For structural induction on trees, we assume the inductive hypothesis holds for the children of the root of a tree T. Then we relate the children to the root somehow, and that is what proof is meant to involve. We can assume properties of the children that apply recursively downward. From that, we can assume that values of whatever type we wish percolate upward to the children of the root of T and go from there.

1.

a) We prove the following statement by structual induction on T, the root of a tree.

STATEMENT $S(T)$: Calling `preorder` on a tree T prints the labels of T in preorder.

BASIS. The basis is where T is a single node. Then line (1) prints the label of T, line (2) gets the leftmost child which is `NULL`, and thus line (3) fails, stopping execution of `preorder`.

INDUCTION. Suppose T is not a leaf. Then T has at least one child. Assume by the inductive hypothesis that `preorder` prints the labels of the children of T in preorder. Clearly the label of the root of T is printed by line (1). This proves the inductive step. We conclude that $S(T)$ is true for all labeled trees T. ✦

b) We prove the following statement by structural induction on T, the root of a tree.

STATEMENT $S(T)$: Calling `postorder` on a tree T prints the labels of T in postorder.

BASIS. Consider when T is a leaf. Line (1) assigns `c` and line (2) fails. All that is left is for line (5) to print the label of T.

INDUCTION. Suppose T is not a leaf. Then T has at least one child. Assume by the inductive hypothesis that `postorder` prints the labels of the children of T in postorder. After the labels of the children of T have been printed, then lastly on line (5), the label of the root of T is printed. This is the correct behavior for postorder, and proves the inductive step. Therefore $S(T)$ holds for all labeled trees T. ✦

2. We prove the following statement by induction on n, the number of nodes a tree has.

STATEMENT $S(n)$: If a tree T has n nodes with each having a branching factor b, then there are $1 + (b-1)n$ `NULL` pointers among its nodes.

BASIS. Suppose $n = 1$. Then T is a leaf. Thus there are $1 + (b-1)1 = b$ `NULL` pointers in the root of T.

INDUCTION. Suppose $n \geq 1$. Consider that T has $n+1$ nodes. Thus T has at least one child. Assume by the inductive hypothesis that all but one leaf of T, that being n nodes, together have $1 + (b-1)n$ `NULL` pointers among them. Therefore all the nodes of T have a total of $1 + (b-1)(n+1) = b(n+1) - n$ `NULL` pointers. This proves the inductive step. We conclude that $S(n)$ holds. ✦

3. We shall prove the following statement by structural induction.

STATEMENT $S(T)$: The number of nodes in T is 1 more than the sum of the degrees of the nodes.

BASIS. Where T has only a leaf, the degree is 0.

INDUCTION. Suppose the root n of T has children $c_1, c_2, \ldots, c_k$. Let $deg(c)$ be the sum of the degrees of the nodes in the tree rooted at c. Let $nodes(c)$ be the number of nodes in the tree rooted at c. We know that the number of nodes in T is $1 + \sum_{i=1}^{k} nodes(c_i)$. We know that the sum of the degrees of the nodes in T is $(\sum_{i=1}^{k} deg(c_i)) + k$. By the inductive hypothesis we must have that

$$\begin{aligned} 1 + \sum_{i=1}^{k} nodes(c_i) &= 1 + \left(\sum_{i=1}^{k} deg(c_i) + 1 \right) \\ &= 1 + \left(\sum_{i=1}^{k} deg(c_i) \right) + k. \end{aligned}$$

This proves the inductive step. ✦

4. We shall prove the following statement by structural induction.

STATEMENT $S(T)$: The number of leaves in T is 1 more than the number of nodes that have right siblings.

BASIS. Where T has only one node, there are no nodes with right siblings and only 1 leaf.

INDUCTION. Suppose the root n of T has children $c_1, c_2, \ldots, c_k$. Let $leaves(c)$ be the number of leaves in the tree rooted at c. Let $nwrs(c)$ be the number of nodes with right siblings in the tree rooted at c. We know that there are $\sum_{i=1}^{k} leaves(c_i)$ leaves in T. We also assume that the root of a tree has no right sibling, as given by n. But nodes $c_1, c_2, \ldots, c_{k-1}$ have right siblings. Therefore there are $(\sum_{i=1}^{k} nwrs(c_i)) + (k-1)$ nodes with right siblings in T. By the inductive hypothesis, we must have that

$$\begin{aligned} \sum_{i=1}^{k} leaves(c_i) &= \left(\sum_{i=1}^{k} nwrs(c_i) - 1 \right) + (k-1) \\ &= \left(\sum_{i=1}^{k} nwrs(c_i) \right) - 1. \end{aligned}$$

This proves the inductive step. ✦

5. We shall prove the following statement by structural induction.

STATEMENT $S(T)$: In a leftmost-child-right-sibling data structure, the number of `NULL` pointers in T is 1 more than the number of nodes.

BASIS. Suppose T is a single node n. The 2 pointers of n are `NULL`. Thus the basis is true.

INDUCTION. Suppose T has more than one node. Let n be the root of T and let $c_1, c_2, \ldots, c_k$ be the children of n. Let $np(c)$ be the number of `NULL` pointers in the tree rooted at c. In a leftmost-child-right-sibing tree we assume that the root has a `NULL` right sibling pointer. This is true as given by n. But the first $k-1$ children of n have non-`NULL` right sibling pointers. Thus we need a correction term. There are $(\sum_{i=1}^{k} np(c_i)) - (k-1) + 1$ `NULL` pointers in T. By the inductive hypothesis, the number of `NULL` pointers in the trees rooted at $c_1, c_2, \ldots, c_k$ is 1 more than the number of nodes. Let $nodes(c)$ be the number of nodes in the tree rooted at c. There must be

$$\begin{aligned}\left(\sum_{i=1}^{k} np(c_i)\right) - (k-1) + 1 &= \left(\sum_{i=1}^{k} nodes(c_i) + 1\right) - (k-1) + 1 \\ &= \left(\sum_{i=1}^{k} nodes(c_i)\right) + 2\end{aligned}$$

`NULL` pointers in T, which is 1 more than $1 + \sum_{i=1}^{k} nodes(c_i)$, the number of nodes in T. This proves the inductive step. ✦

6. We shall prove the following statement by structural induction.

STATEMENT $S(T)$: Every tree T in the recursive sense is a tree in the nonrecursive sense.

BASIS. Where T is a single node n, then n is the root and is a tree.