

1.

a) STATEMENT $S(n)$:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

BASIS. The basis is $S(1)$. That is $\sum_{i=1}^1 i = 1(1+1)/2 = 1$. This is indeed true and thus the basis of $S(n)$ holds.

INDUCTION. Let $n \geq 1$. We must prove that $S(n)$ implies $S(n+1)$. To prove $S(n+1)$, write

$$\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}. \quad (1)$$

The left side of Equation (1) is defined in terms of the inductive hypothesis $S(n)$. That is, we have

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^n i + n + 1. \quad (2)$$

By the inductive hypothesis, the right side of Equation (2) is $n(n+1)/2 + n + 1$, which is equal to the right side of (1). We have thus proved Equation (1), which is $S(n+1)$, in terms of $S(n)$. Therefore $S(n)$ is true for $n \geq 1$. ♦

b) STATEMENT $S(n)$:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

BASIS. The basis is $S(1)$. We substitute $n = 1$ and find

$$\sum_{i=1}^1 i^2 = \frac{1(1+1)(2+1)}{6}. \quad (3)$$

The summation on the left side of Equation (3) is equal to 1, and the right side of (3) is also 1. Thus we have proved the basis of $S(n)$.

INDUCTION. We must prove that $S(n)$ implies $S(n+1)$. Let $n \geq 1$ and write

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \quad (4)$$

Since $S(n+1)$ is defined in terms of $S(n)$, then we can write

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^n i^2 + (n+1)^2, \quad (5)$$

where the term $(n+1)^2$ is added to the summation. By the inductive hypothesis the right side of Equation (5) is equal to the right side of (4). Thus we have proven that $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ♦

c) STATEMENT $S(n)$:

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

BASIS. The basis is $S(1)$. That is

$$\sum_{i=1}^1 i^3 = 1.$$

The summation has value 1 and is equal to the right side of the basis. Hence the basis of $S(n)$ is proven.

INDUCTION. We must prove the statement $S(n+1)$. Let $n \geq 1$. Write

$$\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4}. \quad (6)$$

By the inductive hypothesis, we decompose the summation on the left side of Equation (6) and get

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^n i^3 + (n+1)^3. \quad (7)$$

The right side of Equation (7) is equal to the right side of Equation (6). Hence $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ♦

d) STATEMENT $S(n)$:

$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{(n+1)}$$

BASIS. The basis is $S(1)$. Write

$$\sum_{i=1}^1 \frac{1}{i(i+1)} = \frac{1}{(1+1)}. \quad (8)$$

This summation has value 1/2, which satisfies the basis $S(1)$.

INDUCTION. We must show that $S(n)$ implies $S(n+1)$. That is, if $S(n)$ is true and $S(n+1)$ is derived from $S(n)$, then $S(n+1)$ is true. Let $n \geq 1$ and write $S(n+1)$:

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{((n+1)+1)}. \quad (9)$$

The left side of Equation (9) is composed of the summation in $S(n)$ and one other term, namely $1/(n+1)(n+2)$. We expand this and get

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^n \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)}. \quad (10)$$

By $S(n)$, the summation on the right side of Equation (10) is equal to $n/(n+1)$. Thus the right side of (10) has value $(n+1)/(n+2)$, which is equal to the right side of Equation (9). Hence $S(n)$ implies $S(n+1)$ meaning $S(n)$ is true for all $n \geq 1$.

◆

We change the style of the proofs from hereon.

2. We prove the following statement $S(n)$ by induction on n , for $n \geq 1$.

STATEMENT $S(n)$:

$$\sum_{j=1}^n t_j = n(n+1)(n+2)/6$$

That is, the summation of triangular numbers, or a sum of sums.

BASIS. We take the basis to be $S(1)$ because $j = 1$. We have

$$\sum_{j=1}^1 t_j = 1.$$

There is only one term in the summation, namely $1(1+1)/2 = 1$. Thus the basis of $S(n)$ is true.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$, which is

$$\sum_{j=1}^{n+1} t_j = (n+1)(n+2)(n+3)/6.$$

We decompose the summation in terms of $S(n)$ to get

$$\sum_{j=1}^n t_j = n(n+1)(n+2)/6 + t_{n+1}.$$

We know that the term $t_{n+1} = (n+1)(n+2)/2$. Hence the right side is

$$\begin{aligned} & n(n+1)(n+2)/6 + (n+1)(n+2)/2 \\ &= n(n+1)(n+2)/6 + 3(n+1)(n+2)/6 \\ &= (n+1)(n+2)(n+3)/6. \end{aligned}$$

This expression is equal to the right side of $S(n+1)$. Thus we have shown that $S(n+1)$ is true. Therefore $S(n)$ is true for all $n \geq 1$. ◆

3.

- a) 01101 has three 1's, so its parity is odd.
- b) 111000111 has six 1's, so its parity is even.
- c) 010101 has three 1's, so its parity is odd.

4. We shall prove the following statement $S(n)$ by induction on n , analogous to Example 2.6 in the book. Written below is essentially the template for the proof but I used it as the actual proof instead.

STATEMENT $S(n)$: If C is any error-detecting set of strings of length n using the digits 0, 1, and 2, then C cannot have more than 3^{n-1} strings for any $n \geq 1$.

BASIS. We take the basis to be $S(1)$. Then C is an error-detecting set of strings of length 1. There are three possible sets C , those are $\{0\}$, $\{1\}$, and $\{2\}$. If C has more than one element then C would not be error-detecting, thus the three sets above are the only ones possible. Therefore C cannot have more than $3^{1-1} = 1$ string. This proves the basis.

INDUCTION. Assume that $n \geq 1$ and that $S(n)$ is true. We must prove $S(n+1)$. That is, an error-detecting set C of strings of length $n+1$ cannot have more than 3^n strings.

Divide C into three sets C_0, C_1, C_2 , each being the set of strings in C that begin with 0, 1, 2 respectively. Then remove the leading digit from each string in C_0, C_1, C_2 to coerce the sets into having strings of length n . Since C is error-detecting, then so are C_0, C_1, C_2 . Apply the inductive hypothesis to C_0, C_1, C_2 to prove that they each cannot have more than 3^{n-1} strings. Since every string in C is in either C_0, C_1, C_2 , then C cannot have more than $3 \cdot 3^{n-1} = 3^n$ strings. This concludes the proof. ♦

5. We prove the following statement by induction on n .

STATEMENT $S(n)$: There is an error-detecting set of strings of length n for any $n \geq 1$, using the digits 0, 1, and 2, that has 3^{n-1} strings.

BASIS. The basis is $S(1)$. That is, there is an error-detecting set of strings of length 1, using the digits 0, 1, and 2, that has $3^{1-1} = 1$ string. There are three strings of length one, those being 0, 1, and 2. However only one can be in an error-detecting set, meaning this set has one string. Thus there is such a set and the basis is proven.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$. That is, there is an error-detecting set of strings of length $n+1$, using the digits 0, 1, and 2, that has 3^n strings.

Assume by the inductive hypothesis that there are three error-detecting sets of strings C_0, C_1, C_2 of length n that have 3^{n-1} strings. Let D_0, D_1, D_2 be the sets of strings with 0, 1, and 2 added as the leading digit to each string in C_0, C_1, C_2 respectively. Thus D_0, D_1, D_2 have strings of length $n+1$ and are error-detecting since C_0, C_1, C_2 are error-detecting.

Let C be the set containing all the strings in D_0, D_1, D_2 . Thus C is an error-detecting set, because D_0, D_1, D_2 are, having strings of length $n+1$. We conclude that C has $3 \cdot 3^{n-1} = 3^n$ strings. We have proven $S(n+1)$ hence $S(n)$ holds for $n \geq 1$. ♦

6. We shall prove the following statement by induction on k . The statement is a combination of exercises 2.3.4 and 2.3.5 in the book. We must prove the existence of a set and what its maximum number of elements are.

STATEMENT $S(k)$: There is an error-detecting set of strings of length n , using k different symbols as “digits,” for any $k \geq 2$, with k^{n-1} strings, but no such set of strings with more than k^{n-1} strings.

We do not prove $S(1)$ as the basis. A string is not error-detecting if it is composed of one symbol, no matter the length. We also assume a sensible length (for those strings of length $n \geq 2$, since those of length 1). But strings can only be error-detecting if they use $k \geq 2$ symbols. This is why we prove by induction on k and not n , because the length does not matter.

BASIS. For the basis we take $k = 2$. Choose a length n , say 2. Then there is an error-detecting set of strings of length 2, using 2 different symbols, with $2^{2-1} = 2$ strings. A possible set is $\{00, 11\}$. If we add a third string to this set then it would no longer be error-detecting. Thus it cannot have more than 2 strings. This proves the basis of $S(k)$.

INDUCTION. Assume $S(k)$ is true and $k \geq 2$. We must prove $S(k + 1)$. That is, there is an error-detecting set of strings of length n , using $k + 1$ different symbols as “digits,” with $(k + 1)^{n-1}$ strings, but no such set of strings with more than $(k + 1)^{n-1}$ strings.