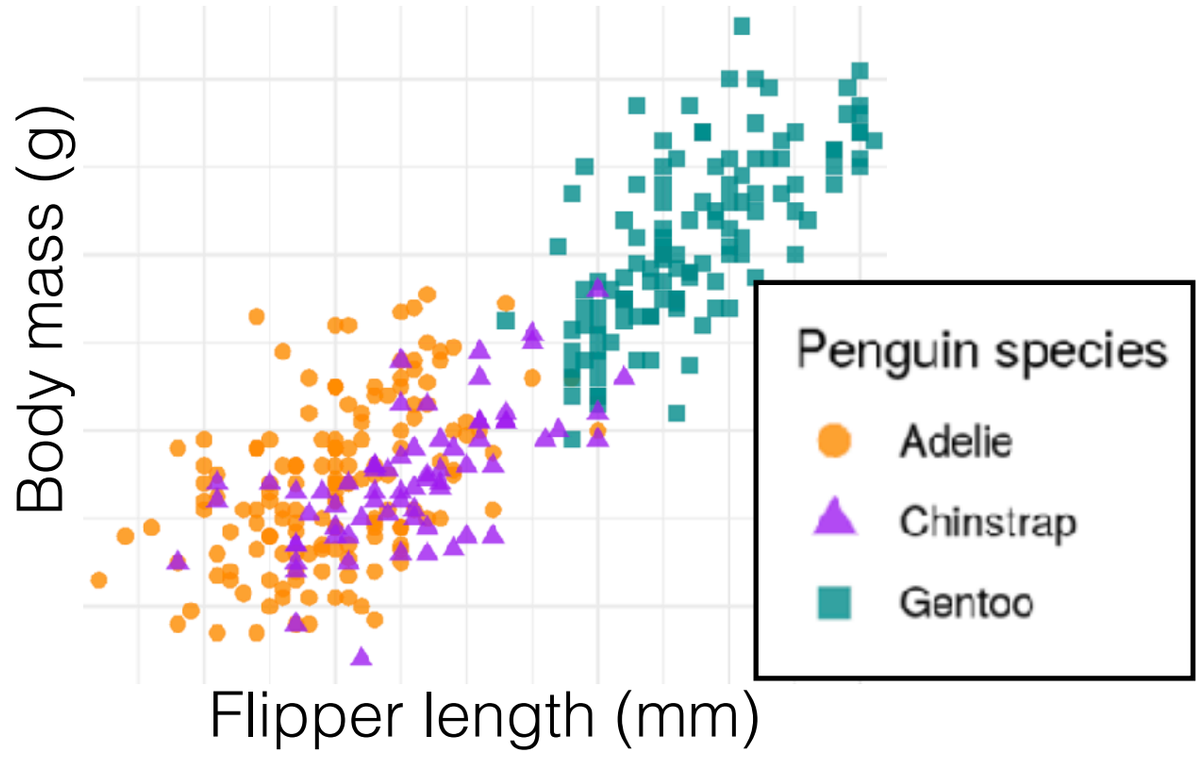
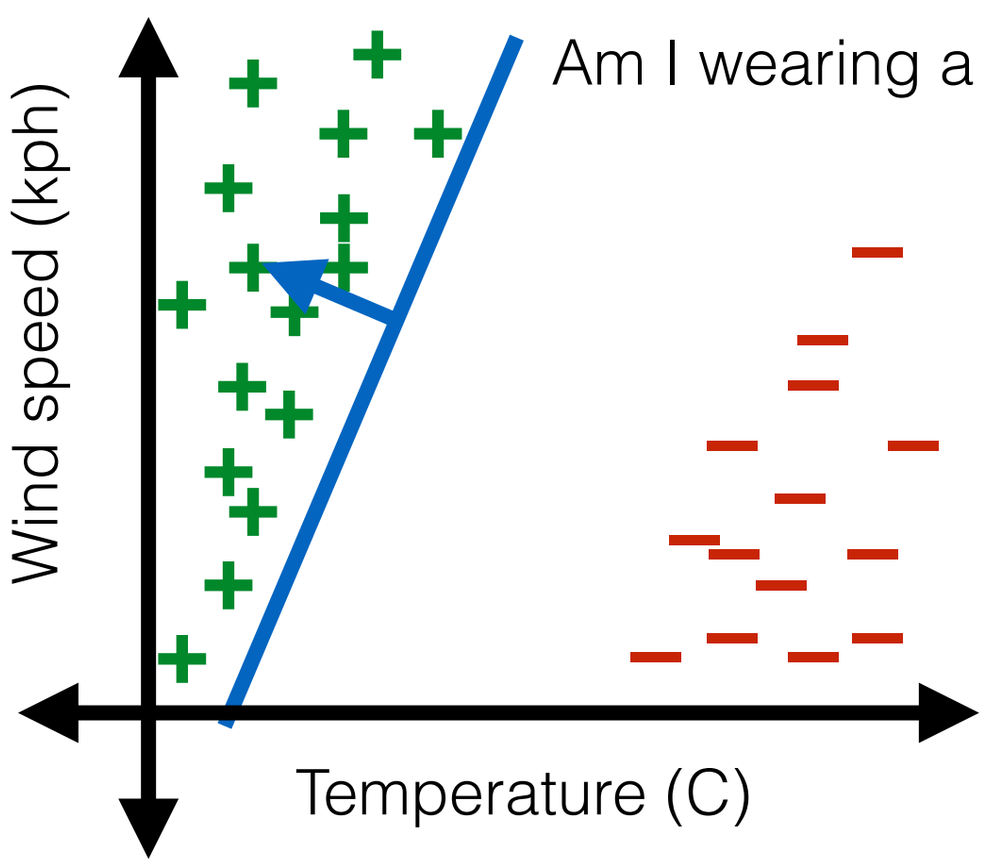
Recall

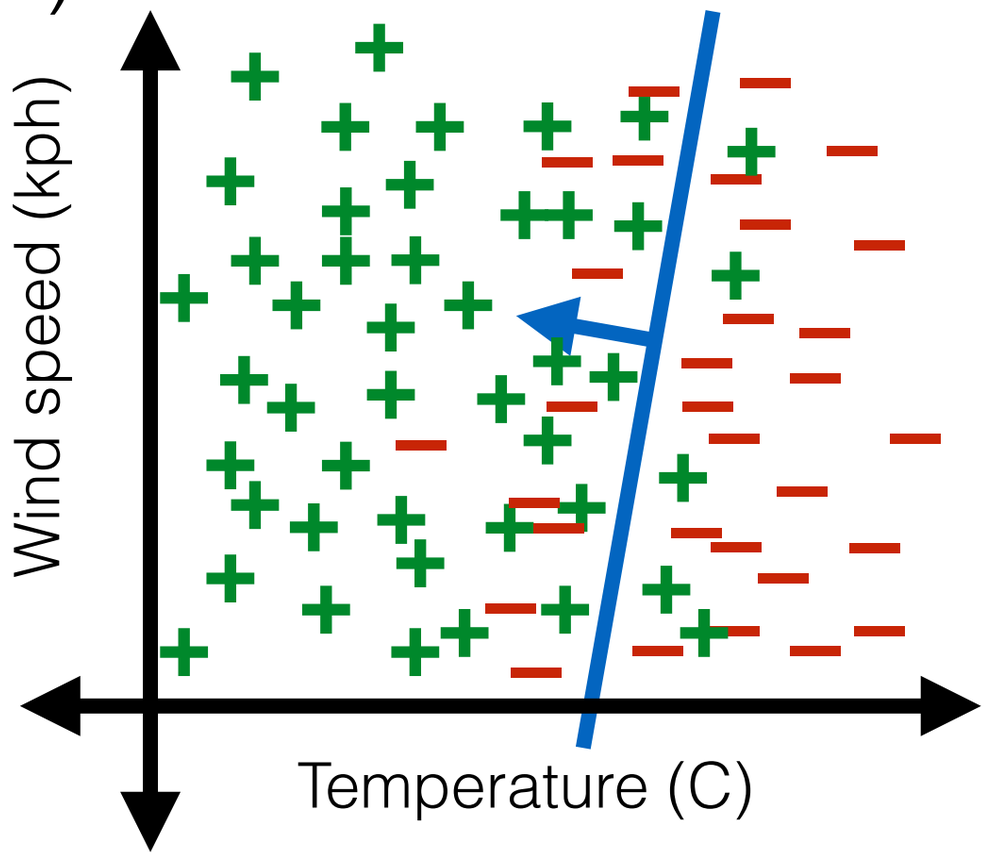
- Perceptron struggles with data that's not linearly separable



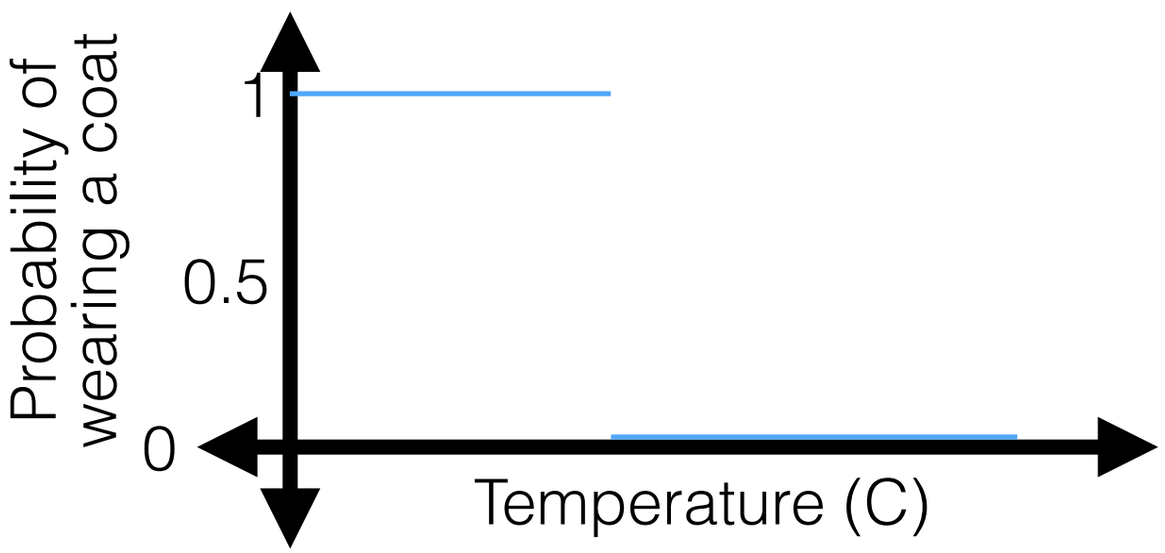
Notice

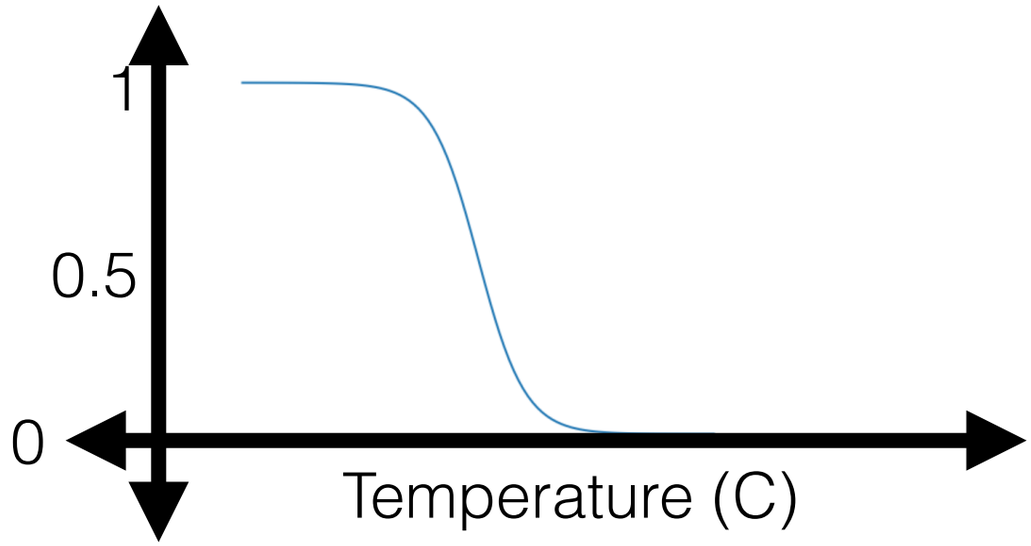
- Perceptron doesn't have a notion of uncertainty (how well do we know what we know?)

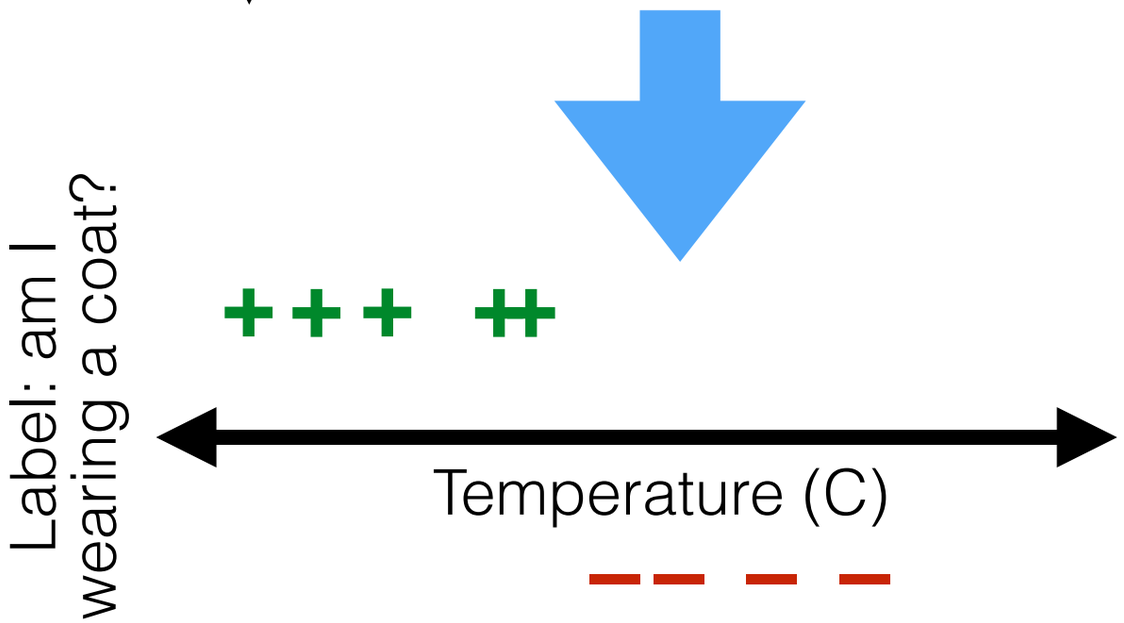


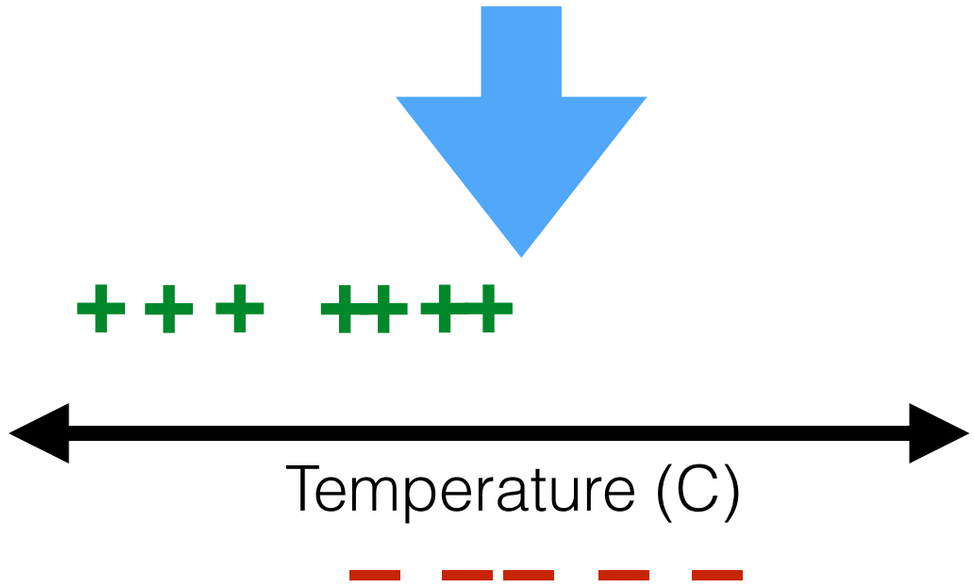


Capturing uncertainty



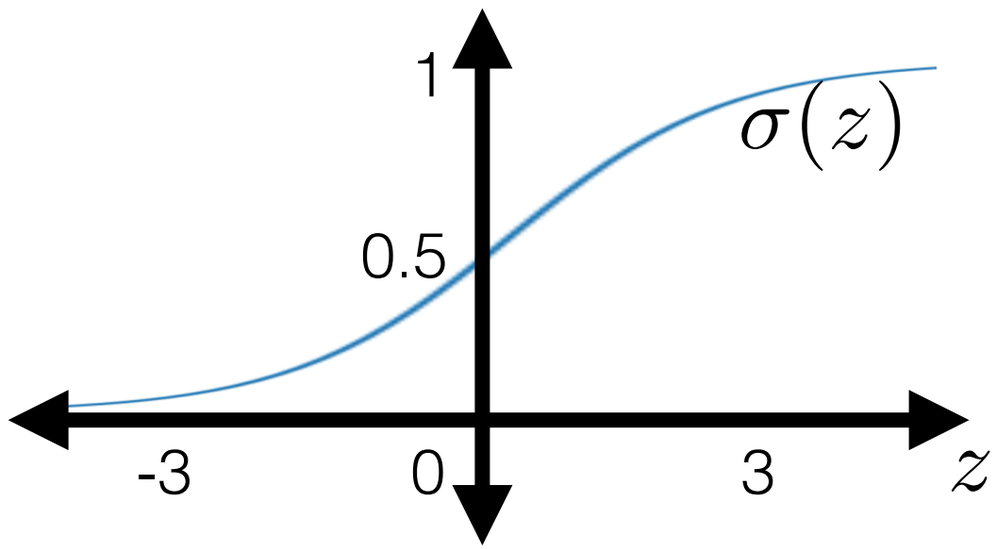






- How to make this shape?
  - Sigmoid/logistic function

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$

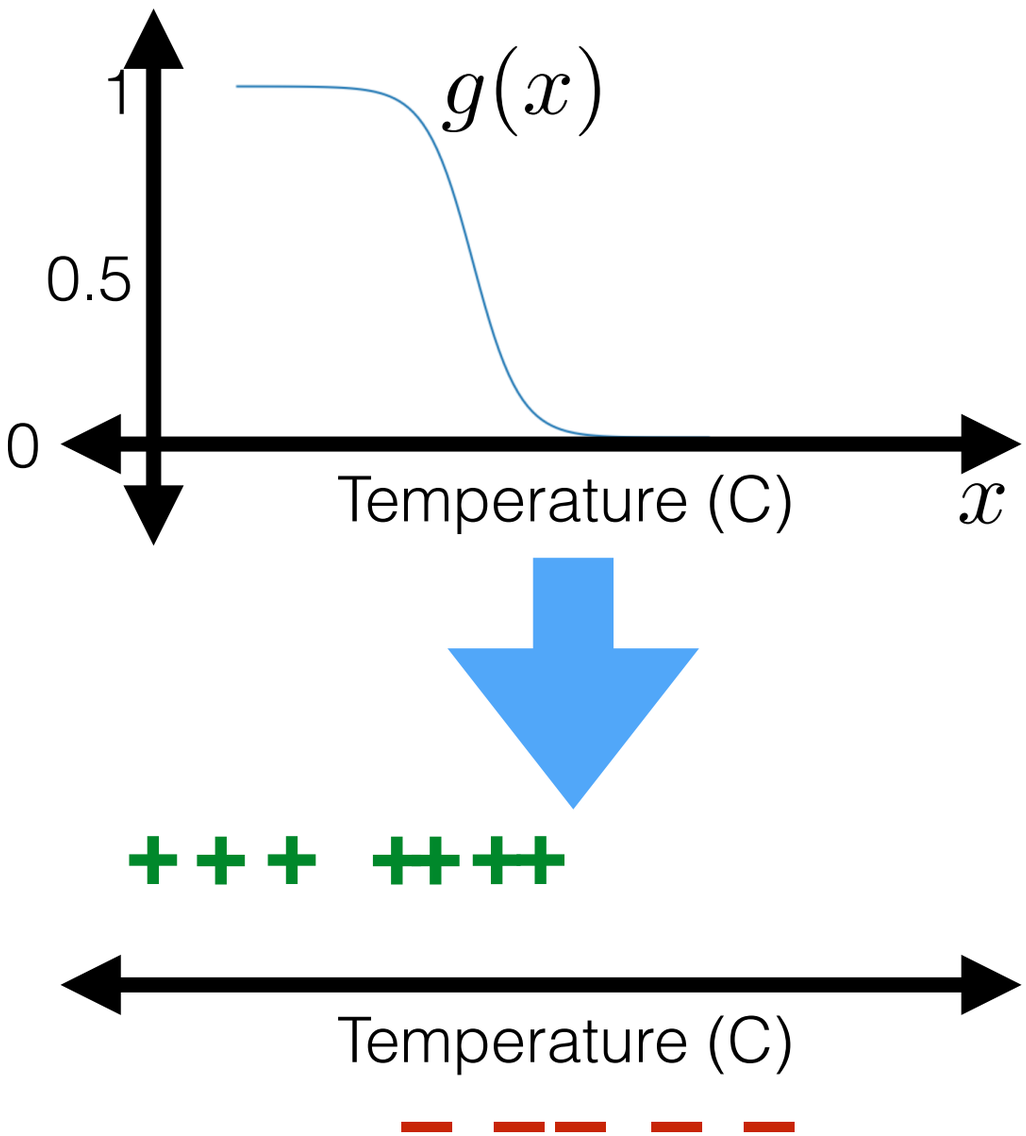


Capturing uncertainty

1 feature:

$$g(x) = \sigma(\theta x + \theta_0)$$

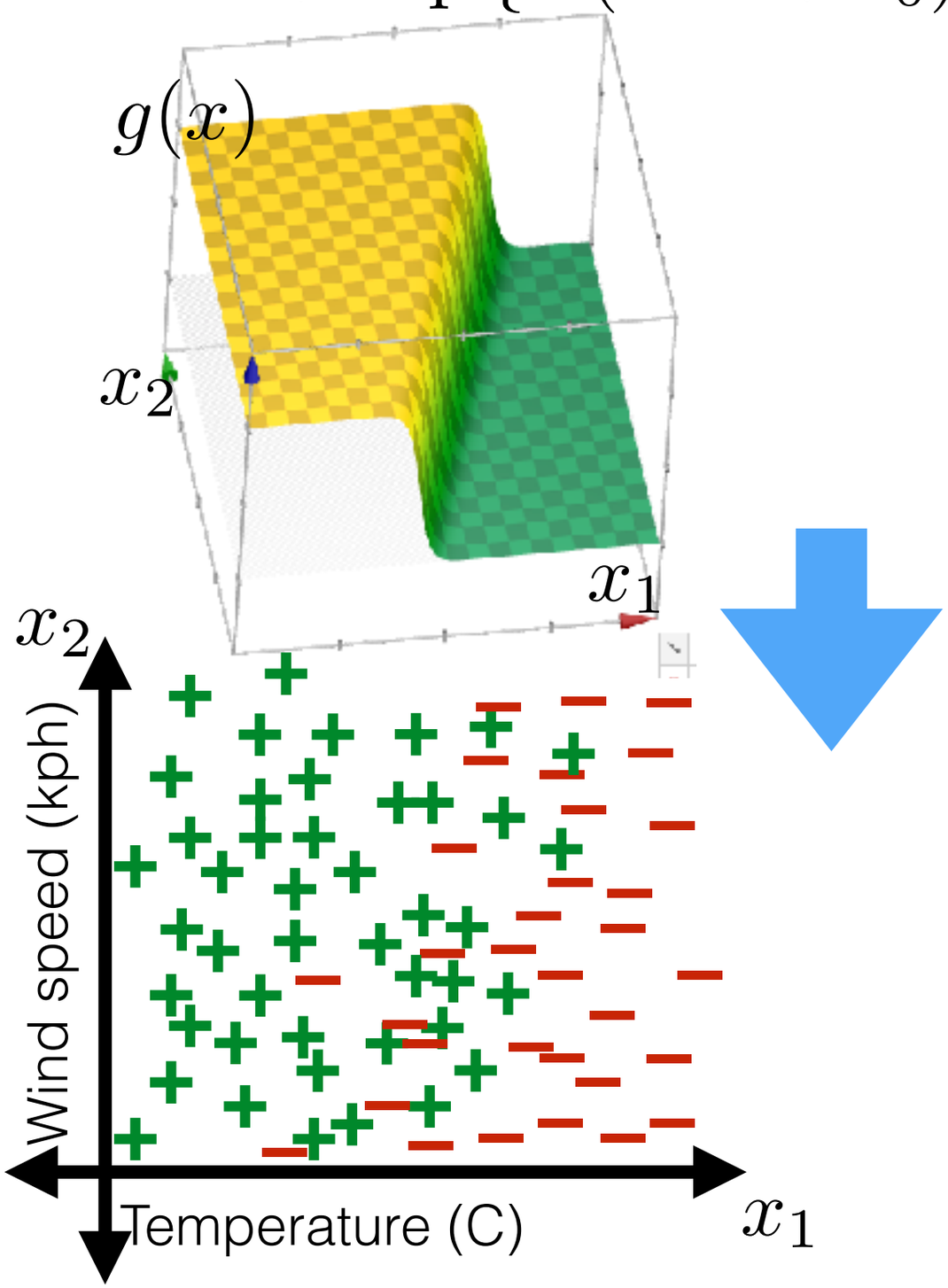
$$= \frac{1}{1 + \exp\{-(\theta x + \theta_0)\}}$$



2 features:

$$g(x) = \sigma(\theta^\top x + \theta_0)$$

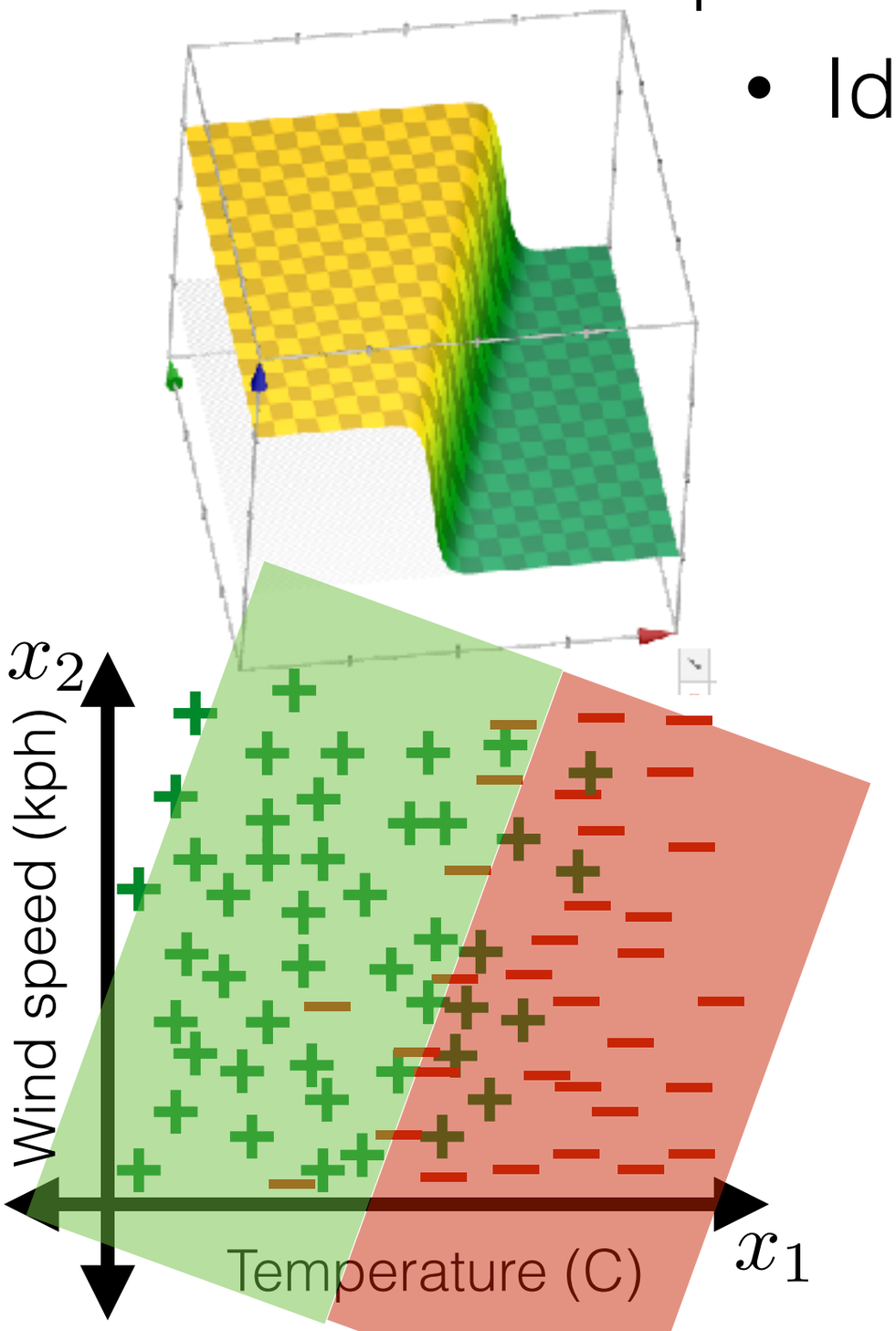
$$= \frac{1}{1 + \exp\{-(\theta^\top x + \theta_0)\}}$$



Linear logistic classification

aka logistic regression

- How do we learn a classifier (i.e. learn θ, θ_0)?
- How do we make predictions?



- Idea: predict +1 if: probability > 0.5

$$\sigma(\theta^\top x + \theta_0) > 0.5$$

$$\frac{1}{1 + \exp\{-(\theta^\top x + \theta_0)\}} > 0.5$$

$$\exp\{-(\theta^\top x + \theta_0)\} < 1$$

$$\theta^\top x + \theta_0 > 0$$

- Same hypothesis class as before! But we will get:
  - Uncertainties
  - Quality guarantees when data not linearly separable

Linear logistic classification

aka logistic regression

- How do we learn a classifier (i.e. learn θ, θ_0)?

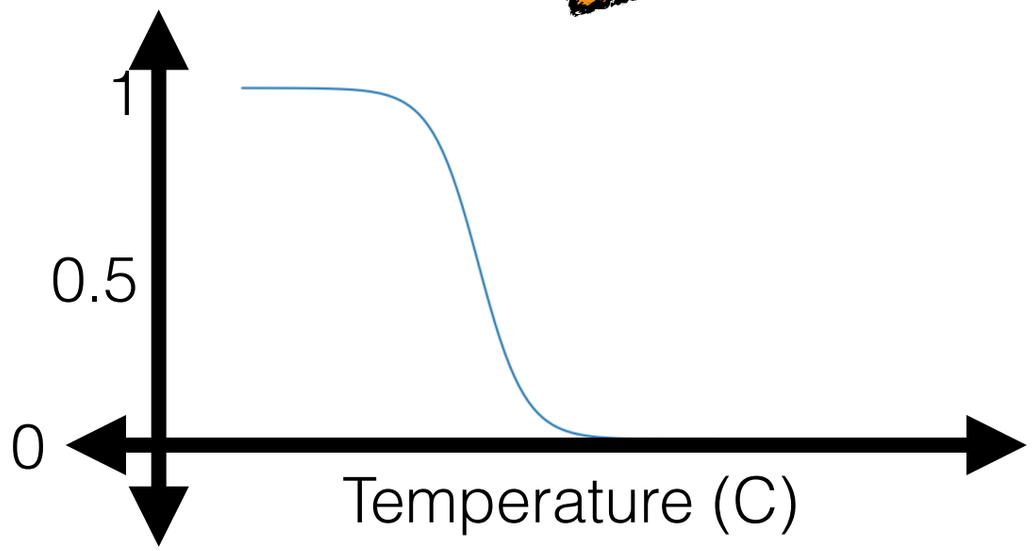
Probability(data)

$$= \prod_{i=1}^{n} \text{Probability}(\text{data point } i)$$

$$[\text{Let } g^{(i)} = \sigma(\theta^\top x^{(i)} + \theta_0)\,]$$

$$= \prod_{i=1}^{n} \begin{cases} g^{(i)} \text{ if } y^{(i)} = +1 \\ (1 - g^{(i)}) \text{ else} \end{cases}$$

$$= \prod_{i=1}^{n} (g^{(i)})^{\mathbf{1}\{y^{(i)}=+1\}} (1 - g^{(i)})^{\mathbf{1}\{y^{(i)}\neq+1\}}$$



Loss(data) = -(1/n) * log probability(data)

$$= \frac{1}{n} \sum_{i=1}^{n} -\left(\mathbf{1}\{y^{(i)} = +1\} \log g^{(i)} + \mathbf{1}\{y^{(i)} \neq +1\} \log(1 - g^{(i)})\right)$$

Negative log likelihood loss (*g* for guess, *a* for actual):

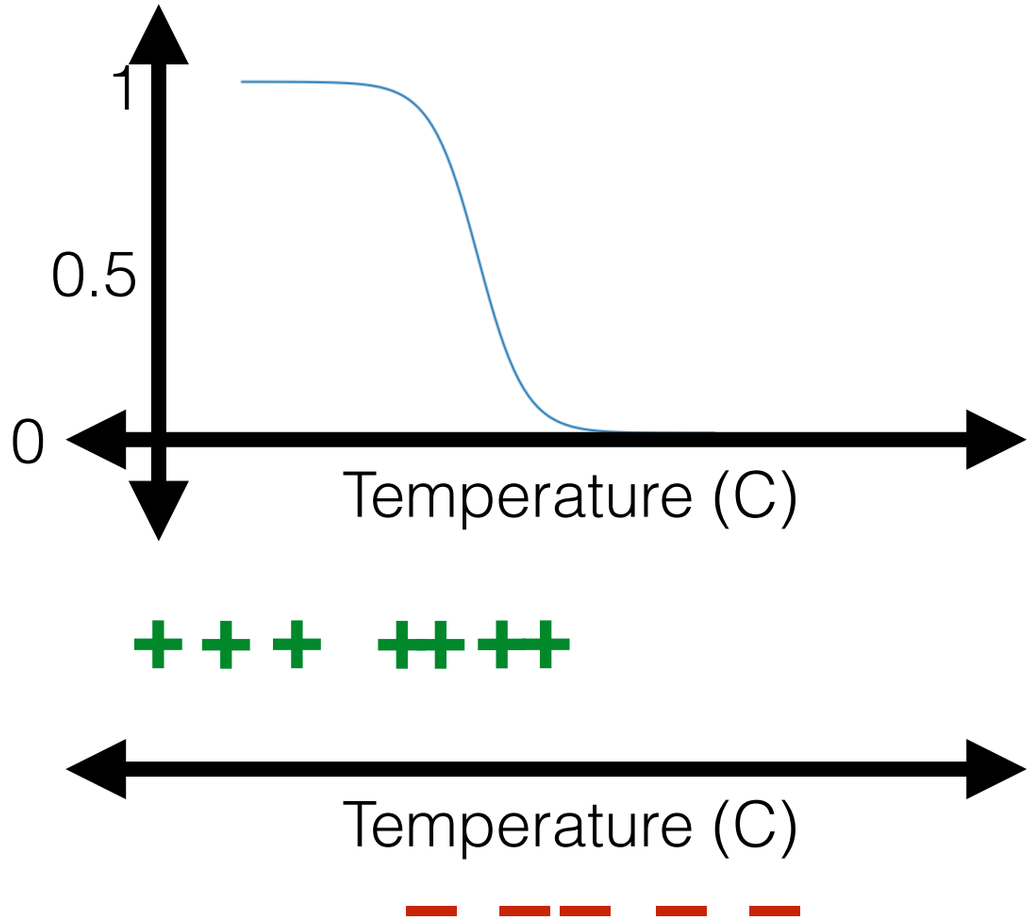
$$-L_{\text{nll}}(g, a) = (1\{a = +1\} \log g + 1\{a \neq +1\} \log(1 - g))$$

Linear logistic classification

aka logistic regression

- How do we learn a classifier (i.e. learn θ, θ_0)?
- Want to find parameter values to minimize average (negative log likelihood) loss across the data

$$J_{\text{lr}}(\Theta) = J_{\text{lr}}(\theta, \theta_0) = \frac{1}{n} \sum_{i=1}^{n} L_{\text{nll}}(\sigma(\theta^\top x^{(i)} + \theta_0), y^{(i)})$$



Gradient descent

- Gradient $\nabla_\Theta f = \left[\frac{\partial f}{\partial \Theta_1}, \ldots, \frac{\partial f}{\partial \Theta_m}\right]^\top$
  - with $\Theta \in \mathbb{R}^m$

`Gradient-Descent` $(\Theta_{\text{init}}, \eta, f, \nabla_\Theta f, \epsilon)$

`Initialize` $\Theta^{(0)} = \Theta_{\text{init}}$

`Initialize t = 0`

`repeat`

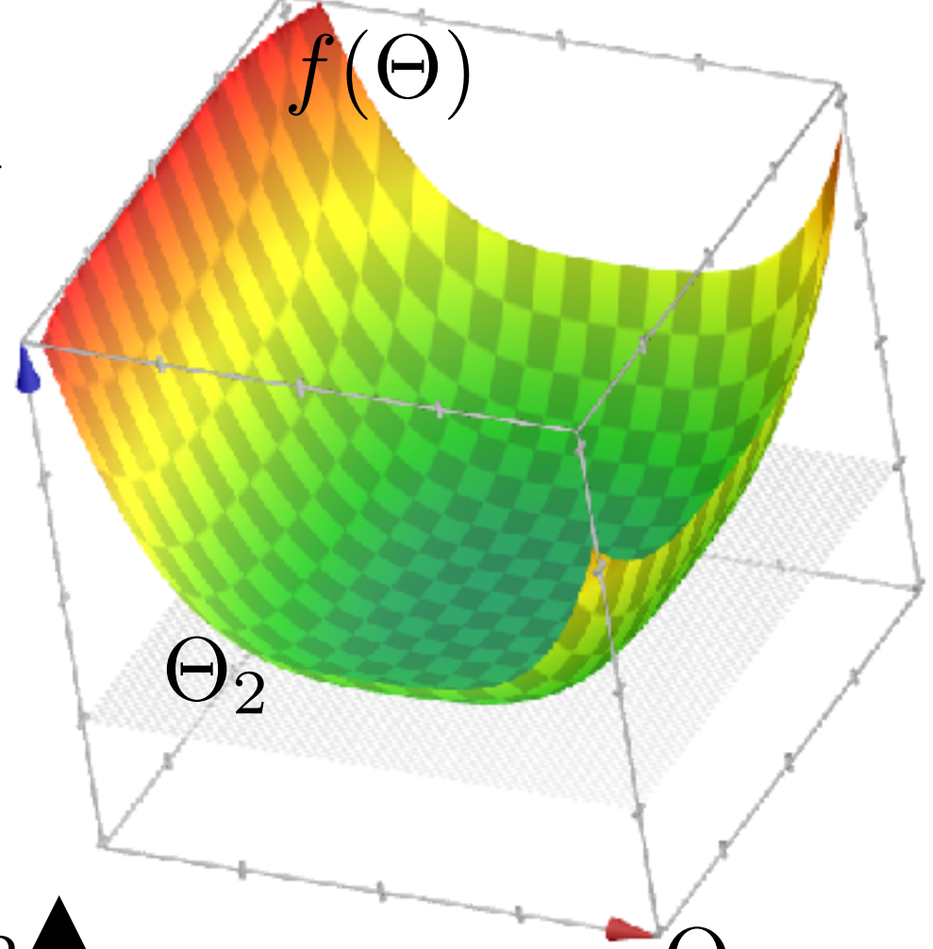
`t = t + 1`

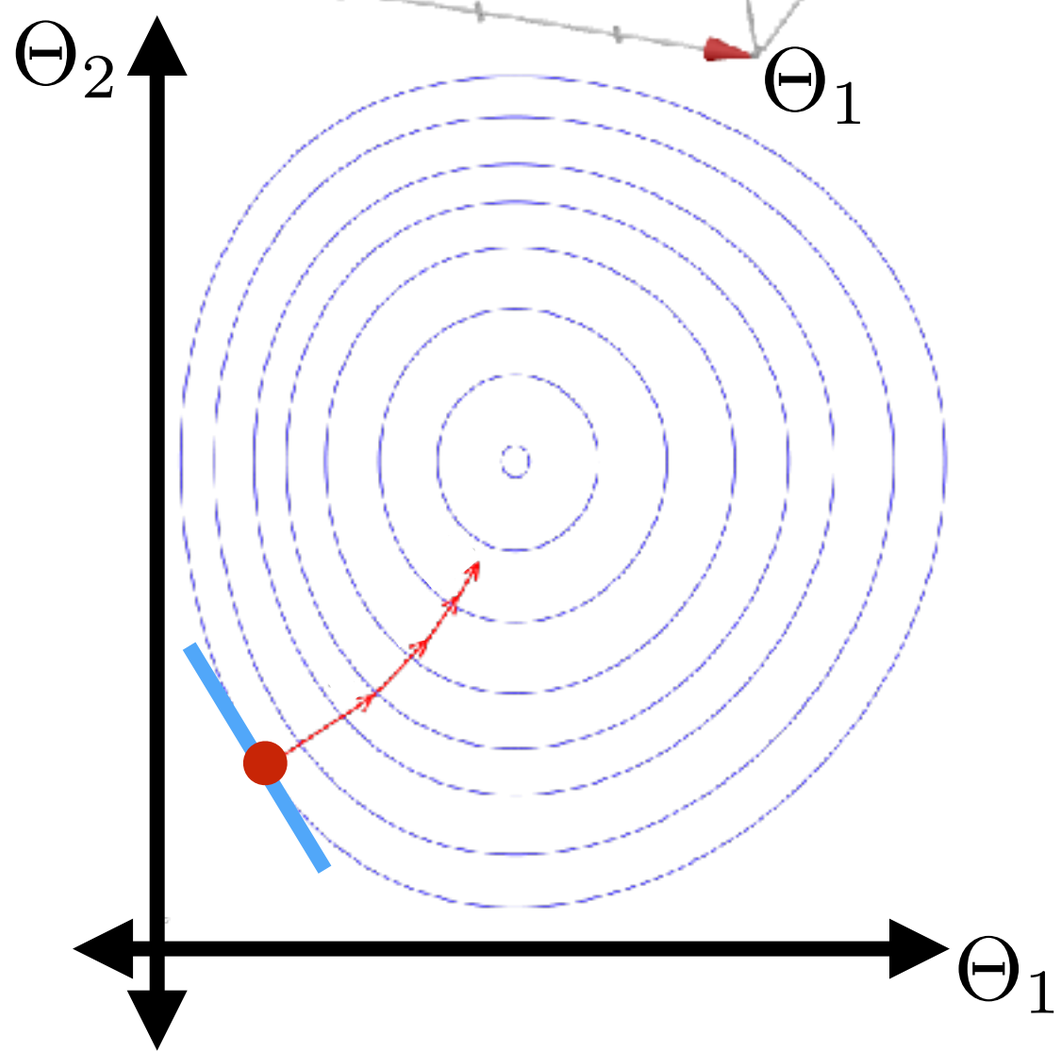
$\Theta^{(t)} = \Theta^{(t-1)} - \eta \nabla_\Theta f(\Theta^{(t-1)})$

`until` $\left| f(\Theta^{(t)}) - f(\Theta^{(t-1)}) \right| < \epsilon$

`Return` $\Theta^{(t)}$

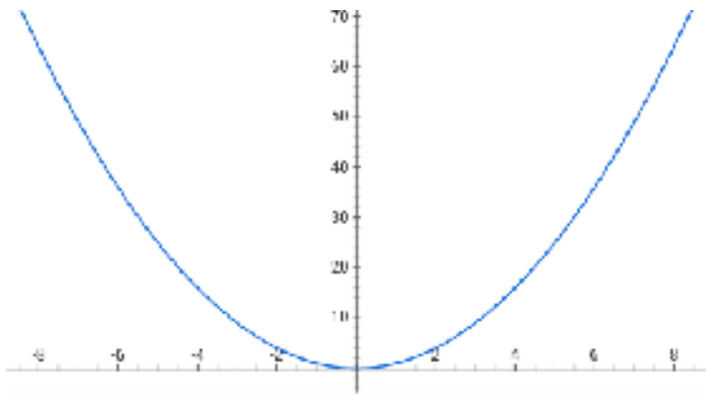
- Other possible stopping criteria:
  - Max number of iterations *T*
  - $|\Theta^{(t)} - \Theta^{(t-1)}| < \epsilon$
  - $\|\nabla_\Theta f(\Theta^{(t)})\| < \epsilon$

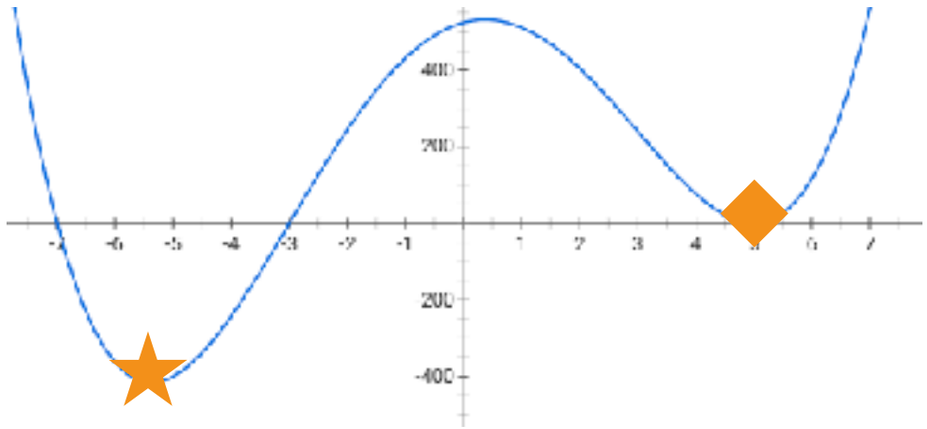


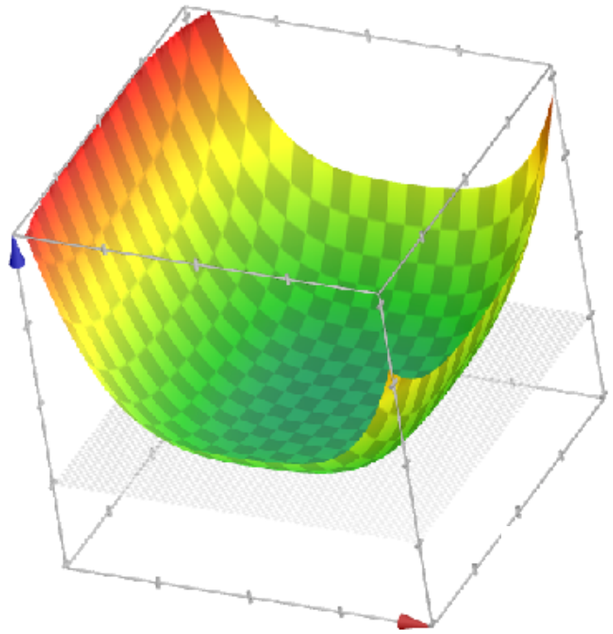


Gradient descent properties

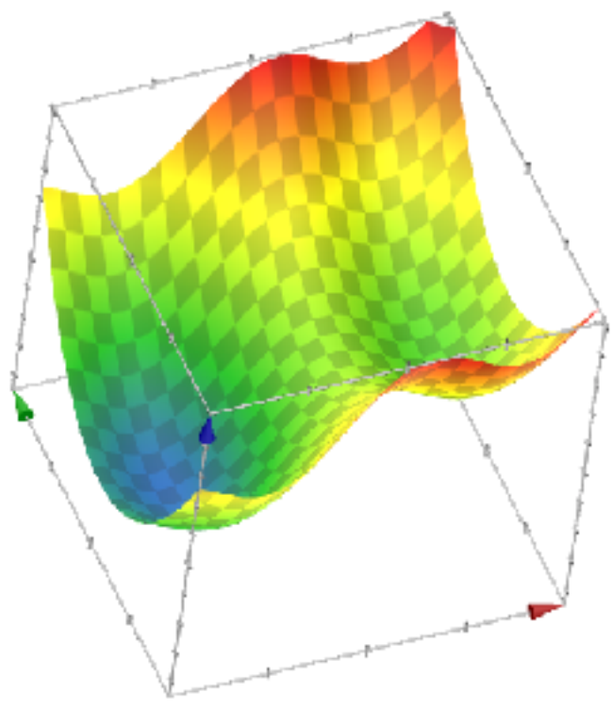
- A function *f* on $\mathbb{R}^m$ is convex if any line segment connecting two points of the graph of *f* lies above or on the graph





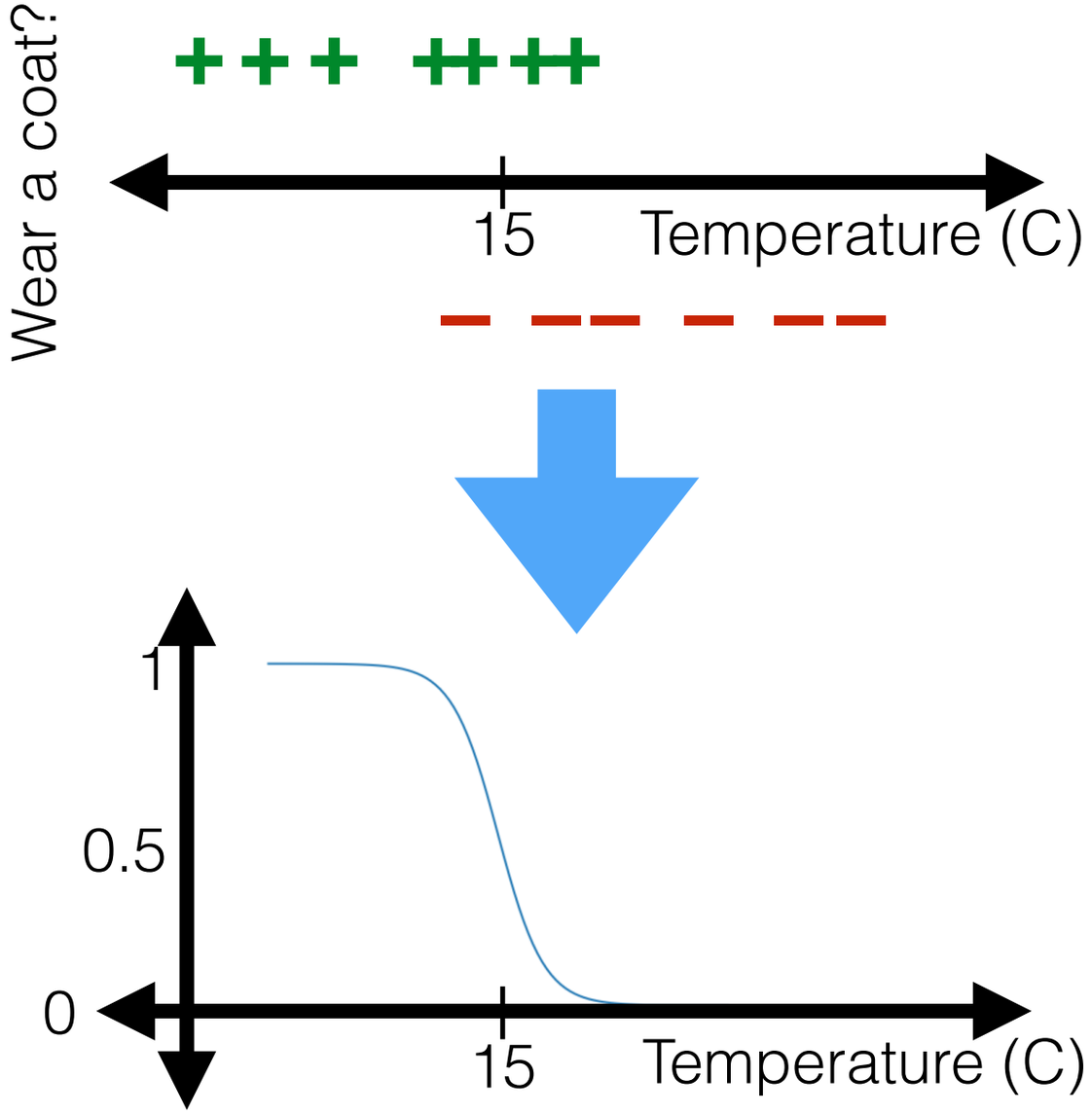


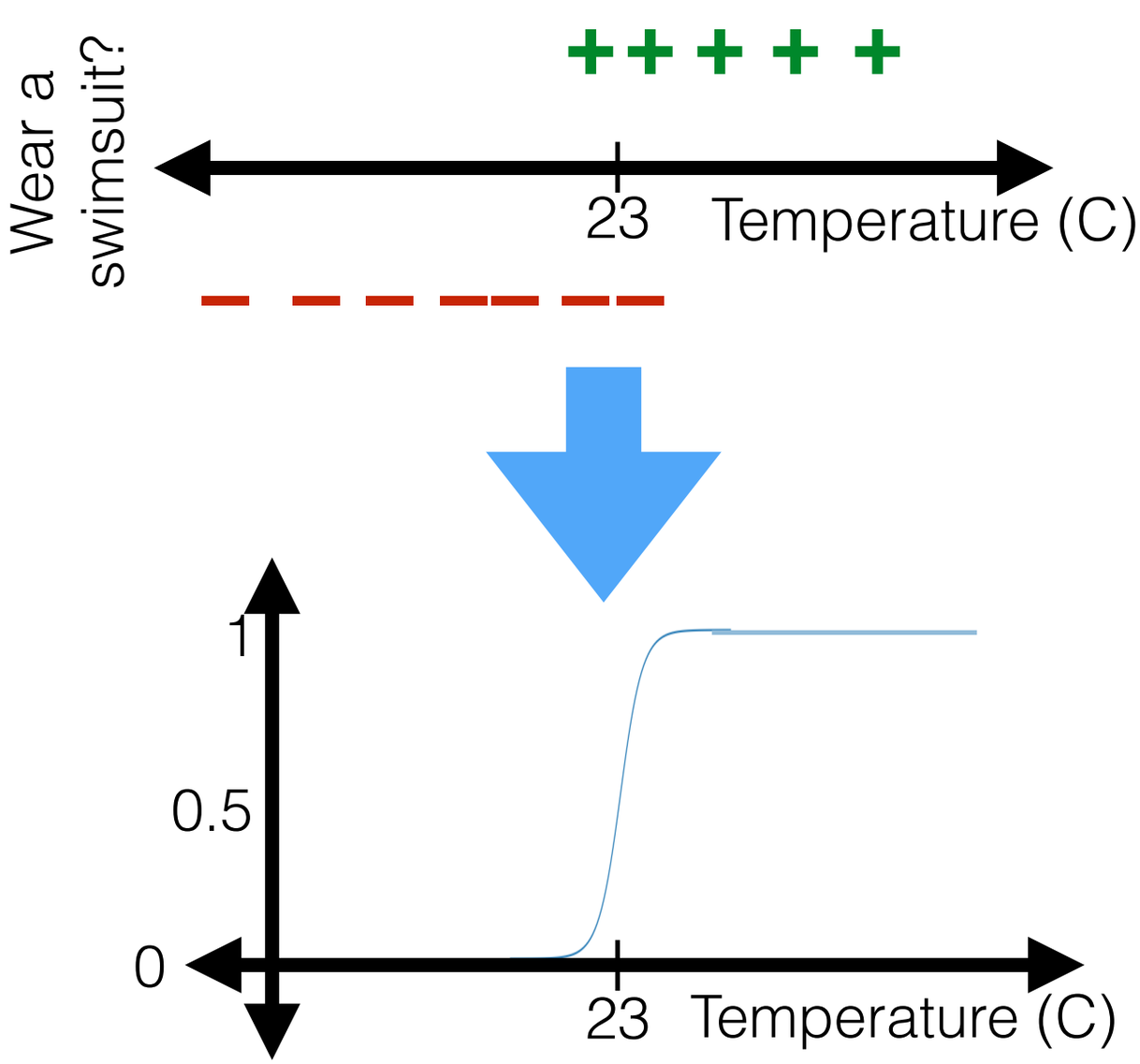
- **Theorem**: Gradient descent performance
  - **Assumptions**: (Choose any $\tilde{\epsilon} > 0$)
    - *f* is sufficiently "smooth" and convex
    - *f* has at least one global optimum
    - η is sufficiently small
  - **Conclusion**: If run long enough, gradient descent will return a value within $\tilde{\epsilon}$ of a global optimum Θ



Gradient descent for logistic regression

- Loss $J_{\text{lr}}(\Theta) = J_{\text{lr}}(\theta, \theta_0)$ is differentiable & convex
- Run `Gradient-Descent` ($\Theta_{\text{init}}, \eta, J_{lr}, \nabla_\Theta J_{lr}, \epsilon$)





Gradient descent for logistic regression

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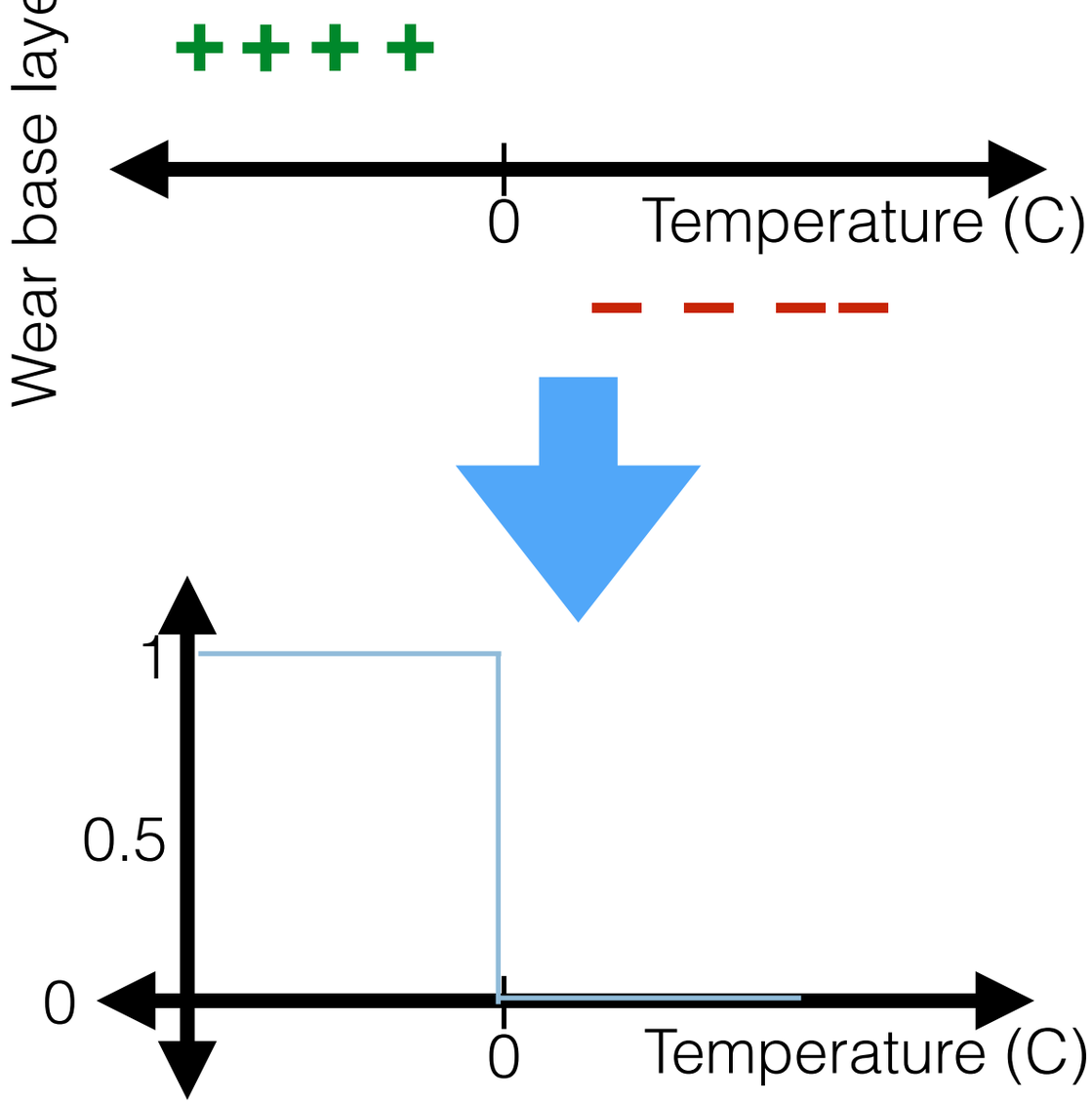
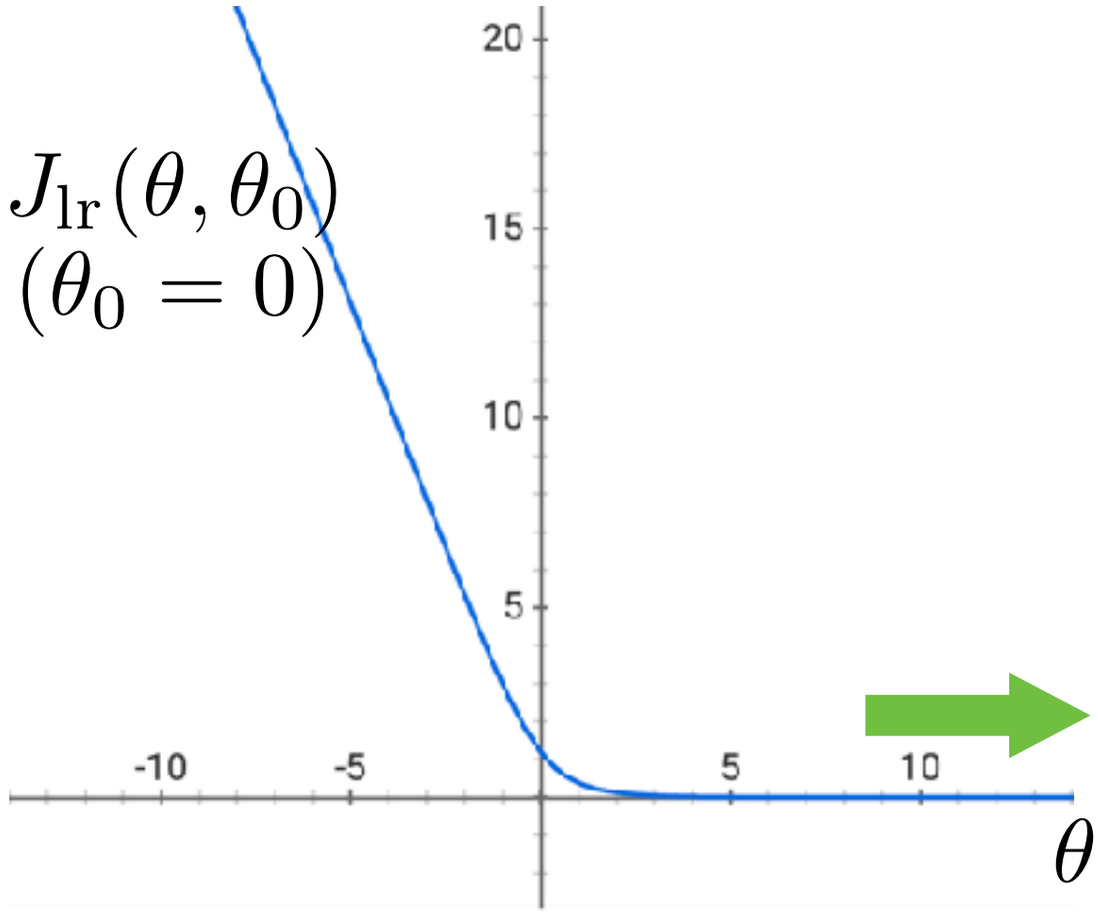
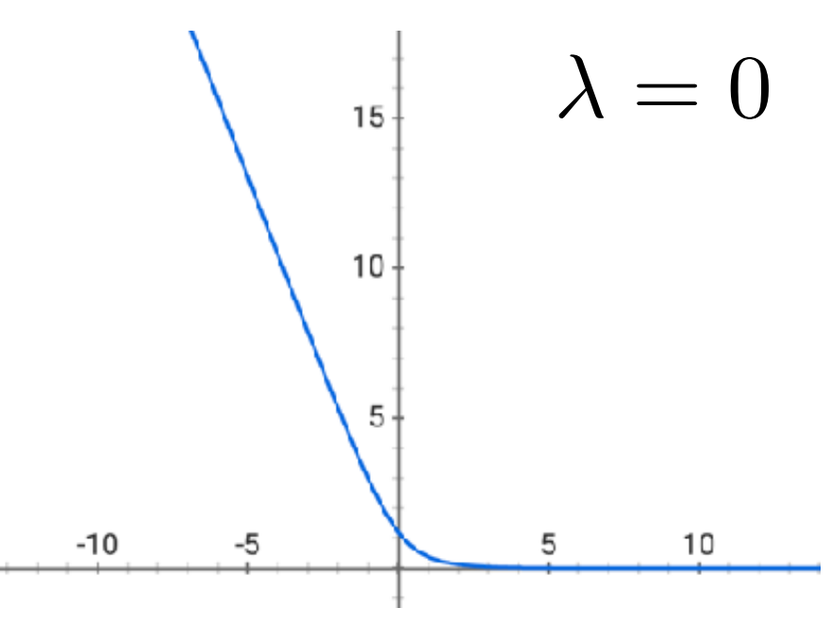


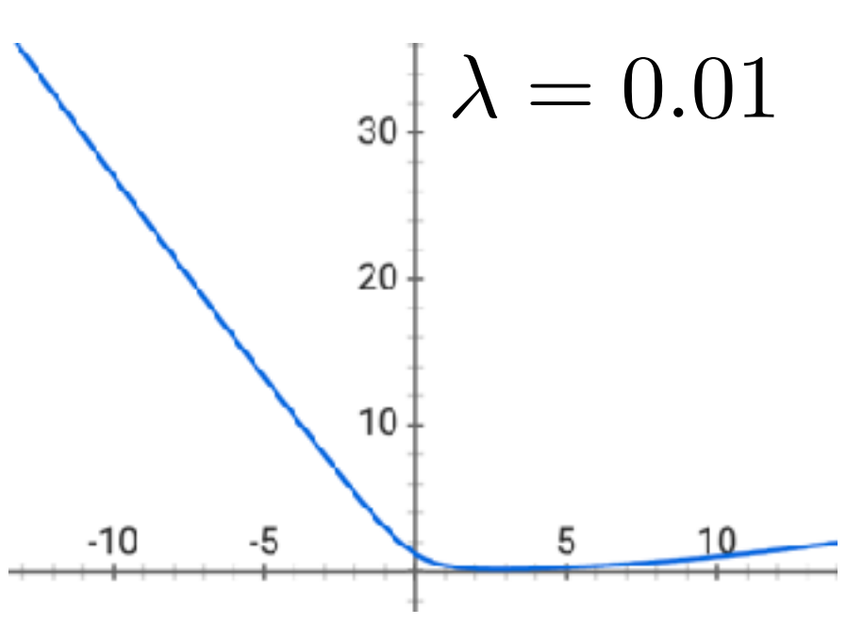


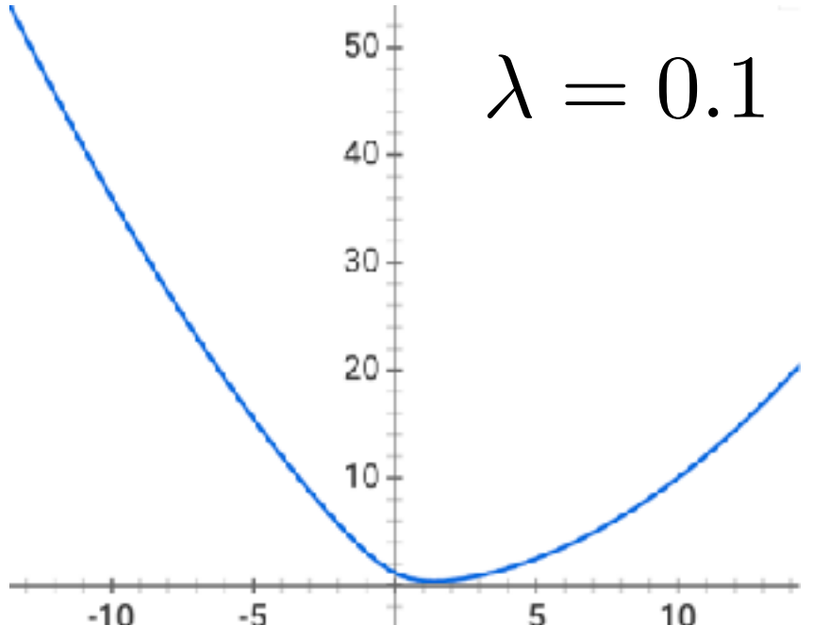
Logistic regression loss revisited

$$J_{\text{lr}}(\Theta) = J_{\text{lr}}(\theta, \theta_0)$$
$$= \frac{1}{n} \sum_{i=1}^{n} L_{\text{nll}}(\sigma(\theta^\top x^{(i)} + \theta_0), y^{(i)}) + \lambda \|\theta\|^2 \quad (\lambda \geq 0)$$

- A "regularizer" or "penalty" $R(\theta) = \lambda \|\theta\|^2$
- Penalizes being overly certain
- Objective is still differentiable & convex (gradient descent)







- How to choose hyperparameters? One option: consider a handful of possible values and compare via CV

Logistic regression learning algorithm

LR-Gradient-Descent($\theta_{\text{init}}, \theta_{0,\text{init}}, \eta, \epsilon$)

Initialize $\theta^{(0)} = \theta_{\text{init}}$

Initialize $\theta_0^{(0)} = \theta_{0,\text{init}}$

Initialize t = 0

repeat

t = t + 1

$$\theta^{(t)} = \theta^{(t-1)} - \eta\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\sigma(\theta^{(t-1)\top} x^{(i)} + \theta_0^{(t-1)}) - y^{(i)}\right] x^{(i)} + 2\lambda\theta^{(t-1)} \right\}$$

$$\theta_0^{(t)} = \theta_0^{(t-1)} - \eta\left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\sigma(\theta^{(t-1)\top} x^{(i)} + \theta_0^{(t-1)}) - y^{(i)}\right] \right\}$$

until $|J_{\text{lr}}(\theta^{(t)}, \theta_0^{(t)}) - J_{\text{lr}}(\theta^{(t-1)}, \theta_0^{(t-1)})| < \epsilon$

Return $\theta^{(t)}, \theta_0^{(t)}$

Exactly gradient descent with *f* given by logistic regression objective