1.

a) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

BASIS. The basis is $S(1)$. That is $\sum_{i=1}^{1} i = 1(1+1)/2 = 1$. This is indeed true and thus the basis of $S(n)$ holds.

INDUCTION. Let $n \geq 1$. We must prove that $S(n)$ implies $S(n+1)$. To prove $S(n+1)$, write

$$\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}. \tag{1}$$

The left side of Equation (1) is defined in terms of the inductive hypothesis $S(n)$. That is, we have

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1. \tag{2}$$

By the inductive hypothesis, the right side of Equation (2) is $n(n+1)/2 + n + 1$, which is equal to the right side of (1). We have thus proved Equation (1), which is $S(n+1)$, in terms of $S(n)$. Therefore $S(n)$ is true for $n \geq 1$. ✦

b) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

BASIS. The basis is $S(1)$. We substitute $n = 1$ and find

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}. \tag{3}$$

The summation on the left side of Equation (3) is equal to 1, and the right side of (3) is also 1. Thus we have proved the basis of $S(n)$.

INDUCTION. We must prove that $S(n)$ implies $S(n+1)$. Let $n \geq 1$ and write

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \tag{4}$$

Since $S(n+1)$ is defined in terms of $S(n)$, then we can write

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2, \tag{5}$$

where the term $(n+1)^2$ is added to the summation. By the inductive hypothesis the right side of Equation (5) is equal to the right side of (4). Thus we have proven that $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

c) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

BASIS. The basis is $S(1)$. That is

$$\sum_{i=1}^{1} i^3 = 1.$$

The summation has value 1 and is equal to the right side of the basis. Hence the basis of $S(n)$ is proven.

INDUCTION. We must prove the statement $S(n+1)$. Let $n \geq 1$. Write

$$\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4}. \tag{6}$$

By the inductive hypothesis, we decompose the summation on the left side of Equation (6) and get

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3. \tag{7}$$

The right side of Equation (7) is equal to the right side of Equation (6). Hence $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

d) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{(n+1)}$$

BASIS. The basis is $S(1)$. Write

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{(1+1)}. \tag{8}$$

This summation has value $1/2$, which satisfies the basis $S(1)$.

INDUCTION. We must show that $S(n)$ implies $S(n+1)$. That is, if $S(n)$ is true and $S(n+1)$ is derived from $S(n)$, then $S(n+1)$ is true. Let $n \geq 1$ and write $S(n+1)$:

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{((n+1)+1)}. \tag{9}$$

The left side of Equation (9) is composed of the summation in $S(n)$ and one other term, namely $1/(n+1)(n+2)$. We expand this and get

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)}. \tag{10}$$

By $S(n)$, the summation on the right side of Equation (10) is equal to $n/(n+1)$. Thus the right side of (10) has value $(n+1)/(n+2)$, which is equal to the right side of Equation (9). Hence $S(n)$ implies $S(n+1)$ meaning $S(n)$ is true for all $n \geq 1$. ✦

We change the style of the proofs from hereon.

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2. We prove the following statement $S(n)$ by induction on n, for $n \geq 1$.

STATEMENT $S(n)$:

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6$$

That is, the summation of triangular numbers, or a sum of sums.

BASIS. We take the basis to be $S(1)$ because $j = 1$. We have

$$\sum_{j=1}^{1} t_j = 1.$$

There is only one term in the summation, namely $1(1+1)/2 = 1$. Thus the basis of $S(n)$ is true.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$, which is

$$\sum_{j=1}^{n+1} t_j = (n+1)(n+2)(n+3)/6.$$

We decompose the summation in terms of $S(n)$ to get

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6 + t_{n+1}.$$

We know that the term $t_{n+1} = (n+1)(n+2)/2$. Hence the right side is

$$\begin{aligned}
&n(n+1)(n+2)/6 + (n+1)(n+2)/2 \\
&= n(n+1)(n+2)/6 + 3(n+1)(n+2)/6 \\
&= (n+1)(n+2)(n+3)/6.
\end{aligned}$$

This expression is equal to the right side of $S(n+1)$. Thus we have shown that $S(n+1)$ is true. Therefore $S(n)$ is true for all $n \geq 1$. ✦

3.

a) 01101 has three 1's, so its parity is odd.
b) 111000111 has six 1's, so its parity is even.
c) 010101 has three 1's, so its parity is odd.

4. We shall prove the following statement $S(n)$ by induction on n, analogous to Example 2.6 in the book. Written below is essentially the template for the proof but I used it as the actual proof instead.

STATEMENT $S(n)$: If C is any error-detecting set of strings of length n using the digits $0, 1,$ and 2, then C cannot have more than 3^{n-1} strings for any $n \geq 1$.

BASIS. We take the basis to be $S(1)$. Then C is an error-detecting set of strings of length 1. There are three possible sets C, those are $\{0\}, \{1\}$, and $\{2\}$. If C has more than one element then C would not be error-detecting, thus the three sets above are the only ones possible. Therefore C cannot have more than $3^{1-1} = 1$ string. This proves the basis.

INDUCTION. Assume that $n \geq 1$ and that $S(n)$ is true. We must prove $S(n+1)$. That is, an error-detecting set C of strings of length $n+1$ cannot have more than 3^n strings.

Divide C into three sets C_0, C_1, C_2, each being the set of strings in C that begin with $0, 1, 2$ respectively. Then remove the leading digit from each string in C_0, C_1, C_2 to coerce the sets into having strings of length n. Since C is error-detecting, then so are C_0, C_1, C_2. Apply the inductive hypothesis to C_0, C_1, C_2 to prove that they each cannot have more than 3^{n-1} strings. Since every string in C is in either C_0, C_1, C_2, then C cannot have more than $3 \cdot 3^{n-1} = 3^n$ strings. This concludes the proof. ✦

5. We prove the following statement by induction on n.

STATEMENT $S(n)$: There is an error-detecting set of strings of length n for any $n \geq 1$, using the digits $0, 1,$ and 2, that has 3^{n-1} strings.

BASIS. The basis is $S(1)$. That is, there is an error-detecting set of strings of length 1, using the digits 0, 1, and 2, that has $3^{1-1} = 1$ string. There are three strings of length one, those being 0, 1, and 2. However only one can be in an error-detecting set, meaning this set has one string. Thus there is such a set and the basis is proven.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$. That is, there is an error-detecting set of strings of length $n+1$, using the digits 0, 1, and 2, that has 3^n strings.

Assume by the inductive hypothesis that there are three error-detecting sets of strings C_0, C_1, C_2 of length n that have 3^{n-1} strings. Let D_0, D_1, D_2 be the sets of strings with 0, 1, and 2 added as the leading digit to each string in C_0, C_1, C_2 respectively. Thus D_0, D_1, D_2 have strings of length $n+1$ and are error-detecting since C_0, C_1, C_2 are error-detecting.

Let C be the set containing all the strings in D_0, D_1, D_2. Thus C is an error-detecting set, because D_0, D_1, D_2 are, having strings of length $n+1$. We conclude that C has $3 \cdot 3^{n-1} = 3^n$ strings. We have proven $S(n+1)$ hence $S(n)$ holds for $n \geq 1$. ✦

6. We shall prove the following statement by induction on n. The statement is a combination of exercises 2.3.4 and 2.3.5 in the book. We must prove the existence of a set and what its maximum number of elements are. We prove analogously to these exercises.

STATEMENT $S(n)$: There is an error-detecting set of strings of length n, using k different symbols as "digits," for any $k \geq 2$, with k^{n-1} strings, but no such set of strings with more than k^{n-1} strings.

We prove by induction on n, not k. The statement is dependent on the length of the strings, not the number of symbols used. Some intuition about why we do not do induction on k is because in earlier proofs we do not always use all the symbols provided for a string. If there is a string of length one, then it does not matter how many symbols we use.

BASIS. We prove the basis $S(1)$, that is for strings of length 1. There is such a set satisfying $S(1)$. Select any symbol k_0. Then there is the set $\{k_0\}$ with $k^{1-1} = 1$ string. If we add any other string to this set using any symbol then the set would no longer be error-detecting, so it cannot have more than one string.

INDUCTION. Assume $S(n)$ is true and $n \geq 1$. We now prove $S(n+1)$. That is, there is an error-detecting set of strings C of length $n+1$, using k different symbols as "digits," with k^n strings, but no such set of strings with more than k^n strings.

We prove the existence of C. By $S(n)$, assume there are k error-detecting sets of strings $C_1, \ldots, C_k$ of length n using k different symbols, with k^{n-1} strings. Prepend each string in $C_1, \ldots, C_k$ with the symbol k_m where m is each set's respective index. The sets retain their error-detecting property and are now of strings of length $n+1$. Let C be the set containing all the strings in $C_1, \ldots, C_k$. Then C is an error-detecting set of strings of length $n+1$ using k different symbols.

Now we prove that C cannot have more than k^n strings. Since $C_1, \ldots, C_k$ have at most k^{n-1} strings by the inductive hypothesis, and that all their strings are in C, then C has at most $k \cdot k^{n-1} = k^n$ strings. Thus C has no more than k^n strings.

Together, the set satisfying the inductive step exists, with k^n strings, and has no more than k^n strings. Therefore $S(n+1)$ is true. We conclude that $S(n)$ is true for $n \geq 1$. ✦

7. We shall prove the following statement by induction on n.

STATEMENT $S(n)$: If $n \geq 1$, the number of strings using the digits 0, 1, and 2, with no two consecutive places holding the same digit, is $3 \times 2^{n-1}$.

We prove this claim by induction on the length of the strings n. Suppose we take the basis to be $S(0)$. The number of strings with no two consecutive places holding the same digit is $3 \cdot 2^{-1} = 3/2$. But there cannot be half-strings. Thus we do not use $S(0)$ as the basis.

BASIS. We take $S(1)$ as the basis. We must show that there are $3 \cdot 2^{1-1} = 3$ strings of length one with no two consecutive places holding the same digit, using the digits 0, 1, and 2. Those strings are 0, 1, and 2. This proves the basis.

INDUCTION. Assume $n \geq 1$ and $S(n)$ is true. Then we must prove $S(n+1)$. That is, there are $3 \cdot 2^n$ strings of length $n+1$ with no two consecutive places holding the same digit, using the digits 0, 1, and 2.

By the inductive hypothesis there is the set of $3 \cdot 2^{n-1}$ strings D with no two consecutive places holding the same digit using the digits 0, 1, and 2. Divide D

into three sets of strings D_0, D_1, D_2 with their strings beginning with 0, 1, and 2 respectively. That is, D_0, D_1, D_2 each have 2^{n-1} strings of length n.

Let C_0 be the set of strings containing all the strings in D_0 prepended with 1 and all the strings in D_0 prepended with 2. Similarly, let C_1 and C_2 be the sets of strings containing all those in D_1 and D_2 but prepended with 0, 2, and 0, 1, respectively. Then the three sets C_0, C_1, C_2 each have twice the number of strings in either of the subsets of D, and have strings of length $n+1$.

Thus we have $3 \cdot 2 \cdot 2^{n-1} = 3 \cdot 2^n$ strings together in C_0, C_1, C_2. This proves the inductive step. Therefore $S(n)$ is true for $n \geq 1$. ✦

8. We prove that the ripple-carry algorithm is correct by induction.

STATEMENT $S(i)$: Under ripple-carry addition, the sum of the tails of length i for the two addends equals the number whose binary representation is the carry bit followed by the i bits of answer generated.

BASIS. Take the basis $S(1)$, which is that the sum of the right-most digits for the two addends equals the number whose binary representation is the carry bit followed by the 1 bit of answer generated. There are three bits to consider, the bits in the addends and the carry bit. If none of the bits are 1 then the sum is 0. If one of the bits is 1 then the sum is 1. If two of the bits are 1 then the sum is 0 with a carry bit of 1. If the three of the bits are 1 then the sum is 1 with a carry bit of 1. This is indeed the binary representation of the sum of the right-most digits.

INDUCTION. We shall prove $S(i+1)$. That is, the sum of the first $i+1$ digits for the two addends equals the number whose binary representation is the carry bit followed by the $i+1$ bits of answer generated. By the inductive hypothesis, that is we assume the first i bits of the answer have been generated, we take the carry bit and add it to the $i+1$-th digits in the two addends. Thus the inductive step is proven. Therefore $S(i)$ is true for $i \geq 1$. ✦

9. We prove the following statement by induction on n.

STATEMENT $S(n)$:

$$\sum_{i=0}^{n-1} ar^i = \frac{(ar^n - a)}{(r-1)}$$

BASIS. We take the basis $S(1)$ and assume $r \neq 1$. Then i equals the upper bound so there is only one term in the summation. The summation equals a and the right side of the equation is equal to $a(r^1 - 1)/(r-1) = a$. This proves the basis.

INDUCTION. Assume $n \geq 1, r \neq 1$ and $S(n)$ is true. We now prove $S(n+1)$. That is, we must prove

$$\sum_{i=0}^{n} ar^i = \frac{(ar^{n+1} - a)}{(r-1)}.$$

We can decompose the summation on the left side in terms of $S(n)$. Thus we have

$$\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n-1} ar^i + ar^n.$$

We expand the summation on the right side and get

$$
\begin{aligned}
&\frac{ar^n - a}{r-1} + ar^n \\
&= \frac{ar^n - a + ar^n(r-1)}{r-1} \\
&= \frac{ar^n - a + ar^{n+1} - ar^n}{r-1} \\
&= \frac{ar^{n+1} - a}{r-1}.
\end{aligned}
$$

Hence the right side of the equation is equal to the right side of $S(n+1)$. We have proved the inductive step. Therefore $S(n)$ is true for $n \geq 1$. ✦

10.

a) We prove the following formula by induction on n. STATEMENT $S(n)$:

$$\sum_{i=0}^{n-1} a + bi = n(2a + (n-1)b)/2.$$

BASIS. We take the basis $S(1)$. Thus there is only one term in the summation. That term is a. Substitute n for 1 on the right side to get $1(2a + (1-1)b)/2 = 2a/2 = a$. Thus the basis is true.

INDUCTION. Assume $S(n)$ is true. We must prove $S(n+1)$. That is

$$\sum_{i=0}^{n} a + bi = (n+1)(2a + bn)/2.$$

Decompose the summation in terms of $S(n)$ to get

$$\sum_{i=0}^{n} a + bi = (\sum_{i=0}^{n-1} a + bi) + (a + bn).$$

Expand the right side to get

$$
\begin{aligned}
&(\sum_{i=0}^{n-1} a + bi) + (a + bn) \\
&= n(2a + (n-1)b)/2 + (a + bn) \\
&= n(2a + (n-1)b)/2 + 2(a + bn)/2 \\
&= (n(2a + (n-1)b) + 2(a + bn))/2 \\
&= (2an + n(n-1)b + 2a + 2bn)/2 \\
&= (2a(n+1) + n(n-1)b + 2bn)/2 \\
&= (2a(n+1) + bn(n-1+2))/2 \\
&= (2a(n+1) + bn(n+1))/2 \\
&= (n+1)(2a + bn)/2.
\end{aligned}
$$

This is equal to the right side of $S(n+1)$. Hence $S(n+1)$ is proven and $S(n)$ is true for $n \geq 1$. ✦

b) We show how Exercise 2.3.1(a) is an example of this formula.

Take the summation $\sum_{i=1}^{n} i$. We can rewrite it as $\sum_{i=1}^{n} a + bi$ where $a = 0$ and $b = 1$. We (optionally) do not take a lower bound of 0 for the second summation because the value of that term is 0. We now prove that the value of the first summation, which is $n(n+1)/2$, is equal to the value of the second summation, which is $(n+1)(2a+bn)/2$. Write

$$
\begin{aligned}
&n(n+1)/2 \\
&= (n+1)(2(0)+(1)n)/2 \\
&= (n+1)n/2.
\end{aligned}
$$

Thus we have shown that $\sum_{i=1}^{n} i$ is an example of $\sum_{i=1}^{n} a + bi$.

11. We informally prove that induction starting at 1 "works" on the assumption that induction starting at 1 works.

a) We do this analogously to the book. Suppose $S(n)$ were not true for at least one value of n. Let a be the greatest nonnegative integer for which $S(a)$ is false. If $a = 1$, then we contradict the basis, so a must be greater than 1. If $a > 1$, and a is the greatest nonnegative integer for which $S(a)$ is false, then $S(a+1)$ must be true. The inductive step, with n replaced by a, tells us that $S(a)$ implies $S(a+1)$. Since $S(a+1)$ is true, because $a+1 > a$, and that $S(a)$ implies $S(a+1)$, then $S(a)$ is true. But $S(a)$ is not true, so this is a contradiction. Since we assumed there were nonnegative values of n for which $S(n)$ is false and derived a contradiction, $S(n)$ must therefore be true for any $n \geq 0$.

b) Suppose we want to show that $S(n)$ is true for all nonnegative integers. If $n = 1$, then we invoke the basis, which we assume to be true. Let a be an arbitrary nonnegative integer. We must show that if $n = a$, then $S(a)$ is true. We claim that $S(a)$ is true if $S(a-1)$ is true. If the truthfulness of $S(a-1)$ is not self-evident, then we find the truthfulness of $S(a-2)$. We continue this until we get to $S(1)$, which is evidently true. Thus $S(2)$ is true, and $S(3)$ is true, and so on, and $S(a)$ is true. Since $n = a$, then for any n we have that $S(n)$ is true.

12. We prove the following statement by induction on n.

STATEMENT $S(n)$: The code consisting of the odd-parity strings of length n detects errors.

BASIS. Let $S(1)$ be the basis. There is only one such possible string in this code, which is 1, and is error-detecting. This proves the basis.

INDUCTION. Assume $S(n)$ is true and $n \geq 1$. We must prove $S(n+1)$. That is, the code consisting of the odd-parity strings of length $n+1$ detects errors.

By the inductive hypothesis, the code consisting of the odd-parity strings of length n detects errors. We attach a bit to the strings to make them of length $n+1$. The bit must be a 0 so the strings are still of odd-parity, and retain their error-detecting property. This proves the inductive step, hence $S(n)$ is true for $n \geq 1$. ✦