1.

a) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

BASIS. The basis is $S(1)$. That is $\sum_{i=1}^{1} i = 1(1+1)/2 = 1$. This is indeed true and thus the basis of $S(n)$ holds.

INDUCTION. Let $n \geq 1$. We must prove that $S(n)$ implies $S(n+1)$. To prove $S(n+1)$, write

$$\sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}. \tag{1}$$

The left side of Equation (1) is defined in terms of the inductive hypothesis $S(n)$. That is, we have

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1. \tag{2}$$

By the inductive hypothesis, the right side of Equation (2) is $n(n+1)/2 + n + 1$, which is equal to the right side of (1). We have thus proved Equation (1), which is $S(n+1)$, in terms of $S(n)$. Therefore $S(n)$ is true for $n \geq 1$. ✦

b) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

BASIS. The basis is $S(1)$. We substitute $n = 1$ and find

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}. \tag{3}$$

The summation on the left side of Equation (3) is equal to 1, and the right side of (3) is also 1. Thus we have proved the basis of $S(n)$.

INDUCTION. We must prove that $S(n)$ implies $S(n+1)$. Let $n \geq 1$ and write

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \tag{4}$$

Since $S(n+1)$ is defined in terms of $S(n)$, then we can write

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2, \tag{5}$$

where the term $(n+1)^2$ is added to the summation. By the inductive hypothesis the right side of Equation (5) is equal to the right side of (4). Thus we have proven that $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

c) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

BASIS. The basis is $S(1)$. That is

$$\sum_{i=1}^{1} i^3 = 1.$$

The summation has value 1 and is equal to the right side of the basis. Hence the basis of $S(n)$ is proven.

INDUCTION. We must prove the statement $S(n+1)$. Let $n \geq 1$. Write

$$\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4}. \tag{6}$$

By the inductive hypothesis, we decompose the summation on the left side of Equation (6) and get

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3. \tag{7}$$

The right side of Equation (7) is equal to the right side of Equation (6). Hence $S(n)$ implies $S(n+1)$. Therefore $S(n)$ holds for $n \geq 1$. ✦

d) STATEMENT $S(n)$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{(n+1)}$$

BASIS. The basis is $S(1)$. Write

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{(1+1)}. \tag{8}$$

This summation has value 1/2, which satisfies the basis $S(1)$.

INDUCTION. We must show that $S(n)$ implies $S(n+1)$. That is, if $S(n)$ is true and $S(n+1)$ is derived from $S(n)$, then $S(n+1)$ is true. Let $n \geq 1$ and write $S(n+1)$:

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{((n+1)+1)}. \tag{9}$$

The left side of Equation (9) is composed of the summation in $S(n)$ and one other term, namely $1/(n+1)(n+2)$. We expand this and get

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)}. \tag{10}$$

By $S(n)$, the summation on the right side of Equation (10) is equal to $n/(n+1)$. Thus the right side of (10) has value $(n+1)/(n+2)$, which is equal to the right side of Equation (9). Hence $S(n)$ implies $S(n+1)$ meaning $S(n)$ is true for all $n \geq 1$. ✦

We change the style of the proofs from hereon.

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2. We prove the following statement $S(n)$ by induction on n, for $n \geq 1$.

STATEMENT $S(n)$:

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6$$

That is, the summation of triangular numbers, or a sum of sums.

BASIS. We take the basis to be $S(1)$ because $j = 1$. We have

$$\sum_{j=1}^{1} t_j = 1.$$

There is only one term in the summation, namely $1(1+1)/2 = 1$. Thus the basis of $S(n)$ is true.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$, which is

$$\sum_{j=1}^{n+1} t_j = (n+1)(n+2)(n+3)/6.$$

We decompose the summation in terms of $S(n)$ to get

$$\sum_{j=1}^{n} t_j = n(n+1)(n+2)/6 + t_{n+1}.$$

We know that the term $t_{n+1} = (n+1)(n+2)/2$. Hence the right side is

$$\begin{aligned} &n(n+1)(n+2)/6 + (n+1)(n+2)/2 \\ &= n(n+1)(n+2)/6 + 3(n+1)(n+2)/6 \\ &= (n+1)(n+2)(n+3)/6. \end{aligned}$$

This expression is equal to the right side of $S(n+1)$. Thus we have shown that $S(n+1)$ is true. Therefore $S(n)$ is true for all $n \geq 1$. ✦

3.

a) 01101 has three 1's, so its parity is odd.
b) 111000111 has six 1's, so its parity is even.
c) 010101 has three 1's, so its parity is odd.

4. We shall prove the following statement $S(n)$ by induction on n, analogous to Example 2.6 in the book. Written below is essentially the template for the proof but I used it as the actual proof instead.

STATEMENT $S(n)$: If C is any error-detecting set of strings of length n using the digits $0, 1,$ and 2, then C cannot have more than 3^{n-1} strings for any $n \geq 1$.

BASIS. We take the basis to be $S(1)$. Then C is an error-detecting set of strings of length 1. There are three possible sets C, those are $\{0\}, \{1\}$, and $\{2\}$. If C has more than one element then C would not be error-detecting, thus the three sets above are the only ones possible. Therefore C cannot have more than $3^{1-1} = 1$ string. This proves the basis.

INDUCTION. Assume that $n \geq 1$ and that $S(n)$ is true. We must prove $S(n+1)$. That is, an error-detecting set C of strings of length $n+1$ cannot have more than 3^n strings.

Divide C into three sets C_0, C_1, C_2, each being the set of strings in C that begin with $0, 1, 2$ respectively. Then remove the leading digit from each string in C_0, C_1, C_2 to coerce the sets into having strings of length n. Since C is error-detecting, then so are C_0, C_1, C_2. Apply the inductive hypothesis to C_0, C_1, C_2 to prove that they each cannot have more than 3^{n-1} strings. Since every string in C is in either C_0, C_1, C_2, then C cannot have more than $3 \cdot 3^{n-1} = 3^n$ strings. This concludes the proof. ✦

5. We prove the following statement by induction on n.

STATEMENT $S(n)$: There is an error-detecting set of strings of length n for any $n \geq 1$, using the digits $0, 1,$ and 2, that has 3^{n-1} strings.

BASIS. The basis is $S(1)$. That is, there is an error-detecting set of strings of length 1, using the digits 0, 1, and 2, that has $3^{1-1} = 1$ string. There are three strings of length one, those being 0, 1, and 2. However only one can be in an error-detecting set, meaning this set has one string. Thus there is such a set and the basis is proven.

INDUCTION. Assume that $n \geq 1$ and $S(n)$ is true. We must prove $S(n+1)$. That is, there is an error-detecting set of strings of length $n+1$, using the digits 0, 1, and 2, that has 3^n strings.

Assume by the inductive hypothesis that there are three error-detecting sets of strings C_0, C_1, C_2 of length n that have 3^{n-1} strings. Let D_0, D_1, D_2 be the sets of strings with 0, 1, and 2 added as the leading digit to each string in C_0, C_1, C_2 respectively. Thus D_0, D_1, D_2 have strings of length $n+1$ and are error-detecting since C_0, C_1, C_2 are error-detecting.

Let C be the set containing all the strings in D_0, D_1, D_2. Thus C is an error-detecting set, because D_0, D_1, D_2 are, having strings of length $n+1$. We conclude that C has $3 \cdot 3^{n-1} = 3^n$ strings. We have proven $S(n+1)$ hence $S(n)$ holds for $n \geq 1$. ✦

6. We shall prove the following statement by induction on n. The statement is a combination of exercises 2.3.4 and 2.3.5 in the book. We must prove the existence of a set and what its maximum number of elements are. We prove analogously to these exercises.

STATEMENT $S(n)$: There is an error-detecting set of strings of length n, using k different symbols as "digits," for any $k \geq 2$, with k^{n-1} strings, but no such set of strings with more than k^{n-1} strings.

We prove by induction on n, not k. The statement is dependent on the length of the strings, not the number of symbols used. Some intuition about why we do not do induction on k is because in earlier proofs we do not always use all the symbols provided for a string. If there is a string of length one, then it does not matter how many symbols we use.

BASIS. We prove the basis $S(1)$, that is for strings of length 1. There is such a set satisfying $S(1)$. Select any symbol k_0. Then there is the set $\{k_0\}$ with $k^{1-1} = 1$ string. If we add any other string to this set using any symbol then the set would no longer be error-detecting, so it cannot have more than one string.

INDUCTION. Assume $S(n)$ is true and $n \geq 1$. We now prove $S(n+1)$. That is, there is an error-detecting set of strings C of length $n+1$, using k different symbols as "digits," with k^n strings, but no such set of strings with more than k^n strings.

We prove the existence of C. By $S(n)$, assume there are k error-detecting sets of strings $C_1, \ldots, C_k$ of length n using k different symbols, with k^{n-1} strings. Prepend each string in $C_1, \ldots, C_k$ with the symbol k_m where m is each set's respective index. The sets retain their error-detecting property and are now of strings of length $n+1$. Let C be the set containing all the strings in $C_1, \ldots, C_k$. Then C is an error-detecting set of strings of length $n+1$ using k different symbols.

Now we prove that C cannot have more than k^n strings. Since $C_1, \ldots, C_k$ have at most k^{n-1} strings by the inductive hypothesis, and that all their strings are in C, then C has at most $k \cdot k^{n-1} = k^n$ strings. Thus C has no more than k^n strings.

Together, the set satisfying the inductive step exists, with k^n strings, and has no more than k^n strings. Therefore $S(n+1)$ is true. We conclude that $S(n)$ is true for $n \geq 1$. ✦

7. We shall prove the following statement by induction on n.