, thus property (3) is satisfied. This proves the inductive step. We conclude that $S(n)$ is true for $n \geq 1$. ✦

5. Suppose the graph were a tree. Then it would have a root; it does not. Thus all three properties do not apply. Therefore the graph is not a tree.