1. We "pull out" each of the operands of $E = (u+v) + ((w + (x+y)) + z)$. We perform this arbitrarily from left to right.

By the associative law, E can be transformed into $u + (v + ((w + (x+y)) + z))$. Thus we have $E = u + E_1$ where $E_1 = v + ((w + (x+y)) + z)$. We trivially pull out v from E_1 to get an expression of the form $v + E_2$ where $E_2 = (w + (x+y)) + z$. With the associative law we transform E_2 into an expression of the form $w + E_3$ where $E_3 = (x+y) + z$. Similarly we transform E_3 into an expression of the form $x + E_4$ where $E_4 = y + z$. We transform E_4 into an expression of the form $y + E_5$ where $E_5 = z$. The sequence of transformations is

$$\begin{aligned}
&(u+v) + ((w + (x+y)) + z)\\
&u + (v + ((w + (x+y)) + z))\\
&u + (v + (w + ((x+y) + z)))\\
&u + (v + (w + (x + (y+z)))).
\end{aligned}$$

2.

a) We transform $E = w + (x + (y+z))$ into $F = ((w+x)+y)+z$. We do so by "pulling out" one operand a from both expressions, which are equivalent, and then repeating with the next operand until none are left.

We first choose to "pull out" w from both expressions first. This is already so for E. For F we follow the sequence

$$((w+x)+y)+z \to (w + (x+y)) + z \to w + ((x+y)+z). \tag{1}$$

Now to transform $E_1 = x + (y+z)$ into $F_1 = (x+y)+z$. We "pull out" x which is already accomplished for E_1. For F_1 we perform the transformation

$$(x+y)+z \to x + (y+z). \tag{2}$$

We "pull out" y next from $E_2 = y+z$ and $F_2 = y+z$. This is done so trivially.

We now transform what is left of the expressions E_2 and F_2 without y. Consider the expressions $E_3 = z$ and $F_3 = z$. E_3 naturally transforms into F_3. Furthermore, $E_2 = y + E_3$ can transform into $F_2 = y + F_3$, and $E_1 = x + E_2$ can transform into $F_1 = x + F_2$. Finally, $E = w + E_1$ can transform into $F = w + F_1$, and we are done. The sequence of transformations is

$w + (x + (y+z))$	Expression E
$w + ((x+y)+z)$	(2) in reverse
$(w + (x+y)) + z$	Middle of (1) in reverse
$((w+x)+y)+z$	Expression F, beginning of (1) in reverse

b) We transform $E = (v+w) + ((x+y)+z)$ into $F = ((y+w) + (v+z)) + x$.

We "pull out" v first from both expressions. The sequences of transformations for E and F respectively are

$$(v+w) + ((x+y)+z) \to v + (w + ((x+y)+z)) \tag{3}$$

and

$$
\begin{aligned}
&((y+w)+(v+z))+x \to (((y+w)+v)+z)+x \\
&\to ((v+(y+w))+z)+x \to (v+((y+w)+z))+x \\
&\to v+(((y+w)+z)+x).
\end{aligned} \tag{4}
$$

We "pull out" w from the subexpressions $w+((x+y)+z)$ and $((y+w)+z)+x$:

$$((y+w)+z)+x \to ((w+y)+z)+x \to (w+(y+z))+x \to w+((y+z)+x). \tag{5}$$

We shall "pull out" x from the subexpressions $(x+y)+z$ and $(y+z)+x$:

$$(x+y)+z \to x+(y+z) \tag{6}$$

and

$$(y+z)+x \to x+(y+z). \tag{7}$$

We then "pull out" y from the subexpressions $y+z$ and $y+z$. We are then left with the operand z in both expressions, which means we can transform one expression into the other. Thus $y+A_1$ can transform into $y+B_1$ if we consider $A_1 = z = B_1$. By successively letting the subexpressions of E and F (starting with z) being added to y, x, w, v in order, we transform E into F. The sequence of transformations is

$(v+w)+((x+y)+z)$	Expression E
$v+(w+((x+y)+z))$	(3)
$v+(w+(x+(y+z)))$	(6)
$v+(w+((y+z)+x))$	(7) in reverse
$v+((w+(y+z))+x)$	Middle-right of (5) in reverse
$v+(((w+y)+z)+x)$	Middle-left of (5) in reverse
$v+(((y+w)+z)+x)$	Beginning of (5) in reverse
$(v+((y+w)+z))+x$	Middle-right of (4) in reverse
$((v+(y+w))+z)+x$	Middle of (4) in reverse
$(((y+w)+v)+z)+x$	Middle-left of (4) in reverse
$((y+w)+(v+z))+x$	Expression F, beginning of (4) in reverse

3. We shall prove the following statement by complete induction on n, the number of occurrences of operators in an expression.

STATEMENT $S(n)$: Let E be an expression with operators $+, -, *,$ and $/$. If E has n operator occurrences, then E has $n+1$ operands.

We choose zero as the basis because it is the least nonnegative number. By induction, the intuitive basis of one would be proved as well.

BASIS. Let $n = 0$. Then E has 1 operand, hence $S(0)$ is true.

INDUCTION. Assume $n \geq 0$ and $S(0), S(1), \ldots, S(n)$ are true. We shall prove $S(n+1)$. We assume that E has at least one operator, therefore E has at least two operands. Let the operands of E be the expressions E_1 and E_2. Since E has exactly $n+1$ operators, then either E_1 or E_2 has at most n operators, but not both. We apply the inductive hypothesis to E_2, meaning it has $n+1$ operands. Thus E_1 has only one operand, because E_1 has no operators. Together, E has $n+2$ operands. This proves the inductive step, and we conclude that $S(n)$ for all $n \geq 0$.✦

We should have written that E_1 has n_1 operator occurrences and E_2 has n_2 operator occurrences and together there are $n_1 + n_2 = n$ operator occurrences. We also could have used a symbol to represent the operator in E, like θ.

6. We prove by complete induction the following statement on n, the length of the expression E.

STATEMENT $S(n)$: An expression E of length n having all binary operators has an odd length.

BASIS. Let $n = 1$. The expression E is only an operand, hence $S(1)$ is true.

INDUCTION. Assume $n \geq 1$ and $S(i)$ for $i = 1, 2, \ldots, n$. We shall prove $S(n+1)$. Let E be an expression with length $n+1$ that can be written in the form $E_1 \theta E_2$, where E_1 and E_2 are expressions and θ is a binary operator. If n is odd, then we apply the inductive hypothesis, and find that E