1. We know that for any function f that $f(n)$ is $O(f(n))$. Thus we need not consider the $O(f_i(n))$ for $f_i(n)$.

$f_1(n)$ is $O(f_2(n))$, $O(f_3(n))$, and $O(f_4(n))$. The witnesses $c = 1$ and $n_0 = 0$ prove these three relationships.

$f_2(n)$ is not $O(f_1(n))$, $O(f_3(n))$, or $O(f_4(n))$.
Suppose there are witnesses n_0 and c such that $f_2(n) \leq cf_1(n)$ for all $n \geq n_0$. Pick n_1 to be at least $2c$ or n_0. Since $n_1 \geq n_0$ then the inequality holds, because it holds for all $n \geq n_0$. If we divide both sides by n_1^2, we have $n_1 \leq c$. But n_1 is at least $2c$, and n_1 cannot be less than c and greater than $2c$. Therefore witnesses that show $f_2(n)$ to be $O(f_1(n))$ do not exist.
Suppose there are witnesses n_0 and c such that $f_2(n) \leq cf_3(n)$ for all $n \geq n_0$. Pick n_1 to be at least $2c+1$ or n_0. We know that $f_2(n_1) \leq cf_3(n_1)$ because $n_1 \geq n_0$. However n_1 is at least an odd number, thus $f_3(n_1) = n_1^2$. If we divide this inequality by n_1^2 we have $n_1 \leq c$. But n_1 cannot be at least $2c + 1$ and at most c. Therefore witnesses that show $f_2(n)$ to be $O(f_3(n))$ do not exist.
Suppose there are witnesses n_0 and c such that $f_2(n) \leq cf_4(n)$ for all $n \geq n_0$. Suppose n is prime. From $n^3 \leq cn^2$ we determine $n \leq c$, which means that c changes its value for each $n \geq n_0$. But c is a constant, and thus we have a contradiction.

$f_3(n)$ is not $O(f_1(n))$, but $f_3(n)$ is $O(f_2(n))$ and $O(f_4(n))$.
Suppose there are witnesses n_0 and c such that $f_3(n) \leq cf_1(n)$ for all $n \geq n_0$. Suppose n is even. Then the inequality is $n^3 \leq cn^2$. This implies that $n \leq c$ for all $n \geq n_0$ which contradicts our assumption that c is a constant.
We can show that $f_3(n)$ is $O(f_2(n))$ and $O(f_4(n))$ by choosing witnesses $n_0 = 3$ and $c = 1$.

$f_4(n)$ is not $O(f_1(n))$ or $O(f_3(n))$, but is $O(f_2(n))$.
Suppose there are witnesses n_0 and c such that $f_4(n) \leq cf_1(n)$ for all $n \geq n_0$. Suppose n is composite. Then $n^3 \leq cn^2$ implies $n \leq c$ for all $n \geq n_0$. But this contradicts the assumption that c is constant.
We can show that $f_4(n)$ is $O(f_2(n))$ by choosing witnesses $n_0 = 0$ and $c = 1$.
Suppose there are witnesses n_0 and c such that $f_4(n) \leq cf_3(n)$ for all $n \geq n_0$. Suppose n is odd and composite. Then the inequality for this particular n is $n^3 \leq cn^2$. This implies that $n \leq c$ for all $n \geq n_0$, but c is a constant. Hence we have a contradiction.

2. We would rather not start approximating things and just say "good enough."

a) Let $n_0 = 0$ and $c = 1000$.

b) Let $n_0 = 3$ and $c = 39$ (bad answer).

c) Let $n_0 = 0$ and $c = 2^{10}$.

d) Let $n_0 = 32$ and $c = 1$.

3. We must have the inequality $f(n) + g(n) \leq c'g(n)$. Then $f(n) \leq (c' - 1)g(n)$. Choose $n_0 = 0$ and $c = c' - 1 = 1$. Then the inequality holds and thus $f(n) + g(n)$ is $O(g(n))$. ✦

4. We state that $f(n) \leq cg(n)$ and $g(n) \leq cf(n)$ for all $n \geq n_0$. We can assume that the highest power of f and g are equal. It is not true that $f(n) = g(n)$ must always hold. Example, $n \leq n + 1$ and $n + 1 \leq 2n$.