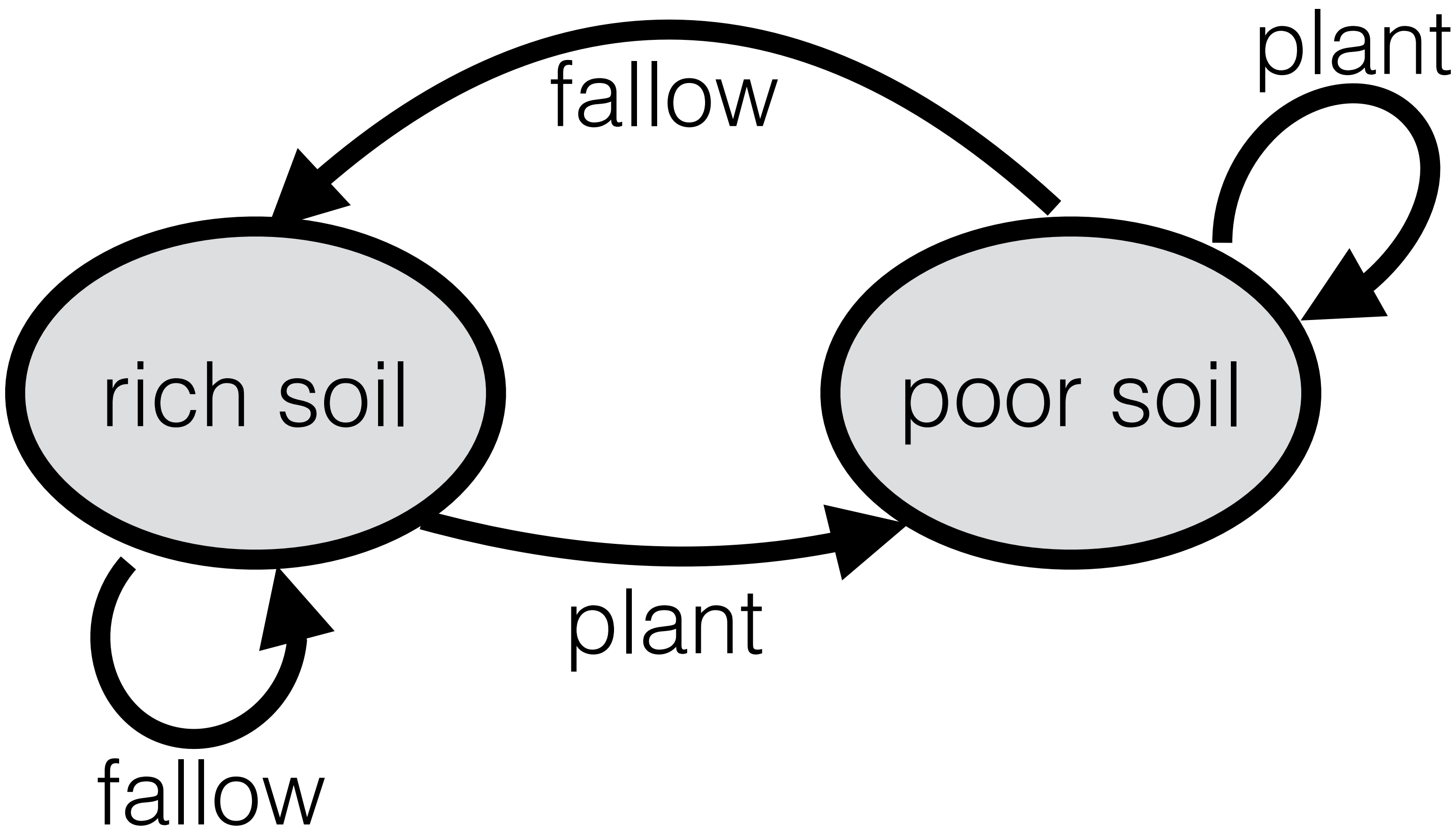
State Machine

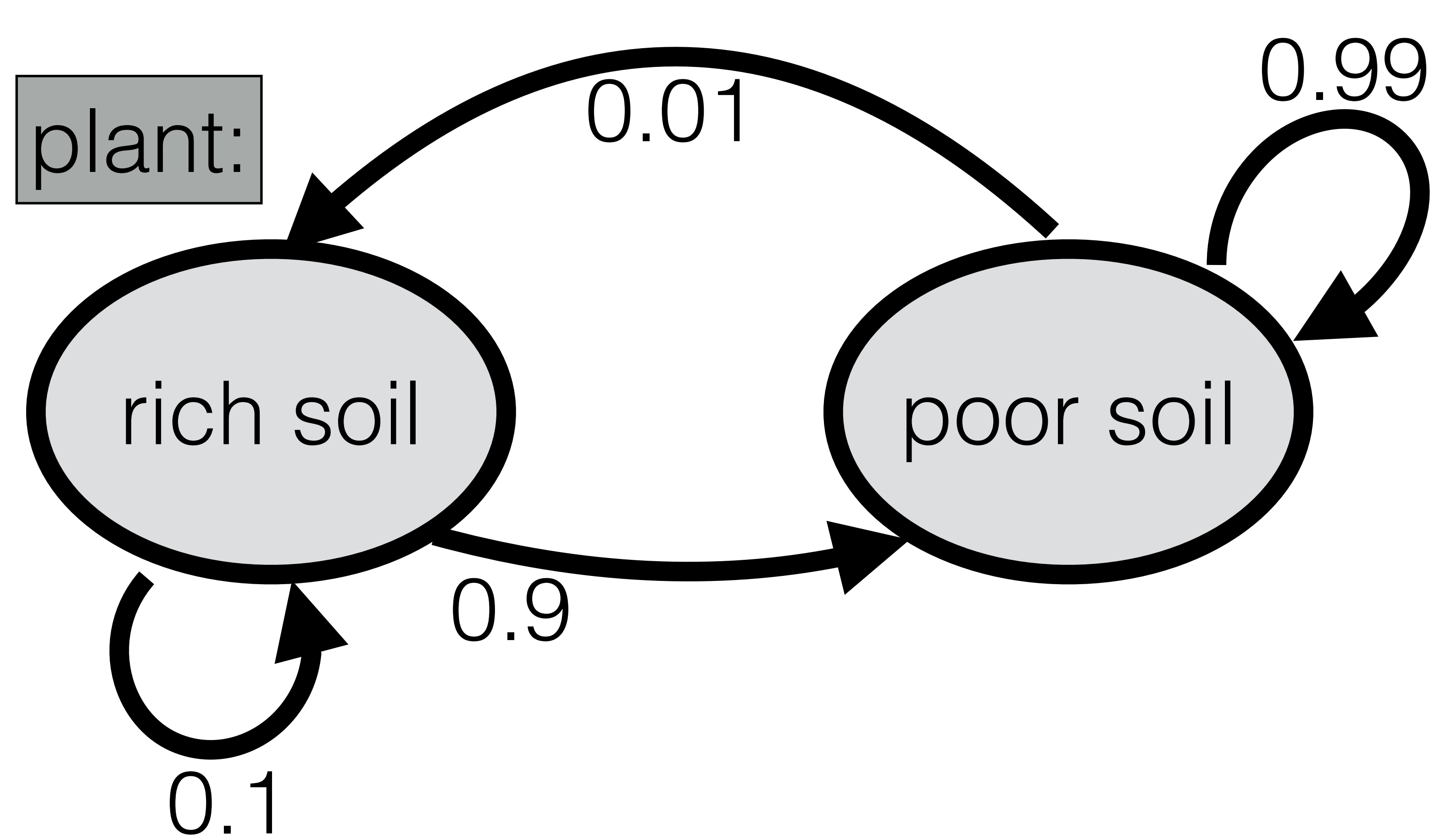
- $\mathcal{S}$ = set of possible states
- $\mathcal{X}$ = set of possible inputs
- $s_0 \in \mathcal{S}$: initial state
- f : $\mathcal{S} \times \mathcal{X} \to \mathcal{S}$: transition function
- $\mathcal{Y}$: set of possible outputs
- g : $\mathcal{S} \to \mathcal{Y}$: output function
  - e.g. g(s) = s
  - e.g. g(s) = soil-moisture-sensor(s)



Example

$$
\begin{aligned}
s_0 &= \text{rich} \\
s_1 &= f(s_0, \text{plant}) = \text{poor}; \\
y_1 &= g(s_1) = \text{poor} \\
s_2 &= f(s_1, \text{fallow}) = \text{rich}; \\
y_2 &= g(s_2) = \text{rich}
\end{aligned}
$$

- $\mathcal{S}$ = set of possible states
- $\mathcal{X}$ = set of possible inputs
- $s_0 \in \mathcal{S}$: initial state
- T transition model
- $R : \mathcal{S} \times \mathcal{X} \to \mathbb{R}$: reward function
  - e.g. R(rich, plant) = 100 bushels; R(poor, plant) = 10 bushels; R(rich, fallow) = R(poor, fallow) = 0 bushels



- Transition matrix for "plant" action:

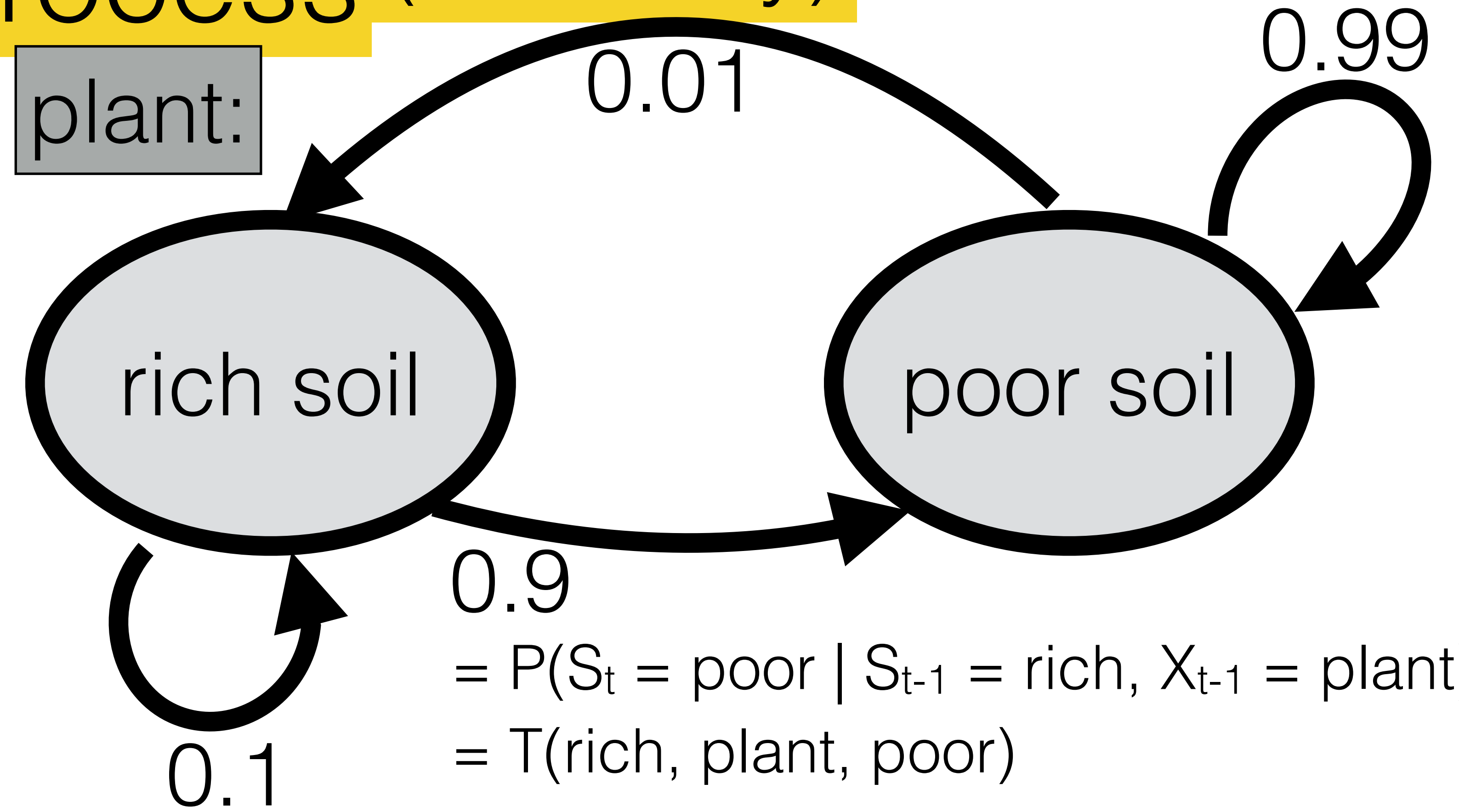
end state

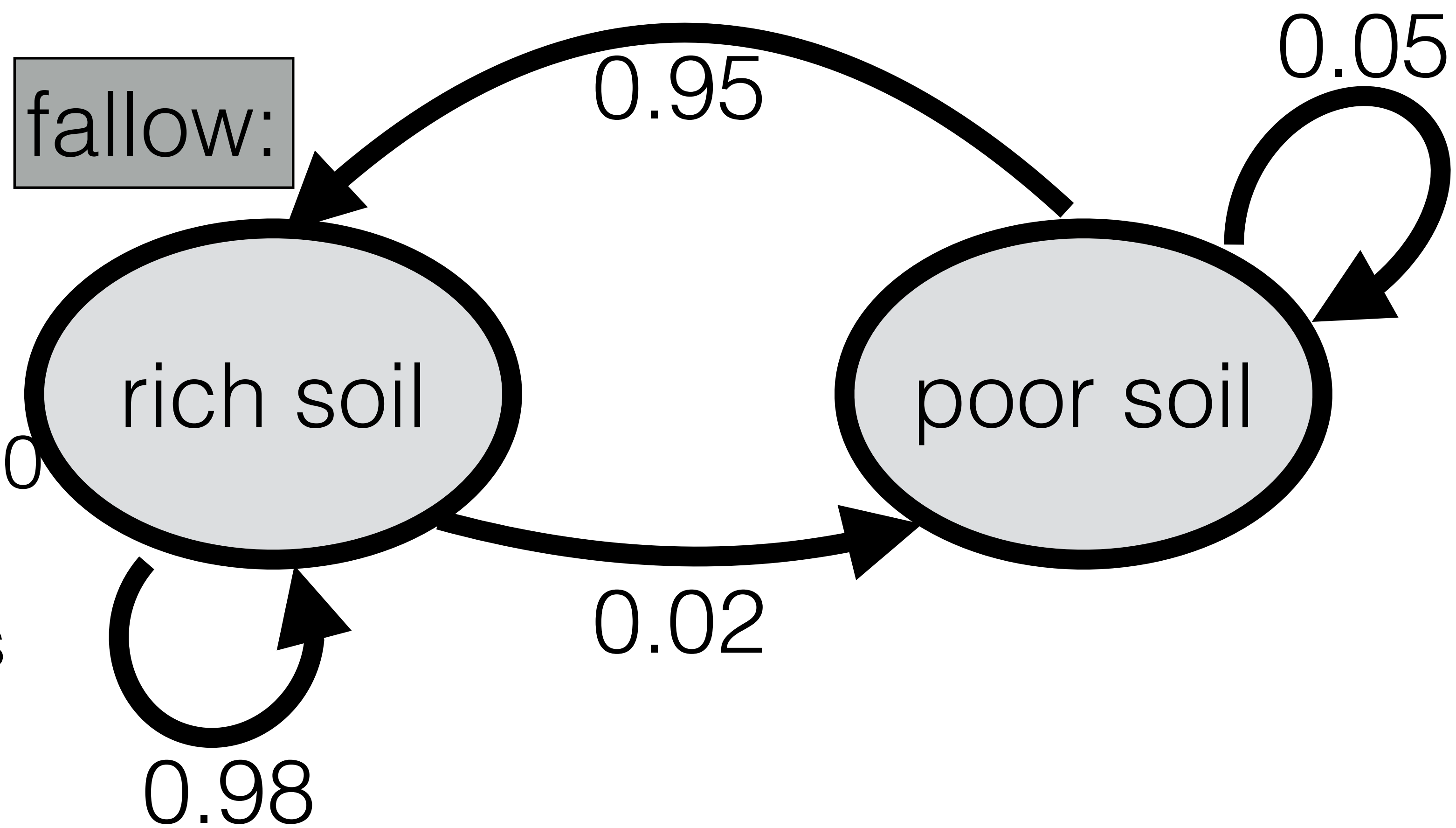
start state

$$\begin{array}{cc} & \begin{array}{cc} \text{rich} & \text{poor} \end{array} \\ \begin{array}{c} \text{rich} \\ \text{poor} \end{array} & \left[\begin{array}{cc} 0.1 & 0.9 \\ 0.01 & 0.99 \end{array}\right] \end{array}$$

Markov Decision Process (basically)

- $\mathcal{S}$ = set of possible states
- $\mathcal{X}$ = set of possible inputs
- $s_0 \in \mathcal{S}$: initial state
- $T : \mathcal{S} \times \mathcal{X} \times \mathcal{S} \rightarrow \mathbb{R}$: transition model
- $R : \mathcal{S} \times \mathcal{X} \rightarrow \mathbb{R}$: reward function
    - e.g. R(rich, plant) = 100 bushels; R(poor, plant) = 10 bushels; R(rich, fallow) = R(poor, fallow) = 0 bushels





Markov Decision Process

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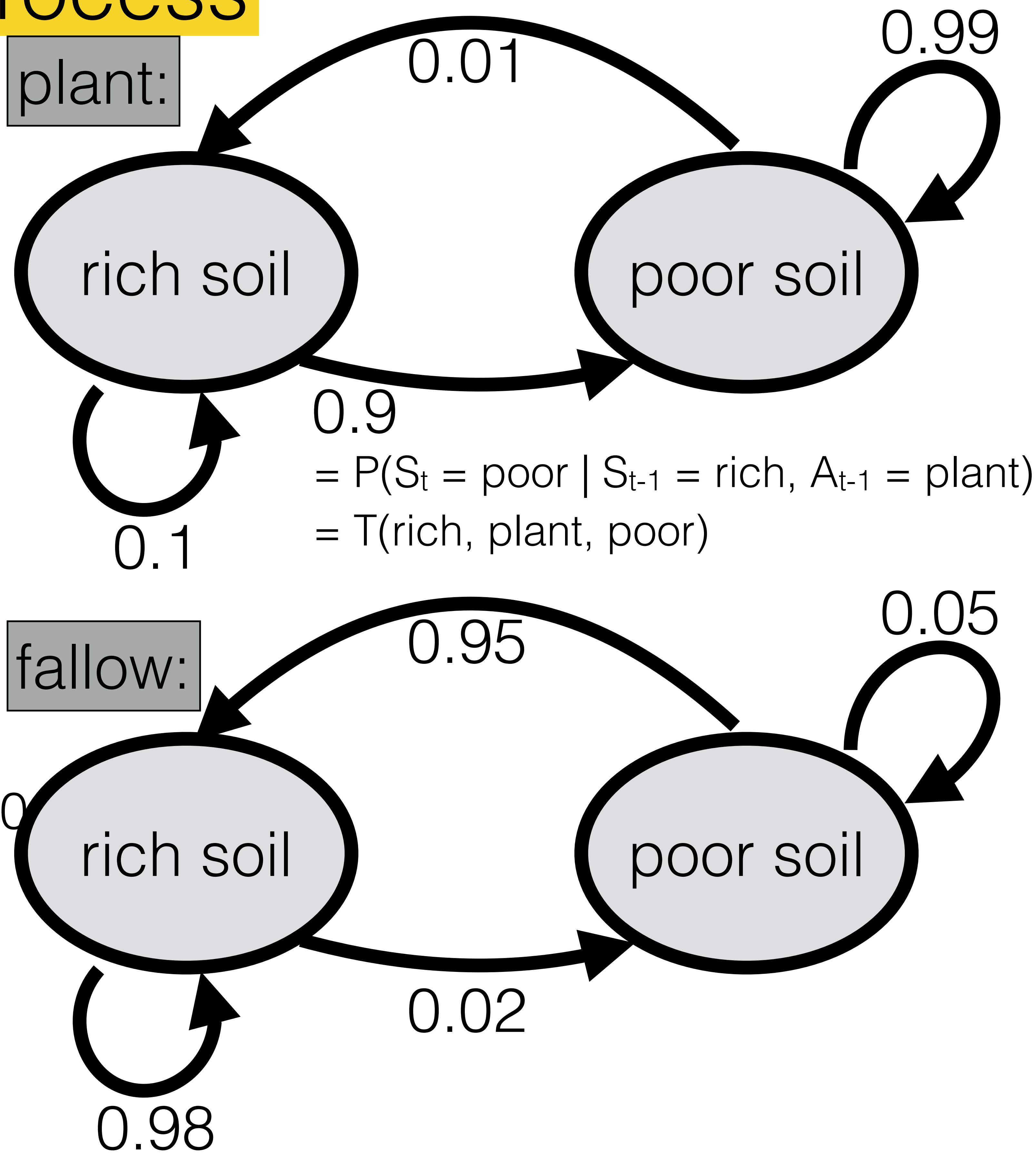
- A discount factor



$= P(S_t = \text{poor} \mid S_{t-1} = \text{rich}, A_{t-1} = \text{plant})$
$= T(\text{rich, plant, poor})$

Markov Decision Process

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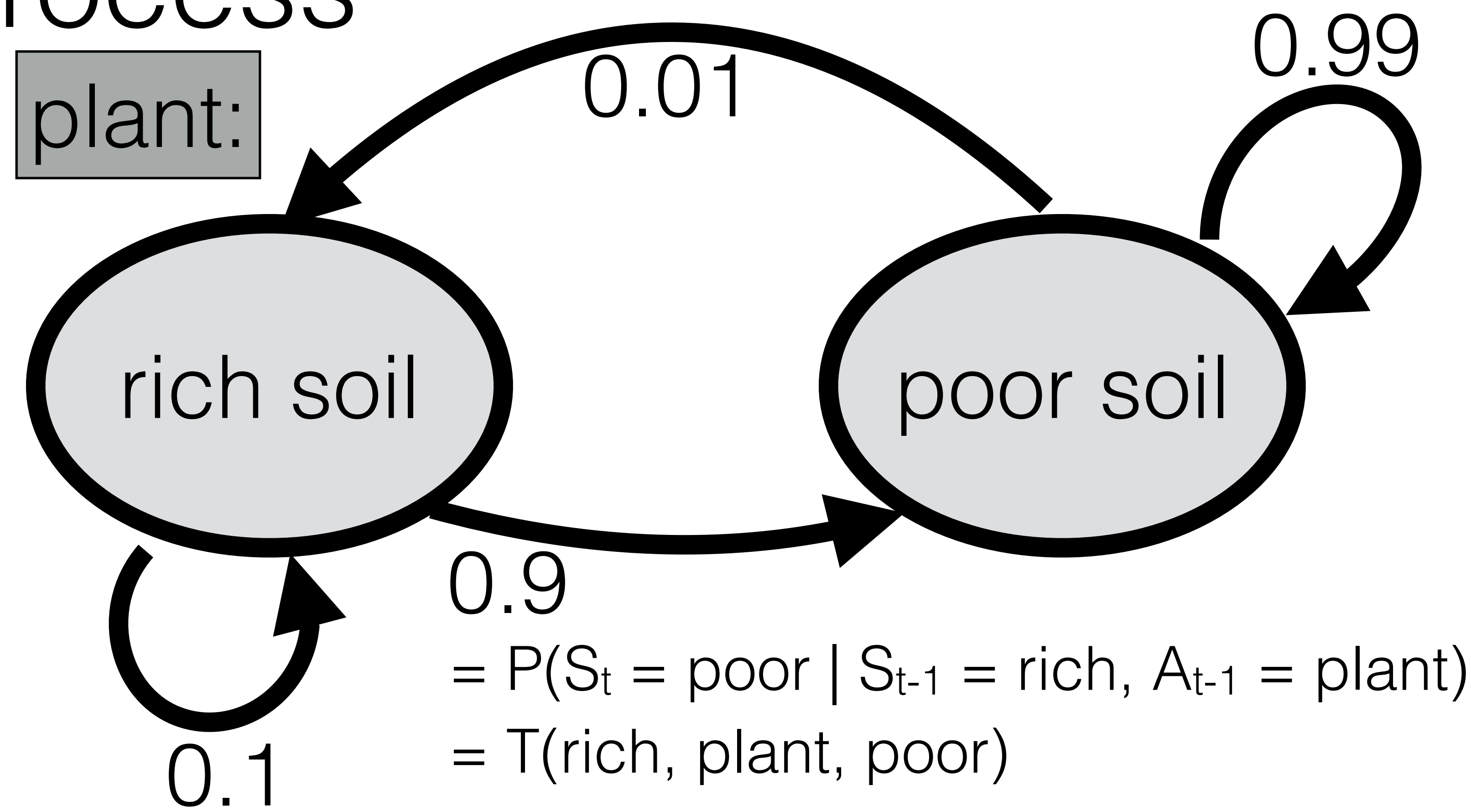
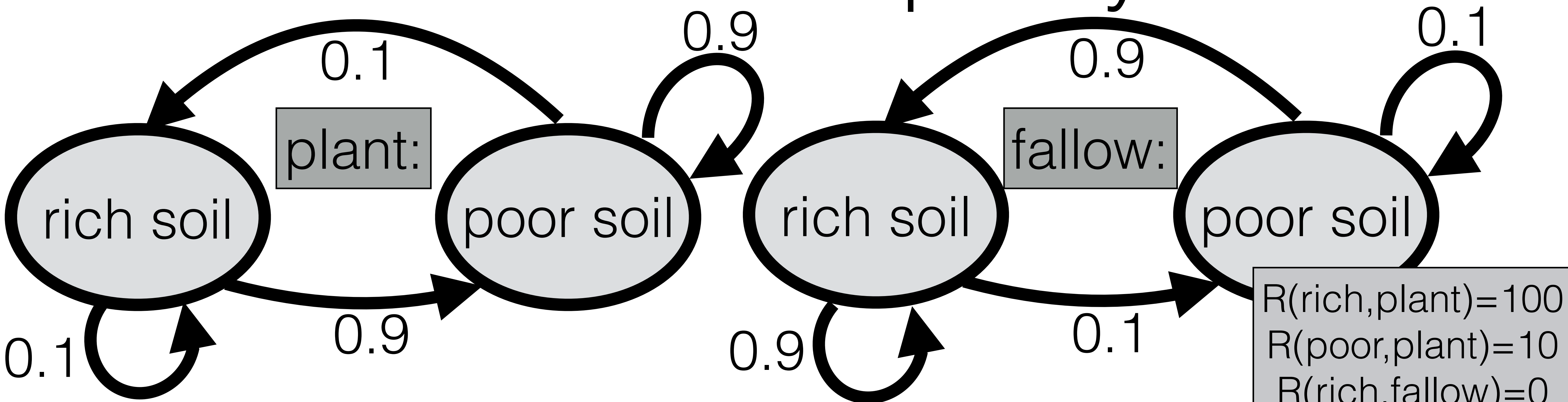


- Definition: A **policy** $\pi : \mathcal{S} \to \mathcal{A}$ specifies which action to take in each state
- Question 1: what's the "value" of a policy?
- Question 2: what's the best policy?

What's the value of a policy?



- *h*: horizon (e.g. how many growing seasons left)
- $V_\pi^h(s)$: value (expected reward) with policy π starting at *s*

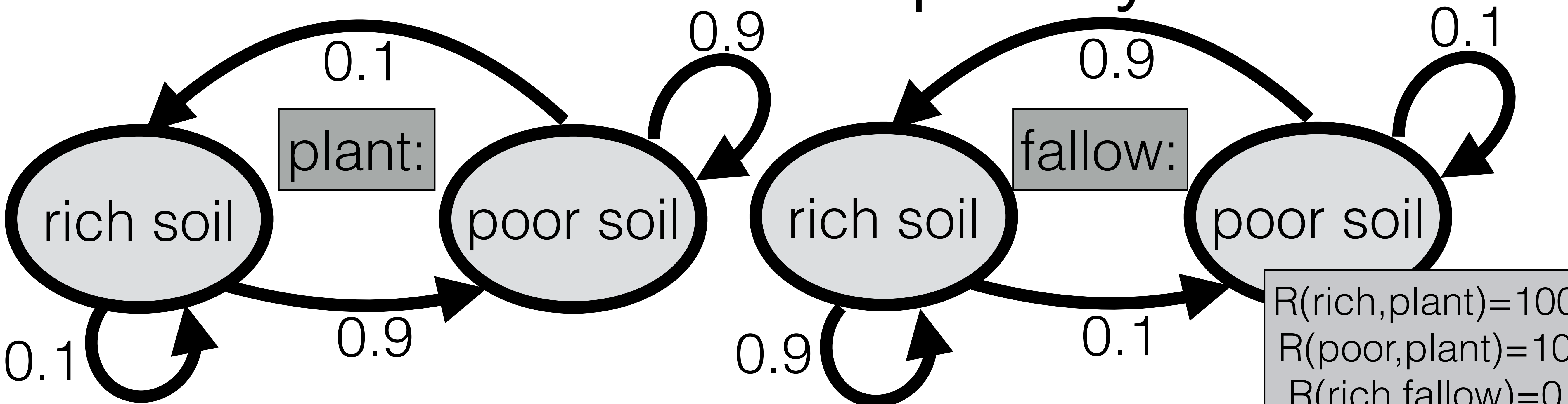
Dueling farmers! π_A : always plant; π_B: plant if rich, else fallow

$$V_\pi^0(s) = 0; \; V_\pi^h(s) = R(s, \pi(s)) + \sum_{s'} T(s, \pi(s), s') \cdot V_\pi^{h-1}(s')$$

$$V_{\pi_A}^1(\text{rich}) = 100; V_{\pi_A}^1(\text{poor}) = 10; V_{\pi_B}^1(\text{rich}) = 100; V_{\pi_B}^1(\text{poor}) = 0$$

$$V_{\pi_A}^2(\text{rich}) = R(\text{rich}, \pi_A(\text{rich})) + T(\text{rich}, \pi_A(\text{rich}), \text{rich}) V_{\pi_A}^1(\text{rich})$$
$$+ T(\text{rich}, \pi_A(\text{rich}), \text{poor}) V_{\pi_A}^1(\text{poor})$$
$$= 100 + (0.1)(100) + (0.9)(10)$$
$$= 119$$

What's the value of a policy?



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$$V_\pi^0(s) = 0; V_\pi^h(s) = R(s, \pi(s)) + \sum_{s'} T(s, \pi(s), s') \cdot V_\pi^{h-1}(s')$$

$$V_{\pi_A}^1(\text{rich}) = 100; V_{\pi_A}^1(\text{poor}) = 10; V_{\pi_B}^1(\text{rich}) = 100; V_{\pi_B}^1(\text{poor}) = 0$$

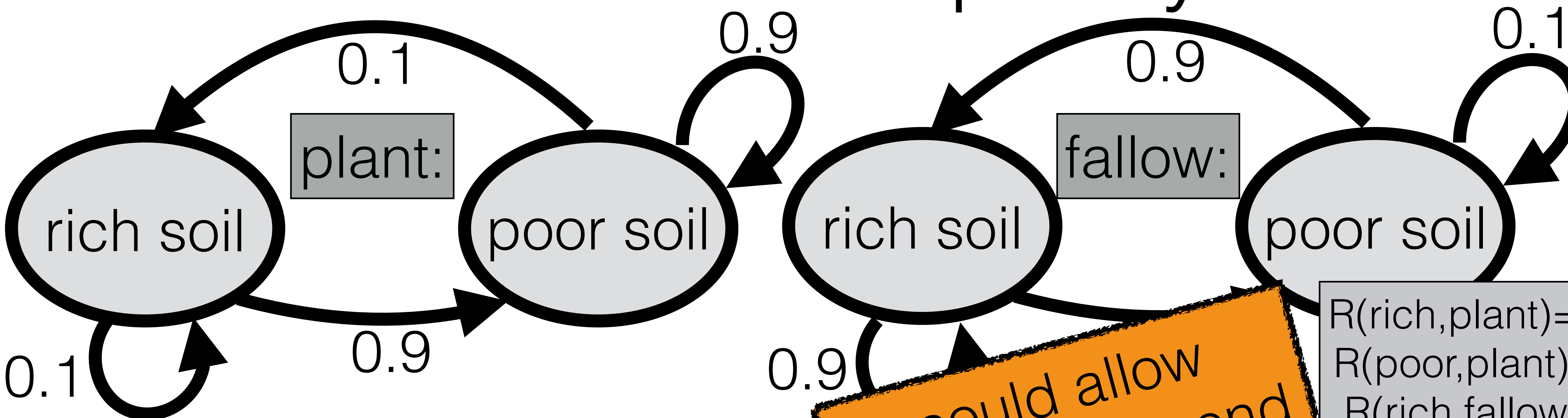
$$V_{\pi_A}^2(\text{rich}) = 119; V_{\pi_A}^2(\text{poor}) = 29; V_{\pi_B}^2(\text{rich}) = 110; V_{\pi_B}^2(\text{poor}) = 90$$

$$V_{\pi_A}^3(\text{rich}) = 138; V_{\pi_A}^3(\text{poor}) = 48; V_{\pi_B}^3(\text{rich}) = 192; V_{\pi_B}^3(\text{poor}) = 108$$

Who wins? $\pi_A >_{h=1} \pi_B; \pi_A <_{h=3} \pi_B;$ No policy wins for *h* = 2

I.e. at least as good at all states and strictly better for at least one state

What's the value of a policy?



R(rich,plant)=100
R(poor,plant)=10
R(rich,fallow)=0
R(poor,fallow)=0

- *h*: horizon (e.g. how many gro
- $V_\pi^h(s)$: value (expected reward ing at *s*

could allow policy to depend on horizon ("non-stationary")

Dueling farmers! π_A : always plan nt if rich, else fallow

$$V_{\pi}^0(s) = 0; V_{\pi}^h(s) = R(s, \pi_h(s)) + \sum_{s'} T(s, \pi_h(s), s') \cdot V_{\pi}^{h-1}(s')$$

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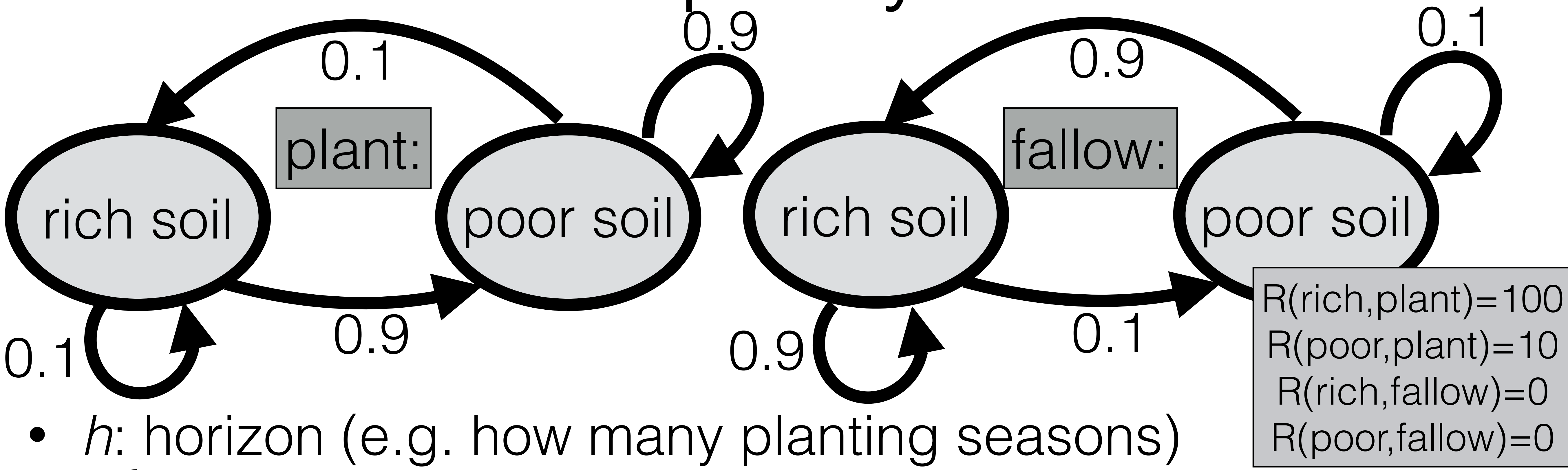
Who wins? $\pi_A >_{h=1} \pi_B ; \pi_A <_{h=3} \pi_B$ value of delayed gratification

I.e. at least as good at all states and strictly better for at least one state

What's the best policy?



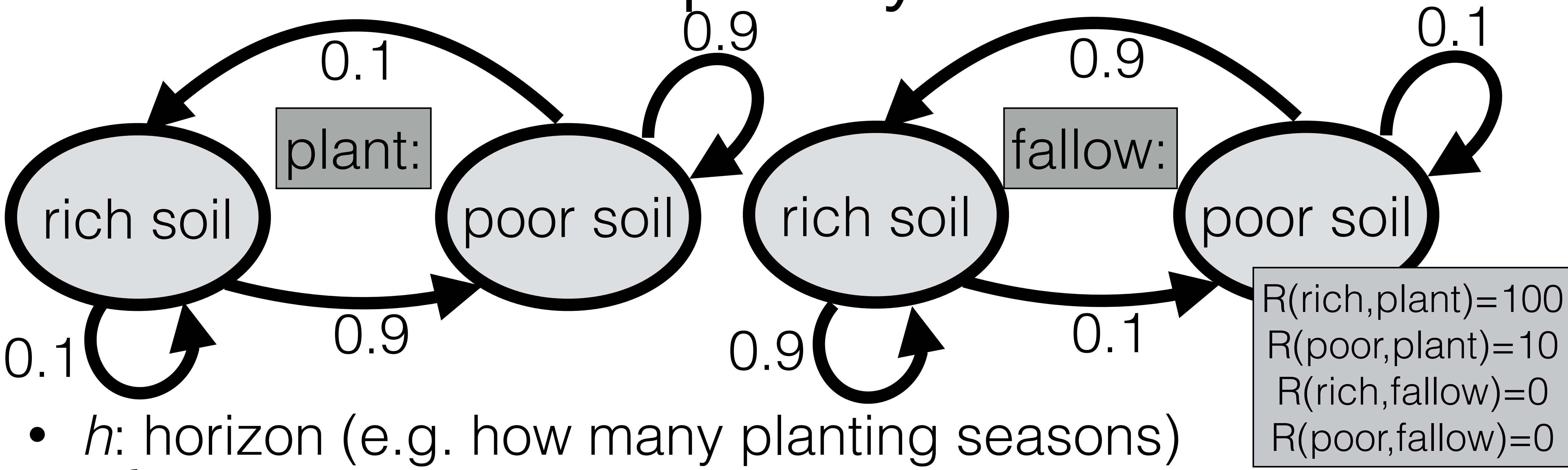
- *h*: horizon (e.g. how many planting seasons)
- $Q^h(s,a)$: expected reward of starting at *s*, making action *a*, and then making the "best" action for the *h*-1 steps left
- With *Q*, can find **an optimal policy**: $\pi^*_h(s) = \arg\max_a Q^h(s,a)$

Compare to: $V^h_\pi(s)$

Note: there can be more than one optimal policy

Note: the optimal policy may be non-stationary

What's the best policy?



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$$Q^1(\text{rich},\text{plant}) = 100; Q^1(\text{rich},\text{fallow}) = 0;$$
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$$Q^2(\text{rich},\text{plant}) = 119; Q^2(\text{rich},\text{fallow}) = 91;$$
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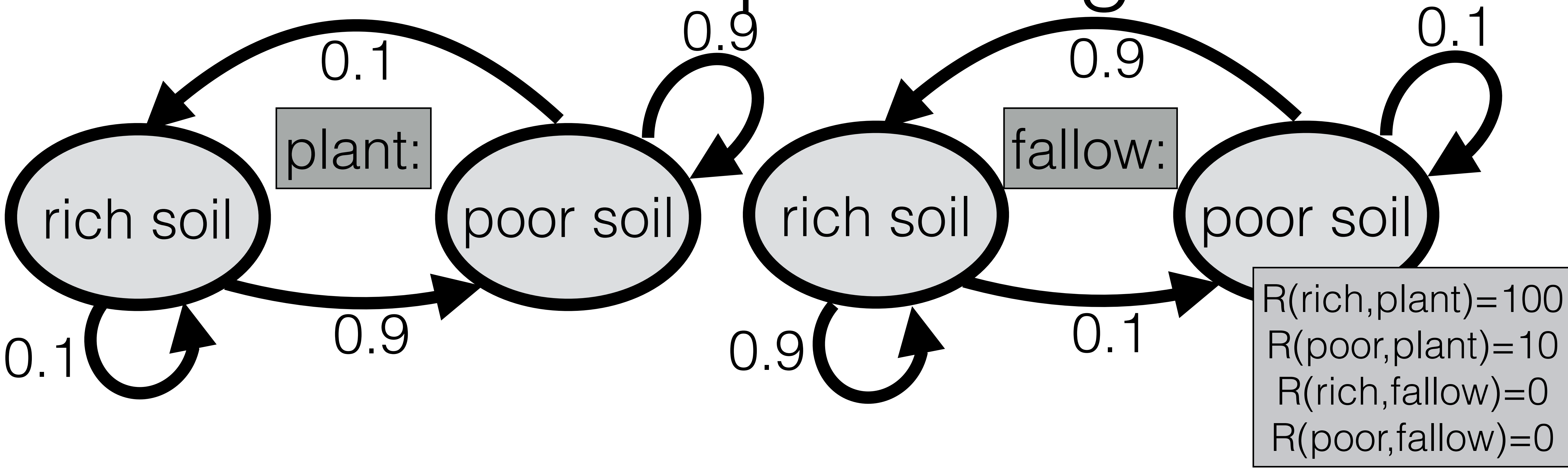
"finite-horizon value iteration"

What's best? Any *s*, $\pi_1^*(s) = \text{plant}$; $\pi_2^*(\text{rich}) = \text{plant}, \pi_2^*(\text{poor}) = \text{fallow}$

What if I don't stop farming?



- Problem: 1,000 bushels today > 1,000 bushels in ten years
  - A solution: **discount factor** $\gamma : 0 < \gamma < 1$
  - Value of 1 bushel after *t* time steps: γ^t bushels
  - Example: What's the value of 1 bushel per year forever?
    $V = 1 + \gamma + \gamma^2 + \cdots = 1 + \gamma(1 + \gamma + \gamma^2 + \cdots) = 1 + \gamma V$
    $V = 1/(1-\gamma)$ E.g. $\gamma = 0.99 \Rightarrow V = 1/0.01 = 100$ bushels
- $V_\pi(s)$: expected reward with policy π starting at state *s*

$$V_\pi(s) = R(s, \pi(s)) + \gamma \sum_{s'} T(s, \pi(s), s') V_\pi(s')$$

  - $|\mathcal{S}|$ linear equations in $|\mathcal{S}|$ unknowns

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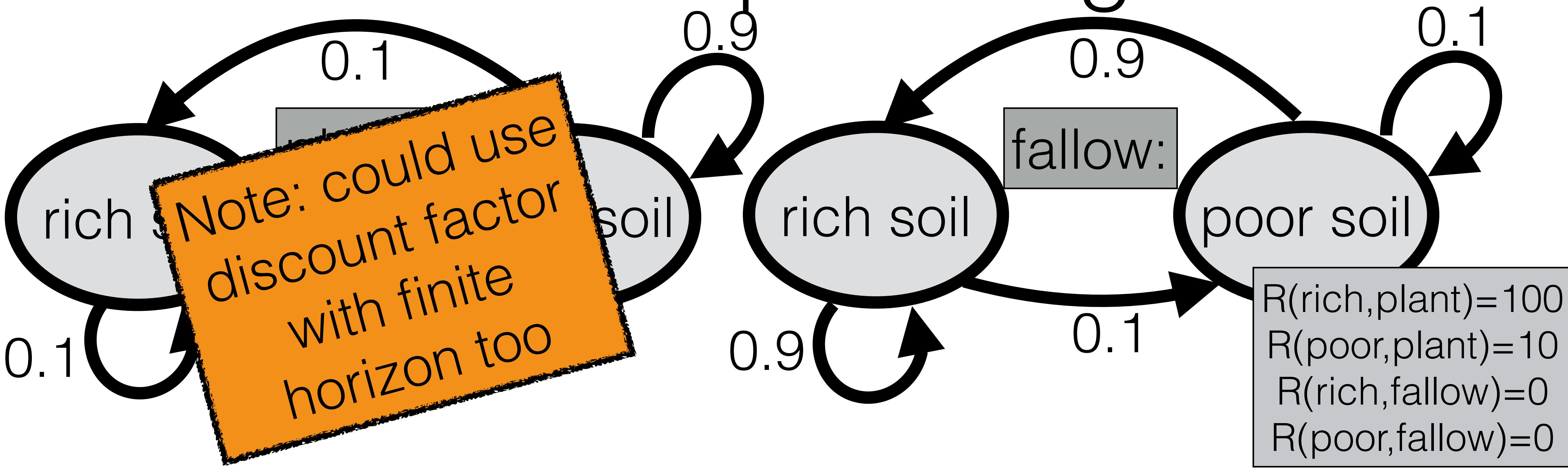

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