Dimensionality Reduction and State-Space Systems: Forecasting the Term Structure of Interest Rates Using the Diebold-Li Model

I have fit the Diebold-Li model of yield curve to a time series of monthly values of yield curves derived from the US Treasury yields ranging from 3 months to 30 years. Firstly, I have representated the Diebold-Li model in a parametric state-space form. Having estimated, simulated, smoothed and forecasted using the state-space models (SSM) as described in Diebold, Rudebusch, Aruoba (2006), I compared the results with those from the two-step method described in Diebold and Li (2006). Lastly, I have simulated the yield curve using Monte Carlo and forecasted using the minimum mean square error (MMSE) technique embedded in the SSM functionalities from the Econometrics toolbox.

The Diebold-Li Model of the Yield Curve

A variant of the Nelson-Siegel model, the Dynamic Nelson Siegel (DNS) or the Diebold-Li model is a reparameterization of the original equation. Given the time to maturity τ, and date t when the yield is observed, the yield $y_t(\tau)$ from the Diebold-Li model is a function of four parameters:

$$y_t(\tau) = L_t + S_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \qquad (1)$$

Here, L_t is the level or the the long-term factor as it influences long-term bonds, S_t is the slope or short-term factor, and C_t is the curvature or the medium-term factor. λ determines the maturity at which the loading on the curvature is maximized, and governs the exponential decay rate of the model.

After specifying the parametric form of an SSM, I have used MLE to estimate the model's parameters, caliberated the smoothed and filtered states by forward and backward recursion, respectively. Thereafter, Iacquired the optimal forecasts of the observed data and latent states, and employed Monte Carlo to simulate the trajectories of the yield curve and latent states.

In SSM, the state equation describes how the state evolves, while the measurement or observation equation describes how the measurable variables relate to the state variables. Given the state vector x_t and observation vector y_t, the parametric form of the state-space model is expressed in the following linear state-space representation:

$$x_t = A_t x_{t-1} + B_t u_t$$

$$y_t = C_t x_t + D_t \epsilon_t$$

where u_t and ϵ_t are uncorrelated, unit-variance white noise vector processes. In the aforementioned equations, the first two equations are the *state equation* and *measurement equation,* respectively. The parameters of the model are matrices A_t, B_t, C_t and D_t that correspond the *state transition*, *state disturbance loading*, *measurement sensitivity*, and *observation innovation* matrices, respectively.

Diebold-Li Model in the State-Space System

The level, slope and curvature in the Diebold-Li model follow a first order vector autoregressive process or VAR(1), which has a state-space representation as follows:

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix}$$

The corresponding measurement equation is:

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_M) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1 - e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1 - e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1 - e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1 - e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ & \vdots & \\ 1 & \frac{1 - e^{-\lambda\tau_M}}{\lambda\tau_M} & \frac{1 - e^{-\lambda\tau_M}}{\lambda\tau_M} - e^{-\lambda\tau_M} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \epsilon_t(\tau_1) \\ \epsilon_t(\tau_2) \\ \vdots \\ \epsilon_t(\tau_M) \end{pmatrix}$$

In matrix notation, we can write the state-space system for the three-dimensional vector of the mean adjusted factors f_t and observed yields y_t:

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t$$

$$y_t = \Lambda f_t + \epsilon_t$$

η_t and e_t are orthogonal to each other and follow white noise processes and we can write their distribution as:

$$\begin{pmatrix} \eta_t \\ \epsilon_t \end{pmatrix} \sim WN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix} \right)$$

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathrm{WN}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$$

$$\begin{bmatrix}\eta_t\\ \epsilon_t\end{bmatrix} \sim \text{WN}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\Sigma_\eta & 0\\ 0 & \Sigma_\epsilon\end{bmatrix}\right)$$

The factor disturbances η_t are correlated with each other, causing the covariance matrix Q to be non-diagonal. In juxtaposition, the disturbances e_t in the observation equation are uncorrelated, yielding a diagonal covariance matrix H.

The mean-adjusted factors describe the latent states:

$$x_t = f_t - \mu$$

The yields are demeaned or adjusted for intercept

$$y'_t = y_t - \Lambda\mu$$

Susbtituting these in the aforementioned equations, the state-space system of the Diebold-Li model can be characterized as:

$$x_t = Ax_{t-1} + \eta_t$$

$$y'_t = y_t - \Lambda\mu = \Lambda x_t + e_t$$

In simplistic terms, we can describes the state-space system as follows:

$$x_t = Ax_{t-1} + Bu_t$$

$$y_t = Cx_t + D\epsilon_t$$

The state transition matrix A matches with that of the matrix A in the Diebold-Li formulation, while Λ is equivalent to the state measurement sensitivity matrix C in the SSM formulation.

Unlike the straightforward relationship of state transition and measurement sensitivity matrices with those in the Diebold-Li state-space system, the disturbances have a more sophisticated relationship, leading to a subtle parameterization of the B and D matrices. We observe that that the disturbances of the Diebold-Li state space system equate with those in the SSM functionality.

$$\eta_t = Bu_t$$

$$e_t = D\epsilon_t$$

As the disturbances u_t and ϵ_t are unit-variance, uncorrelated, white noise vector processes, their covariance matrices are identity matrices. Therefore, we relate the parameters of the SSM function ot the covariance of the Diebold-Li model such that:

$$Q = BB'$$

$$H = DD'$$

Representing the Diebold-Li model in the state-space system, I have created the SSM implicitely by enumerating a parameter mapping function whereby a vector of input parameters is mapped to parameters A, B, C, and D of the model. I have employed a mapping function as the three latent factors in the Diebold-Li model are demeaned, which forms a common regression component. Additionally, the mapping function ensures that the $Q = BB'$ is symmetric, the covariance matrix $H = DD'$ is diagonal. Finally, the mapping function estimates the decay parameter λ. Implicitely, the mapping function specifies the model. This is advantageous as we can estimate convoluted models and impose multiple constraints on the parameters.

Moreover, I have demeaned the yields y_t as the SSM functions such as `filter`, `smooth`, `forecast`, and `simulate`, assume that observations are already manually demeaned. While estimating, a parameter mapping function deflates the yields observed. The upside is that it maintains the three-dimensionality of the latent factors in the state vector. Nonetheless, the downside is that I have to separately deflate and subsequently inflate the yields to offset the adjustment from estimating on the demeaned yields.

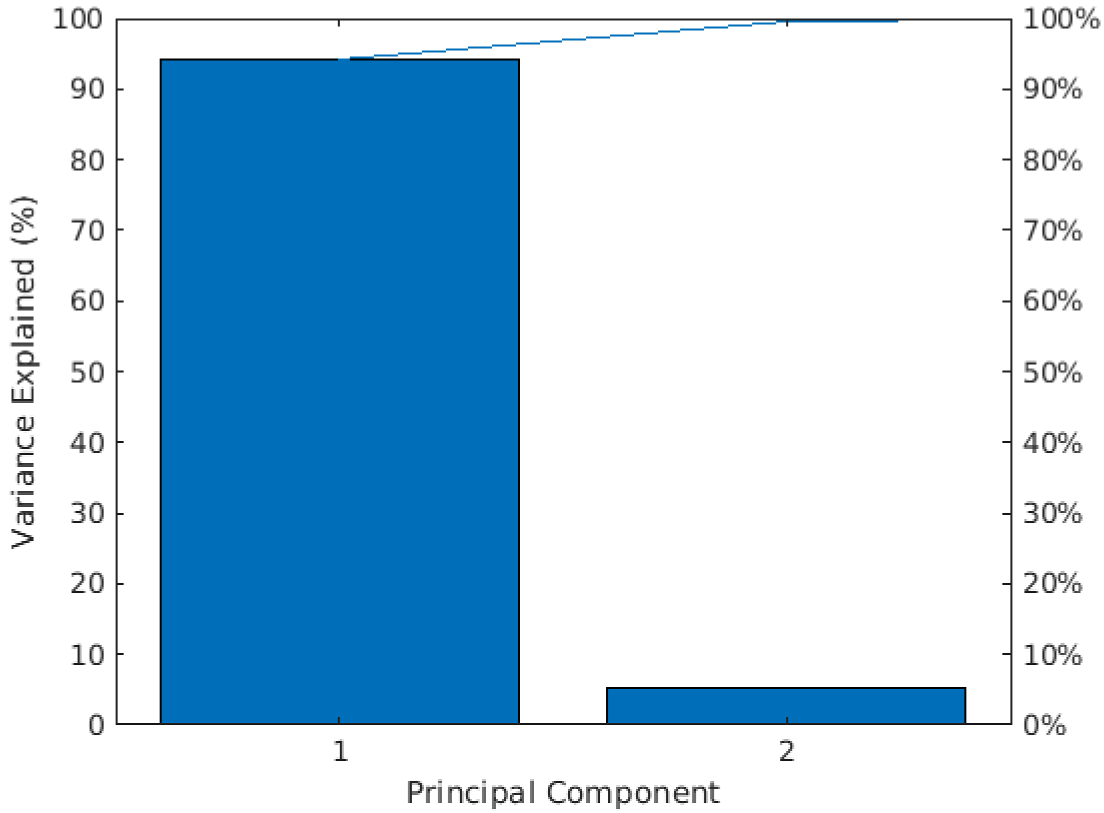
The Yield Curve Data

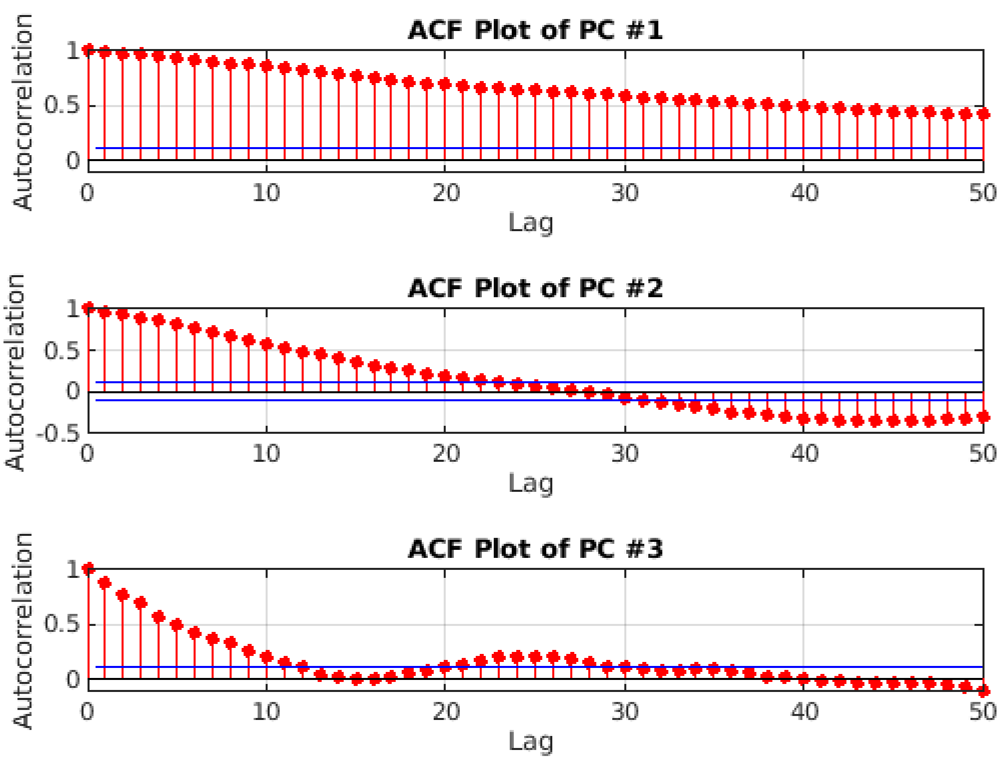
I have aggregated the monthly data on the treasury yields of various maturities, ranging from 3 month to 30 years from FRED. Depicting a snapshot of the current yield levels in the market, each of the ten series has both a cross-sectional and a temporal dimension.

```
% has data from October 31, 1993 to August 30, 2020 collected from FRED
% no missing values
tres = readtable("data_df.csv");

% extract the "Dates" column and store it as "dates" table
dates = table2array(tres(:, 'Date'));

% extract column names of table "tres"
tres.Properties.VariableNames;

% removes the "Date" column
tresc= removevars(tres,{'Date'});
data = table2array(tresc);

% DataTable: has columns' names of yields of varying ttm
% and row names for each date

% yields of 10 term to maturity (TTM) in months
ttm =[3; 6; 12; 24; 36; 60; 84; 120; 240; 360];

% yields = dataset on 10 different maturities for in-sample modeling
yields = data(1:end,:);      % in-sample yields for estimation
```

```
score = 324×10
    3.6445    1.4172   -0.0164    0.0355   -0.0126    0.0789    0.0092   -0.0125 ···
    4.4332    1.8032   -0.0508    0.0471   -0.0388    0.0345   -0.0272   -0.0296
    4.4024    1.9380   -0.1615   -0.0237   -0.0087    0.0338   -0.0238    0.0054
    4.0180    1.7597   -0.0877   -0.0298    0.0158    0.0004   -0.0449   -0.0311
    5.6163    1.9656   -0.1863   -0.0073    0.0431    0.0006    0.0218   -0.0057
    7.0827    2.4914   -0.5446   -0.0647   -0.0176   -0.0590   -0.0134    0.0225
    8.3686    2.2893   -0.6293   -0.1596   -0.0483   -0.0892   -0.0103    0.0421
    9.0822    2.0705   -0.4930   -0.2202   -0.0656   -0.0780    0.0362    0.0080
    9.4717    2.3287   -0.6343   -0.2947   -0.0494   -0.0862    0.0150    0.0492
    8.9989    1.9298   -0.4652   -0.1895   -0.0133   -0.1137    0.0023    0.0002
      ⋮

explained = 10×1
   94.2829
    5.2684
    0.3561
    0.0477
    0.0186
    0.0153
    0.0038
    0.0033
    0.0028
    0.0010
avg = 1×10
    2.3889    2.5141    2.6354    2.8809    3.0744    3.4426    3.7441    3.9800 ···
```

	PC1	PC2	PC3
1	3.6445	1.4172	-0.016
2	4.4332	1.8032	-0.050
3	4.4024	1.9380	-0.161
4	4.0180	1.7597	-0.087
5	5.6163	1.9656	-0.186
6	7.0827	2.4914	-0.544
7	8.3686	2.2893	-0.629


Variance Explained (%)
Principal Component
100
90
80
70
60
50
40
30
20
10
0
100%
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
1
2


ACF Plot of PC #1
Autocorrelation
Lag
ACF Plot of PC #2
Autocorrelation
Lag
ACF Plot of PC #3
Autocorrelation
Lag
1
0.5
0
-0.5
0
10
20
30
40
50

Two-Step Diebold-Li Model with Fixed λ

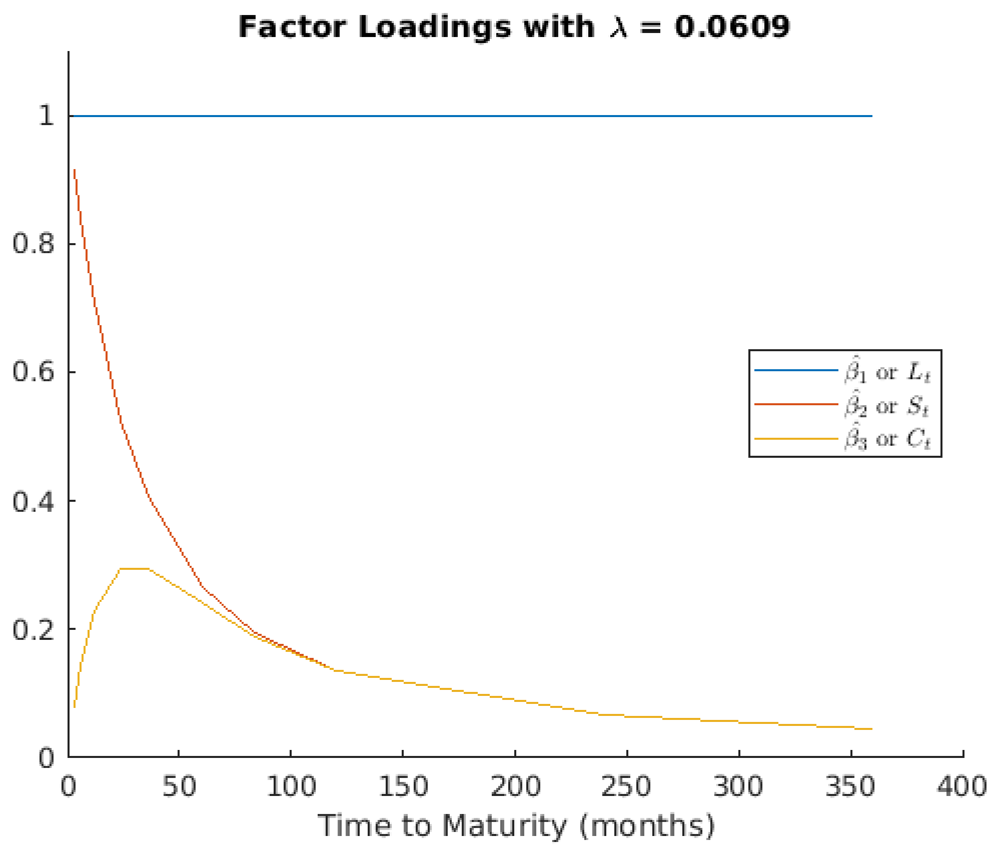
Finding the ex-ante value of the latent factors is tantamount to finding the ex-ante values of the yield curve as it is a function of the factors. Then, I have constructed the Diebold-Li model with the explanatory variables as factor forecasts. The two-step approach of fitting the Diebold and Li model and estimating the parameters of their yield curve is:

- Firstly, I have kept λ fixed and estimated the level, slope, and curvature parameters for each of the daily values of the yield curve. We can consider the regression coefficients from OLS as the as the three factors. Repeating this process for all observed yield curves yields a three-dimensional matrix of estimates of the three latent factors. Keeping λ fixed simplifies the estimation method from non-linear least squares to OLS, which creates a static Nelson Siegel model at each month.
- Secondly, I have fitted $\mathrm{VAR}(1)$ to the time series of factors derived in the first step.

In the Nelson-Siegel model, λ_t determines the time to maturity at which the loadings on the curvature, or the medium-term factor are maximized. These are usually yields that mature in 24 to 36 months. Dielbold and Li (2005) set $\lambda_t = 0.0609$ for all t. This is the value at which the loading on the curvature (medium-term factor) is maximized occurs at 30 months. Below, I have shown the first-step in the Diebold-Li model, and accumulated the coefficients (factor loadings) and the residuals from the OLS model. The factor loadings on β_1, β_2 and β_3 govern the level, slope and curvature, respectively, of the yield curve. Then, I plotted the factor loadings in one graphs.

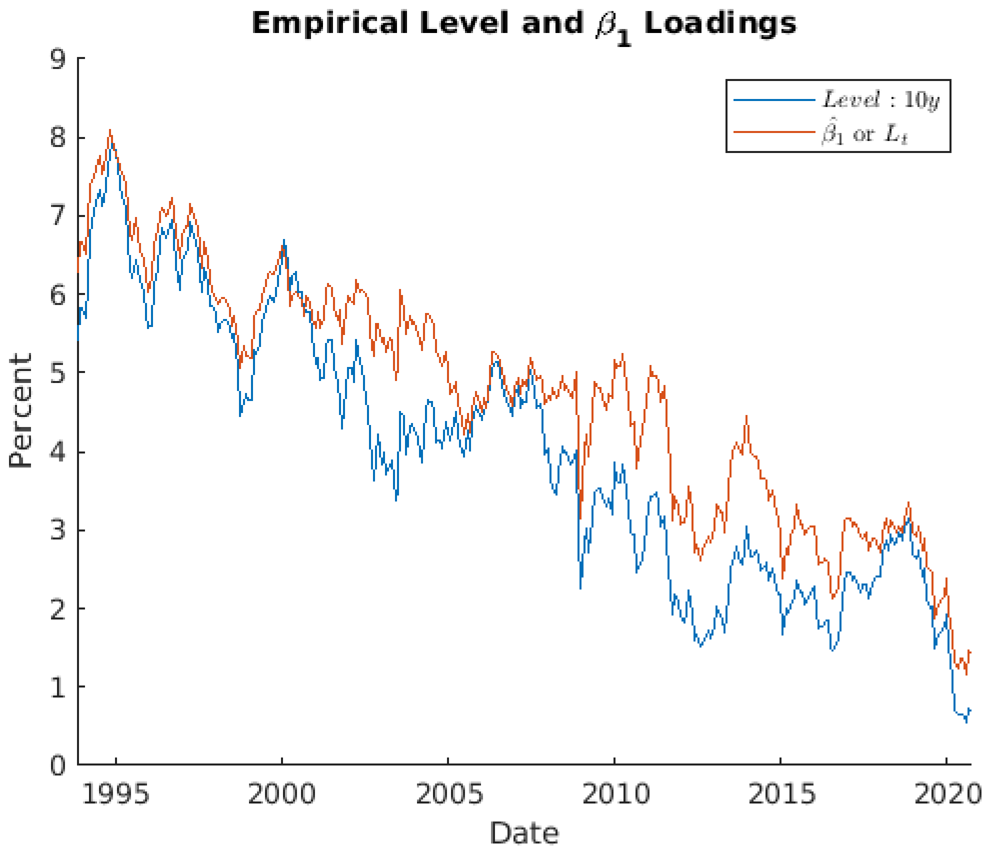
```
lambda0 = 0.0609;
% X is a 10 by 3 matrix: as there is data on 10 different yields and 3 beta
% factors
X = [ones(size(ttm)) (1-exp(-lambda0*ttm))./(lambda0*ttm) ...
    ((1-exp(-lambda0*ttm))./(lambda0*ttm)-exp(-lambda0*ttm))];


% size(yields,1): queries the length of the 1st dim of 'yields" : 324
% beta = zeros(324, 3)
% stores the beta
beta = zeros(size(yields,1),3);

% numel : returns the number of elements in array "ttm" : 10
% resid = zeros(324, 10)
% stores the resid from the OLS model
resid = zeros(size(yields,1),numel(ttm));

% fix lambda and compute the 3 beta params for each monthly observation of the
% yield curve by OLS
% the beta coefficients from the OLS regression are equivalent to the 3
% factors: level, slope, and curvature
% collect the 3D time-series of the betas (estimated factors) by fitting
```

```
% the OLS model on each yield curve observed
% store the betas (regression coefficients) and resid

for i = 1:size(yields,1)
    OLSmod = fitlm(X, yields(i,:)', 'Intercept', false);
    beta(i,:) = OLSmod.Coefficients.Estimate';

    % raw residuals (observed - fitted values) from the OLS model
    resid(i,:) = OLSmod.Residuals.Raw';
end


% Plot the factor loadings
figure
hold on
plot(ttm,X)
title('Factor Loadings with \lambda = 0.0609')
xlabel('Time to Maturity (months)')
ylim([0 1.1])
legend({'$\hat{\beta_1}$ or $L_t$';'$\hat{\beta_2}$ or $S_t$'; ['$\hat{\beta_3}$ ' ...
    'or $C_t$']},'Interpreter', 'latex','location','east')
hold off
```

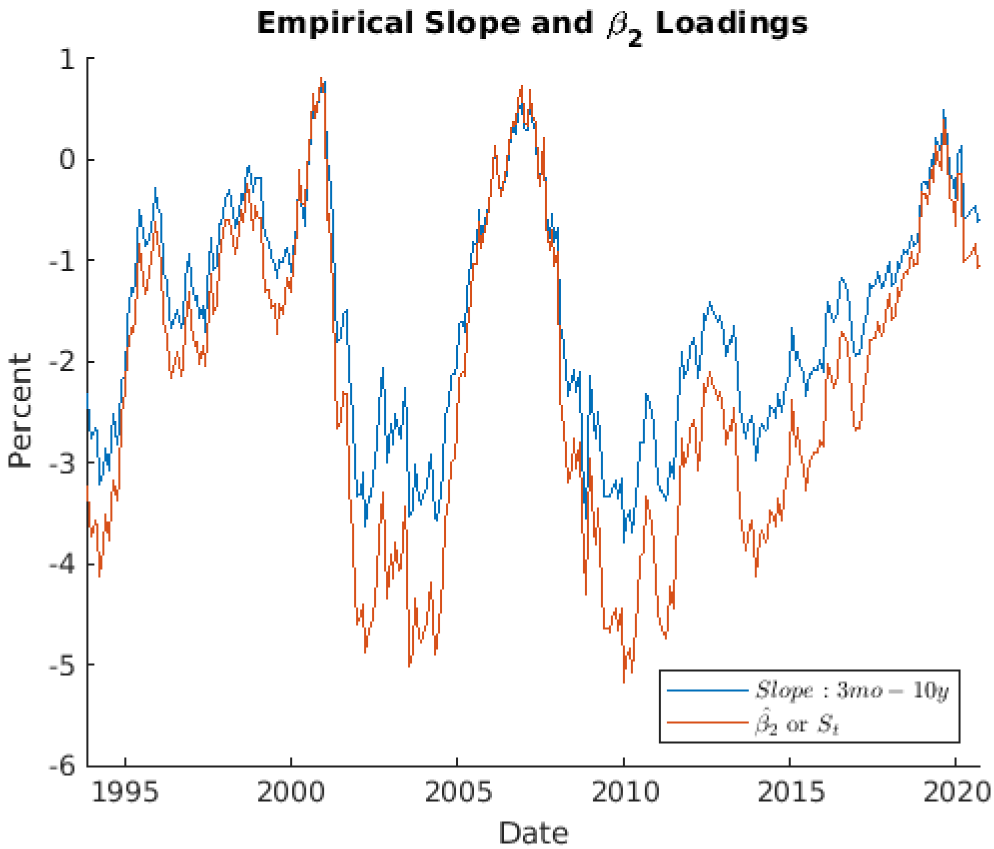


β_1 is a flat line parallel to the maturity axis, representing the constant L_t in the Diebold-Li equation (1). The loadings on β_2 or S_t is a function $\frac{1-e^{\lambda\tau}}{\lambda\tau}$, that starts at 1 approximately but monotonically decays to 0; therefore, it is a convex-shaped curve. Finally, the loadings on β_3 or C_t is a function $\frac{1-e^{\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}$ that starts at 0 and gradually converges to 0, affecting the medium-term yields. λ_t governs the pace of exponential decay wherein small values of λ_t slows the decay and better fits the curve at longer term maturities. This contrasts with large values of λ_t which fastens the rate of recay and better fits shorter-term maturities. Now, I have compared the path of the level, slope and curvature with their corresponding factor loadings from the first step (OLS regresson) in the Diebold-Li model. The 10-year Treasury yield is a measure of level. The slope or the yield spread is the difference between the 3-month and 10-year Treasury yield. Lastly, the curvature is measured as $2 \times 2y - 10y - 3\text{mo}$.

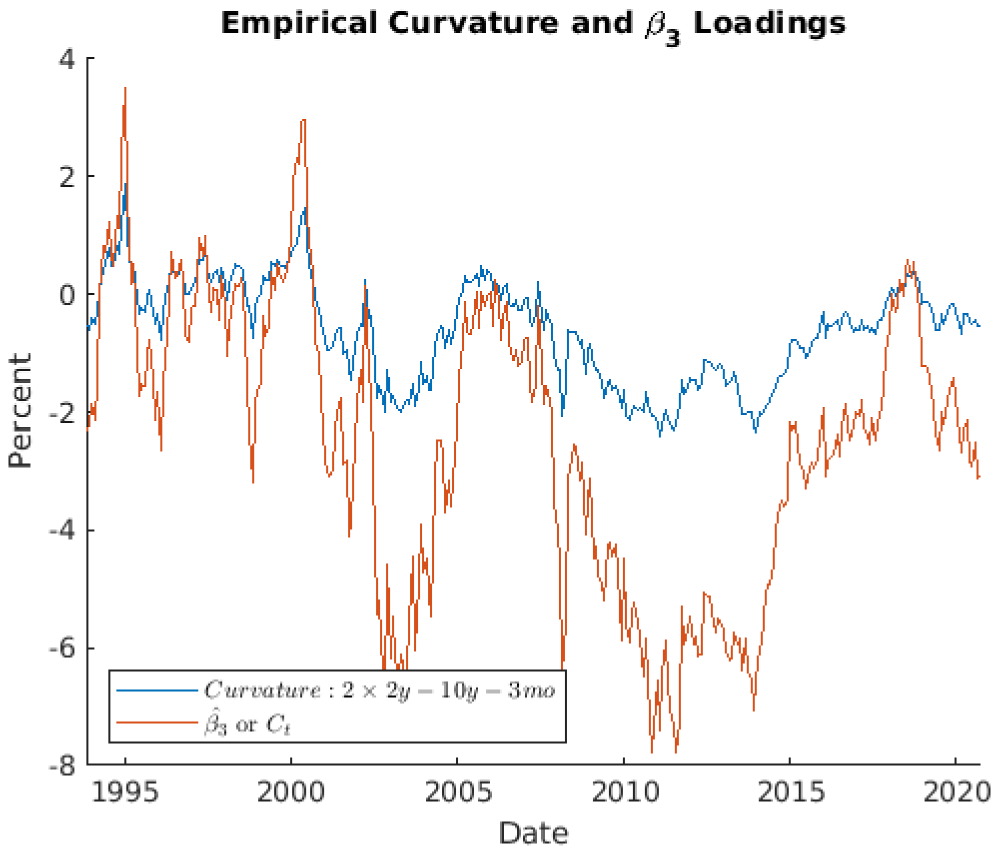
```
% actual level, slope and curvature from the dataset
% measure of level: L_t
tres_level = tres{:, 'tres10y'};

% measure of slope S_t = yield spread
tres_slope = tres{:, 'tres3mo'} - tres{:, 'tres10y'};

% measure of curvature C_t
tres_curvature = 2*tres{:, 'tres2y'} - tres{:, 'tres10y'} - tres{:, 'tres3mo'};

% stack them together in a matrix
compare_mat = [tres_level tres_slope tres_curvature beta];

% Convert the matrix into a table with variable names
compare_tab = array2table(compare_mat,...
    'VariableNames',{'level','slope','curvature', 'beta1', 'beta2', 'beta3'});

% convert the string of dates into datetime format
dates_dt = array2table(datetime(dates,'InputFormat','yyyy-MM-dd'), ...
    'VariableNames',{'Date'});

% stack the datetime dates with the level, slope and curvature in a table
compare_dt = [dates_dt, compare_tab];


% In each separate graph, plot the level, slope and curvature
% alsongside its corresponding factor loadings

figure
hold on
plot(dates, compare_mat(:,[1,4]))
title('Empirical Level and \beta_1 Loadings')
xlabel('Date')
ylabel('Percent')

legend({'$Level: 10y$';'$\hat{\beta_1}$ or $L_t$'},'Interpreter', 'latex', ...
    'location','northeast')
hold off
```




```
figure
hold on

plot(dates, compare_mat(:,[2,5]))
title('Empirical Slope and \beta_2 Loadings')
xlabel('Date')
ylabel('Percent')
legend({'$Slope: 3mo-10y$';'$\hat{\beta_2}$ or $S_t$'},'Interpreter', 'latex', ...
    'location','southeast')
hold off
```




```
figure
hold on

plot(dates, compare_mat(:,[3,6]))
title('Empirical Curvature and \beta_3 Loadings')
xlabel('Date')
ylabel('Percent')
legend({'$Curvature: 2\times2y-10y-3mo$';'$\hat{\beta_3}$ or $C_t$'},'Interpreter', ...
    'latex','location','southwest')
hold off
```



The empirical level, slope and curvature factors, alonside their corresponding factor loadings closely follow each other, as also reflected by the high positive correlation with each other :

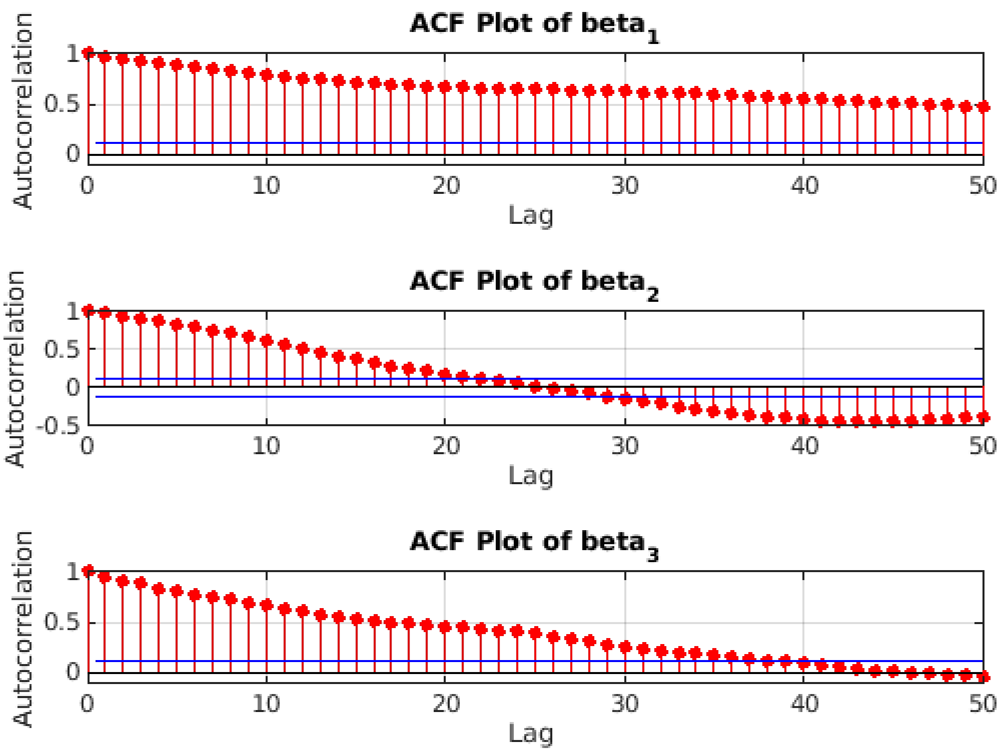
$\rho(10y\,,\text{level}\,) = 0.9615, : \rho(3\text{mo} - 10y\,,\text{slope}\,) = 0.9939, \rho(2 \times 2y - 10y, \text{curvature}\,) = 0.9781.$

```
corr_compare = corrcoef(compare_mat);
rowNames = {'10y','3mo-10y','2*2y-10y-3mo', 'beta 1', 'beta 2', 'beta 3'};
colNames = {'level','slope','curvature', 'beta 1', 'beta 2', 'beta 3'};
corr_compare_tab = array2table(corr_compare,'RowNames',rowNames, ...
    'VariableNames',colNames)
```

corr_compare_tab = 6×6 table

	level	slope	curvature	beta 1	beta 2	beta 3
1 10y	1.0000	0.0950	0.5617	0.9615	0.1405	0.5959
2 3mo-10y	0.0950	1.0000	0.5785	-0.1337	0.9939	0.5379
3 2*2y-10y-3mo	0.5617	0.5785	1.0000	0.3333	0.6413	0.9781
4 beta 1	0.9615	-0.1337	0.3333	1.0000	-0.1002	0.3673
5 beta 2	0.1405	0.9939	0.6413	-0.1002	1.0000	0.6033
6 beta 3	0.5959	0.5379	0.9781	0.3673	0.6033	1.0000

```
figure
hold on % retains current plot while adding new plots
for i=1:3
    subplot(3,1,i)
    autocorr(beta(:,i), 50)
    caption = sprintf('ACF Plot of beta_%d', i);
    title(caption, 'FontSize', 15);

    % label the x-axis and add extra space between two plots
    xlabel("Lag " + newline + "    ")
    ylabel("Autocorrelation")

end
hold off
```




```
% Summary stats of the latent variables from the Diebold-Li model (DNS)


tab_beta=table();
tab_beta.Min=min(beta)';
tab_beta.Mean=mean(beta)';
tab_beta.Max=max(beta)';
tab_beta.Std_dev=std(beta)';
tab_beta.Properties.RowNames = {'beta 1', 'beta 2', 'beta 3'};
```

```
disp_sumstat_beta = {"Summary Statistics of the Latent Variables from DNS:";...
    "----------------------------------";...
    tab_beta};
cellfun(@disp,disp_sumstat_beta)
```

```
Summary Statistics of the Latent Variables from DNS:
----------------------------------
```

	Min	Mean	Max	Std_dev
beta 1	1.1602	4.7137	8.1034	1.5517
beta 2	-5.1804	-2.2325	0.81551	1.5339
beta 3	-7.7825	-2.491	3.5225	2.4178

The descriptive statistics indicates that the curvature factor is most volatile, followed by the level and slope factors, respectively. After fitting the three-dimensional data of factors, I have fit a first-order VAR model on this estimated time series of factors. As the SSM equation works with factors adjusted for mean, I incorporated an additive constant to the $\mathrm{VAR}(1)$ model.

```
% fit a VAR(1) model in the 3D time series of the estimated factors "beta"
% for each of the 324 observations obtained

% "VARmod" stores the estimated parameter values acquired when we fit VAR(1)
% MLE estimates the parameters

VARmod = estimate(varm(3,1), beta);

% Create a  Bayesian VAR model under conjugate priors
% It specifies the prior distribution of the AR coefficient matrices, model constant ve
% and innovations covariance matrix.
% bayesvarm: displays a summary of the prior distributions
% numseries = 3; numlags = 1
BVARmod = estimate(bayesvarm(3,1, 'ModelType','conjugate' ), beta);
```

```
Bayesian VAR under conjugate priors
Effective Sample Size:          323
Number of equations:            3
Number of estimated Parameters: 12
             |   Mean     Std
-------------------------------
 Constant(1) |  0.0957  0.0604
 Constant(2) |  0.1009  0.0739
 Constant(3) | -0.2767  0.1506
 AR{1}(1,1)  |  0.9785  0.0115
 AR{1}(2,1)  | -0.0325  0.0141
 AR{1}(3,1)  |  0.0382  0.0287
 AR{1}(1,2)  |  0.0004  0.0134
 AR{1}(2,2)  |  0.8951  0.0164
 AR{1}(3,2)  |  0.0326  0.0335
 AR{1}(1,3)  |  0.0033  0.0092
 AR{1}(2,3)  |  0.0704  0.0112
 AR{1}(3,3)  |  0.9330  0.0228
   Innovations Covariance Matrix
    |    Y1        Y2        Y3
-----------------------------------
 Y1 |  0.0755   -0.0595    0.0204
```

```
   | (0.0059)  (0.0060)  (0.0104)
Y2 | -0.0595    0.1131   -0.0458
   | (0.0060)  (0.0088)  (0.0129)
Y3 |  0.0204   -0.0458    0.4698
   | (0.0104)  (0.0129)  (0.0366)
```

```
% Display summary statistics from the estimation.
varsum = summarize(VARmod);
bvarsum = summarize(BVARmod);
```

```
             |   Mean     Std
------------------------------
 Constant(1) |  0.0957  0.0604
 Constant(2) |  0.1009  0.0739
 Constant(3) | -0.2767  0.1506
 AR{1}(1,1)  |  0.9785  0.0115
 AR{1}(2,1)  | -0.0325  0.0141
 AR{1}(3,1)  |  0.0382  0.0287
 AR{1}(1,2)  |  0.0004  0.0134
 AR{1}(2,2)  |  0.8951  0.0164
 AR{1}(3,2)  |  0.0326  0.0335
 AR{1}(1,3)  |  0.0033  0.0092
 AR{1}(2,3)  |  0.0704  0.0112
 AR{1}(3,3)  |  0.9330  0.0228
   Innovations Covariance Matrix
    |    Y1        Y2        Y3
----------------------------------
 Y1 |  0.0755   -0.0595    0.0204
    | (0.0059)  (0.0060)  (0.0104)
 Y2 | -0.0595    0.1131   -0.0458
    | (0.0060)  (0.0088)  (0.0129)
 Y3 |  0.0204   -0.0458    0.4698
    | (0.0104)  (0.0129)  (0.0366)
```

```
Acoef = VARmod.AR{1}
```

```
Acoef = 3×3
    0.9961    0.0093   -0.0043
   -0.0274    0.9130    0.0631
    0.0334    0.0246    0.9418
```

The (3×3) matrix A is:

$$A_1^{2-\text{step}} = \begin{bmatrix} 0.9961 & 0.0093 & -0.0043 \\ -0.0274 & 0.913 & 0.0631 \\ 0.0334 & 0.0246 & 0.9418 \end{bmatrix}$$

```
% Extract the variance covariance matrix of the innovations of VAR(1)
sigma_eta = VARmod.Covariance
```

```
sigma_eta = 3×3
    0.0598   -0.0604    0.0199
   -0.0604    0.1027   -0.0495
```

```
    0.0199   -0.0495    0.4678
```

The variance-covariance matrix of the innovations of VAR(1) are:

$$\Sigma_{\eta}^{2-\text{step}} = \begin{bmatrix} 0.0598 & -0.0604 & 0.0199 \\ -0.0604 & 0.1027 & -0.0495 \\ 0.0199 & -0.0495 & 0.4678 \end{bmatrix}$$

Then I have forecasted the time series of the latent factors:

```
beta_table = array2table(beta);
beta_date = table2timetable(beta_table,'RowTimes',dates);

numperiods = 12;
fct_date = dateshift(beta_date.Time(end),'end','month',1:numperiods);
[Forecast,ForecastMSE] = forecast(VARmod,numperiods,beta(end-3:end,:));
```

Extract the main diagonal elements from the matrices in each cell of `ForecastMSE`. Apply the square root of the result to obtain standard errors.

```
extractMSE = @(x)diag(x)';
MSE = cellfun(extractMSE,ForecastMSE,'UniformOutput',false);
SE = sqrt(cell2mat(MSE));
```

The 95% forecast intervals for each of the latent factors are estimated as:

```
mdl = varm(3,1)
```

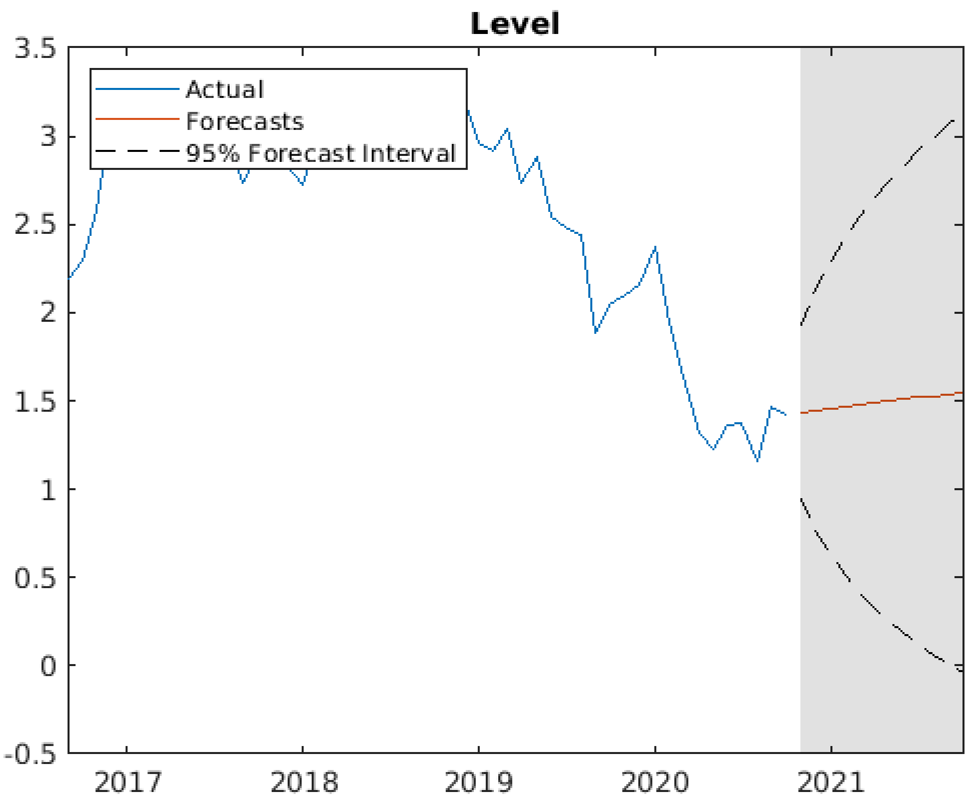
```
mdl = 
  varm with properties:

     Description: "3-Dimensional VAR(1) Model"
     SeriesNames: "Y1"  "Y2"  "Y3"
       NumSeries: 3
               P: 1
        Constant: [3×1 vector of NaNs]
              AR: {3×3 matrix of NaNs} at lag [1]
           Trend: [3×1 vector of zeros]
            Beta: [3×0 matrix]
      Covariance: [3×3 matrix of NaNs]
```

```
ForecastFI = zeros(numperiods,mdl.NumSeries,2);

ForecastFI(:,:,1) = Forecast - 2*SE;
ForecastFI(:,:,2) = Forecast + 2*SE;
```

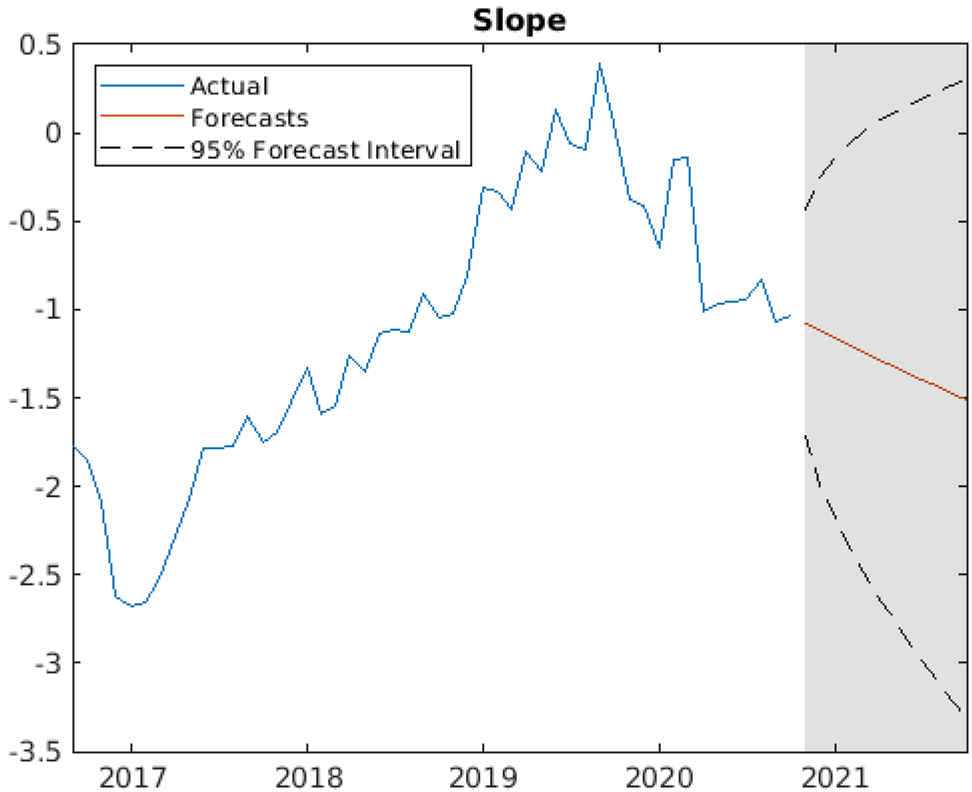
Finally, I have plotted the individual forecasts for 1 year ahead alongside the 95 % forecast intervals.

```
figure;
h1 = plot(beta_date.Time((end-49):end),beta((end-49):end,1));
```

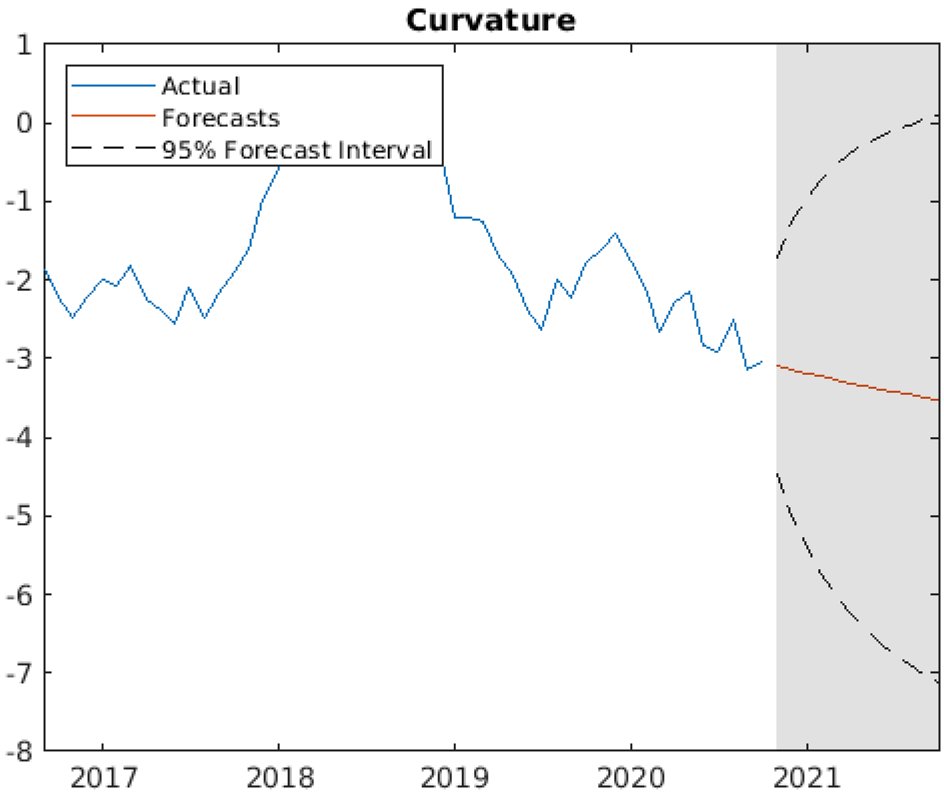
```
hold on;
h2 = plot(fct_date,Forecast(:,1));
h3 = plot(fct_date,ForecastFI(:,1,1),'k--');
plot(fct_date,ForecastFI(:,1,2),'k--');
title('Level');
h = gca;
fill([fct_date(1) h.XLim([2 2]) fct_date(1)],h.YLim([1 1 2 2]),'k',...
    'FaceAlpha',0.1,'EdgeColor','none');
legend([h1 h2 h3],'Actual','Forecasts','95% Forecast Interval',...
    'Location','northwest')
hold off;
```




```
figure;
h1 = plot(beta_date.Time((end-49):end),beta((end-49):end,2));
hold on;
h2 = plot(fct_date,Forecast(:,2));
h3 = plot(fct_date,ForecastFI(:,2,1),'k--');
plot(fct_date,ForecastFI(:,2,2),'k--');
title('Slope');
h = gca;
fill([fct_date(1) h.XLim([2 2]) fct_date(1)],h.YLim([1 1 2 2]),'k',...
    'FaceAlpha',0.1,'EdgeColor','none');
legend([h1 h2 h3],'Actual','Forecasts','95% Forecast Interval',...
    'Location','northwest')
hold off;
```




```
figure;
h1 = plot(beta_date.Time((end-49):end),beta((end-49):end,3));
hold on;
h2 = plot(fct_date,Forecast(:,3));
h3 = plot(fct_date,ForecastFI(:,3,1),'k--');
plot(fct_date,ForecastFI(:,3,2),'k--');
title('Curvature');
h = gca;
fill([fct_date(1) h.XLim([2 2]) fct_date(1)],h.YLim([1 1 2 2]),'k',...
    'FaceAlpha',0.1,'EdgeColor','none');
legend([h1 h2 h3],'Actual','Forecasts','95% Forecast Interval',...
    'Location','northwest')
hold off;
```



One-Step Kalman Filter with λ as a Random Variable

By specifying a parameter mapping function, I have estimated the Diebold-Li model implicity. This mapping function maps a parameter vector to SSM model parameters, deflates the observations to account for the means of each factor, and imposes constraints on the covariance matrices.

```
% create an SSM model
% pass the parameter mapping function "Example_DieboldLi" to the "ssm"
% function
% the input vector "params" calls the mapping function
% other input arguments to mapping function: information on yields and time
% to maturity
% link to the mapping function:
% https://www.mathworks.com/matlabcentral/fileexchange/47479-data_dieboldli-zip


model = ssm(@(params)dns(params,yields,ttm));
```

Then, the Kalman filter in the state-space model helps to output maximum likelihood estimates, optimal smoothed and filtered estimates of the latent factors. However, the Kalman filter can be very sensitive to the initial values of the parameters if we use it to compute the MLE of the SSM model. So, I have initialized the estimation by inputting the results from the Diebold-Li two-step method to acquire the intial transition matrix.

Encoded in a column vector are the initial values passed in the SSM `estimate.` While the matrix A of the SSM model is a 3-by-3 AR coefficient matrix of the $VAR(1)$ model, it is stacked in a column vector of 9 elements. Likewise, the matrix B of the is a 3-by-3 matrix such that we can decompose the postive-definite symmetric matrix Q into a product of two matrices: B and its transpose : $Q = BB'$. Herein, B is the lower Cholesky factor of Q. To ensure that the off-diagonal elements of the matrix Q is non-diagonal i.e. the elements are correlated with each other, I have allocated six elements of the lower Cholesky factor B as a column vector of inital parameter values. Then, I initialized the elements of the initial parameters' vector by taking a square root of the variance of the estimated innovations from the VAR(1) model. Thus, I have assumed diagonality in the covariance matrix Q. As before, I have arranged the initial parameter vector in a manner such that the elements of B along and below the diagonal are stacked as a column vector.

Just as with matrix B, I have constrained the matrix D to be diagonal. This is because the covariance matrix H in the Diebold-Li model is also diagonal, implying that $H = DD'$. Hence, the vector of initial parameters is the square root of the diagonal elements of the sample covariance matrix of innovations from $VAR(1)$.

Therefore, the elements of the initial parameter vector associated with D are set to the square root of the diagonal elements of the sample covariance matrix of the resid of the $VAR(1)$ model, "one such element for each of the 10 time to maturity of the yields are arranged column-wise.

Unlike other aforementioned matrices in the state-space system, the matrix C is indirectly estimated. Computed internally by the mapping function, it is a parametarized function of the estimated decay rate parameter λ. λ is initialized to 0.0609 as Diebold, Li and Rudebusch (2006) did. Lastly, I've set the sample average of the three OLS regression coefficients (in the first step) as the vector of initial latent factor parameters.

In total, the numerical optimization estimates 29 free parameters. These are as as follows:

9 parameters from the 3-by-3 matrix A;

3 parameters from the 3-by-1 vector of means μ;

1 parameter from the measurement matrix Λ, which is the exponential decay rate λ;

6 parameters from the transition disturbance matrix Q : for each of the three latent factors, I computed the error variance : $\sigma_l^2, \sigma_s^2, \sigma_c^2$; and three covariance terms : $\sigma_{\mathrm{lc}}, \sigma_{\mathrm{ls}}, \sigma_{\mathrm{sc}}$;

10 parameters from the measurement disturbance covariance matrix H : for each of the ten yields, I computed the error variances as above.

```
% Get the VAR(1) matrix (stored as a cell array)
% A0: 3-by-3 coefficient matrix at lag 1
A0 = VARmod.AR{1};
A0_bvar = BVARmod.AR{1};
```

```
% Columnwise stacking: stacked the above 3-by-3 coefficient matrix into a
% 9-by-1 column vector
A0 = A0(:);
A0_bvar = A0_bvar(:);

% Extract the variance covariance matrix of the innovations of VAR(1)
Q0 = VARmod.Covariance
```

```
Q0 = 3×3
    0.0598   -0.0604    0.0199
   -0.0604    0.1027   -0.0495
    0.0199   -0.0495    0.4678
```

```
Q0_bvar = BVARmod.Covariance
```

```
Q0_bvar = 3×3
    0.0755   -0.0595    0.0204
   -0.0595    0.1131   -0.0458
    0.0204   -0.0458    0.4698
```

```
% B: take the square root of the diagonal elements of Q and other elements
% other elements of Q are 0

B0 = [sqrt(Q0(1,1)); 0; 0; sqrt(Q0(2,2)); 0; sqrt(Q0(3,3))];
B0_bvar = [sqrt(Q0_bvar(1,1)); 0; 0; sqrt(Q0_bvar(2,2)); 0; sqrt(Q0_bvar(3,3))];

% H0: 10-by-10 covariance matrix of the residuals from the OLS model
H0 = cov(resid);

% diag(H0): stores the elements of the main diagonal of the 10-by-10
% matrix D0 as a column vector:
% it is the square root of the diagonal elements of
D0 = sqrt(diag(H0));

% Average of the 3 beta coefficients
mu0 = mean(beta)'
```

```
mu0 = 3×1
    4.7137
   -2.2325
   -2.4910
```

```
% each of the 4 params is a vector, respectively
% param0 stack these 4 vectors into a column vector one below the other
% (vectorize)
% param0: the initial parameter column vector.
% lamda is set to a pre-specified value
% lamda is stored in the last element of param0
param0 = [A0; B0; D0; mu0; lambda0];
```

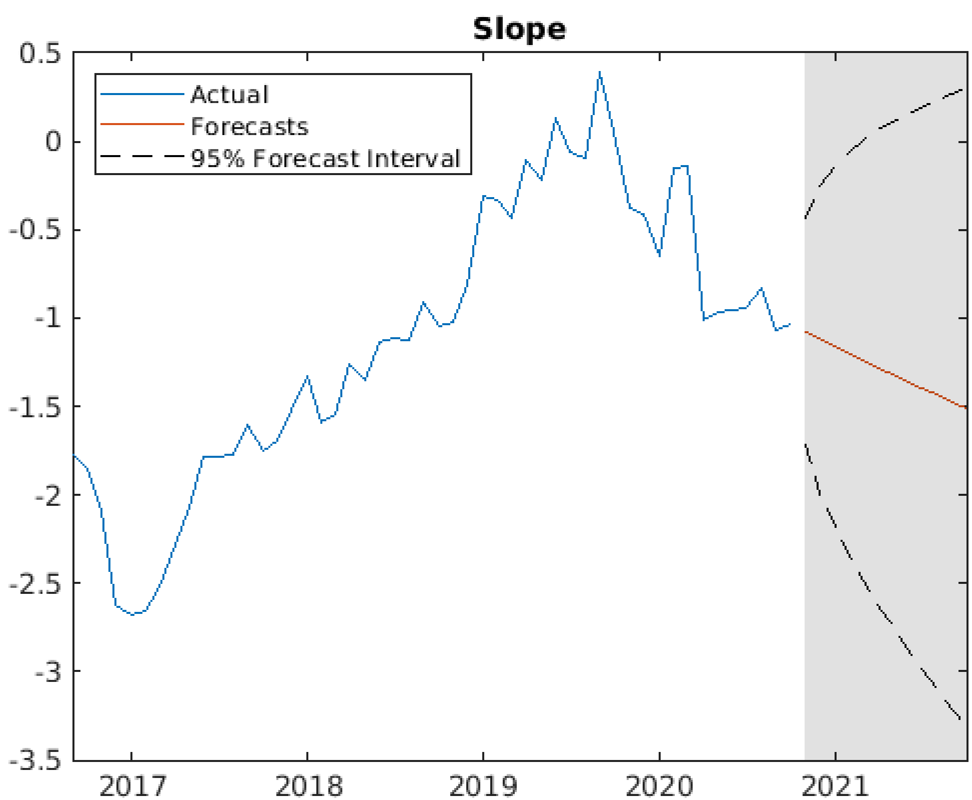
After caliberating the initial values, I have established the optimization parameters and estimated the model via the Kalman filter. The Kalman filter outputs the maximum likelihood estimates, optimal filtered and smoothed estimates of the 29 free parameters. As the covariance matrix $H = DD'$ is diagonal, i.e. the observation innovations are uncorrelated, I have treated the multivariate series as a univariate series. This hastens the estimation's runtime performance, and enhances numerical stability.

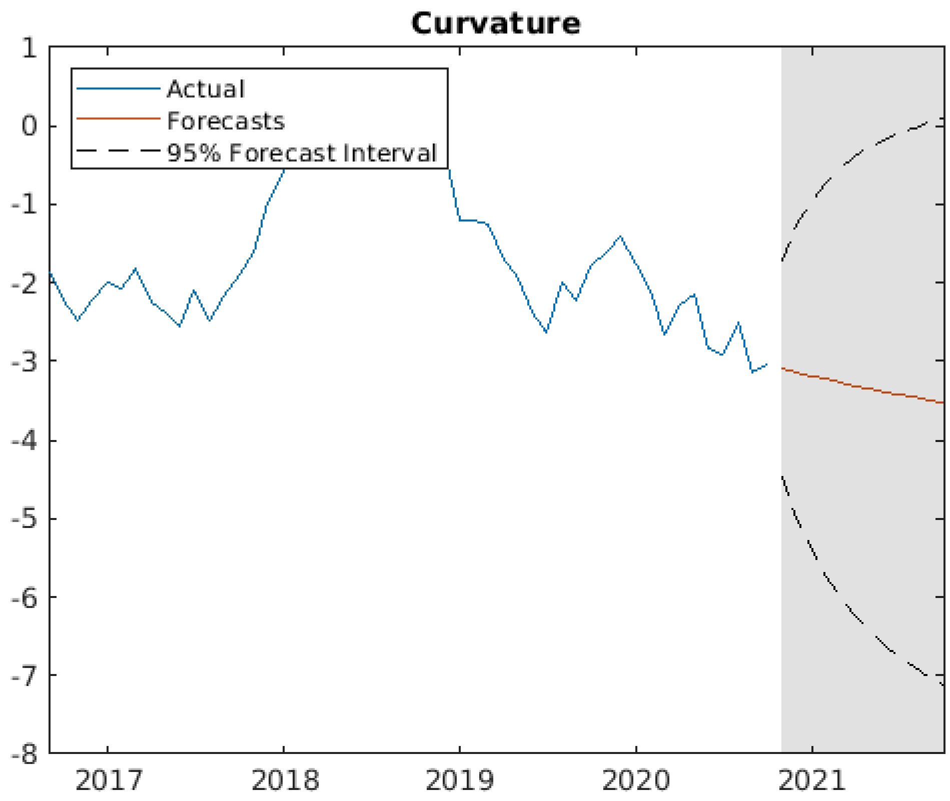
```
% create the options for the optimization solver
% MaxFunEvals: maximum number of times the routine evaluates the objective function,
% each time with the updated parameter estimates.
% fminunc: maximum likelihood
% maximum number of function evaluations = 30,000
% MaxIter: maximum number of iterations are the iterations of the fminunc function
% itself.
% tolerance is a threshold which, if crossed, stops the iterations of a solver.
% TolX: lower bound on the step size
% If the fminunc solver attempts to take a step that is smaller than TolX,
% the iterations end.
% TolFun: lower bound on the change in the value of the objective function during
% a step.
% PlotFcn: plot various measures of progress during the execution of a solver.

options = optimoptions('fminunc','MaxFunEvals',30000,'algorithm','quasi-newton', ...
    'TolFun' ,1e-9,'TolX',1e-9,'MaxIter',1100,'Display','off',...
    'PlotFcn',{@optimplotx,@optimplotfval,@optimplotfirstorderopt});

% "estimate" function estimates the SSM model ("model") by the Kalman filter
% SSMmod stores the state-space equation, observation equation,
% means and the covariance matrix of the initial states
% params: 29-by-1 vector of the updated parameters from SSM
% treat the multivariate series as "univariate": sequential filtering
% estimate the asymptotic covariance by taking the outer product of
% gradients (OPG)

[SSMmod,params] = estimate(model,yields,param0,'Display','full', ...
    'options',options,'Univariate',true);
```

```
____________________________________________________________
   Diagnostic Information

Number of variables: 29

Functions
Objective:                            @(c)-fML(c,Mdl,Y,Predictors,unitFlag,sqrtFlag,mexFlag,mexTvFlag,tol,
Gradient:                             finite-differencing
Hessian:                              finite-differencing (or Quasi-Newton)


Algorithm selected
   quasi-newton


____________________________________________________________
   End diagnostic information
                                                                First-order
 Iteration  Func-count       f(x)        Step-size          optimality
     0           30         -688.369                          9.83e+04
     1          120         -1467.71      1.01772e-07         4.87e+04
     2          150         -1666.84                1          2.1e+04
     3          180          -1710.9                1         6.67e+03
```

4	210	-1747.49	1	6.1e+03
5	240	-1890.78	1	1.46e+04
6	270	-1977.46	1	1.29e+04
7	300	-2029.07	1	6.85e+03
8	330	-2043.97	1	1.91e+03
9	360	-2046.81	1	1.09e+03
10	390	-2048.34	1	1.21e+03
11	420	-2052.24	1	2.46e+03
12	450	-2060.3	1	3.58e+03
13	480	-2075.18	1	3.83e+03
14	510	-2088.81	1	2.05e+03
15	540	-2097.06	1	982
16	570	-2103.63	1	2.04e+03
17	600	-2104.51	1	931
18	630	-2105.8	1	657
19	660	-2108.91	1	3.79e+03

Iteration	Func-count	f(x)	Step-size	First-order optimality
20	690	-2113.87	1	6.91e+03
21	720	-2124.87	1	1.04e+04
22	750	-2140.06	1	1.02e+04
23	780	-2148.83	1	5.39e+03
24	810	-2152	1	2.05e+03
25	840	-2153.45	1	868
26	870	-2154.11	1	1.44e+03
27	900	-2156.43	1	3.25e+03
28	930	-2160.76	1	5.12e+03
29	960	-2168.56	1	6.56e+03
30	990	-2175.61	1	5.33e+03
31	1020	-2180.49	1	1.66e+03
32	1050	-2181.64	1	1.04e+03
33	1080	-2182.31	1	1.1e+03
34	1110	-2185.31	1	3.17e+03





Iteration	Func-count	f(x)	Step-size	First-order optimality
35	1140	-2190.62	1	4.87e+03
36	1170	-2200.12	1	6.29e+03
37	1200	-2207.92	1	4.2e+03
38	1230	-2210.47	1	1.52e+03
39	1260	-2211.2	1	532
Iteration	Func-count	f(x)	Step-size	First-order optimality
40	1290	-2211.5	1	641
41	1320	-2212.47	1	745
42	1350	-2214.96	1	815
43	1380	-2220.41	1	873
44	1410	-2229.61	1	917
45	1440	-2234.69	1	857
46	1470	-2235.89	1	821
47	1500	-2236.85	1	700
48	1530	-2237.28	1	636
49	1560	-2238.44	1	494
50	1590	-2240.47	1	514
51	1620	-2243.99	1	539
52	1650	-2248.16	1	606
53	1680	-2250.4	1	633
54	1710	-2250.71	1	418
55	1740	-2250.82	1	375
56	1770	-2251.18	1	296
57	1800	-2251.75	1	287
58	1830	-2252.56	1	263
59	1860	-2253.05	1	172
Iteration	Func-count	f(x)	Step-size	First-order optimality
60	1890	-2253.17	1	161
61	1920	-2253.2	1	155
62	1950	-2253.25	1	148
63	1980	-2253.35	1	150
64	2010	-2253.61	1	181

65	2040	-2254.15	1	260
66	2070	-2255.08	1	269
67	2100	-2256.02	1	163
68	2130	-2256.38	1	125
69	2160	-2256.43	1	124
70	2190	-2256.46	1	122
71	2220	-2256.53	1	114
72	2250	-2256.64	1	108
73	2280	-2256.79	1	109
74	2310	-2256.97	1	140
75	2340	-2257.06	1	89
76	2370	-2257.11	1	79.8
77	2400	-2257.14	1	98.5
78	2430	-2257.16	1	108
79	2460	-2257.18	1	112
Iteration	Func-count	f(x)	Step-size	First-order optimality
80	2490	-2257.21	1	111
81	2520	-2257.28	1	100
82	2550	-2257.39	1	90.3
83	2580	-2257.51	1	80
84	2610	-2257.57	1	72.4
85	2640	-2257.59	1	72
86	2670	-2257.59	1	67.8
87	2700	-2257.61	1	59.3
88	2730	-2257.65	1	55
89	2760	-2257.73	1	57.8
90	2790	-2257.88	1	80
91	2820	-2258.07	1	73.1
92	2850	-2258.19	1	48.5
93	2880	-2258.22	1	48.1
94	2910	-2258.22	1	48.2
95	2940	-2258.23	1	48.3
96	2970	-2258.24	1	48.4
97	3000	-2258.26	1	48.5
98	3030	-2258.32	1	48.3
99	3060	-2258.46	1	64
Iteration	Func-count	f(x)	Step-size	First-order optimality
100	3090	-2258.73	1	84.3
101	3120	-2259.12	1	134
102	3150	-2259.39	1	156
103	3180	-2259.48	1	49.8
104	3210	-2259.49	1	40.1
105	3240	-2259.5	1	39.7
106	3270	-2259.51	1	39.6
107	3300	-2259.55	1	38.5
108	3330	-2259.64	1	35
109	3360	-2259.85	1	29.2
110	3390	-2260.24	1	49.7
111	3420	-2260.69	1	64.7
112	3450	-2260.95	1	59.7
113	3480	-2261.02	1	49.7
114	3510	-2261.02	1	49.1
115	3540	-2261.03	1	49.4
116	3570	-2261.04	1	49.3
117	3600	-2261.08	1	48.3
118	3630	-2261.15	1	44.5
119	3660	-2261.31	1	52.5
Iteration	Func-count	f(x)	Step-size	First-order optimality
120	3690	-2261.53	1	37.7
121	3720	-2261.69	1	22.1
122	3750	-2261.74	1	17
123	3780	-2261.74	1	16.6

```
   124       3810        -2261.74               1           16.6
   125       3840        -2261.74               1           16.6
   126       3870        -2261.75               1           18.9
   127       3900        -2261.75               1           38.2
   128       3930        -2261.76               1           67.7
   129       3960        -2261.78               1            108
   130       3990        -2261.84               1            148
   131       4020        -2261.91               1            145
   132       4050        -2261.96               1           91.5
   133       4080        -2261.98               1           32.9
   134       4110        -2261.99               1           20.1
   135       4140        -2261.99               1           19.4
   136       4170        -2261.99               1           31.1
   137       4200           -2262               1           59.6
   138       4230        -2262.02               1            105
   139       4260        -2262.07               1            169
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
   140       4290        -2262.17               1            242
   141       4320        -2262.33               1            271
   142       4350        -2262.48               1            181
   143       4380        -2262.53               1           60.3
   144       4410        -2262.54               1           60.8
   145       4440        -2262.54               1           60.7
   146       4470        -2262.55               1           60.1
   147       4500        -2262.57               1           64.4
   148       4530        -2262.63               1            110
   149       4560        -2262.77               1            179
   150       4590        -2263.12               1            280
   151       4620        -2263.88               1            405
   152       4650        -2265.07               1            481
   153       4680        -2266.03               1            337
   154       4710        -2266.47               1           64.5
   155       4740        -2266.53               1           50.8
   156       4770        -2266.53               1           49.5
   157       4800        -2266.54               1           49.2
   158       4830        -2266.55               1             48
   159       4860        -2266.58               1           45.8
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
   160       4890        -2266.64               1           49.7
   161       4920        -2266.79               1           71.5
   162       4950        -2267.05               1             73
   163       4980        -2267.28               1             39
   164       5010        -2267.36               1           36.5
   165       5040        -2267.37               1           36.8
   166       5070        -2267.38               1           36.9
   167       5100        -2267.38               1           36.9
   168       5130        -2267.39               1           36.9
   169       5160        -2267.42               1           36.6
   170       5190        -2267.48               1           35.6
   171       5220        -2267.62               1           60.5
   172       5250        -2267.88               1           86.4
   173       5280        -2268.18               1           91.8
   174       5310        -2268.38               1           52.7
   175       5340        -2268.44               1           33.3
   176       5370        -2268.44               1           32.2
   177       5400        -2268.44               1           31.7
   178       5430        -2268.45               1           30.8
   179       5460        -2268.46               1           29.3
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
   180       5490        -2268.49               1           26.9
   181       5520        -2268.56               1           23.2
   182       5550        -2268.73               1           30.8
```

183	5580	-2268.99	1	34.1
184	5610	-2269.26	1	26.4
185	5640	-2269.38	1	23.4
186	5670	-2269.4	1	22.9
187	5700	-2269.4	1	23.4
188	5730	-2269.4	1	23.5
189	5760	-2269.4	1	23.5
190	5790	-2269.41	1	23.4
191	5820	-2269.42	1	22.9
192	5850	-2269.44	1	30.1
193	5880	-2269.48	1	42.4
194	5910	-2269.55	1	44.4
195	5940	-2269.6	1	28.4
196	5970	-2269.62	1	18.4
197	6000	-2269.62	1	18.7
198	6030	-2269.62	1	18.4
199	6060	-2269.62	1	18.1

Iteration	Func-count	f(x)	Step-size	First-order optimality
200	6090	-2269.62	1	17.6
201	6120	-2269.62	1	16.7
202	6150	-2269.63	1	15
203	6180	-2269.64	1	12
204	6210	-2269.68	1	17.2
205	6240	-2269.75	1	20.5
206	6270	-2269.83	1	15.9
207	6300	-2269.88	1	9.09
208	6330	-2269.89	1	9.29
209	6360	-2269.89	1	9.3
210	6390	-2269.89	1	9.29
211	6420	-2269.89	1	9.27
212	6450	-2269.89	1	9.24
213	6480	-2269.89	1	9.13
214	6510	-2269.89	1	8.89
215	6540	-2269.9	1	8.25
216	6570	-2269.91	1	12.3
217	6600	-2269.93	1	16
218	6630	-2269.95	1	14.3
219	6660	-2269.96	1	5.47

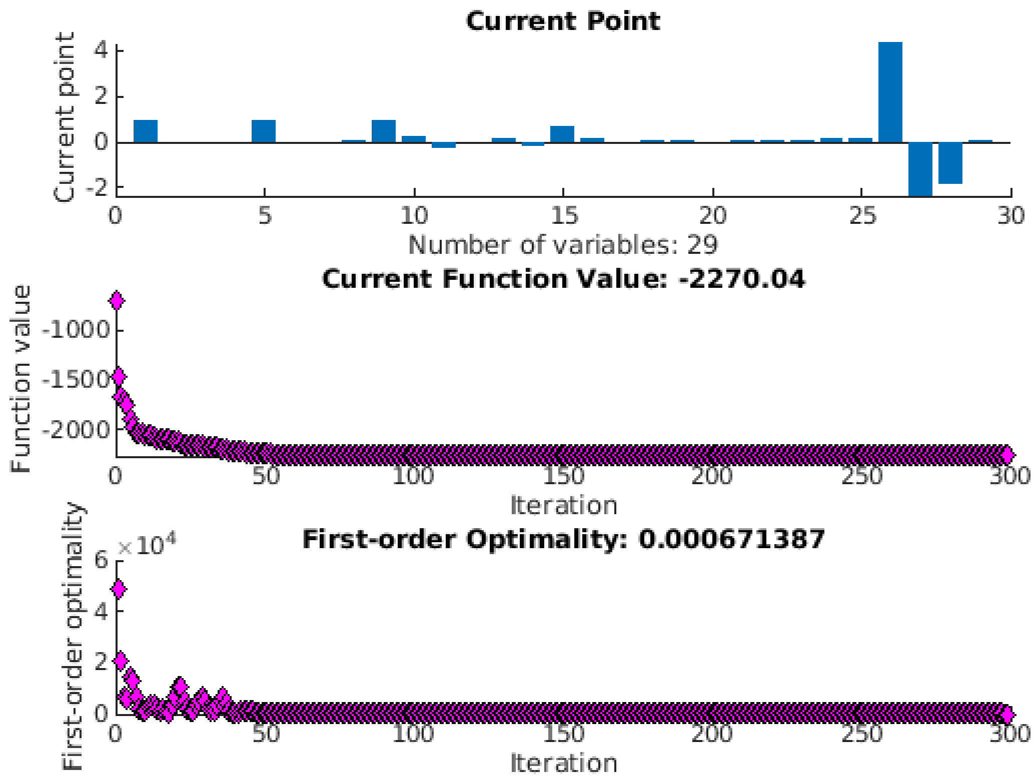
Iteration	Func-count	f(x)	Step-size	First-order optimality
220	6690	-2269.96	1	1.36
221	6720	-2269.96	1	1.36
222	6750	-2269.96	1	1.37
223	6780	-2269.96	1	1.38
224	6810	-2269.96	1	1.39
225	6840	-2269.96	1	1.39
226	6870	-2269.96	1	1.4
227	6900	-2269.96	1	1.41
228	6930	-2269.96	1	2.46
229	6960	-2269.96	1	3.97
230	6990	-2269.96	1	5.59
231	7020	-2269.96	1	6.07
232	7050	-2269.96	1	4.07
233	7080	-2269.96	1	1.43
234	7110	-2269.96	1	0.291
235	7140	-2269.96	1	0.292
236	7170	-2269.96	1	0.292
237	7200	-2269.96	1	0.292
238	7230	-2269.96	1	0.292
239	7260	-2269.96	1	0.292

Iteration	Func-count	f(x)	Step-size	First-order optimality
240	7290	-2269.96	1	0.292
241	7320	-2269.96	1	0.292

242	7350	-2269.96	1	0.466
243	7380	-2269.96	1	0.763
244	7410	-2269.96	1	1.27
245	7440	-2269.96	1	2.06
246	7470	-2269.96	1	3.34
247	7500	-2269.97	1	5.35
248	7530	-2269.97	1	8.41
249	7560	-2269.98	1	12.5
250	7590	-2269.99	1	16.3
251	7620	-2270.02	1	16.2
252	7650	-2270.04	1	10.4
253	7680	-2270.04	1	2.99
254	7710	-2270.04	1	0.297
255	7740	-2270.04	1	0.258
256	7770	-2270.04	1	0.257
257	7800	-2270.04	1	0.257
258	7830	-2270.04	1	0.258
259	7860	-2270.04	1	0.259

Iteration	Func-count	f(x)	Step-size	First-order optimality
260	7890	-2270.04	1	0.259
261	7920	-2270.04	1	0.256
262	7950	-2270.04	1	0.259
263	7980	-2270.04	1	0.338
264	8010	-2270.04	1	0.349
265	8040	-2270.04	1	0.226
266	8070	-2270.04	1	0.0679
267	8100	-2270.04	1	0.025
268	8130	-2270.04	1	0.0249
269	8160	-2270.04	1	0.0249
270	8220	-2270.04	0.197628	0.0249
271	8250	-2270.04	1	0.0249
272	8280	-2270.04	1	0.0248
273	8310	-2270.04	1	0.0249
274	8340	-2270.04	1	0.0251
275	8370	-2270.04	1	0.0251
276	8400	-2270.04	1	0.0249
277	8430	-2270.04	1	0.0249
278	8490	-2270.04	10	0.0253
279	8520	-2270.04	1	0.0251

Iteration	Func-count	f(x)	Step-size	First-order optimality
280	8550	-2270.04	1	0.025
281	8580	-2270.04	1	0.0247
282	8610	-2270.04	1	0.0247
283	8640	-2270.04	1	0.0245
284	8670	-2270.04	1	0.0425
285	8700	-2270.04	1	0.0634
286	8730	-2270.04	1	0.0815
287	8760	-2270.04	1	0.103
288	8790	-2270.04	1	0.119
289	8820	-2270.04	1	0.123
290	8850	-2270.04	1	0.109
291	8880	-2270.04	1	0.0724
292	8910	-2270.04	1	0.0827
293	8940	-2270.04	1	0.0804
294	8970	-2270.04	1	0.0819
295	9000	-2270.04	1	0.0351
296	9030	-2270.04	1	0.00656
297	9060	-2270.04	1	0.000916
298	9120	-2270.04	0.381109	0.00058
299	9300	-2270.04	0.468445	0.000671




```
Local minimum possible.

fminunc stopped because it cannot decrease the objective function
along the current search direction.

<stopping criteria details>
Warning: Covariance matrix of estimators cannot be computed precisely due to inversion difficulty.
 Check parameter identifiability. Also try different starting values and other options to compute
 the covariance matrix.
Method: Maximum likelihood (fminunc)
Sample size: 324
Logarithmic  likelihood:      2270.04
Akaike   info criterion:     -4482.09
Bayesian info criterion:     -4372.45
       |     Coeff         Std Err       t Stat      Prob
--------------------------------------------------------
 c(1)  |  0.98903         0.01312    75.40552   0
 c(2)  | -0.02014         0.01670    -1.20633  0.22770
 c(3)  |  0.02555         0.04292     0.59523  0.55169
 c(4)  |  0.00250         0.01572     0.15895  0.87371
 c(5)  |  0.93378         0.01728    54.04903   0
 c(6)  |  0.02398         0.03077     0.77932  0.43580
 c(7)  |  0.00374         0.01042     0.35917  0.71947
 c(8)  |  0.05828         0.01211     4.81329  0.00000
 c(9)  |  0.92883         0.02534    36.66087   0
 c(10) |  0.25785         0.00925    27.86503   0
 c(11) | -0.24548         0.01664   -14.74876   0
 c(12) |  0.00352         0.04479     0.07848  0.93745
 c(13) |  0.20475         0.00640    32.00650   0
 c(14) | -0.13988         0.04249    -3.29205  0.00099
 c(15) |  0.69513         0.02401    28.95071   0
 c(16) |  0.14203         0.00429    33.13842   0
 c(17) | -0.00000      1721.61156    -0.00000  1.00000
 c(18) |  0.07404         0.00284    26.02499   0
```

```
 c(19) |  0.06073          0.00372    16.32884   0
 c(20) | -0.00000       7187.95637    -0.00000  1.00000
 c(21) |  0.06436          0.00443    14.51922   0
 c(22) |  0.06501          0.00394    16.47932   0
 c(23) |  0.05757          0.00588     9.79841   0
 c(24) |  0.20640          0.01238    16.67541   0
 c(25) |  0.19054          0.01162    16.40246   0
 c(26) |  4.35486          1.33131     3.27111  0.00107
 c(27) | -2.35135          1.22411    -1.92086  0.05475
 c(28) | -1.87201          1.44069    -1.29939  0.19382
 c(29) |  0.04560          0.00029   159.16548   0
       |
       |   Final State      Std Dev     t Stat    Prob
 x(1)  | -3.09106          0.07839   -39.43100   0
 x(2)  |  1.30219          0.07037    18.50463   0
 x(3)  | -0.10587          0.14730    -0.71869  0.47233
SSMmod =
State-space model type: ssm

State vector length: 3
Observation vector length: 10
State disturbance vector length: 3
Observation innovation vector length: 10
Sample size supported by model: Unlimited

State variables: x1, x2,...
State disturbances: u1, u2,...
Observation series: y1, y2,...
Observation innovations: e1, e2,...

State equations:
x1(t) = (0.99)x1(t-1) + (2.50e-03)x2(t-1) + (3.74e-03)x3(t-1) + (0.26)u1(t)
x2(t) = -(0.02)x1(t-1) + (0.93)x2(t-1) + (0.06)x3(t-1) - (0.25)u1(t) + (0.20)u2(t)
x3(t) = (0.03)x1(t-1) + (0.02)x2(t-1) + (0.93)x3(t-1) + (3.52e-03)u1(t) - (0.14)u2(t)
        + (0.70)u3(t)

Observation equations:
y1(t) = x1(t) + (0.93)x2(t) + (0.06)x3(t) + (0.14)e1(t)
y2(t) = x1(t) + (0.87)x2(t) + (0.11)x3(t) - (6.19e-08)e2(t)
y3(t) = x1(t) + (0.77)x2(t) + (0.19)x3(t) + (0.07)e3(t)
y4(t) = x1(t) + (0.61)x2(t) + (0.27)x3(t) + (0.06)e4(t)
y5(t) = x1(t) + (0.49)x2(t) + (0.30)x3(t) - (4.75e-09)e5(t)
y6(t) = x1(t) + (0.34)x2(t) + (0.28)x3(t) + (0.06)e6(t)
y7(t) = x1(t) + (0.26)x2(t) + (0.23)x3(t) + (0.07)e7(t)
y8(t) = x1(t) + (0.18)x2(t) + (0.18)x3(t) + (0.06)e8(t)
y9(t) = x1(t) + (0.09)x2(t) + (0.09)x3(t) + (0.21)e9(t)
y10(t) = x1(t) + (0.06)x2(t) + (0.06)x3(t) + (0.19)e10(t)

Initial state distribution:

Initial state means
 x1  x2  x3
  0   0   0

Initial state covariance matrix
     x1     x2     x3
 x1   3.27  -0.67  1.09
 x2  -0.67   2.72  1.89
 x3   1.09   1.89  4.67

State types
     x1          x2          x3
 Stationary  Stationary  Stationary
params = 29×1
    0.9890
```

```
  -0.0201
   0.0255
   0.0025
   0.9338
   0.0240
   0.0037
   0.0583
   0.9288
   0.2578
     ⋮
```

```
% Obtain the estimated lambda or decay rate
lambda = params(end);

% Obtain the estimated factor means
mu = params(end-3:end-1)';
```

Contrasting the Estimated Parameters

Below I have contrasted the results yielded from the two-step and SSM methods to analyze the extent of similarities in the results from the aforementioned techniques. Besides, the juxtaposition informs how apt is the use of the two-step approach in establishing the initial parameter values needed to estimate the state-space model.

First, I have compared the $AR(1)$ coefficient matrix from the VAR model with the state transition matrix A of the SSM model.

```
% cellfun: applies the function "disp" to the contents of each cell of cell
% array "display_vars"
% then cellfun concatenates the outputs from "disp" into the output array
% The input argument "disp" is a function handle to a function

disp_ssmA = {"SSM State Transition Matrix (A):";...
    "--------------------------------";...
    SSMmod.A};
cellfun(@disp,disp_ssmA)
```

```
SSM State Transition Matrix (A):
--------------------------------
    0.9890    0.0025    0.0037
   -0.0201    0.9338    0.0583
    0.0255    0.0240    0.9288
```

```
disp_varA = {"Two-Step State Transition Matrix (A) from VAR(1):";...
    "--------------------------------";...
    VARmod.AR{1}};
cellfun(@disp,disp_varA)
```

```
Two-Step State Transition Matrix (A) from VAR(1):
--------------------------------
    0.9961    0.0093   -0.0043
   -0.0274    0.9130    0.0631
    0.0334    0.0246    0.9418
```

```
disp_bvarA = {"Two-Step State Transition Matrix (A) from BVAR(1):";...
    "--------------------------------";...
    BVARmod.AR{1}};
cellfun(@disp,disp_bvarA)
```

```
Two-Step State Transition Matrix (A) from BVAR(1):
--------------------------------
    0.9785    0.0004    0.0033
   -0.0325    0.8951    0.0704
    0.0382    0.0326    0.9330
```

The matrix A from the one-step Kallman filter approach:

$$A_{1-\text{step}} = \begin{bmatrix} 0.9890 & 0.0025 & 0.0037 \\ -0.0201 & 0.9338 & 0.0583 \\ 0.0255 & 0.0240 & 0.9288 \end{bmatrix}$$

For comparison, I have displayed the two coefficient matrices side-by-side:

$$A_1^{2-\text{step}} = \begin{bmatrix} 0.9961 & 0.0093 & -0.0043 \\ -0.0274 & 0.913 & 0.0631 \\ 0.0334 & 0.0246 & 0.9418 \end{bmatrix}, A_{1-\text{step}} = \begin{bmatrix} 0.9890 & 0.0025 & 0.0037 \\ -0.0201 & 0.9338 & 0.0583 \\ 0.0255 & 0.0240 & 0.9288 \end{bmatrix}$$

Mostly, the results are in agreement. Notably, the large positive diagonal elements of the SSM matrix and the two-step transition matrices A connote high level of persistent dynamics prevalent in each latent factor. Simultaneously, the very low values of othe off-diagonal elements conveys weak covariance between the factors.

Secondly, I compared the covariance matrix of the innovation $Q = BB'$ from the state disturbance loading matrix B with the covariance matrix of the innovations from $VAR(1)$ model. The results from both methods are very similar, and the volatility in the state transition shock or the estimated variance (main diagonal elements) of the curvature is the highest, followed by that of slope and curvature, respectively.

```
disp_ssmB = {"SSM State Error Loading Matrix (B):";...
    "-----------------------------------------";...
    SSMmod.B};
cellfun(@disp,disp_ssmB)
```

```
SSM State Error Loading Matrix (B):
-----------------------------------------
    0.2578         0         0
```

```
   -0.2455    0.2047         0
    0.0035   -0.1399    0.6951
```

```
disp_ssmQ = {"SSM State Disturbance Covariance Matrix (Q = BB''):";...
    "--------------------------------------------------";...
    SSMmod.B * SSMmod.B'};
cellfun(@disp,disp_ssmQ)
```

```
SSM State Disturbance Covariance Matrix (Q = BB''):
--------------------------------------------------
    0.0665   -0.0633    0.0009
   -0.0633    0.1022   -0.0295
    0.0009   -0.0295    0.5028
```

```
disp_varQ = {"Two-Step State Disturbance Covariance Matrix (Q) from VAR(1):";...
    "-------------------------------------------------";...
    VARmod.Covariance};
cellfun(@disp,disp_varQ)
```

```
Two-Step State Disturbance Covariance Matrix (Q) from VAR(1):
-------------------------------------------------
    0.0598   -0.0604    0.0199
   -0.0604    0.1027   -0.0495
    0.0199   -0.0495    0.4678
```

```
disp_bvarQ = {"Two-Step State Disturbance Covariance Matrix (Q) from BVAR(1):";...
    "-------------------------------------------------";...
    BVARmod.Covariance};
cellfun(@disp,disp_bvarQ)
```

```
Two-Step State Disturbance Covariance Matrix (Q) from BVAR(1):
-------------------------------------------------
    0.0755   -0.0595    0.0204
   -0.0595    0.1131   -0.0458
    0.0204   -0.0458    0.4698
```

The matrix Σ_η from the one-step Kalman filter approach:

$$\Sigma_\eta^{1-\text{step}} = \begin{bmatrix} 0.0665 & -0.0633 & 0.0009 \\ -0.0633 & 0.1022 & -0.0295 \\ 0.0009 & -0.0295 & 0.5028 \end{bmatrix}$$

For comparison, I have displayed the variance-covariance matrix of the innovations from VAR(1) with those from the one-step Kalman filter side-by-side:

$$\Sigma_\eta^{2-\text{step}} = \begin{bmatrix} 0.0598 & -0.0604 & 0.0199 \\ -0.0604 & 0.1027 & -0.0495 \\ 0.0199 & -0.0495 & 0.4678 \end{bmatrix}, \quad \Sigma_\eta^{1-\text{step}} = \begin{bmatrix} 0.0665 & -0.0633 & 0.0009 \\ -0.0633 & 0.1022 & -0.0295 \\ 0.0009 & -0.0295 & 0.5028 \end{bmatrix}$$

The lower triangular matrix B is:

$$Z = \begin{bmatrix} 0.2578 & 0 & 0 \\ -0.2455 & 0.2047 & 0 \\ 0.0035 & -0.1399 & 0.6951 \end{bmatrix}$$

Finally, I have estimated means of the level, slope and curvature factors calculated in both approaches. The average values are very close to each other in both the methods, albeit the mean of curvature is slightly smaller in the two-step approach (-2.491) as opposed to that of SSM (-1.872).

```
disp_ssm_fmean = {"SSM Factor Means:";...
    "-----------------";...
    mu};
cellfun(@disp,disp_ssm_fmean)
```

```
SSM Factor Means:
-----------------
    4.3549   -2.3513   -1.8720
```

```
disp_2step_fmean = {"Two-Step Factor Means:";...
    "----------------------";...
    mu0'};
cellfun(@disp,disp_2step_fmean)
```

```
Two-Step Factor Means:
----------------------
    4.7137   -2.2325   -2.4910
```

Contrasting the Derived Factors

The variables primarily of interest in projecting the ex-ante path of the yield curve are the Diebold-Li model's unobserved factors . While maximum likelihood estimates the unknown parameters,filtering, smoothing and forecasting measure the unobserved states. I have analyzed the states inferred from each approach. The coefficients from the OLS regression in the first step are the latent states (factors). Kalman smoother extracts optimal values of the three factors.The the `smooth` function in the SSM model carries out the Kalman smoothing process wherein the smoothed states for all $t = 1, 2, ...T$ are:

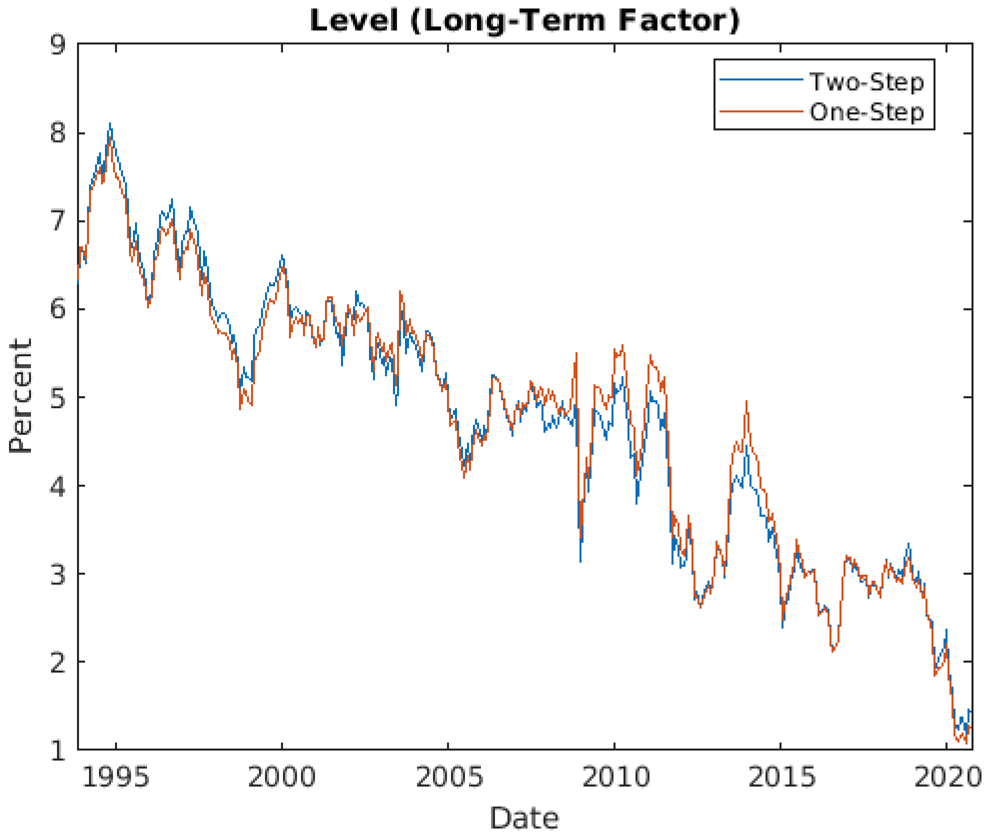
$$E[x_t | y_T, ..., y_1]$$

However, before invoking the `smooth` function, during the estimation, the SSM formulation considers the offset adjustments made to the yields observed. In particular, the parameter mapping function deflates the original yields while estimating. Further, rather than working with the original yields, it works with the offset-adjusted yields : $y'_t = y_t - \Lambda\mu$. The SSM model estimated is oblivious of the adjusted yields as only the mapping function contains the adjustments. Thereby, the other SSM functions such as `smooth`, `filter`, `simulate`, and `forecast` must take into consideration the regression components related to the measurement equation's predictors. Consequently, I have demeaned the original yields before calling the `smooth` function to ascertain the estimated states. To deflate, I subtracted the intercept corresponding to the estimated offset, $C\mu = \Lambda\mu$, as it counteracts the offset adjustment during the estimation process. Yet, the inferred states are associated with the demeaned, although the actual states (the level, slope, and curvature factors) are variables of interest, rather than their corresponding offset-adjusted values. Therefore, I have readjusted the demeaned states after smoothing by adding the estimated mean, μ, to the factors.

```
intercept = SSMmod.C * mu';

% demeaned the yields  as the SSM functions such as filter,  smooth, forecast,
% and simulate, assume that observations are already manually demeaned.
% offset: Truncate vectors by removing or keeping beginning or ending values

% bsxfun: apply element-wise operation to two arrays with implicit expansion enabled
% subtract the intercept from the yields to get the demeaned yields
def_yields = bsxfun(@minus,yields,intercept');

% returns smoothed states "def_states" by performing backward recursion on
% the fully specified state-space model "SSMmod"
% "smooth" applies the standard Kalman filter using "SSMmod"
% and the responses observed: "def_yields".
def_states = smooth(SSMmod,def_yields);


% add the estimated means "mu" to the demeaned states (factors)
est_states = bsxfun(@plus,def_states,mu);
```

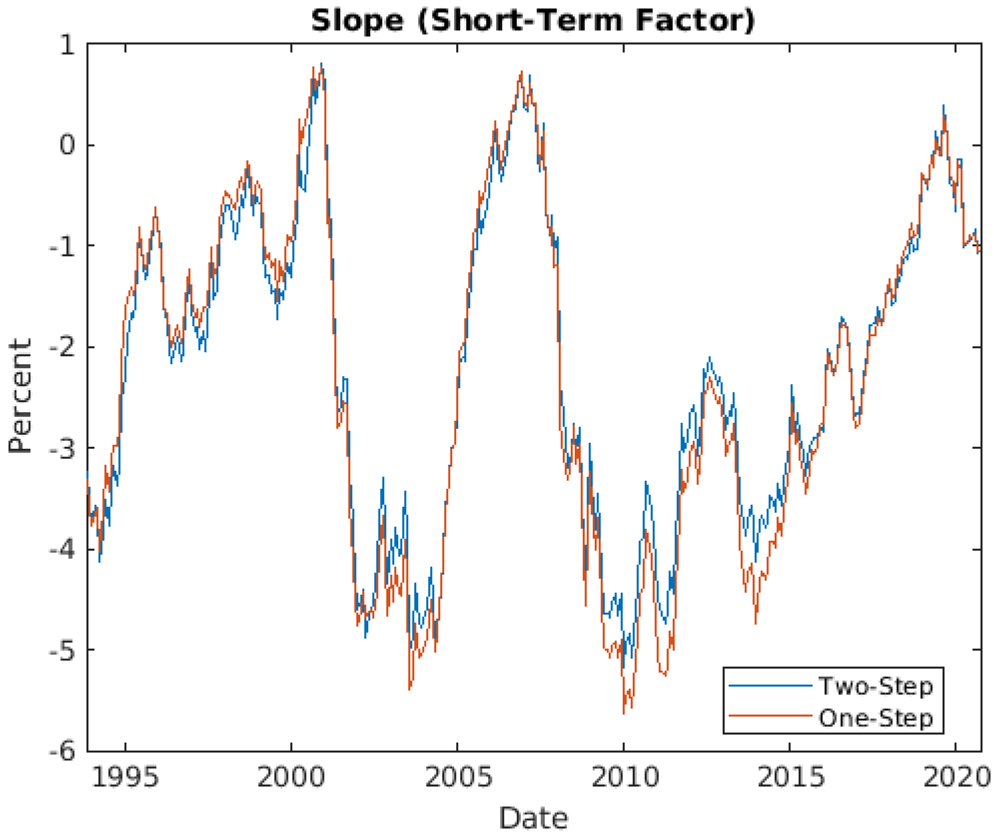
After inferring the states, I have separately plotted the latent factors : level, slope and curvature latent derived from the two-step approach and the state-space model. The long-term factor or level is:

```
figure
plot(dates, [beta(:,1) est_states(:,1)])
title('Level (Long-Term Factor)')
ylabel('Percent')
xlabel('Date')
legend({'Two-Step','One-Step'},'location','best')
```



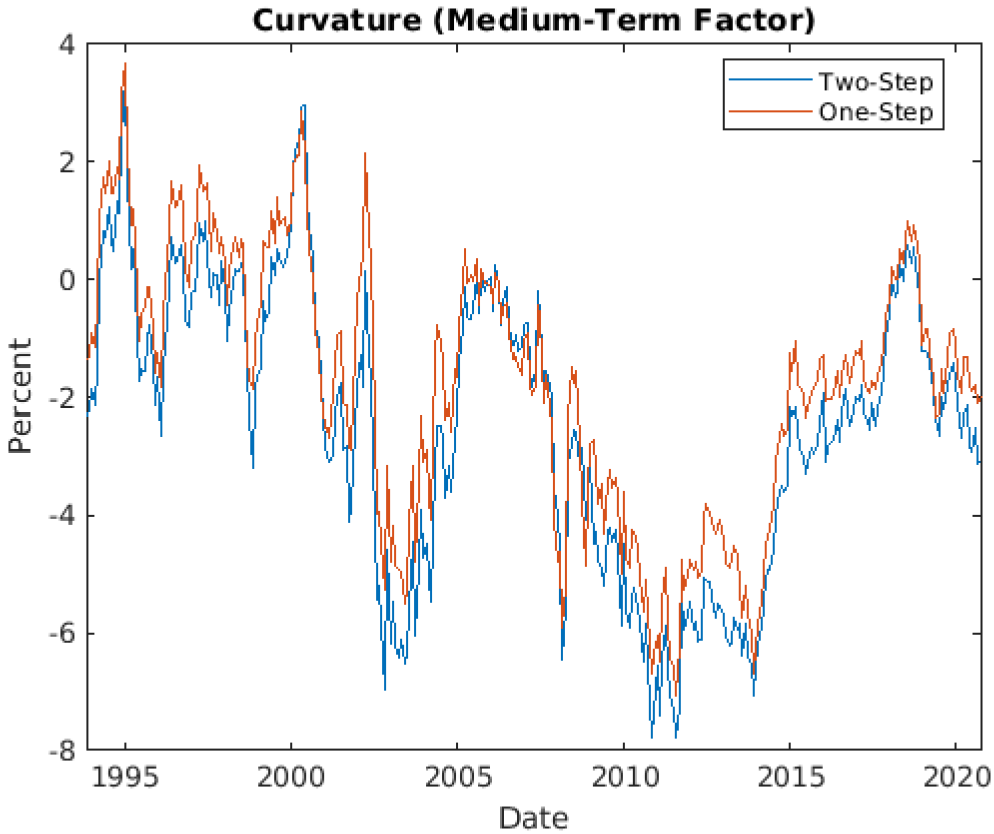
The short-term factor or slope is:

```
figure
plot(dates, [beta(:,2) est_states(:,2)])
title('Slope (Short-Term Factor)')
ylabel('Percent')
xlabel('Date')
legend({'Two-Step','One-Step '},'location','best')
```



The medium-term factor or curvature is:

```
figure
plot(dates, [beta(:,3) est_states(:,3)])
title('Curvature (Medium-Term Factor)')
ylabel('Percent')
xlabel('Date')
legend({'Two-Step','One-Step '},'location','best')
```



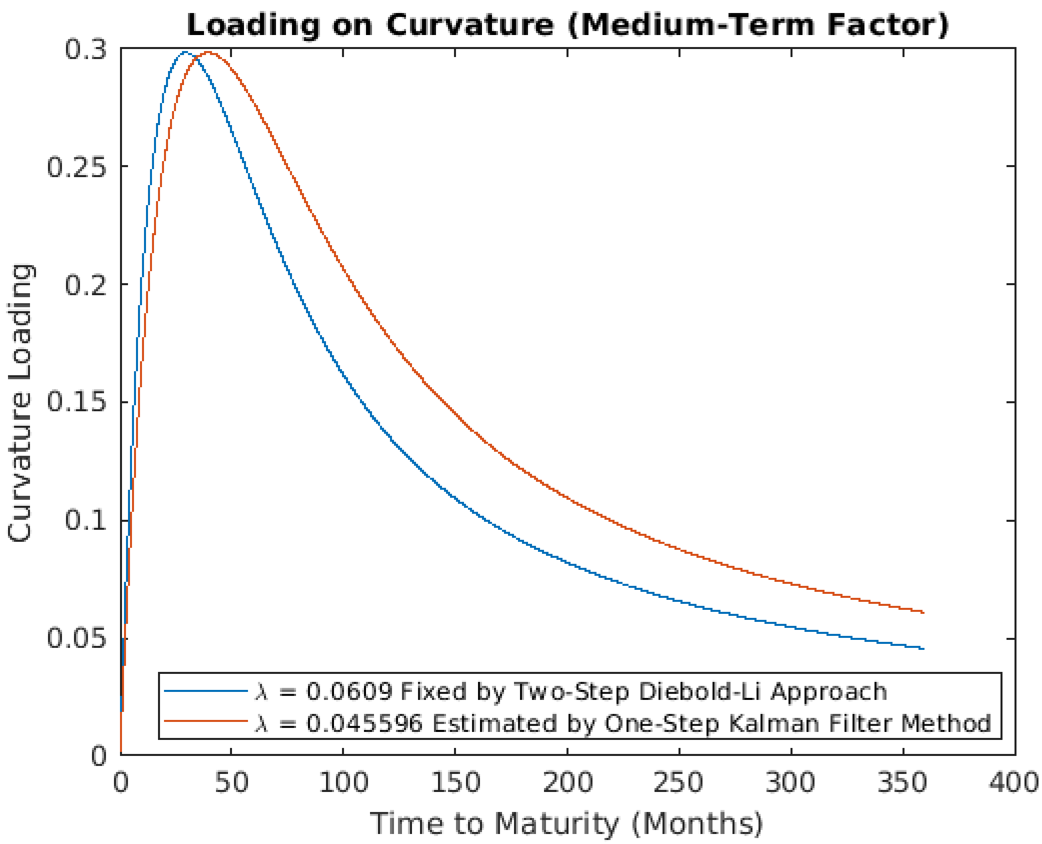
From the separate graphs of all the three factors, the level factor shows considerable persistance as the trajectory indicates a long-term downward trend. Alternatively, the slope and curvature factors are more volatile as they fluctuate from trough to crests, taking in both negative and positive values. Linked with the curvature factor, the estimated decay rate parameter λ is 0.0456. Unlike the constant decay rate applied in the two-step approach where λ is 0.0609, the decay rate parameter estimated from the one-step method is lower.

```
display_vars = {"SSM Rate of Decay (lambda):";...
    "------------------------";...
    lambda};

% apply the function "disp" to each cell of cell array "display_var"
% "disp": displays the value of lamda
cellfun(@disp,display_vars)
```

```
SSM Rate of Decay (lambda):
------------------------
    0.0456
```

λ determines the time to maturity at which the loading on the curvature is maximized. I have plotted the loading on curvature linked to each value of λ. Setting λ arbitrarity to 0.0609 maximizes the loading on the curvature at precisely 28.58 months. As opposed to the two-step approach, when λ is estimated to be 0.0456 in the one-step approach, the curvature on loading is maximized a decade later in exactly 39.5 months. We can attribute the interpretation of curvature as a medium-term factor to the hump-shape (concave looking) curves as a function of the time to maturity for both values of λ.

```
% max(ttm) = 360 and 360*12 = 4320
tau = 0:(1/12):max(ttm);    % time to maturity (months)

% store 2 lambda values from the two-step and SSM approaches
decay = [lambda0 lambda];

% numel(tau): number of elements in "tau" = 4321
% zeros(4321, 2) = 4321-by-2 matrix of 0s
loading = zeros(numel(tau), 2);

% loadings on the medium-term factor: curvature
for i = 1:numel(tau)
    loading(i,:) = ((1-exp(-decay*tau(i)))./(decay*tau(i))-exp(-decay*tau(i)));
end

figure
plot(tau,loading)
title('Loading on Curvature (Medium-Term Factor)')
xlabel('Time to Maturity (Months)')
ylabel('Curvature Loading')
legend({'\lambda = 0.0609 Fixed by Two-Step Diebold-Li Approach ', ...
    ['\lambda = ' num2str(lambda) ' Estimated by One-Step Kalman Filter Method'],}, ...
    'location','best')
```



Albeit differences in the path of the curvature's loading from both the methods, which are promiment after the loadings reach the apex, the factors derived from both methods are reasonably similar. I think I would prefer to use the one-step SSM wherein I simultaneously estimated all the parameters. A caveat of the two-step approach is that the second step does not take into account the parameters estimated and the uncertainity associated with extracting signals in the first step.

Now, I have contrasted the standard deviations and means of the residuals of the measurement equations from the two methods, and expressed the results in basis points (bps). In the SSM framework, the state measurement sensitivity matrix C is equivalent to the factor loadings matrix Λ.

```
% residuals from the measurement equation
resid_SSM = yields - est_states*SSMmod.C';

% residuals from the two-step approach
resid_2step = yields - beta*X';

% Average and standard deviation of the residuals from the
% One-step Kalman filter approach
resid_avg_ssm = 100*mean(resid_SSM)';
resid_sd_ssm = 100*std(resid_SSM)';
```

```
% Average and standard deviation of the residuals from the
% Two-step Diebold-Li approach
resid_avg_2step = 100*mean(resid_2step)';
resid_sd_2step = 100*std(resid_2step)';

display_vars = {"  -------------------------------------------------";...
    "              One-Step             Two-Step";...
    "          -------------------  ------------------";...
    "                          Standard             Standard";...
    "  Maturity    Mean    Deviation    Mean   Deviation";...
    "  (Months)   (bps)      (bps)     (bps)     (bps)  ";...
    "  --------  --------  ---------  -------  ---------";...
    [ttm resid_avg_ssm resid_sd_ssm resid_avg_2step resid_sd_2step]};

cellfun(@disp,display_vars)
```

```
  -------------------------------------------------
            One-Step             Two-Step
          -------------------  ------------------
                     Standard            Standard
  Maturity    Mean   Deviation    Mean   Deviation
  (Months)   (bps)     (bps)     (bps)     (bps)
  --------  --------  ---------  -------  ---------
    3.0000   -7.9165   11.8059   -8.2772    6.9310
    6.0000    0.0000    0.0000    2.8505    4.3342
   12.0000    1.6919    7.2086    7.3354    8.1979
   24.0000    3.2513    5.1182    7.1949    5.7332
   36.0000    0.0000    0.0000   -0.3532    3.5134
   60.0000   -1.3002    6.1814   -7.6432    6.5867
   84.0000    1.2382    5.8963   -6.6773    7.8875
  120.0000   -2.0432    3.8820   -8.9523    5.5131
  240.0000   12.9364   15.1194   10.1659    8.7315
  360.0000    5.4692   17.2032    4.3565   10.8041
```

```
disp_sumstat_beta = {"Summary Statistics of the Latent Variables from DNS:";...
    "----------------------------------";...
    tab_beta};
cellfun(@disp,disp_sumstat_beta)
```

Summary Statistics of the Latent Variables from DNS:

	Min	Mean	Max	Std_dev
beta 1	1.1602	4.7137	8.1034	1.5517
beta 2	-5.1804	-2.2325	0.81551	1.5339
beta 3	-7.7825	-2.491	3.5225	2.4178

```
% Average value of standard deviation from SSM
avg_sd_ssm = mean(resid_sd_ssm);

% Average value of standard deviation from the two-step approach
avg_sd_2step = mean(resid_sd_2step);

disp_avg_ssm = {"Average value of standard deviation from One-Step:";...
    "----------------------------------";...
    avg_sd_ssm};
```

```
cellfun(@disp,disp_avg_ssm)
```

```
Average value of standard deviation from One-Step:
----------------------------------
    7.2415
```

```
disp_avg_2step = {"Average value of standard deviation from Two-Step:";...
    "----------------------------------";...
    avg_sd_2step};
cellfun(@disp,disp_avg_2step)
```

```
Average value of standard deviation from Two-Step:
----------------------------------
    6.8233
```

Though SSM fits better for maturities ranging from 6 months to 10 years, the deviation is extrmeley high for the shortest (3 month) and long term maturities (20 and 30 years). These extreme values in standard deviation push the average value of the standard deviation of the residuals from SSM larger (7.2415),than that from the two-step approach (6.8233). So we can reckon that the the yield curve fitted with the constant λ from the two-step approach generates more accurate, and hence, is preferable to the SSM.

Monte Carlo Simulations and Forecasts

The SSM function enables us to simulate the path of the yield curve by Mone Carlo method and and construct minimum mean square error (MMSE) forecasts. As the Diebold-Li model is founded upon the non-linear combination of the unobserved estimated factors, forecasting the factors is tantamount to forecasting the yield curve. Yet, because I estimated the one-step method after making the offset adjustment, so the estimates are based on the demeaned yields. Therefore, I have applied the `forecast` function on the demeaned yields to calculate the MMSE foecasts of the demeaned yields 12 months into the future. Then, I added the estimated offset $C\mu$ to the demeaned yields to obtain the ex-ante foreasts of actual yields and stored it in `forecast_yields.` Since the forecast horizon is set to 12, then for each of the 10 different maturities, `forecast_yields` is a (12×10) matrix of forecasts for each time to maturity.

```
horizon = 12; % forecast horizon (months)

% call the forecast function on the demeaned yields to compute the MMSE forecasts
[forecast_def_yields,mse] = forecast(SSMmod,horizon,def_yields);

% add the estimated offset C*mu to the demeaned yields: forecasts of yields
forecast_yields = bsxfun(@plus,forecast_def_yields,intercept');
```

After enumerating the deterministic MMSE forecasts, I have approximated the same results using Mone Carlo simulation. To ensure that the simulation takes into account the latest information, I initialized covariance matrix and the mean vector of the initial states (factors) from the estimated one-step approach. Thereby, I procured the smoothed states via backward recursion.

```
% call the smooth function to acquire the smoothed states
% backward recursion  yields the smoothed states
[~,~,results] = smooth(SSMmod,def_yields);

% initialize the mean vector at the end of the historical dataset
SSMmod.Mean0 = results(end).SmoothedStates;
cov0 = results(end).SmoothedStatesCov;

% initialize the covariance matrix
SSMmod.Cov0 = (cov0 + cov0')/2;
```

Thereafter, I computed the ex-ante yield curve by Monte Carlo simulation $100,000$ times. Each simulation projects the yields of ten different maturities, twelve months into the future, creating a $100,000 \times 12 \times 10$ dimension of `sim_def_yields.` As before, I have added the factor offsets from the demeaned yields to get the unnormalized yields.

```
rng('default') % rng: random number generator

% repeat the simulation 100,000 times
% each path is the ex-ante (simulated) yield curve of the future
nPaths = 100000;
% 100,000 sims of 10 demeaned yields (columns), 12 months in the future (rows)
sim_def_yields = simulate(SSMmod, horizon, nPaths);

% add the estimated offset C*mu to the demeaned yields: forecasts of yields
sim_yields = bsxfun(@plus, sim_def_yields, intercept');
```

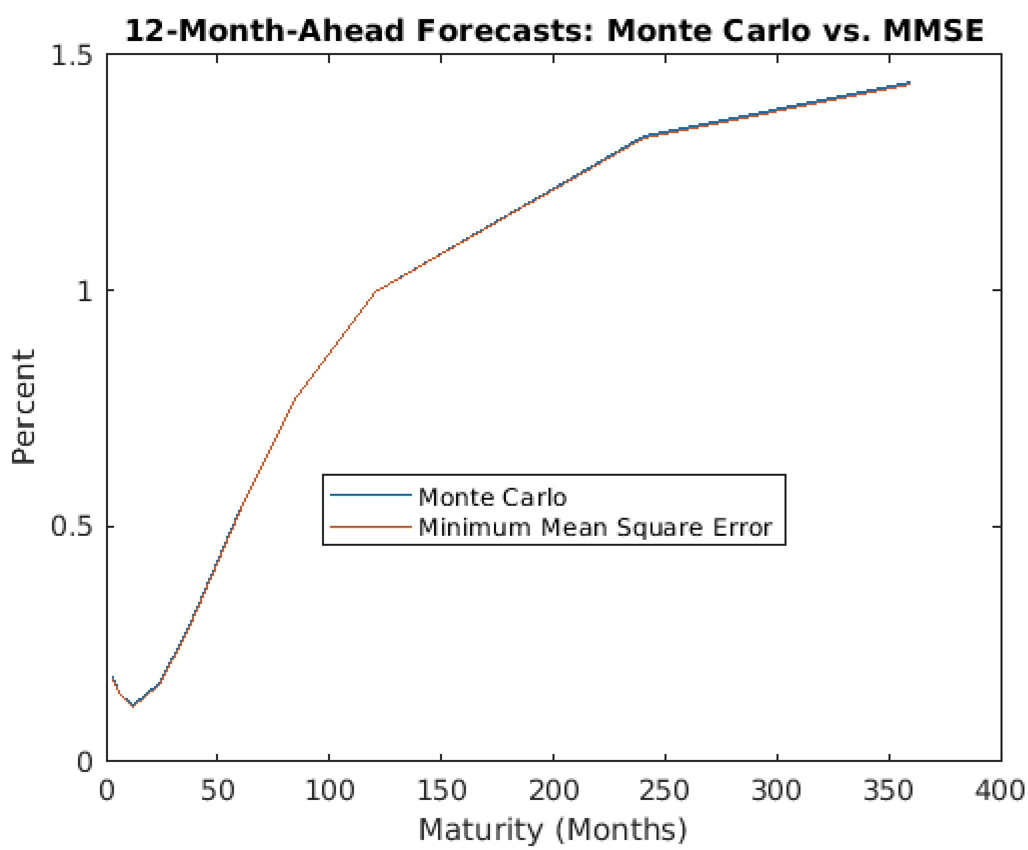
Analogous to the MMSE forecasts and standard errors, I caliberated sample standard deviation and mean of the 100,000 trials. I reordered the matrix of simulated yields to calculate the sample means and standard deviations. The matrix encompasses 100,000 rows, 12 columns, and 10 pages.

```
% sim_yields: matrix of simulated yields: 100,000 rows, 12 columns and 10 pages
sim_yields = permute(sim_yields,[3 1 2]); % re-order for convenience
```

```
forecasts = zeros(horizon,numel(ttm));
std_errors = zeros(horizon,numel(ttm));

for i = 1:numel(ttm)
    forecasts(:,i) = mean(sim_yields(:,:,i));
    std_errors(:,i) = std(sim_yields(:,:,i));
end
```

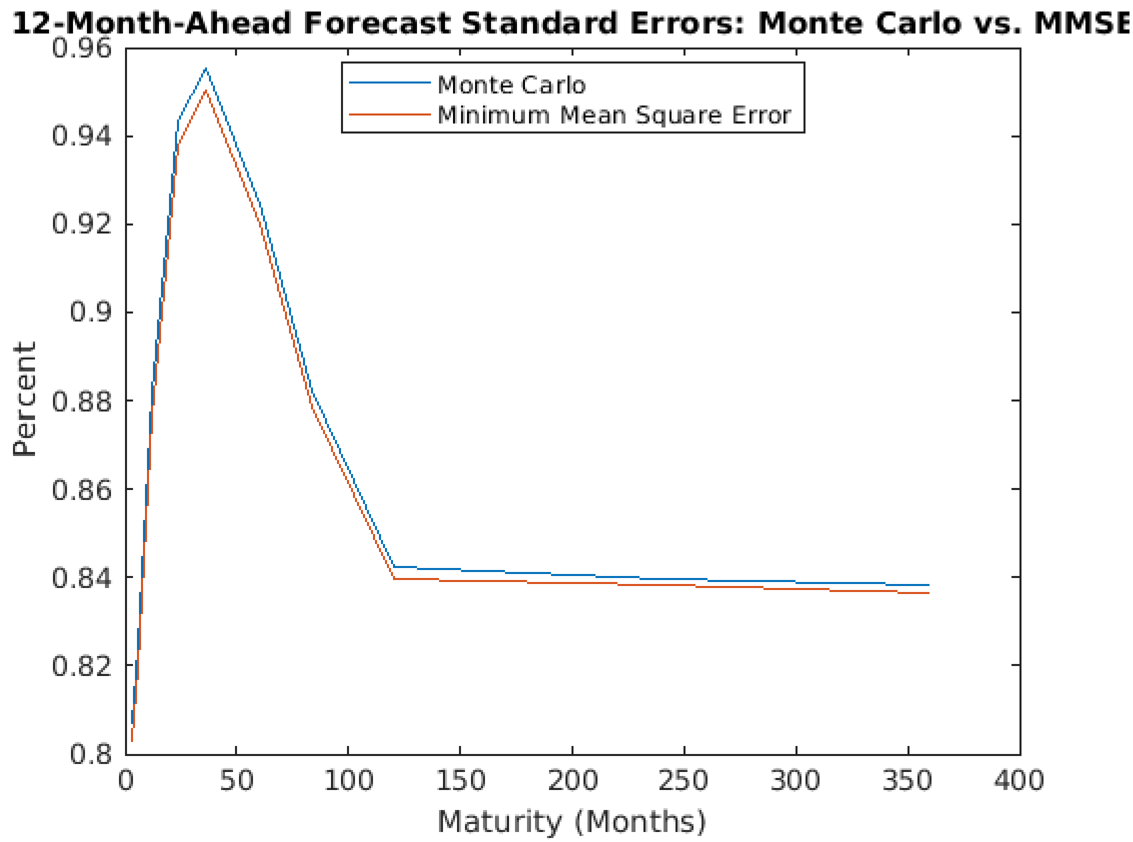
Visually comparing the projections and standard errors from MMSE and Monte Carlo methods convey idential paths of the yield curve.

```
figure
plot(ttm, [forecasts(horizon,:)' forecast_yields(horizon,:)'])
title('12-Month-Ahead Forecasts: Monte Carlo vs. MMSE')
xlabel('Maturity (Months)')
ylabel('Percent')
legend({'Monte Carlo','Minimum Mean Square Error'},'location','best')
```




```
figure
plot(ttm, [std_errors(horizon,:)' sqrt(mse(horizon,:))'])
title('12-Month-Ahead Forecast Standard Errors: Monte Carlo vs. MMSE')
xlabel('Maturity (Months)')
```

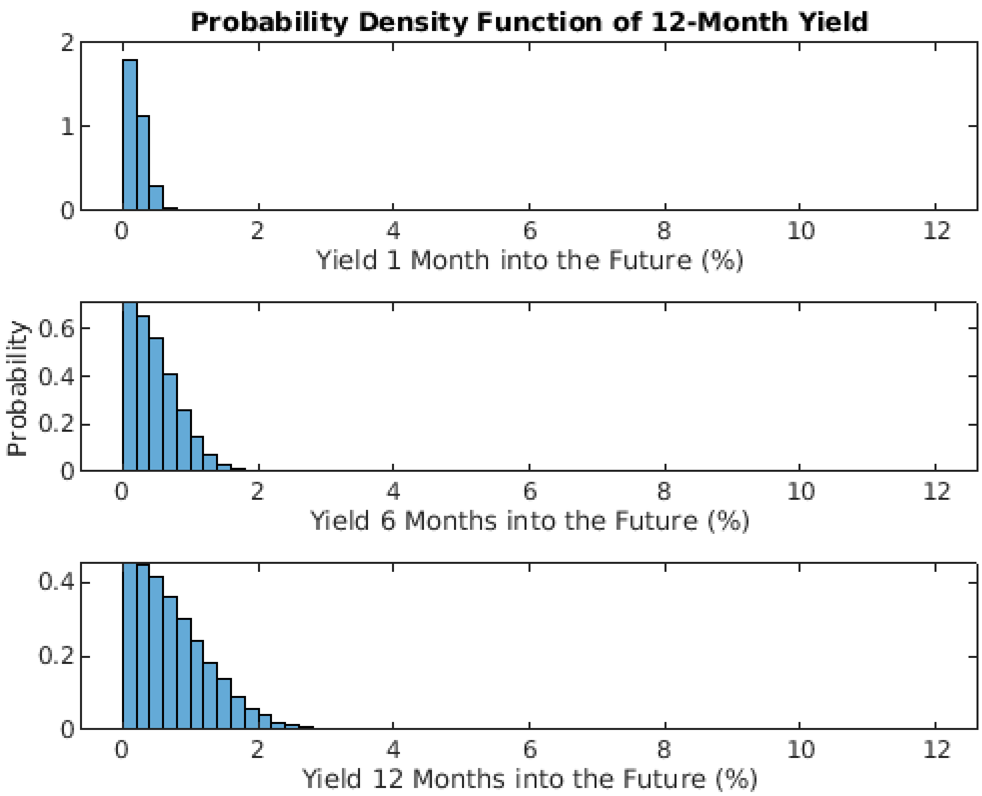
```
ylabel('Percent')
legend({'Monte Carlo','Minimum Mean Square Error'},'location','best')
```



Beyond the mean and standard errors, Monte Carlo simulations provide insight into the distribution of yields. As a corollary, we can gauge the distribution of other macroeconomic variables that depend on the yield curve. Next, the plots represent the distibution of the simulated 1 year yield at one, six, and twelve months into the future.

```
index12 = find(ttm == 12);  % page index of 12-month yield
bins = 0:0.2:12;

% 12-month yield forecasted 1 month into the future
figure
subplot(3,1,1)
histogram(sim_yields(:,1,index12), bins, 'Normalization', 'pdf')
title('Probability Density Function of 12-Month Yield')
xlabel('Yield 1 Month into the Future (%)')

% 12-month yield forecasted 6 months into the future
subplot(3,1,2)
histogram(sim_yields(:,6,index12), bins, 'Normalization', 'pdf')
xlabel('Yield 6 Months into the Future (%)')
```

```
ylabel('Probability')

% 12-month yield forecasted 12 months into the future
subplot(3,1,3)
histogram(sim_yields(:,12,index12), bins, 'Normalization', 'pdf')
xlabel('Yield 12 Months into the Future (%)')
```



Summary

I have formulated the Diebold-Li’s two-step approach into a state-space representation by defining the state and measurement equations, and estimated the parameters via the Kalman filter. The One-Step is a stochastic, discrete-time model with two equations: state and measurement equations. The state equation characterizes the transition of the latent (unobserved) states, and the observation equation connects the states to the observed data on yield curve in this analysis. It expresses how we can indirectly caliberate the latent process at each time period. From the time-series of various yield curves, I inferred the three latent states or factors – level, slope, and curvature, that drive the evolution of the term structure of interest rates. Then, applying the functionalities of the One-Step , I estimated the parameters using the Kalman filter, smoothed, forecasted and constructed Monte Carlo simulations.

References

Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics, 130*(2), 337-364. doi:10.1016/j.jeconom.2005.03.005

Diebold, F. X., Rudebusch, G. D., & Aruoba, S. B. (2006). The macroeconomy and the yield curve: A dynamic latent factor approach. *Journal of Econometrics, 131*(1-2), 309-338. doi:10.1016/j.jeconom.2005.01.011

Diebold, Francis X., and Glenn D. Rudebusch. Yield Curve Modeling and Forecasting: The Dynamic Nelson-Siegel Approach. *Princeton University Press,* 2013.

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https://www.mathworks.com/help/econ/using-the-kalman-filter-to-estimate-and-forecast-the-diebold-li-model.html**

*In particular, I acknowledge the signficant help taken from the last website, but I have made several additions and modifications. The website served as a useful guide in estimating the state space models for the first time.