1.

a) Choose $n_0 = 0$ and $c = 1$. Because $a \leq b$, then $n^a \leq cn^b$. Therefore n^a is $O(n^b)$.

b) Suppose there are witnesses n_0 and $c > 0$ such that n^a is $O(n^b)$ if $a > b$. Then $n^a \leq cn^b$ for all $n \geq n_0$. But n^a/n^b becomes arbitrarily large as n increases since $a > b$. Thus $n^a/n^b \leq c$ contradicts c being constant.

c) Choose $n_0 = 0$ and $c = 1$. Since $1 < a \leq b$, then $a^n \leq cb^n$. Therefore a^n is $O(b^n)$.

d) Suppose there are witnesses n_0 and $c > 0$ such that $a^n \leq cb^n$ if $1 < b < a$ for all $n \geq n_0$. From the big-oh inquality we must have $1 \leq cb^n/a^n$. But b^n/a^n approaches 0 as n increases. For large n we would have $1 \leq 0$, a contradiction.

e) Exponentials grow faster than polynomials. We can likely choose $c = 1$, but the general n_0 would be too difficult to find, however, n_0 must be large. Therefore n^a is $O(b^n)$ for any a and for any $b > 1$.

f) Suppose there are witnesses n_0 and $c > 0$ such that $a^n \leq cn^b$ for any b and for any $a > 1$ for all $n \geq n_0$. But a^n/n^b becomes arbitrarily large as n increases. Therefore $a^n/n^b \leq c$ contradicts c being a constant.

2. We must have the inequality $f(n) + g(n) \leq c \cdot \max(f(n), g(n))$ for some c. By Exercise 3.4.3, if $f(n) \leq g(n)$ then we must have $O(g(n))$. If $g(n) \leq f(n)$ then we must have $O(f(n))$. Thus it is true that we must have big-oh of the maximum of the two functions.

3. We have that $T(n) \leq cf(n)$ for all $n \geq n_0$ and some $c > 0$, and $g(n) \geq 0$. Multiply both sides by $g(n)$ to get $g(n)T(n) \leq cg(n)f(n)$, which holds for all $n \geq n_0$ and some $c > 0$. Therefore $g(n)T(n)$ is $O(g(n)f(n))$.

4. We are given $S(n) \leq c_1 f(n)$ for $c_1 > 0$ for all $n \geq n_1$ and $T(n) \leq c_2 g(n)$ for $c_2 > 0$ for all $n \geq n_2$. We can multiply them to get $S(n)T(n) \leq c_1 c_2 f(n)g(n)$ for some positive numbers c_1 and c_2. Let $c = c_1 c_2$ and n_2 be the larger of n_1 and n_0. Then the inequality holds for c and for all $n \geq n_2$. Therefore $S(n)T(n)$ is $O(f(n)g(n))$.

5. We have that $f(n) \leq cg(n)$ for all $n \geq n_0$ and some $c > 0$.