1.

STATEMENT $S(i)$: If $1 \leq i < n$, then

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

BASIS. The basis is $i = 1$, as given by the lower bound of the interval. Then we have the inductive definition of the recurrence hence the basis holds.

INDUCTION. If $i \geq n$ then there is nothing to prove. Suppose that $i+1 < n$. From the inductive hypothesis we expand $T(n-i)$ to get

$$\begin{aligned} T(n) &= T(n-i-1) \sum_{j=0}^{i-1} g(n-j) + g(n-i) \\ &= T(n-i-1) + \sum_{j=0}^{i} g(n-j). \end{aligned}$$

This is the statement $S(i+1)$ and we have proved the inductive step. We conclude $S(i)$ is true for $1 \leq i < n$. ✦

2.

a) We have

$$\begin{aligned} T(n) &= T(n-1) + g(n) \\ &= T(n-2) + g(n-1) + g(n) \\ &= T(n-3) + g(n-2) + g(n-1) + g(n). \end{aligned}$$

Thus we have

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j).$$

But we know this already by Exercise 3.11.1.

Let $i = n-1$. Then we have the recurrence in terms of the basis, which is

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j). \tag{1}$$

But $g(n) = n^2$. This is a summation of squares. By Exercise 2.3.1(b) we know that

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6.$$

We adjust the bounds to get the equivalent summation

$$\sum_{j=1}^{n} j^2 = \sum_{j=0}^{n-1} (n-j)^2.$$

But this has the extra term for when $j = n-1$, that is $g(1)$. We subtract 1 and get

$$T(n) = a + \sum_{j=0}^{n-1} (n-j)^2 - 1,$$

the same as Equation (1). We could have just taken the summation and decomposed it with the $(n-2)$th index then wrote the remaining summation as the summation of squares. But now that we have the last term separate, we rewrite the summation in terms of the identity

$$T(n) = a + n(n+1)(2n+1)/6 - 1.$$

Clearly $T(n)$ is cubic, hence $T(n)$ is $O(n^3)$ if $g(n) = n^2$.

b) We have

$$T(n) = a + \sum_{j=0}^{n-2} g(n-j)$$

where $g(n) = n^2 + 3n$. Then we have two summations, one of the n^2 terms and the other of $3n$ terms. Write

$$T(n) = a + \sum_{j=0}^{n-2} (n-j)^2 + \sum_{j=0}^{n-2} 3j.$$