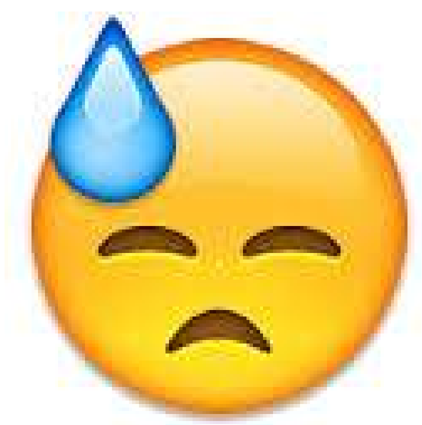
CREDIT TO ASHISH JAIN

APPLIED MATHEMATICS II



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Unit - I

MATRICES

Review of Matrices :-

Matrix - Set of simultaneous linear equation arranged in rows and columns in rectangular brackets.

$x + 2y + 3z + 4t = 0$
$3x + 4y + 6z + 7t = 0$
$4x + 5y + 7z + 8t = 0$

$$\underset{\substack{\swarrow \quad \downarrow \\ \text{Row} \quad \text{Column}}}{A_{3\times 4}} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 6 & 7 \\ 4 & 5 & 7 & 8 \end{bmatrix} \end{array} \begin{array}{l} \to a_{13} \\ \to a_{24} \\ \underbrace{3\times 4}_{\text{order}} \end{array}$$

a_{33}

$$= [a_{ij}] \quad \begin{array}{l} i = 1 \text{ to } 3 \\ j = 1 \text{ to } 4 \end{array}$$

Rank of Matrix :-

(I) operation on $R_1 \longrightarrow R_1 \longleftrightarrow R_2$

R_{12}

(operation on i^{th} Row) $R_i th \longrightarrow R_i \longleftrightarrow R_j$

$R_i \longrightarrow R_{ij}$

operation on $C_1 \longrightarrow C_1 \longleftrightarrow C_2$

$C_1 \longrightarrow C_{12}$

$C_i th \longrightarrow C_i \longleftrightarrow C_j$

$C_i \longrightarrow C_{ij}$

(II) $R_i \longrightarrow \frac{1}{K} R_i \quad (K \neq 0)$

$C_i \longrightarrow \frac{1}{K} C_i \quad (K \neq 0)$

(III) $R_i \pm kR_j \quad (k \neq 0)$
$C_i \pm kC_j \quad (k \neq 0)$

$A = B$ (Equal Matrix)-
i) Order same.
ii) Corresponding elements are same

Equivalent Matrix $(A \sim B)$
A matrix which is obtained with the help of elementary transformation is called Equivalent Matrix.

Rank :-
A matrix is said to be of rank 'r' when
i) It has atleast one non-zero minor of order 'r'.
ii) Every minor of order higher than 'r' are zero.
i.e. The rank of a matrix is the largest order of any non-zero minor of the matrix.

NOTE:- 1) If a matrix has non-zero minor of order 'r' - its rank is $\geq r$.

2) If all minors of a matrix of order '(r+1)' are zero then its rank $\leq r$.

Rank of a matrix is denoted by
$$\rho(A) = r$$

→ Rank of a matrix ~~is equal to~~
= Total no. of non-zero rows in upper-triangular matrix

NOTE: In this method, we apply only elementary row transformations. So, this method is also called as row-rank method.

Q.-1) Find the rank of the given matrix:-

$$A = \begin{bmatrix} 2 & -5 & 1 & 2 \\ -1 & 2 & 3 & -2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

Sol^n:-

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$ Applying at $R_2 \to R_1 \leftrightarrow R_2$
$R_1 \to R_{12}$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

Since, all minors greater than order 2 are zero.

$\therefore$ Rank of matrix $= 2$

(Alt: By upper Δ matrix)

$$\rho(A) = 2$$

Q-2) Find rank of matrix by converting it into upper triangular matrix or by row-rank method:-

$$A = \begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$$

Solⁿ:-

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 3 & -4 & -1 & 2 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix} \quad R_1 \to R_{12}$$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & -37 & -10 & -1 \\ 0 & -66 & -20 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 - 9R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & 24/5 & 12/25 \\ 0 & 0 & 32/5 & 16/25 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - \frac{37}{25}R_2 \\ R_4 \to R_4 - \frac{66}{25}R_2 \end{matrix}$$

$\Rightarrow R_4 \to R_4 - \frac{4}{3}R_3$

$\because$ 3 Non-zero Rows in upper Δ Matrix

$\Rightarrow \rho(A) = 3$

$$\begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & 24/5 & 12/25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

-2
$5 - 15$
-2
$-4 - 7\times 3$
$-4 - 21$ $\quad 9 - 9$
$-1 - 9$
$2 - 3 \quad 7 - 27$
$-3 - 63 \quad 5 -$
$R_3 \to R_3 - \frac{37}{25}R_2$
$R_4 \to R_4 - \frac{66}{25}R_2$
$-37 + \frac{37}{25}\times 25$
$-10 + \frac{10\times 37}{25}$ $\quad -100+$
$-10 + \frac{-50+74}{5}$
$= \frac{24}{5}$
$-1 + \frac{37}{25}$
$\frac{-25+37}{25}$
$\frac{12}{25}$
$-20 + \frac{-100+74}{5}$
$-\frac{36}{5}$
$-2 + \frac{37}{25}$
$-20 + \frac{10\times 66}{25}$
$-100+$
$\frac{-100+132}{5}$
$\frac{41}{25}$
$-2 + \frac{66}{25}\times 1$
$\frac{-25+66}{25}$
$\frac{32}{5}$

Normal Form of a Matrix -
Every non-zero matrix of order $m \times n$ with rank 'r' can be reduced by a sequence of elementary transformation to any one of the following forms:-

1) $[I_r]$

2) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

3) $\begin{bmatrix} I_r & 0 \end{bmatrix}$

4) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

These are normal forms of for given matrix A & 'r' so obtained is called rank of matrix A.

↳ Equivalence of Matrix :-
Let matrix B of order $m \times n$ is obtained from matrix A, then A is called equivalent to B, i.e. $A \sim B$. Matrices A & B have same rank & can be expressed as $B = PAQ$, where P & Q are Non-singular matrices.
If A is order $m \times n$, then P has order $m \times m$ and Q has order $n \times n$ such that $B = PAQ$

Process of finding matrices P & Q :- (Non-Singular)
Let A be a matrix of order $m \times n$.

(I) $B = A = I_{m \times m} A I_{n \times n}$

(II) Transform matrix A to normal form using elemen transformations.

(III) Elementary row transformation is applied simultane to A & $I_{m \times m}$ i.e. the Pre-factor matrix.

(IV) Elementary column operation applied to A also ap to $I_{n \times n}$ i.e. Post-factor Matrix.

(V) Finally we find $B = PAQ$ where B is normal form of A & P and Q are non-singular matrices.

(Q-1) Reduce matrix A to its normal form where

$$A = \begin{bmatrix} 2 & 4 & 3 & 4 \\ 1 & 2 & -1 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

solⁿ:-

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix} \quad R_1 \to R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \\ C_4 \to C_4 - 4C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -3 & 5 & 0 \end{bmatrix} \quad C_2 \to C_2 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 2 & 5 & 0 \end{bmatrix} C_2 \to C_2 + C_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

Q.2) For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

find non-singular matrices P & Q such that PAQ is in the normal form & if possible, then find its inverse, otherwise give reason.

Sol^n:-

$$A_{3\times3} = I_{3\times3} \, A \, I_{3\times3}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad |P| = 1 \neq 0$$

$$Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad |Q| \neq 0$$

$\cancel{A^{-1} = QP}$, A^{-1} does not excist

$\because |A| = 0$.

Q-3) If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}; also find a non-singular matrices P & Q such (Normal form).

Sol:-

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1+2

$R_2 \to R_2 - 4R_3$

-3+4
-3-(-1)4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4-4

$\cancel{R_3 \to R_3 + R_2}$ $\quad R_3 \to R_3 + R_2$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8 \\ 0 & 0 & -7 \end{bmatrix}} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \cancel{A \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$C_3 \to C_3 + 8C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ 2 & 3 & 20 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3\left(\frac{-1}{7}\right)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8/7 \\ 0 & 0 & -1/7 \end{bmatrix}}_{Q}$$

$A^{-1} = Q.P$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8/7 \\ 0 & 0 & -1/7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{Q}$$

$A^{-1} = Q.P$

$= P$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

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→ **Solution of Systems of Linear Equations by Rank Theory :-**

Let a system of 'm' linear equations in 'n' unknowns can be written as follows -

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m$$

where, $a_{11}, a_{12}, \cdots, a_{mn}$ are coefficients of $x_1, x_2 \cdots$

$x_1, x_2, \cdots, x_n$ are unknowns

& $b_1, b_2, \cdots, b_m$ are constants

This system of linear eq^n in matrix notation is written as:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Since, we want to apply rank theory. So, we construct a new matrix called augmented Matrix with the help of the element of A & B.

$$C = [A, B]$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}, & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n}, & b_2 \\ \vdots & \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}, & b_m \end{bmatrix}$$

$\Rightarrow$ A non-homogeneous system $AX = B$ is said to be consistent if there exist a solution & if there is no sol^n, then system is inconsistent.

1) If $\rho(A) \neq \rho(C)$, system is inconsistent.

2) If $\rho(A) = \rho(C) = n$, then system has unique solution.

3) If $\rho(A) = \rho(C) \leq n$, then system has infinite no. of solutions

Q-1) Examine whether the given system of equations is consistent if so, find the solution.

$$2x + 3y - 2z = 2$$
$$x + y + z = 6$$
$$5x + y + 2z = 13$$

Sol^n:-

This system of eq^n's can be written in matrix form:- $AX = B$

where,

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 1 & 1 \\ 5 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 6 \\ 13 \end{bmatrix} \quad \text{--- } (1)$$

If we want to apply Rank theory, then we have to construct an augmented Matrix $C = [A, B]$

$$C = \begin{bmatrix} 2 & 3 & -2 & , & 2 \\ 1 & 1 & 1 & , & 6 \\ 5 & 1 & 2 & , & 13 \end{bmatrix}$$

$R_1 \rightarrow R_1 \leftrightarrow R_2$

R_{12}

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1, & 6 \\ 2 & 3 & -2, & 2 \\ 5 & 1 & 2, & 13 \end{array}\right]$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1, & 6 \\ 0 & 1 & -4, & -10 \\ 0 & -4 & -3, & -17 \end{array}\right]$$

$R_3 \to R_3 + 4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1, & 6 \\ 0 & 1 & -4, & -10 \\ 0 & 0 & -19, & -57 \end{array}\right]$$

$R_3 \to -\frac{1}{19} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1, & 6 \\ 0 & 1 & -4, & -10 \\ 0 & 0 & 1, & 3 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5, & 16 \\ 0 & 1 & -4, & -10 \\ 0 & 0 & 1, & 3 \end{array}\right]$$

$R_1 \to R_1 - 5R_3, \quad R_2 \to R_2 + 4R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0, & 1 \\ 0 & 1 & 0, & 2 \\ 0 & 0 & 1, & 3 \end{array}\right]$$

$\rho(C) = 3, \quad \rho(A) = 3$

$\rho(C) = \rho(A) = 3$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x = 1, \; y = 2, \; z = 3$$

Q-2) Find for what values of λ & μ, the system of linear equations:

$$x + y + z = 6$$
$$x + 2y + 5z = 10$$
$$2x + 3y + \lambda z = \mu$$

i) a unique solution

ii) No solution

iii) Infinite solutions

Also find the solutions for $\lambda = 2$ & $\mu = 8$.

Sol:-

$$AX = B$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & \lambda \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & , 6 \\ 1 & 2 & 5 & , 10 \\ 2 & 3 & \lambda & , \mu \end{bmatrix}$$

$$R_2 \to R_2 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & , 6 \\ 0 & 1 & 4 & , 4 \\ 2 & 3 & \lambda & , \mu \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & , 6 \\ 0 & 1 & 4 & , 4 \\ 0 & 1 & \lambda-2 & , \mu-12 \end{bmatrix}$$

$|x = 8, y =$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & , 6 \\ 0 & 1 & 4 & , 4 \\ 0 & 0 & \lambda-6 & , \mu-16 \end{bmatrix}$$

i) Unique

$$\rho(A) = \rho(C)$$

$\lambda \neq 6$, $\cancel{\mu \neq 16}$ $\to$ unique

ii) No $\lambda = 6, \mu \neq 16$

iii) Infinite $\lambda = 6, \mu = 16$

$\lambda = 2, \mu = 8$

$$\begin{bmatrix} 1 & 1 & 1 & , 6 \\ 0 & 1 & 4 & , 4 \\ 0 & 0 & -4 & , -8 \end{bmatrix}$$

Eigen Values or characteristic Roots :-

1) Let A be any square matrix then matrix $A-\lambda I$ is known as characteristic matrix, where I = Identity Matrix and λ = Scalar quantity.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

2) Characteristic Polynomial :-
The determinant $|A-\lambda I|$ when expanded with gives us characteristic polynomial.

$$\begin{aligned}
|A-\lambda I| &= (2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1[2-3+\lambda] \\
&= (2-\lambda)[6+\lambda^2-5\lambda-2] - 2[1-\lambda] + \lambda - 1 \\
&= (2-\lambda)[\lambda^2-5\lambda+4] - 2 + 2\lambda + \lambda - 1 \\
&= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 3 + 3\lambda \\
&= -\lambda^3 + 7\lambda^2 - 11\lambda + 5
\end{aligned}$$

3) $|A-\lambda I| = 0 \Rightarrow$ Characteristic Equation

4) Characteristic Roots or Eigen Values -
The roots of the eqn $|A-\lambda I| = 0$ are called characteristic roots or Eigen values.

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$
$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$
$$\lambda^2(\lambda-1) - 6\lambda(\lambda-1) + 5(\lambda-1) = 0$$
$$(\lambda^2 - 6\lambda + 5)(\lambda-1) = 0$$
$$(\lambda-5)(\lambda-1)(\lambda-1) = 0$$
$$\lambda = 1, 1, 5$$

Properties:-

- Sum of Eigen Values is always equal to Trace of Matrix.
- Product of Eigen Values is always equal to Determinant of Matrix ($|A|$).
- Eigen values of Transpose are equal to given Matrix.
- Eigen values for A^{-1} are reciprocal of Eigen values for A i.e. $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n}$
- $A^m : \lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$
- $kA : k\lambda_1, k\lambda_2, k\lambda_3, \ldots, k\lambda_n$

Cayley - Hamilton Theorem :-

Every square matrix satisfies its own Characteristic eqn.

If $|A - \lambda I| = (-1)^n \{\lambda^n + a_1\lambda^{n-1} + \ldots + a_n\} = 0$

be the characteristic polynomial of $A_{n\times n}$, then the matrix eqn satisfies:

$$\Rightarrow \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$$
$$\Rightarrow A^n + a_1A^{n-1} + \ldots + a_nI = 0$$

Q1) Find char. eqn & characteristic roots of matrix A

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Verify Cayley - Hamilton Theo. & hence P.T.

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Sol^n:-

Let the charc. eq^n for the given matrix is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)\left[(2-\lambda)^2 - 1\right] = 0$$

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4 - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda = 1, 1, 3$$

$$\lambda^3 - 4\lambda^2 + 3\lambda - \lambda^2 + 4\lambda - 3 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \quad \text{--- (1)}$$

Acc. to Cayley - Hamilton theo. this eq^n is satisfied by 'A'.

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (2)}$$

$\cancel{A^8 - 5A^7 + 7I}$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 2+1+1 & 2+2 \\ 0 & 1 & 0 \\ 2+2 & 1+1+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10+4 & 5+4+4 & 5+8 \\ 0 & 1 & 0 \\ 8+5 & 4+4+5 & 4+10 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$(A^8 - 5A^7 + 7A^6 - 3A^5) + A(A^3 - 5A^2 - 7A - 3I)$$
$$+ A^2 + A + I$$

$$= 0 + A.0 + A^2 + A + I$$

$$= \begin{bmatrix} 5+2+1 & 4+1+0 & 4+1+0 \\ 0+0+0 & 1+1+1 & 0+0+0 \\ 4+1+0 & 4+1+0 & 5+2+1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = 0 \times A^{-1}$$

$$A^2 - 5A + 7 - 3A^{-1} = 0$$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7)$$

Complex Matrices -

Conjugate of a complex Matrix:-

Conjugate of a complex matrix is formed by replacing the elements of a matrix by their respective conjugate numbers is called the conjugate of the matrix and it is denoted by $\bar{A}$.

$$A = [a_{ij}], \quad \bar{A} = [\bar{a}_{ij}]$$

→ Transpose of conjugate of a matrix A is denoted by A^{θ} or A^{*}.

→ A square matrix $A = [a_{ij}]$ is said to be ~~Herm~~ Hermitian if (i, j)th element of A is equal to the conjugate complex of (j, i)th element of A.

$$A = (\bar{A})' = A^{\theta}$$

or $A^{\theta} = A$

Hermitian: $A' = \bar{A}$

→ Skew - Hermitian Matrix -

A square matrix A is said to be skew-hermitian matrix if the (i, j)th element of A is equal to the negative of the conjugate complex of the (j, i)th element of A i.e. $a_{ij} = -\bar{a}_{ji}$ | $A' = -\bar{A}$

→ Unitary Matrix:-

A square matrix A is said to be unitary matrix A if

$$AA^{\theta} = A^{\theta}A = I$$

Q-1) Pro

Sol:-

Q-2)

Sol:-

Prepare 5 questions on each

Q-1) Prove that $A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is harmetion matrix.

Sol:-

$$\bar{A} = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$$

$$\bar{A}^{\theta} = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$$

Q-2) Prove that $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -\cancel{4i}-2i \end{bmatrix}$ is skew-harmesion matrix

Sol:-

$$\begin{bmatrix} -i & -3+2i & +2\mp i \\ +3+2i & 0 & -3\mp 4i \\ -2\mp i & +3\mp 4i & -2i \end{bmatrix}$$

$$\begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$$

Unit - II

DIFFERENTIAL EQUATIONS

Differential Equation:-
An equation which involves differential coefficients is known as differential equation.

- Partial DE → More than one independent variable
- Ordinary DE → One dependent & one independent

$$\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = nu$$

Order and Degree of Differential Equation:-

Prepare Ques. on formation of Ordinary Differential eq^n.

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D.E. of first order and first degree:-

Exact Differential Equation:-

An eq^n of the form $Mdx + Ndy = 0$ is said to be Exact DE if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\int M dx$ + $\int$ (those terms of N which do not contains x) dy = C
y as const.

Q-1) Solve $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

Sol:-

Let the given differential eq^n is

$$(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0 \quad \text{--- (1)}$$

Now, comparing eq^n (1) with $Mdx + Ndy = 0$

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$

$$N = 2x^3y - 3x^2y^2 - 5y^4$$

Now differentiate M partially wrt y

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

Differentiate N partially wrt x

$$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ given eq^n is exact DE. and its sol^n is given be

$\int M dx$ + $\int$ (those terms of N which donot contai
$y \rightarrow$ as const. = C

$$\int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int (-5y^4)\,dy = C$$

y as const.

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = C$$

Q.3) Solve $(1+\log xy)\,dx + \left[1+\frac{x}{y}\right]dy = 0$

Sol:- $M = 1+\log xy$, $N = 1+\frac{x}{y}$

$\frac{\partial M}{\partial y} = \frac{1}{xy}\cdot x$ $\quad\quad \frac{\partial N}{\partial x} = \frac{1}{y}$

$= \frac{1}{y}$

$\int (1+\log xy)\,dx + \int 1.\,dy$

$\Rightarrow x + \int \underset{II}{1}.\underset{I}{\log xy}\,dx + y = c$

$\Rightarrow x + \log xy \cdot x - \int \frac{1}{xy}\cdot y \cdot x\,dx + y = c$

$\Rightarrow x + x\log xy - \frac{x^2}{2y} + y = c$

Equations Reducible to Exact DE :-

Sometimes a DE which is not exact may become so on multiplication by a suitable function known as the integrating factor.

Rules-

1) If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = F(x)$ is a function of 'x' alone say $F(x)$.

So, Integrating Factor $= e^{\int F(x)\cdot dx}$

3) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ is a function of 'y' alone, say $f(y)$, then Integrating factor

$$= e^{\int f(y)dy}$$

3) If M is of the form, $M = yf_1(xy)$ and N is of the form, $N = xf_2(xy)$, then

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

4) If we have $x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'y\,dx + b'x\,dy) = 0$, in this form then integrating factor is given as-

$$IF = x^h y^k$$

where, h, k is given by

$$\frac{m+h+1}{a} = \frac{n+k+1}{b}$$

and

$$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

5) If the given eq^n $M\,dx + N\,dy = 0$ is homogeneous eq^n an $Mx + Ny \neq 0$, then

$$IF = \frac{1}{Mx + Ny}$$

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Q-1) $(2x\log x - xy)dy + 2y\,dx = 0$

Sol:-

$$\cancel{M = 2x\log x - xy}, \quad \cancel{N = 2y}$$

$$\cancel{\frac{\partial M}{\partial y} = -x}, \quad \cancel{\frac{\partial N}{\partial x} = 0}$$

$$\cancel{-x \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-x}{2y}}$$

$$M = 2y \quad , \quad N = 2x\log x - xy$$

$$\frac{\partial M}{\partial y} = 2 \quad , \quad \frac{\partial N}{\partial x} = 2\left[\frac{x}{x} + \log x\right] - y$$

$$= 2 + 2\log x - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2 - 2 - 2\log x + y = -(2\log x - y)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\cancel{(2\log x - y)}}{x\cancel{(2\log x - y)}} = \frac{-1}{x}$$

$$IF = e^{\int \frac{1}{x} dx} = \frac{1}{x}$$

$$\frac{2y}{x} dx + \frac{(2x\log x - xy)}{x} dy = 0$$

$$\frac{2y}{x} dx + (2\log x - y) dy = 0$$

$$M_1 = \frac{2y}{x} \quad , \quad N_1 = 2\log x - y$$

$$\frac{\partial M_1}{\partial y} = \frac{2}{x} \quad , \quad \frac{\partial N_1}{\partial x} = \frac{2}{x}$$

$\int_{y = const.} M_1 dx + \int$(those terms of N which do not contain x) $dy = c$

$$\int_{y = const.} \frac{2y}{x} dx + \int -y \, dy = c$$

$$\Rightarrow 2y \ln x - \frac{y^2}{2} = c$$

Q-2) Solve $(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy=0$

Sol:-

$$M = xy^2+2x^2y^3$$

$$\frac{\partial M}{\partial y} = 2xy+6x^2y^2 \quad , \quad N = x^2y - x^3y^2$$

$$, \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

$$M = y\,(xy+2(xy)^2) \quad , \quad N = x\,(xy-(xy)^2)$$

$$IF = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(xy+2(xy)^2) - xy(xy-(xy)^2)}$$

$$= \frac{1}{xy[\cancel{xy}+2(xy)^2-\cancel{xy}+(xy)^2]}$$

$$= \frac{1}{xy\,[3(xy)^2]} = \frac{1}{3x^3y^3}$$

$$\cancel{xy^2}\ \frac{\cancel{xy}\,(y+2xy^2)}{\cancel{3x^3y^3}\ 3x^2y^2}\,dx + \frac{\cancel{xy}\,(x-x^2y)}{\cancel{3x^3y^3}\ 3x^2y^2}\,dy = 0$$

$$\left(\frac{y+2xy^2}{3x^2y^2}\right)dx + \left(\frac{x-x^2y}{3x^2y^2}\right)dy = 0$$

$$M_1 = \frac{y+2xy^2}{3x^2y^2} \quad , \quad N_1 = \frac{x-x^2y}{3x^2y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{3x^2y^2[1+4xy]-[y+2xy^2]6x^2y}{9x^4y^4} = \frac{3x^2y^2+\cancel{12x^3y^3}-6x^2y^2-\cancel{12x^3y^3}}{9x^4y^4}$$

$$= \frac{-3x^2y^2}{9x^4y^4} = \frac{-1}{3x^2y^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{3x^2y^2[1-2xy]-[x-x^2y]6xy^2}{9x^4y^4} = \frac{3x^2y^2-\cancel{6x^3y^3}-6x^2y^2+\cancel{6x^3y^3}}{9x^4y^4}$$

$$\int\left(\frac{y+2xy^2}{3x^2y^2}\right)dx + \int\left(\frac{-1}{3y}\right)dy = c$$

$$\Rightarrow \frac{y}{3y}\int x^{-2}dx + \frac{2}{3x}\int dx - \frac{1}{3}\ln y = c$$

$$\Rightarrow \frac{-1}{3xy} + \frac{2}{3} - \frac{1}{3}\ln y = c$$

Q-3) Solve $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$

Sol:-

$$M = y^3 - 2x^2y \quad , \quad N = 2xy^2 - x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \quad , \quad \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$Mx + Ny$$

$$= xy^3 - 2x^3y + 2xy^3 - x^3y \neq 0$$

Homogeneous.

$$IF = \frac{1}{Mx+Ny}$$

$$= \frac{1}{xy^3 - 2x^3y + 2xy^3 - x^3y}$$

$$(y^3dx + 2xy^2dy) + (-2x^2y\,dx - x^3dy) = 0$$

$$\Rightarrow x^0y^2(1.y\,dx + 2.x\,dy) + x^2y^0(-2.y\,dx - 1.x\,dy) = 0$$

$$x^h y^k$$

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$x^m y^n (ay\,dx + bx\,dy) + x^{m'} y^{n'} (a'y\,dx + b'x\,dy) = 0$

$m = 0,\ n = 2,\ a = 1,\ b = 2$

$m' = 2,\ n' = 0,\ a' = -2,\ b' = -1$

$$\frac{m+h+1}{a} = \frac{n+k+1}{b}$$

$$\frac{0+h+1}{1} = \frac{2+k+1}{2}$$

$$2(h+1) = k+3$$

$$2h+2 = k+3$$

$$k = (2h-1)$$

$$\frac{m'+h+1}{a'} = \frac{n'+k+1}{b'}$$

$$\frac{2+h+1}{-2} = \frac{0+k+1}{-1}$$

$$\frac{h+3}{2} = \frac{k+1}{1}$$

$$\frac{h+3}{2} = k+1$$

$$2h+3 = 2(2h)$$

$$h+3 = 4h$$

$$h = 1$$

$$k = 1$$

$$IF = xy$$

$$(xy^4 - 2x^3y^2)\,dx + (2x^2y^3 - x^4y)\,dy = 0$$

$$M_1 = xy^4 - 2x^3y^2 \quad , \quad N_1 = 2x^2y^3 - x^4y$$

$$\frac{\partial M_1}{\partial y} = 4xy^3 - 4x^3y \quad , \quad \frac{\partial N_1}{\partial x} = 4xy^3 - 4x^3y$$

$\int_{y=\text{const}} M\,dx + \int$(those terms of N, which do not contains x) $dy = c$

$$\Rightarrow \int_{y=\text{const}} (xy^4 - 2x^3y^2)\,dx + \int 0\,dy = c$$

$$\Rightarrow \frac{x^2y^4}{2} - \frac{\cancel{2}x^4y^2}{\cancel{4}2} = c$$

Q-4) Solve: $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

Sol:-

$\cancel{x=}$ $y = tx$

$\frac{dy}{dx} = t + x\frac{dt}{dx}$

$t + x\frac{dt}{dx} = \frac{x^3+t^3x^3}{x.t^2x^2} = \frac{1+t^3}{t^2}$

$x\frac{dt}{dx} = \frac{1+t^3}{t^2} - t = \frac{1+\cancel{t^3}-\cancel{t^3}}{t^2}$

$x\frac{dt}{dx} = \frac{1}{t^2}$

$\int t^2 dt = \int \frac{dx}{x}$

$\frac{t^3}{3} = \ln x + c$

$\frac{y^3}{3x^3} = \ln x + c$

Alternate: $(x^3+y^3)\,dx + (-xy^2)\,dy = 0$

$M = x^3+y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2+y^2}{-xy^2} = \frac{-4}{x}$

$IF = e^{-4\int\frac{1}{x}dx} = e^{-4\ln x} = x^{-4}$

$\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx + \frac{y^2}{x^3}dy = 0$

$(1+y^3)\int\frac{1}{x^5}$

Equations of first order and higher degree:-

The DE which involves $\frac{dy}{dx}$ in higher degree and $\frac{dy}{dx}$ will be denoted by 'p'.

$$\frac{dy}{dx} = p \quad , \quad F(x, y, p) = 0$$

Equations solvable for 'p':-

$$x^2 = 1 + p^2$$
$$p^2 = x^2 - 1$$
$$p = \pm\sqrt{x^2-1}$$
$$\frac{dy}{dx} = \pm\sqrt{x^2-1}$$
$$dy = (\pm\sqrt{x^2-1})\,dx$$
$$y = \pm\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right)\right] + c$$
$$y = \pm\frac{x}{2}\sqrt{x^2-1} \mp \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + c$$

Equations solvable for 'y':-

$$y = (x-a)p - p^2 \quad \text{—— (1)}$$

Differentiate eq^n (1) w.r.t. x

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p - 2p\frac{dp}{dx}$$

$$\cancel{p} = (x-a)\frac{dp}{dx} + \cancel{p} - 2p\frac{dp}{dx}$$

$$\cancel{(x-a)\frac{dp}{dx}} - \quad \frac{dp}{dx}(x-a-2p) = 0$$

$$\frac{dp}{dx} = 0 \quad \Rightarrow P = \text{constant}$$

Now substitute this 'p' in given eq^n:-

$$y = (x-a)c - c^2$$

$$p = \frac{x-a}{2}$$

$$y = (x-a)\left(\frac{x-a}{2}\right) - \frac{(x-a)^2}{4} \quad \times$$

[∵ Integrating const. is not present]

~~Clairauts~~ Clairaut's Equations:-

The equation $y = p(x) + F(p)$ is known as Clairaut's eq^n.

$$y = p(x) + F(p) \quad \text{--- (1)}$$

Differentiate given eq^n w.r.t x.

$$\frac{dy}{dx} = p + \cancel{F}\, x\frac{dp}{dx} + F'(p)\frac{dp}{dx}$$

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + F'(p)\frac{dp}{dx}$$

$$\frac{dp}{dx}\left[x + F'(p)\right] = 0$$

$$\frac{dp}{dx} = 0 \Rightarrow p = a$$

$$\boxed{y = ax + F(a)} \longrightarrow \text{Solution of Clairaut's } Eq^n$$

1) Solve:- $y = px + p^3$

$$\frac{dy}{dx} = \cancel{p} + x\frac{dp}{dx} + 3p^2\frac{dp}{dx}$$

$$\frac{dp}{dx}\left[x + 3p^2\right] = 0$$

$$\frac{dp}{dx} = 0$$

$$p = a$$

$$y = ax + a^3$$

Linear DE of Higher Order:-

If the degree of dependent variable and all derivatives is one such DE is known as linear differential equation while if the degree of dependent variable and all its derivatives are greater than one such DE are known as non-linear DE.

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + ay = x \text{ — Linear}$$

$$\frac{d^3y}{dx^3} + p\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + ay^2 = R(x) \text{ — Non-linear}$$

Linear DE of n^{th} order with constant coefficients:-

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = R(x) \rightarrow$$ std. form of II order Eq^n

$$a_0\frac{d^3y}{dx^3} + a_1\frac{d^2y}{dx^2} + a_2\frac{dy}{dx} + a_3y = R(x)$$

$$\vdots$$

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = \cancel{\phi(x)}\ \phi(x)$$

$$\frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} = D^2y$$

$$\frac{d^3y}{dx^3} = D^3y \quad \cdots \quad \frac{d^ny}{dx^n} = D^ny$$

$$\frac{1}{D} \rightarrow \int dx$$

$$\frac{1}{D^2} \rightarrow \iint dx$$

$$\vdots$$

$$\frac{1}{D^n} \rightarrow \iint \cdots \int dx$$

$$C.S. = C.F. + P.I$$
$$y = C.F. + P.I.$$

$$a_0 D^n y + a_1 D^{n-1} y + \dots + a_n y = \phi(x) \quad \text{—} (1)$$
$$\text{—} (2)$$
$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = 0 \quad \text{—} (3)$$

<u>Method</u> <u>for</u> <u>finding</u> <u>Complementary</u> <u>func^n</u>:-

1) To find CF replace ~~the~~ RHS of eq^n (1) or (2) by zero.

2) For finding auxiliary eq^n replace D by 'm' and 'y' by '1' in eq^n (3)

$$D \to m, \; y \to 1$$
$$a_0 m^n + a_1 m^{n-1} + \dots + a_n = 0 \quad \text{—} (4)$$

3) If roots of auxiliary eq^n are real & distinct

$$m = m_1, m_2, \dots, m_n$$
$$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$
If $m = m_1 = m_2 = \dots = m_n$
$$CF = (c_1 + x c_2 + x^2 c_3 + x^3 c_4 + \dots + c_n x^{n-1}) e^{mx}$$

$$1, 1, 1, 4$$
$$CF = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = (c_1 + x c_2 + x^2 c_3) e^x + c_4 e^{4x}$$

$$\alpha \pm i\beta$$

$$CF = e^{\alpha x} [c_1 \cos\beta x + c_2 \sin\beta x]$$

$$F(D).y = \phi(x)$$

$$P.I. = \frac{1}{F(D)}.\phi(x)$$

(I) $\phi(x) = e^{ax}$

$$P.I. = \frac{1}{F(D)}.e^{ax}$$

$$= \frac{1}{F(a)}.e^{ax} \quad [F(a) \neq 0]$$

If $F(a)=0$, then rule fails

$$P.I. = x.\frac{1}{F'(a)}.e^{ax} \quad [F'(a) \neq 0]$$

$$= x^2.\frac{1}{F''(a)}.e^{ax} \quad [F''(a) \neq 0]$$

(II) $\phi(x) = x^n$

$$P.I. = \frac{1}{F(D)}.x^n$$

$$= [\underbrace{F(D)}]^{-1} x^n$$

$\rightarrow$ Expand by Binomial Theorem

(III) $\phi(x) = \sin ax / \cos ax$

$$P.I. = \frac{1}{F(D^2)} \sin ax / \cos ax$$

$$= \frac{1}{F(-a^2)}.\sin ax/\cos ax \quad [F(-a^2) \neq 0]$$

(IV) $\phi(x) = e^{ax}\phi_1(x)$

$$P.I. = \frac{1}{F(D)}.e^{ax}.\phi_1(x)$$

$$= e^{ax}.\frac{1}{F(D+a)}.\phi_1(x)$$

(I) $\phi(x) =$ Any function of $x = \phi_1(x)$

$$\frac{1}{(D+a)} \cdot \phi_1(x) = e^{-ax} \int e^{ax} \cdot \phi_1(x) \cdot dx$$

$$\frac{1}{(D-a)} \cdot \phi_1(x) = e^{ax} \int e^{-ax} \phi_1(x)\, dx$$

Q-1) solve

$$(D^4 - 3D^2 - 4) \cdot y = 5\sin 2x - e^{-2x}$$

Sol:-

Let given DE is

$$(D^4 - 3D^2 - 4) \cdot y = 5\sin 2x - e^{-2x} \quad \text{— (1)}$$

For finding CF, we replace RHS by 0 & D by m and y by 1. So, required auxiliary eqn is as below

$$(m^4 - 3m^2 - 4) = 0$$
$$m^4 - 4m^2 + m^2 - 4 = 0$$
$$m^2(m^2 - 4) + 1(m^2 - 4) = 0$$
$$m = 2, -2, i, -i$$

$$C.F. = c_1 e^{2x} + c_2 e^{-2x} + e^{0.x}[c_3 \cos x + c_4 \sin x]$$
$$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x$$

$$P.I. = \frac{1}{(D^4 - 3D^2 - 4)} [5\sin 2x - e^{-2x}]$$
$$= \frac{5\sin 2x}{D^4 - 3D^2 - 4} - \frac{e^{-2x}}{D^4 - 3D^2 - 4}$$

$$PI = PI_1 + PI_2$$

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$$PI_1 = \frac{1}{(D^4-3D^2-4)} \cdot 5\sin 2x$$

$$= 5 \cdot \frac{1}{(-4)(-4)-3(-4)-4} \cdot \sin 2x$$

$$= 5 \cdot \frac{1}{16+12-4} \cdot \sin 2x$$

$$= \frac{5}{24} \sin 2x$$

$D^4 = D^2 \cdot D^2$
$D^3 = D^2 \cdot D$

$\frac{1}{F(D^2)} \sin ax$

$= \frac{1}{F(-a^2)} \sin(ax)$

$$PI_2 = \frac{1}{(D^4-3D^2-4)} \cdot e^{-2x}$$

$$= \frac{1}{(-2)^4 - 3(-2)^2 - 4} \cdot e^{-2x}$$

$$= \frac{1}{0} \cdot e^{-2x} \text{ [Rule failed]}$$

$$= x \cdot \frac{1}{(4D^3-6D)} \cdot e^{-2x}$$

$$= x \cdot \frac{1}{4\times -8 + 12} \cdot e^{-2x}$$

$$= \frac{-x}{20} \cdot e^{-2x}$$

$$PI = \frac{5}{24} \sin 2x + \frac{x}{20} e^{-2x}$$

$$C.S. = C.F. + P.I.$$
$$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos x + c_4 \sin x + P.I$$

Q.7 Solve $y'' - 2y' + 2y = x + e^x \cos x$

Sol.-

$D^2y - 2Dy + 2y = x + e^x \cos x$

$m^2y - 2my + 2y = 0$

$y[m^2 - 2m + 2] = 0$

$m = \frac{2 \pm \sqrt{4-8}}{2}$

$= 1 \pm i$

$C.F. = e^x [c_1 \cos x + c_2 \sin x]$

$P.I. = \frac{1}{(D^2 - D + 2)} [x + e^x \cos x]$

$= \frac{1}{(D^2-2D+2)} x + \frac{1}{(D^2-2D+2)} \cdot e^x \cos x$

$P.I._1 = \frac{1}{(D^2-2D+2)} \cdot x$

$Dx = 1$

$\frac{1}{x^n} D^n = \dots$

$f(D) = 0$

$[f(m)]^{-1} x^n$

$D^2(x) = D(1) = 0$

$= \frac{1}{(2-2D+D^2)} \cdot x$

$= \frac{1}{2} \cdot \frac{1}{\left(1 - D + \frac{D^2}{2}\right)} \cdot x$

$= \frac{1}{2} \cdot \left[\left(1 - \left(D - \frac{D^2}{2}\right)\right)\right]^{-1} x$

$= \frac{1}{2} \left[1 + \left(D - \frac{D^2}{2}\right) + \left(D - \frac{D^2}{2}\right)^2 + \dots \right] x$

$= \frac{1}{2} \left[1 + D - \frac{D^2}{2} - D^2 + D^3 - \frac{D^4}{4} + \dots \right] x$

$= \frac{1}{2} \left[x + Dx - \frac{3D^2}{2} x + D^3 x + \dots \right]$

$= \frac{1}{2} (x + 1)$

$$P.I._2 = \frac{1}{(D^2-2D+2)} \cdot e^x \cos x$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} \cdot \cos x$$

$$= e^x \cdot \frac{1}{D^2+2D+1-2D-2+2} \cdot \cos x$$

$$= e^x \cdot \frac{1}{(D^2+1)} \cdot \cos x$$

$$= e^x \frac{1}{-1^2+1} \cos x \quad \text{(Rule failed)}$$

$$= x e^x \frac{1}{2D} \cos x$$

$$= \frac{1}{2} x e^x \sin x$$

$$PI = \frac{1}{2}\left[x+1+xe^x \cancel{\cos x}\ \sin x\right]$$

Q-3) Solve $\dfrac{d^3y}{dx^3} + y = \cos^2\left(\dfrac{x}{2}\right) + e^{-x}$

Sol:-

$$D^3y + y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$$

$$m^3+1=0$$

$$m^3 = -1$$

$$m = \cancel{-1},\ -1, -\omega, -\omega^2$$

$$\omega = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$CF = m^3+1 = (m+1)(m^2-m+1) = 0$$

$$CF = C_1 e^{-x} + e^{x/2}\left[C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right]$$

$$PI = \frac{1}{(D^3+1)} \cdot \cos^2\frac{x}{2} + \frac{1}{(D^3+1)} \cdot e^{-x}$$

$$PI = PI_1 + PI_2$$

$$PI_1 = \frac{1}{(D^2-1)} \cos^2 \frac{x}{2}$$

$$= \frac{1}{(D^2-1)} \frac{(1+\cos x)}{2}$$

$$= \frac{1}{2} \left[\frac{1}{(D^2-1)} e^{0x} + \frac{1}{(D^2-1)} \cos x \right]$$

$$= \frac{1}{2} \left[\frac{1}{1} + \frac{1}{-D+1} \cos x \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{(1-D)} \times \frac{(1+D)}{(1+D)} \cos x \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{2} (\cos x + D \cos x) \right]$$

$$= \frac{1}{2} \left[-1 - \frac{1}{2} (\cos x - \sin x) \right] = -\frac{1}{2} - \frac{1}{4} (\cos x - \sin x)$$

<u>Method</u> of <u>variation</u> of <u>Parameters</u>:-

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x)$$

$$(a_0 D^2 + a_1 D + a_2) y = \phi(x)$$

$$CF = c_1 y_1(x) + c_2 y_2(x)$$

$$PI = u y_1(x) + v y_2(x)$$

$$u = \int \frac{-y_2(x) \cdot \phi(x)}{y_1(x) \cdot y_2'(x) - y_1'(x) \cdot y_2(x)} dx$$

$$v = \int \frac{y_1(x) \phi(x)}{y_1(x) y_2'(x) - y_2(x) \cdot y_1'(x)} dx$$

Q.1) Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ —— (1)

Sol:-

$D^2y +$ $D^2y + y = 0$

$m^2 + 1 = 0$

$m = i, -i$

$CF = c_1 \cos x + c_2 \sin x$ —— (2)

$y_1(x) = \cos x \quad , y_2(x) = \sin x$

$y_1'(x) = -\sin x, \ y_2'(x) = \cos x$

$PI = u y_1(x) + v y_2(x)$ —— (3)

$$u = \int \frac{-\sin x \cdot \operatorname{cosec} x}{\cos x \cdot \cos x + \sin x \cdot \sin x} dx = -x$$

$$v = \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx = \int \frac{\cos x}{\sin x} dx = \int \cot x \, dx$$
$$= \log \sin x$$

$P.I. = -x \cos x + \log \sin x \cdot \sin x$

Alt.

$$P.I. = \frac{1}{(D^2+1)} \cdot \operatorname{cosec} x$$

$$= \frac{1}{(D+i)(D-i)} \operatorname{cosec} x$$

$$= \frac{1}{-2i}\left[\frac{1}{(D+i)} - \frac{1}{(D-i)}\right] \operatorname{cosec} x \longrightarrow$$

$$\frac{1}{(D+i)} \operatorname{cosec} x = e^{-ix} \int e^{ix} \operatorname{cosec} x \cdot dx$$

Unit 3

PARTIAL DIFFERENTIAL EQUATION

The differential equation involved one or more partial derivatives are called partial differential equation.

$$z = z(x,y) \quad ; x, y \text{ - Independent variable}$$

$$\frac{\partial z}{\partial x} = p \quad , \frac{\partial z}{\partial y} = q$$

First order differential eq^n involves p, q

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y \partial x} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

Second order differential eq^n involves r, s, t.

Formation of partial differential equation:-

1) By elimination of arbitrary constants:

$$\phi(x,y,z,a,b) = 0 \quad \text{—— (1)}$$

If No. of constants = No. of no. of independent variables First order eq^n is formed

$$\phi(x,y,z,p,q) = 0 \quad \text{—— (2)}$$

NOTE -

If the no. of arbitrary constant is greater than the no. of independent variable then the order ~~number~~ of partial differential equation will be more than one.

2) By elimination of arbitrary function:

Let u, v be two given function of x, y, z

$$u = u(x,y,z)$$

$$v = v(x,y,z)$$

Connected by relation $\quad F(u,v) = 0$ —— (1)

where, F is an arbitrary function then linear parti

eq^n is given as

$$P.p + Q.q = R \quad — (3)$$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix}$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Q.-1) Form the partial differential eq^n by eliminating arbitrary constants:-

$$z = (x^2+a)(y^2+b) \quad — (1)$$

Sol:-

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[(x^2+a)(y^2+b)\right] = 2x(y^2+b) \quad — (2)$$

$$\frac{\partial z}{\partial y} = 2y(x^2+a) \quad — (3)$$

eq^n (2) × (3)

$$p.q = 4xy(x^2+a)(y^2+b)$$

$$p.q = 4xyz$$

(Q.2) Form the partial differential eqn for

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{--- (1)}$$

Sol:- Here, the total no. of constants is more than total no. of independent variable.

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0$$

$$\frac{2y}{b^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial y}=0$$

$$\frac{x^2}{a^2}+\frac{zx}{c^2}\cdot\frac{\partial z}{\partial x}+\frac{y^2}{b^2}+\frac{zy}{c^2}\cdot\frac{\partial z}{\partial y}=0$$

$$1-\frac{z^2}{c^2}+\frac{z}{c^2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=0$$

$$0-\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}+\frac{z}{c^2}\left(\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\partial x}\right)=0$$

$$z\left(\frac{\partial z}{\partial x}+x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\partial x}\right)-2z\frac{\partial z}{\partial x}=0$$

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\partial x}-\frac{\partial z}{\partial x}=0$$

(Q.3) Form the partial differential eqn :-

$$z=f(x^2-y^2)$$

$$P.p+Q.q=R$$

Sol:-

$$p=\frac{\partial z}{\partial x}=f'(x^2-y^2)\cdot(2x) \quad \text{--- (1)}$$

$$q=\frac{\partial z}{\partial y}=f'(x^2-y^2)\cdot(-2y) \quad \text{--- (2)}$$

eqn (1) ÷ (2)

$$\frac{p}{q}=\frac{-x}{y} \Rightarrow py+qx=0$$

The standard form of Lagrange's
$Pp + Qq$

& find a

auxiliary eqn (any $\frac{dx}{P} = \frac{dy}{Q}$) This eqn can be differential eqn in x & y only. This eqn can be easily solved and we get one soln. Similarly, we take other 2 members for second soln.

$$\frac{dx}{P} = \frac{dz}{R} \quad \text{or} \quad \frac{dy}{Q} = \frac{dz}{R}$$

2) Method of Multipliers:-

In this method, we use multipliers l, m, n (not always constant) and find

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

These multipliers can be selected such that

$$lP + mQ + nR = 0$$

then $\quad l\,dx + m\,dy + n\,dz = 0$

After integration we get one soln.

For II set of soln

$$(l_1, m_1, n_1)$$

$$l_1 dx + m_1 dy + n_1 dz = 0$$

3) Combination of Method I & II

Q.1) Solve $y^2zp + x^2zq = xy^2$ —(1)

Sol:

Compare given eqⁿ with standard form:-

$$Pp + Qq = R$$

$$P = y^2z, \quad Q = x^2z, \quad R = xy^2$$

For finding two solⁿ's we construct the auxiliary eqⁿ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

For I solⁿ, we take first 2 fractions

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} \Rightarrow x^2dx = y^2dy$$

$$\Rightarrow x^3 - y^3 = a = u(x,y,z)$$

For II solⁿ, we take first & third fracⁿ

$$\frac{dx}{y^2z} = \frac{dz}{xy^2}$$

$$xdx = zdz$$

$$x^2 - z^2 = b$$

$$f(x^3 - y^3, x^2 - z^2) = 0$$

Q.2) Solve ~~$xy^2 + z$~~ $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Sol:-

$$Pp + Qq = R$$

$$P = x(y^2+z), \quad Q = -y(x^2+z), \quad R = z(x^2-y^2)$$

$$lP + mQ + nR = 0$$

$$l = x, \quad m = y, \quad n = ~~z~~ -1$$

$$x^2y^2 + \cancel{x^2z} - x^2y^2 - y^2z \mp \cancel{x^2z} + z y^2$$

$$l\,dx + m\,dy + n\,dz = 0$$

$$x\,dx + y\,dy - dz = 0$$

Integrating $\frac{x^2}{2} + \frac{y^2}{2} - z = b \Rightarrow x^2 + y^2 - 2z = b$

Let $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ be II set of multipliers

$$\frac{1}{x} \cdot x(y^2+z) - \frac{y}{y}(x^2+z) + \frac{z}{z}(x^2-y^2) = 0$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\ln x + \ln y + \ln z = \ln a$$

$$xyz = a$$

$$f(xyz, x^2+y^2-2z) = 0$$

Q-3) Solve:-

Sol:- $$y^2p - xyq = x(z-2y)$$

$$y^2p - xyq = x(z-2y)$$

$$Pp + Qq = R$$

$P = y^2$, $Q = -xy$, $R = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{y}\ dy\,(z-2y)$$

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$-x\,dx$$

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$dy\ (z-2y)$

Q) Solve

$$(x^2-yz)p + (y^2-zx)q = z^2-xy$$

Sol:-

$$P = x^2-yz, \quad Q = y^2-zx, \quad R = z^2-xy$$

$\cancel{x^2y - y^2z + y^2z - z^2x + z^2x}$

$d\ (y, z, x)$

l, m, n

$$y\,dx + z\,dy + x\,dz = 0$$

$$yx + zy + xz = a$$

$$\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-zx)} = \frac{dz}{(z^2-xy)}$$

$$\frac{dx-dy}{x^2-y^2-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

$$\frac{dx-dy}{(x-y)} = \frac{dy-dz}{(y-z)} = \frac{dz-dx}{(z-x)}$$

Solution of homogeneous partial DE with constant coefficients:-

$$A_0\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \text{------} +$$

$$A_n D'^n z = \phi(x,y)$$

$$\frac{\partial}{\partial x} = D \quad , D' = \frac{\partial}{\partial y}$$

$$(A_0 D^n + A_1 D^{n-1} D' + \text{----} + D'^n) z = \phi(x,y)$$

$$f(D,D').z = \phi(x,y) \quad \text{—} (1)$$

$$C.S. = C.F. + P.I.$$

$$z = C.F. + P.I.$$

$$a_0\frac{\partial^n y}{\partial x^n} + \text{------} + a_n y = \phi(x)$$

$$\frac{d}{dx} \to D$$

$(a_0 D^n$.

Method for finding Complementary Function:-

The eq^n obtained by replacing D by m and D' by 1 and z by 1 ~~and~~ in $f(D,D').z = 0$ is called the auxiliary eq^n of (1)

Case-(1) Roots of eq^n (2) are real & distinct.

$$A_0 m^n + A_1 m^{n-1} + \text{------} + A_n = 0 \quad \text{—} (2)$$

$$m = m_1, m_2, m_3, \text{------}, m_n$$

$$C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \phi_3(y+m_3x) + \ldots + \phi_n(y+m_nx)$$

Case ② Roots are equal

$$A_0 m^n + A_1 m^{n-1} + \ldots + A_n = 0 \quad \text{— ②}$$

$$m = m_1 = m_2 = m_3 = \ldots = m_n$$

$$C.F. = \phi_1(y+mx) + x\phi_2(y+mx) + x^2\phi_3(y+mx) + \ldots + x^{n-1}\phi_n(y+mx)$$

$$= 0\pm i,\ 0\pm i$$

$$= +i, +i, -i, -i$$

Rules for finding Particular Integral:-

(I) $\phi(x,y) = e^{ax+by}$

$$P.I. = \frac{1}{F(D,D')} \cdot e^{ax+by}$$

$$D \to a,\quad D' \to b$$

$$= \frac{1}{F(a,b)} \cdot e^{ax+by} \qquad [F(a,b) \neq 0]$$

If $F(D,D') = 0$, rule fails

$$P.I. = x \cdot \frac{1}{\underbrace{\frac{\partial F(D,D')}{\partial D}}_{\text{keeping } D' \text{ as constant}}} \cdot e^{ax+by} \qquad \text{(keeping)}$$

(II) $\phi(x,y) = \sin(ax+by) \,/\, \cos(ax+by)$

$$P.I. = \frac{1}{F(D^2, DD', D'^2)} \sin(ax+by) \,/\, \cos(ax+by)$$

$$D^2 \to -a^2,\quad DD' \to -ab,\quad D'^2 \to -b^2$$

$$P.I = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by) / \cos(ax+by)$$

$$[F(-a^2, -ab, -b^2) \neq 0)$$

(III) $\phi(x,y) = x^n y^m$ $[n = m, n < m, n > m]$

$$P.I. = \frac{1}{F(D,D')} x^n . y^m$$

$$= [F(D,D')]^{-1} x^n . y^m$$

Expand $[F(D,D')]^{-1}$ by Binomial Theorem in ascending powers of D or D'.

(IV) $\phi(x,y) = e^{ax+by} . \phi(x,y)$

$$P.I = \frac{1}{F(D,D')} . e^{ax+by} \phi(x,y)$$

$$D \rightarrow D+a \quad , \quad D' \rightarrow D'+b$$

$$= e^{ax+by} . \frac{1}{F(D+a, D'+b)} . \phi(x,y)$$

(V) $\phi(x,y) = \sin ax . \sin by / \cos ax . \cos by$

$$P.I = \frac{1}{F(D^2, D'^2)} \sin ax . \sin by / \cos ax . \cos by$$

$$D^2 \rightarrow -a^2 , \quad D'^2 \rightarrow -b^2$$

$$= \frac{1}{F(-a^2, -b^2)} . \sin ax . \sin by / \cos ax . \cos by$$

$$[F(-a^2, -b^2) \neq 0]$$

General

$$\frac{1}{(D-mD')}$$

After

Q.1) solve

Sol:-

Giv

General formula:-

$$\frac{1}{(D-mD')}\phi(x,y) = \int \phi(x, c-mx)\,dx$$

i.e. put $y = c-mx$

After integration replace $c \to y+mx$

Q.1) Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial y^2} = \cos 4x \cdot \cos 3y$

Sol:- Given eqn can be written as:-

$$(D^2 - 4D'^2)\,z = \cos 4x \cdot \cos 3y$$

$$\cancel{D^2} \quad m^2 - 4 = 0 \quad \Rightarrow m = \pm 2$$

$$CF = \phi_1(y+2x) + \phi_2(y-2x)$$

$$P.I. = \frac{1}{\cancel{(D^2-D'^2)}\,(D^2-4D'^2)} \cos 4x \cdot \cos 3y$$

$$D^2 \to -4^2, \quad D^2 \to -16$$

$$D'^2 \to -3^2, \quad D'^2 \to -9$$

$$= \frac{1}{-16+36} \cdot \cos 4x \cdot \cos 3y$$

$$= \frac{1}{20} \cdot \cos 4x \cdot \cos 3y$$

$$C.S. = C.F. + P.I.$$

$$z = \phi_1(y+2x) + \phi_2(y-2x) + \frac{1}{20}\cos 4x \cdot \cos 3y$$

Q-2) Solve $r - 2s + 2t = \sin(x-y)$

Sol. - Given eqn can be written as-

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = \sin(x-y)$$

$\downarrow$

$$(D^2 - 2DD' + 2D'^2)z = \sin(x-y)$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$CF = \phi_1(y + (1+i)x) + \phi_2(y + (1-i)x)$$

$$P.I. = \frac{1}{(D^2 - 2DD' + 2D'^2)} \sin(x-y)$$

$-2x(-1)(-1)$

$$= \frac{1}{(-1-2-2)} \sin(x-y) = \frac{-1}{5}\sin(x-y)$$

$$C.S. = \phi_1(y+(1+i)x) + \phi_2(y+(1-i)x) - \frac{1}{5}\sin(x-y)$$

Practice Questions on the formulae

Method of Separation of Variables-
In this method, we assume that solⁿ to be the product of two functions each of which involve only one of the variables.

Q-1) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ by the method of separation of variables
where, $u(x,0) = 6e^{-3x}$

Sol:-
Let the given PDE is
$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{— (1)}$$

Let $u(x,t) = X(x).T(t)$
or $u = XT$ — (2)
where, $X = X(x)$ & $T = T(t)$
Now, diff. partially eqⁿ (2) wrt x.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XT)$$
$$= X'T \qquad \left[X' = \frac{dX}{dx}\right] \quad \text{— (3)}$$

$$\frac{dy}{d} \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT)$$
$$= XT' \qquad \left[T' = \frac{dT}{dt}\right] \quad \text{— (4)}$$

Now, substitute $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$ in eqⁿ (1)

$$X'T = 2XT' + XT$$
$$X'T - XT = 2XT'$$
$$(X' - X)T = 2XT'$$
$$\frac{X' - X}{X} = \frac{2T'}{T}$$

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Now, substitute x and T in eq^n (2)

$$u(x,t) = C_1C_2\, e^{(2k+1)x + kt}$$

$$u(x,0) = 6e^{-3x} = C_1C_2\, e^{(2k+1)x}$$

$$C_1C_2 = 6$$

and $2k+1 = -3$

$\Rightarrow k = -2$

$$\Rightarrow \quad u(x,t) = 6\, e^{-3x-2t}$$

Q 2) Solve the eq^n:

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Sol.

Let given PDE is:

$$\frac{\partial^2 u}{\partial x^2} - \frac{2\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \text{--- (1)}$$

Let $u(x,y) = X(x) \cdot Y(y)$

or $u = X \cdot Y$ ——— (2)

where $X = X(x)$

& $Y = Y(y)$

Now,

$$\frac{\partial u}{\partial x} = X'Y \qquad \left[X' = \frac{dX}{dx}\right]$$

$$\& \quad \frac{\partial^2 u}{\partial x^2} = X''Y \qquad \left[X'' = \frac{d^2X}{dx^2}\right]$$

$$\frac{\partial u}{\partial y} = XY' \qquad \left[Y' = \frac{dY}{dy}\right]$$

Substitute all these values in eq^n (1)

$$X''Y - 2X'Y + XY' = 0$$

$$Y(X'' - 2X') + XY' = 0$$

— ③

$$\frac{X''-2X'}{X} = \frac{-Y'}{Y}$$

$$\frac{X''-2X'}{X} = k$$

$$X''-2X' = kX$$

$$\frac{d^2X}{dx^2} - \frac{2dX}{dx} - kX = 0$$

$P.I = 0$ [$\because$ RHS $= 0$]

$\therefore$ Answer will only in form of C.F.

$(D^2 - 2D - k)X = 0$ | $D \to m, X \to 1$

$m^2 - 2m - k = 0$

$$m = \frac{2 \pm \sqrt{4+4k}}{2}$$

$$m = 1 \pm \sqrt{k+1}$$

[we will consider it as a case of Real & distinct roots]

$$C.F. = c_1 e^{(1+\sqrt{k+1})x} + c_2 e^{(1-\sqrt{k+1})x}$$

$\to$ $-Y' = kY$

$$\frac{dY}{dy} = -kY$$

$$\ln Y = -ky + \ln c_3$$

$$Y = c_3 e^{-ky}$$

Unit - 4

STATISTICS & PROBABILITY

→ Conditional Probability] → Prepare yourself
→ Bayes Theorem

Binomial Probability Distribution:-

Let there be 'n' independent trials in an experiment. Let a random variable 'x' denote the no. of successes in these 'n' trials. Let 'p' be the probability of a success and 'q' that of a failure in a single trial such that $p+q=1$. Let the trials be independent & ~~'p' be~~ constant for

$$P(x=r) = {}^nC_r \, P(\underbrace{SSS\ldots S}_{r \text{ times}}\underbrace{FFF\ldots F}_{(n-r)\text{ times}})$$

$$= {}^nC_r \, \underbrace{P(S).P(S).P(S)\ldots P(S)}_{'r'}.\underbrace{P(F).P(F)\ldots P(F)}_{(n-r)\text{ fail}}$$

$$= {}^nC_r \, p^r q^{n-r}$$

where, $r = 0, 1, 2, \ldots n$

Q.) Prove that mean & variance in binomial distribution are np & npq respectively while standard deviation is $\sqrt{npq}$.

Sol:-

We know that,

$$P(r) = {}^nC_r \, q^{n-r} p^r$$

$$\mu = \sum^{n} r\,P(r)$$

$$= \sum_{r=0}^{n} r\,{}^nC_r q^{n-r} p^r$$

$$= 0 + 1.{}^nC_1 p.q^{n-1} + 2{}^nC_2 p^2.q^{n-2} + 3.{}^nC_3 p^3 q^{n-3} + \cdots$$
$$\cdots + n.{}^nC_n.p^n.q^0$$

$$= npq^{n-1} + 2.\frac{n(n-1)}{\lfloor 2} p^2 q^{n-2} + 3 \cdots\cdots + np^n$$

$$= npq^{n-1} + \frac{n(n-1)}{\lfloor 1} p^2 q^{n-2} + \frac{n(n-1)(n-2)}{\lfloor 2} p^3 q^{n-3} + \cdots$$
$$\cdots + np^n$$

$$= np\left\{q^{n-1} + \frac{(n-1)}{\lfloor 1} pq^{n-2} + \cdots\cdots + p^{n-1}\right\}$$

$$= np\left\{{}^{n-1}C_0 p^0 q^{n-1} + {}^{n-1}C_1 pq^{n-2} + \cdots\cdots + {}^{n-1}C_n p^{n-1} q^0\right\}$$

$$= np(p+q)^{n-1}$$

$$= np$$

$$\therefore \mu = np$$

$$\sigma^2 = \sum_{r=0}^{n} r^2 P(r) - \mu^2$$

$$= \sum_{r=0}^{n} \{r + r(r-1)\} P(r) - \mu^2$$

$$= \sum_{r=0} r P(r) + \sum_{r=2}^{n} r(r-1).P(r) - \mu^2$$

Q-9 [illegible]

(i) [illegible]

(ii) [illegible]

[illegible]

Sol:-

[illegible] $= \frac{1}{5}$

$q = \frac{4}{5}$

[illegible]

$n=6$

(i) The probability that not more than 3 will be busy

$$P(x \le 3) = P(0) + P(1) + P(2) + P(3)$$
$$= {}^6C_0 p^0 q^6 + {}^6C_1 p^1 \cdot q^5 + {}^6C_2 p^2 \cdot q^4 + {}^6C_3 p^3 q^3$$
$$= {}^6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right)^1 \left(\frac{14}{15}\right)^5 + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4$$
$$+ {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$$

(ii) $$P(x \ge 3) = P(3) + P(4) + P(5) + P(6)$$
$$= {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3 + {}^6C_4 \left(\frac{1}{15}\right)^4 \left(\frac{14}{15}\right)^2 + {}^6C_5 \left(\frac{1}{15}\right)^5 \left(\frac{14}{15}\right)$$
$$+ {}^6C_6 \left(\frac{1}{15}\right)^6 \left(\frac{14}{15}\right)^0$$

Q.-2) If the probability of hitting a target is 10% and 10 shots are fired independently, what is the probability that the target will be hit atleast once?

Sol:-

$$p = \frac{10}{100} = \frac{1}{10}$$

$$q = \frac{9}{10}$$

$$n = 10$$

$$P(x \ge 1) = P(1) + P(2) + \cdots + P(10)$$
$$= 1 - P(0)$$
$$= 1 - {}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10}$$

Q-3) A student is given a True/False examⁿ with 8 ques. If he corrects atleast 7 ques. he passes the examⁿ. Find the probability that he will pass given that he guesses all questions.

Sol-

$$p = \frac{1}{2}, \quad q = \frac{1}{2}, \quad n = 8$$

$$P(r \geq 7) = P(7) + P(8)$$

Q-4) If 10% of the bolts produced by a machine are defective, Determine the probability that out of 10 bolts chosen at random:

(i) 1 (ii) none (iii) atmost 2 bolts

will be defective

Sol-

$$p = \frac{1}{10}, \quad q = \frac{9}{10}, \quad n = 10$$

(i) $P(r=1) = P(1) = {}^{10}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^9$

(ii) $P(r=0) = P(0) = {}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10}$

(iii) $P(r \leq 2) = P(0) + P(1) + P(2)$

$= {}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10} + {}^{10}C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^9 + {}^{10}C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^8$

Poisson's Probability Distribution:-

Poisson's distribution has a limiting case of Binomial distribution. If the parameters 'n' & 'p' of a binomi distribution are known, we can find the distribution

But in situation where 'n' is very large & 'p' is very small, application of binomial distribution becomes complicated i.e. if we assume $n \to \infty$ & $p \to 0$ such that np always remains finite (say λ), then we get Poisson's approximation to the binomial distribution.

$$np = \lambda$$
$$p = \lambda/n$$

$$P(r) = {}^nC_r\, q^{n-r} p^r$$

$$= {}^nC_r\, (1-p)^{n-r}\, p^r$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n \cdot \left(\frac{\lambda}{n}\right)^r}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{\lfloor r} \left\{ \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\cdots\left(\frac{n-r+1}{n}\right) \right\} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{\lfloor r} \left\{ 1 \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdots\cdots \left(1-\frac{(r-1)}{n}\right) \right\} \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left[1-\frac{\lambda}{n}\right]^r}$$

$$n \to \infty,\ 1-\frac{1}{n},\ 1-\frac{2}{n},\ 1-\frac{3}{n},\ \cdots\cdots,\ 1-\frac{\lambda}{n} \to 1$$

$$1-\frac{\lambda}{n}$$

$$P(r) = \frac{\lambda^r}{\lfloor r} e^{-\lambda} \quad ; \quad \lambda = np$$
$$r = 0, 1, 2, \ldots \infty$$

Q-1) P.T. in P... equal ...

Sol:-

P(

Q-1) P.T. in Poisson probability distribⁿ, mean & variance are equal to 'λ'.

Sol:-

$$P(r) = \frac{\lambda^r}{r!} \cdot e^{-\lambda} \quad ; \quad r = 0, 1, 2, \ldots, \infty$$

$$\mu = \sum_{r=0}^{\infty} r\, P(r)$$

$$= \sum_{r=0}^{\infty} r \frac{\lambda^r}{r!} \cdot e^{-\lambda}$$

$$= e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{(r-1)!}$$

$$= e^{-\lambda} \left\{ \lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{3!} + \cdots\cdots \right\}$$

$$= \lambda e^{-\lambda} \left\{ 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\cdots \right\}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Al. $$\sigma^2 = \sum_{r=0}^{\infty} r^2 p(r) - \mu^2$$

$$=$$

Q-1) Assume that the probability of an individual coal miner ~~[illegible]~~ killed in a mine accident during a year is $\frac{1}{2400}$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners, there will be at least one fatal accident.

Sol:-

$$p = \frac{1}{2400}$$

$$n = 200$$

$$np = 200 \times \frac{1}{2400}$$

$$= \frac{1}{12}$$

$$e^{-\lambda} = e^{-1/12} = 0.083$$

$$P(r) = \frac{\lambda^r}{\lfloor r} \cdot e^{-\lambda}$$

$$= \frac{\left(\frac{1}{12}\right)^r}{\lfloor r} e^{-\lambda} = \frac{\left(\frac{1}{12}\right)^r}{\lfloor r} (0.083)$$

P(atleast one fatal accident) = 1 − P(No fatal accident)

$$= 1 - \frac{1}{1}(0.083)$$

Q-2) If the probability of a bad rxn from a certain injection is 0.0002. Determine the chance that out of 1000 individuals, more than 2 will get a bad rxn.

Sol:-

$n = 1000$

$p = 0.0002$

$\lambda = np = 0.2$

[illegible] — (i)

[illegible] — (ii)

[illegible] — (iii)

[illegible]

[illegible]

NOTE: In case of change of origin

1) If 'n' is odd then we define U as

$$U = \frac{x - (\text{middle term})}{\text{interval}}$$

2) If n is even then we define U as

$$U = \frac{x - (\text{mean of two middle terms})}{\frac{1}{2}(\text{interval})}$$

Q-1) Create a straight line to the given set of data:-

x	y
0	1
1	18
2	33
3	45
4	63

Sol:-

$$y = a + bx \quad — (1)$$

$$\Sigma y = na + b\Sigma x \quad — (2)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad — (3)$$

x	y	xy	x^2
0	1	0	0
1	18	18	1
2	33	66	4
3	45	135	9
4	63	252	16
$\Sigma x = 10$	$\Sigma y = 160$	$\Sigma xy = 471$	$\Sigma x^2 = 30$

$n = 5$

eq^n (2) : $\quad 160 = 5a + 10b$

eq^n (3) : $\quad 471 = 10a + 30b$

Q.2) Create a second degree parabola for the given set of data

x	y
0	1
1	4
2	10
3	17
4	30

Sol:-

$$y = a + bx + cx^2 \quad \text{—(1)}$$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad \text{—(2)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{—(3)}$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{—(4)}$$

x	y	x^2	xy	x^3	x^2y
0	1	0	0	0	0
1	4	1	4	1	4
2	10	4	20	8	40
3	17	9	51	27	153
4	30	16	120	64	480
$\Sigma x=10$	$\Sigma y=62$	$\Sigma x^2=30$	$\Sigma xy=195$	$\Sigma x^3=100$	$\Sigma x^2y=677$

$$62 = 5a + 10b + 30c$$

$$195 = 10a + 30b + 100c$$

Q-3) Create the straight line for following set of data:

x	y
1998	28
1999	38
2000	46
2001	40
2002	56
2003	60

Sol:

$$y = a + bx \quad — ①$$

$n = 6 \rightarrow$ even

$$u = \frac{x - \left(\frac{2000+2001}{2}\right)}{\frac{1}{2} \times 1} = 2x - 4001$$

$$v = y - 46$$

$$v = A + Bu \quad — (a)$$

$$\Sigma v = nA + B\Sigma u \quad — (b)$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 \quad — (c)$$

x	y	$u = 2x - 4001$	$v = y - 46$	uv	u^2
1998	28				
1999	38				
2000	46				
2001	40				
2002	56				
2003	60				
		$\Sigma u = 0$	$\Sigma v = -8$	$\Sigma uv = 208$	$\Sigma u^2 = 70.$

Final Ans:- $y = 5e^{.9428x} =$

CORRELATION

Two variables 'X' and 'Y' are so related that an increase in the one is accompanied by an increase or decrease in the other, then variables are said to be correlated.

e.g:- The of ~~frog~~ varies with the amount of rainfall

Types of correlation:-

1) Positive Correlation:-

$X\uparrow, Y\uparrow$ or $X\downarrow, Y\downarrow$

If an increase in the value of one variable 'X' results in a corresponding increase in the value of 'Y' on an avg. or if a decrease in the value of 'X' results in corresponding decrease in the value of 'Y', then correlation is said to be positive.

2) Negative

3)

If all plotted pts. lie approximately on a st. line then the correlation is said to be linear.

4) Perfect Correlation:-

If deviation of 'X' is proportional to the deviation in other variable 'Y', then correlation is said to be perfec

→ Perfect (-)ve

→ ——— (+)ve

Karl Pearson's Coefficient of Correlation (r):
Karl Pearson's Coefficient 'r' b/w 2 variables x & y is defined as:

$$r = \frac{\Sigma xy}{\sqrt{\ }}$$

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2)(\Sigma Y^2)}} = \frac{P}{\sigma x \cdot \sigma y} = \frac{cov(x,y)}{\sqrt{var\,x}\,\sqrt{var\,y}}$$

where,

$X = x - \bar{x}$

$Y = y - \bar{y}$

i.e. X & Y are the deviations of x & y measured from the respective means

$P =$

Q-1) Calculate the coefficient of corr. b/w marks obtained by 10 students in Economics & Statistics

R.No.	Marks in Eco.	Marks in Stats
1	78	84
2	36	51
3	98	91
4	25	60
5	75	68
6	82	62
7	90	86
8	62	58
9	65	[illegible]

Sol:-

Let the marks of 2 subjects are denoted by 'x' & 'y' respectively.

$$\bar{x} = \frac{\Sigma x}{n} = \frac{650}{10} = 65$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{660}{10} = 66$$

'X' & 'Y' are the deviations of x & y from their respective means.

R.No.	Eco x	Stats y	$X = x - \bar{x}$ $= x - 65$	$Y = y - \bar{y}$ $= y - 66$	XY	X^2	Y^2
1			13	18			
2			-29	-15			
3			33	25			
4			-40	-6			
5			10	2			
6			17	-4			
7			25	20			
8			-3	-8			
9			0	-13			
10			-26				
			$\Sigma X = 0$	$\Sigma Y = 0$	$\Sigma XY =$		

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2)(\Sigma Y^2)}}$$

Regression → Prepare yourself

Unit 3

ALGEBRA & GEOMETRY

Solution of Equations:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots\cdots + a_n$$
$$(a_0 \neq 0)$$

$$f(x) = 0 \Rightarrow a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$
$$x = \alpha, \quad (x-\alpha)$$

General Properties:

1) If α is a root of the equation $f(x)=0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$

2) Every eq^n of the n^{th} degree has 'n' roots. Roots may be real or imaginary.

3) Intermediate value prop:
If $f(a)$ & $f(b)$ have diff. sign, then eq^n $f(x)=0$ has atleast one root b/w $x=a$ & $x=b$.

4) In an eq^n with real coeff. imaginary roots occur in conjugate pair i.e if $\alpha+i\beta$ is a root of the eq^n $f(x)=0$, then $\alpha-i\beta$ must also be its root. Similarly,

5) If $\alpha+\sqrt{\beta}$ is an irrational root of an eq^n, then $\alpha-\sqrt{\beta}$ must also be its root.

Relations b/w Roots & Co-efficients:-

If $\alpha_1, \alpha_2, \text{———}, \alpha_n$ be the roots of the equation
$$a_0x^n + a_1x^{n-1} + \text{———} + a_n = 0$$

$\Sigma\alpha_1 = -\frac{a_1}{a_0}$, $\Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}$, $\Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$,

$\Sigma\alpha_1\alpha_2 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Q-1) Solve the equation: $2x^3 + x^2 - 13x + 6 = 0$

Sol:- $\because$ By trial or inspection, $x=2$ is one of the root of the eqn.

So, $(x-2)$ will be a factor of the $2x^3 + x^2 - 13x + 6$

$$
\begin{array}{r|l}
 & 2x^2 + 5x - 3 \\
(x-2) & 2x^3 + x^2 - 13x + 6 \\
 & \cancel{2x^3} \mp 4x^2 \\
 & \quad 5x^2 - 13x + 6 \\
 & \quad \cancel{5x^2} \mp 10x \\
 & \qquad -3x + 6 \\
 & \qquad \mp 3x \pm 6 \\
 & \qquad\quad 0
\end{array}
$$

Now, for remaining roots equate $2x^2+5x-3$ to zero

$$2x^2 + 5x - 3 = 0$$
$$2x^2 + 6x - x - 3 = 0$$
$$(2x-1)(x+3) = 0$$
$$x = \frac{1}{2}, -3$$

Q-2) Solve the equation; $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ —(1)

Sol:- given that sum of 2 of its roots is zero.

Let $\alpha, \beta, \gamma, \delta$ be the 4 ...

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$\alpha+\beta+\gamma+\delta = 2$

$\alpha\beta \quad \alpha+\beta=0$

$\gamma+\delta=2$

$x^2 -$ (sum of the roots)x + (Product of the roots)

α, β

$x^2-(\alpha+\beta)x+q$

$= x^2+q$

γ, δ

$x^2-(\gamma+\delta)x+p$

$= x^2-2x+p$

Now, $(x^2+q)(x^2-2x+p) = x^4-2x^3+4x^2+6x-21$

$\Rightarrow x^4-2x^3+px^2+qx^2-2qx+pq = x^4-2x^3+4x^2+6x-21$

$\Rightarrow x^4-2x^3+(p+q)x^2-2qx+pq = x^4-2x^3+4x^2+6x-21$

$q=-3$

$p=7$

$x^2+q=0$

$\Rightarrow x^2-3=0$

$\Rightarrow x = \pm\sqrt{3}$

$x^2-2x+7=0$

$x = \dfrac{2\pm\sqrt{4-28}}{2}$

$x = \dfrac{2\pm\sqrt{-24}}{2} = 1\pm\sqrt{6}i$

Q-> Solve the equation : $x^4-2x^3-21x^2+22x+40=0$ — (1) whose roots are in A.P.

Sol:-

Let the roots be $(a-3d), (a-d), (a+d), (a+3d)$

$4a = 2 \Rightarrow a = \frac{1}{2}$

$\left(\frac{1}{2}-3d\right)\left(\frac{1}{2}-d\right)\left(\frac{1}{2}+d\right)\left(\frac{1}{2}+3d\right) = 40$

$\left(\frac{1}{4}-d^2\right)\left(\frac{1}{4}-9d^2\right) = 40 \Rightarrow (1-4d^2)(1-36d^2) = \frac{40}{16}$

$\Rightarrow 1-4d^2-36d^2+144d^4 = 40$

$\frac{2}{3}\frac{5}{6}4$

$144d^2 - 36d^2 - 4d^2 + 15d = 0$

$\left(\frac{1}{4} - t\right)\left(\frac{1}{4} - 9t\right) = 40$

<u>Solution of Cubic Equation by Cardan's Method:</u>

Consider an eq^n:

$$ax^3 + bx^2 + cx + d = 0 \quad \text{--- (1)}$$

$$b/a = l, \quad c/a = m, \quad d/a = n$$

$$x^3 + lx^2 + mx + n = 0$$

To remove the x^2 term by putting $y = x - \left(-\frac{l}{3}\right)$ or

$x = y - \frac{l}{3}$ so that the resulting eq^n is of the fo[rm]

$$y^3 + py + q = 0 \quad \text{--- (2)}$$

To solve eq^n (2) by putting $y = u + v$

$$y^3 = (u+v)^3$$

$$y^3 = u^3 + v^3 + 3uv(u+v)$$

$\Rightarrow$

Comparing

$\therefore u^2$

$$y^3 = u^3 + v^3 + 3uvy$$

$$\Rightarrow \quad y^3 = 3uvy - (u^3+v^3) = 0 \qquad —③$$

Comparing with eq^n ②

$$uv = -p/3$$

$$u^3 + v^3 = -q$$

$$u^3v^3 = -p^3/27$$

$\therefore$ u^3 & v^3 are the roots of the eq^n

$$t^2 - (u^3+v^3)t + u^3v^3 = 0$$

$$t^2 + qt - p^3/27 = 0$$

$$u^3 = \frac{1}{2}\left[-q \pm \sqrt{q^2 + 4p^3/27}\right] = \lambda^3$$

$$v^3 = \frac{1}{2}\left[-q - \sqrt{q^2 + 4p^3/27}\right]$$

$$y = u + v$$

$$= \lambda - \frac{p}{3\lambda},\ \lambda\omega - p\omega^2/3\lambda,\ \lambda\omega^2 - p\omega/3\lambda$$

$$v = \lambda,\ \lambda\omega,\ \lambda\omega^2$$

$$v = -p$$

Q-1) Solve by Cardan's Method :-

$$x^3 - 3x^2 + 12x + 16 = 0 \quad \text{— (1)}$$

Sol- To remove x^2 term we put $y = x - \left(\frac{3}{3}\right) = x - 1$

$$\Rightarrow x = y + 1$$

$$(y+1)^3 - 3(y+1)^2 + 12(y+1) + 16 = 0$$

$$y^3 + 3y^2 + 3y + 1 - 3y^2 - 6y - 3 + 12y + 12 + 16 = 0$$

$$y^3 + 9y + 26 = 0 \quad \text{— (2)}$$

Now, let $y = u + v$

$$y^3 = (u+v)^3$$

$$\Rightarrow y^3 - 3uvy - (u^3 + v^3) = 0 \quad \text{— (3)}$$

Compare eq^n (2) & (3)

$$uv = -3, \quad u^3 + v^3 = -26$$

$$\Downarrow$$

$$u^3v^3 = -27 \;\&\; u^3 + v^3 = -26$$

If u^3 & v^3 are roots, then we have

$$t^2 + 26t - 27 = 0$$

$$t^2 + 27t - t - 27 = 0$$

$$t = 1, -27$$

$$u^3 = t = -27 \Rightarrow u = -3$$

$$v^3 = t = 1 \Rightarrow v = 1$$

$$u = -3, -3\omega, -3\omega^2$$

$$v = 1, \omega, \omega^2$$

$$y = u + v = -2$$

$$x = y + 1 = -1$$

Q-2) Solve

Sol:-

$-3y^2 \cdot \frac{1}{3}$

$+3y\left(\frac{1}{2}\right)$

Q-13

$$
\begin{array}{r|l}
(x+1) & x^3 - 3x^2 + 12x + 16 \quad (x^2 - 4x + 16 \\
 & x^3 + x^2 \\
\hline
 & -4x^2 + 12x + 16 \\
 & -4x^2 - 4x \\
\hline
 & 16x + 16 \\
 & 16x + 16 \\
\hline
 & 0
\end{array}
$$

$x^2 - 4x + 16 = 0$

$x = \frac{4 \pm \sqrt{16 - 64}}{2} = \frac{4 \pm \sqrt{-48}}{2} = 2 \pm 2\sqrt{3}i$

Q - Solve the eq^n: $x^3 + x^2 - 16x + 20 = 0$ by Cardan's method.

Sol:

$x^3 + x^2 - 16x + 20 = 0$ —— (1)

$y = x - \left(-\frac{1}{3}\right) = x - \left(-\frac{1}{3}\right) = x + \frac{1}{3}$

$\Rightarrow \quad x = y - \frac{1}{3}$

$\left(y - \frac{1}{3}\right)^3 + \left(y - \frac{1}{3}\right)^2 - 16\left(y - \frac{1}{3}\right) + 20 = 0$

$y^3 - \frac{1}{27} - y^2 + \frac{y}{3} + y^2 - \frac{2y}{3} + \frac{1}{9} - 16y + \frac{16}{3} + 20 = 0$

Characteristic Function :-

Let X be a universal set & 'A' be any subset of
The characteristic func^n of A is denoted by $X_A(x)$
and it is defined by for each $x \in X$ as charact-
-istic function $X_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$ for any

$x \in X$.

Q. Let the universal set 'X' be the set of all negative no.
and let A is a set containing x.

then write characteristic function for subset A.

Sol:-

'X' is the set of real no. (non-negative) & A =
$A = \{2, 3, 4, 5, 6, 7, 8\}$.

Some Properties:-

1) If A and B are any two subsets of X, then the characteristic function $X_{A \cup B}$ of the union $A \cup B$ of A and B is defined by maximum of characteristic function of A and B

$$X_{A \cup B}(x) = \text{Max}\{X_A(x), X_B(x)\} \ \forall x \in X$$

2) Characteristic func^n of $A \cap B$ of the intersection $A \cap B$ of two sets A & B is defined by min^m of charac. func^n of A, X_B $\forall x \in X$.

$$X_{A \cap B}(x) = \text{Min}\{X_A(x), X_B(x)\} \ \forall x \in X$$

3) If A and B are two sets and $A \subseteq B$ if and only if $\chi_A(x) \leq \chi_B(x)$, $\forall x \in X$.

4) If $\bar{A}$ denote the complement of the set A, then char func^n of $\bar{A}$

$$\chi_{\bar{A}}(x) = 1 - \chi_A(x), \forall x \in X.$$

Membership func^n:-

Func^n similar to the characteristic func^n is called membership func^n. The membership func^n assigned to each element $x \in X$ a number $\mu_A(x)$ in the closed interval $[0,1]$ that characterise the degree of membership of x in A, thus membership func^n are the func^n of the form

$$\mu_A : X \longrightarrow [0,1]$$

The membership func^n of x in A is also called grade of membership or degree of truth of x in A.
If A and B are two sets, then we have

$$\mu_{A \cup B}(x) = \max.[\mu_A(x), \mu_B(x)] \quad , \forall x \in X$$

$$\mu_{A \cap B}(x) = \min [\mu_A(x), \mu_B(x)] \quad , \forall x \in X$$

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \quad , \forall x \in X$$

$$A \subseteq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \ , \forall x \in X$$

Fuzzy set :-

Let 'X' be a universal set, then a fuzzy set A in X is defined as set of ordered pairs

$$A = \{ (x, \mu_A(x)) : x \in X \}$$

where, $\mu_A(x)$ is called the Membership func^n

A

$X = \{x_1, x_2, x_3, \ldots\ldots, x_n\}$

$A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), (x_3, \mu_A(x_3)) \ldots\ldots, (x_n, \mu_A(x_n))\}$

$A = \{(x, \mu_A(x)) : x \in X\}$

$A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2} \ldots\ldots \frac{\mu_A(x_n)}{x_n}\right\}$

Standard Operation on Fuzzy Sets :-

⇒ 1) Complement of a fuzzy set :-

Let 'A' be a fuzzy set defined on a universal set X, the complement of A is denoted by A' or $\bar{A}$ or A^c and it is defined as

$A'(x) = \{(x, \mu_{A'}(x)) : x \in X\}$

where, $\mu_{A'}(x) = 1 - \mu_A(x)$, $\forall x \in X$

2) Subset

3) Union of 2 Fuzzy Set :-

Let A & B be two fuzzy set defined on the universal set X. The union of A & B is denoted by $A \cup B$ which is a fuzzy set

$(A \cup B)(x) = \{(x, \mu_{A \cup B}(x)) ; x \in X\}$

$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)]$, $\forall x \in X$

4) Intersection of 2 Fuzzy sets:-

Let A and B be two fuzzy set defined on a universal set X, the intersecⁿ of A & B is denoted by $A \cap B$

$$(A \cap B)(x) = \{(x, \mu_{A \cap B}(x)) : x \in X\}$$

$$\mu_{A \cap B}(x) = \min [\mu_A(x), \mu_B(x)] \quad \forall x \in X.$$

5) Equality of 2 Fuzzy sets:-

If A & B are two fuzzy set

$A = B$ if and only if $\mu_A(x) = \mu_B(x)$, $\forall x \in X$.

6) De-Morgan's law:-

If A & B are two fuzzy set, then De Morgan's law is

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

7) Difference of 2 fuzzy sets:-

If A & B are 2 fuzzy set, then their diff is denoted by $A-B$ and is

$$A - B = A \cap B'$$

Some Fundamental Prop. of Fuzzy sets:-

If 'X' is universal set, 'ϕ' is the null fuzzy set, A, B, C are 3 fuzzy sets, then

1) Identity law:-

(a) $A \cap X = A$ and $A \cup \phi = A$

(b) $A \cup X = X$ and $A \cap \phi = \phi$

2) Idempotent law:-

$A \cup A = A$, $A \cap A = A$

3) Commutative law:-

$A \cup B = B \cup A$

$A \cap B = B \cap A$

4) Associative law:-

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

5) Distributive law:-

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6) Transitive law:-

If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$

Algebraic Operations on fuzzy set:-

1) Algebraic sum:-

The algebraic sum of 2 fuzzy sets A & B is denoted by A+B and defined as:-

$(A+B)(x) = \{(x, \mu_{A+B}(x)) ; x \in X\}$

Where,

$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$ $\forall x$

2) Algebraic Product :-

The algebraic product of two fuzzy set A and B is ~~defe~~ denoted by $A.B$ and defined as

$$(A.B)(x) = \{(x, \mu_{A.B}(x)) : x \in X\}$$

$$\mu_{A.B}(x) = \mu_A(x) . \mu_B(x) , \forall x \in X$$

Q-1) Let $X = \{x_1, x_2, x_3, x_4\}$ and two fuzzy sets A & B are

$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.7), (x_4, 1)\}$

$B = \{(x_1, 0.6), (x_2, 1), (x_3, 0.4), (x_4, 0.3)\}$

then find $A \cup B$, $A \cap B$. Is $A \subseteq B$?

Sol.- Write union's def^n

$A \cup B(x) = \{(x, \mu_{A \cup B}(x)) : x \in X\}$

where, $\mu_{A \cup B}(x)$ is evaluated as

$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] , \forall x \in X$

$$\cancel{\mu_{A \cup B}(x)} = \mu_{A \cup B}(x_1) = \max[\mu_A(x_1), \mu_B(x_1)] = \max[0.2, 0.6] = 0.6$$

$$\mu_{A \cup B}(x_2) = \max[\mu_A(x_2), \mu_B(x_2)] = \max[0.5, 1] = 1$$

$$\mu_{A \cup B}(x_3) = \max[\mu_A(x_3), \mu_B(x_3)] = \max[0.7, 0.4] = 0.7$$

$$\mu_{A \cup B}(x_4) = \max[\mu_A(x_4), \mu_B(x_4)] = \max[1, 0.3] = 1$$

$(A \cup B) = \{(x_1, 0.6), (x_2, 1), (x_3, \ldots$

$(A \cap B) = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.3)\}$

$A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x), \forall x \in X$

$\mu_A(x_1) \leq \mu_B(x_1)$

$0 \leq 0.6$

$\mu_A(x_2) \leq \mu_B(x_2)$

$0.5 \leq 1$

$\mu_A(x_3) \leq \mu_B(x_3)$

$0.7 \nleq 0.4$

$\mu_A(x_4) \leq \mu_B(x_4)$

$1 \nleq 0.3$

Q-2) Let $X = \{47, 48, 49, 50, 51\}$

$A = \left\{\frac{0.3}{47}, \frac{0.4}{48}, \frac{0.7}{49}, \frac{0.8}{50}, \frac{1}{51}\right\}$

$B = \left\{\frac{1}{47}, \frac{0}{48}, \frac{0.8}{49}, \frac{0.6}{50}, \frac{0.3}{51}\right\}$

Find $A \cup B$ and $A \cap B$.

Sol:-

$A \cup B = \left\{\frac{\max[0.3, 1]}{47}, \frac{\max[0.4, 0]}{48}, \frac{\max[0.7, 0.8]}{49}, \frac{\max[0.8, 0.6]}{50}, \frac{\max[1, 0.3]}{51}\right\}$

$= \left\{\frac{1}{47}, \frac{0.4}{48}, \frac{0.8}{49}, \frac{0.8}{50}, \frac{1}{51}\right\}$

Q-3 Let $A = \{(1, 0.2), (3, 0.6), (4, 1), (5, 0.6)\}$

$B = \{(2, 1), (3, 0.2), (4, 0.5), (6, 1)\}$

Find $A + B$, $A.B$, A^2.