

1. Let  $E$  be the expression  $n - i$ . With each iteration of the loop  $i$  increases by 1. Therefore  $n - i$  decreases by 1 with each pass. Eventually  $E$  will be negative, and the loop will terminate. In particular, when  $n - i \leq -1$ .

We prove the following statement by induction on the value of the variable  $i$ .

STATEMENT  $S(m)$ : If we reach the loop test  $i \leq n$  with the variable  $i$  having the value  $m$ , then the value of the variable **sum** is  $\sum_{k=1}^{m-1} k$ .

BASIS. The basis is  $m = 1$ . When we first enter the loop we reach the test with  $i$  having value 1 and **sum** having value 0. We see that  $\sum_{k=1}^0 k$  is surely 0. So the basis is proven.

INDUCTION. Assume  $S(m)$ . We shall prove  $S(m + 1)$ .

We assume here that we are not entering the loop for the first time. If  $m > n$ , then when  $i$  has the value  $m$  we do not reach the loop test. Thus with  $i$  having the value  $m + 1$ , we do not reach the loop test. In that case  $S(m + 1)$  is trivially true.

If  $m \leq n$ , then we consider what happens when we execute the body of the loop with  $i$  having the value  $m$ . By the inductive hypothesis, **sum** has value  $m(m - 1)/2$  and  $i$  has value  $m$  (yes we repeat that  $i$  has value  $m$  again). After the body of the loop is executed, and when we reach the loop test, **sum** has the value  $\sum_{k=1}^m k = m(m + 1)/2$  and  $i$  has the value  $m + 1$ . We have proven  $S(m + 1)$ , therefore  $S(m)$  holds for  $m \geq 1$ .

We expressed earlier that the loop will terminate when  $n - i \leq -1$ . That is, when  $i$  has the value  $n + 1$ . Thus after the body terminates  $S(n + 1)$  must hold, because we reach the test loop when  $i$  has at most the value  $n + 1$ . This statement says that **sum** has the value  $\sum_{k=1}^n k$ , which is the desired result of the program. ♦