

For structural induction on trees, we assume the inductive hypothesis holds for the children of the root of a tree  $T$ . Then we relate the children to the root somehow, and that is what proof is meant to involve. We can assume properties of the children that apply recursively downward. From that, we can assume that values of whatever type we wish percolate upward to the children of the root of  $T$  and go from there.

1.

a) We prove the following statement by structural induction on  $T$ , the root of a tree.

STATEMENT  $S(T)$ : Calling **preorder** on a tree  $T$  prints the labels of  $T$  in preorder.

BASIS. The basis is where  $T$  is a single node. Then line (1) prints the label of  $T$ , line (2) gets the leftmost child which is NULL, and thus line (3) fails, stopping execution of **preorder**.

INDUCTION. Suppose  $T$  is not a leaf. Then  $T$  has at least one child. Assume by the inductive hypothesis that **preorder** prints the labels of the children of  $T$  in preorder. Clearly the label of the root of  $T$  is printed by line (1). This proves the inductive step. We conclude that  $S(T)$  is true for all labeled trees  $T$ . ♦

b) We prove the following statement by structural induction on  $T$ , the root of a tree.

STATEMENT  $S(T)$ : Calling **postorder** on a tree  $T$  prints the labels of  $T$  in postorder.

BASIS. Consider when  $T$  is a leaf. Line (1) assigns  $c$  and line (2) fails. All that is left is for line (5) to print the label of  $T$ .

INDUCTION. Suppose  $T$  is not a leaf. Then  $T$  has at least one child. Assume by the inductive hypothesis that **postorder** prints the labels of the children of  $T$  in postorder. After the labels of the children of  $T$  have been printed, then lastly on line (5), the label of the root of  $T$  is printed. This is the correct behavior for postorder, and proves the inductive step. Therefore  $S(T)$  holds for all labeled trees  $T$ . ♦

2. We prove the following statement by induction on  $n$ , the number of nodes a tree has.

STATEMENT  $S(n)$ : If a tree  $T$  has  $n$  nodes with each having a branching factor  $b$ , then there are  $1 + (b - 1)n$  NULL pointers among its nodes.

BASIS. Suppose  $n = 1$ . Then  $T$  is a leaf. Thus there are  $1 + (b - 1)1 = b$  NULL pointers in the root of  $T$ .

INDUCTION. Suppose  $n \geq 1$ . Consider that  $T$  has  $n + 1$  nodes. Thus  $T$  has at least one child. Assume by the inductive hypothesis that all but one leaf of  $T$ , that being  $n$  nodes, together have  $1 + (b - 1)n$  NULL pointers among them. Therefore all the nodes of  $T$  have a total of  $1 + (b - 1)(n + 1) = b(n + 1) - n$  NULL pointers. This proves the inductive step. We conclude that  $S(n)$  holds. ♦

3. We shall prove the following statement by structural induction.

STATEMENT  $S(T)$ : The number of nodes in  $T$  is 1 more than the sum of the degrees of the nodes.

BASIS. Where  $T$  has only a leaf, the degree is 0.

INDUCTION. Suppose the root  $n$  of  $T$  has children  $c_1, c_2, \dots, c_k$ . Let  $\deg(c)$  be the sum of the degrees of the nodes in the tree rooted at  $c$ . Let  $\text{nodes}(c)$  be the number of nodes in the tree rooted at  $c$ . We know that the number of nodes in  $T$  is  $1 + \sum_{i=1}^k \text{nodes}(c_i)$ . We know by the inductive hypothesis we must have that

$$\begin{aligned} 1 + \sum_{i=1}^k \text{nodes}(c_i) &= \left( \sum_{i=1}^k \deg(c_i) + 1 \right) \\ &= \left( \sum_{i=1}^k \deg(c_i) \right) + k. \end{aligned}$$

That is, the latter term is the number of nodes in  $T$  excluding  $n$ , and is 1 fewer node than the number of nodes in  $T$ . This proves the inductive step. ♦

4. We shall prove the following statement by structural induction.

STATEMENT  $S(T)$ : The number of leaves in  $T$  is 1 more than the number of nodes that have right siblings.

BASIS. Where  $T$  has only one node, there are no right siblings and only 1 leaf.

INDUCTION.

5. We shall prove the following statement by structural induction.

STATEMENT  $S(T)$ : In a leftmost-child-right-sibling data structure, the number of NULL pointers in  $T$  is 1 more than the number of nodes.

BASIS. Suppose  $T$  is a single node  $n$ . The 2 pointers of  $n$  are NULL. Thus the basis is true.

INDUCTION. Suppose  $T$  has more than one node. Let  $n$  be the root of  $T$  and let  $c_1, c_2, \dots, c_k$  be the children of  $n$ . Let  $np(c)$  be the number of NULL pointers in the tree rooted at  $c$ . In a leftmost-child-right-sibling tree we assume that the root has a NULL right sibling pointer. This is true as given by  $n$ . But the first  $k - 1$  children of  $n$  have non-NULL right sibling pointers. Thus we need a correction term. Therefore there are  $(\sum_{i=1}^k np(c_i)) - (k - 1) + 1$  NULL pointers in  $T$ . By the inductive hypothesis, the number of NULL pointers in the trees rooted at  $c_1, c_2, \dots, c_k$  is 1 more than the number of nodes. Let  $\text{nodes}(c)$  be the number of nodes in the tree rooted at  $c$ . There must be

$$\begin{aligned} \left( \sum_{i=1}^k np(c_i) \right) - (k - 1) + 1 &= \left( \sum_{i=1}^k \text{nodes}(c_i) + 1 \right) - (k - 1) + 1 \\ &= \left( \sum_{i=1}^k \text{nodes}(c_i) \right) + 2 \end{aligned}$$

NULL pointers in  $T$ , which is 1 more than  $1 + \sum_{i=1}^k \text{nodes}(c_i)$ , the number of nodes in  $T$ . This proves the inductive step. ♦