# AE602A - Mathematics for Aerospace Engineers.

## Assignment - 1

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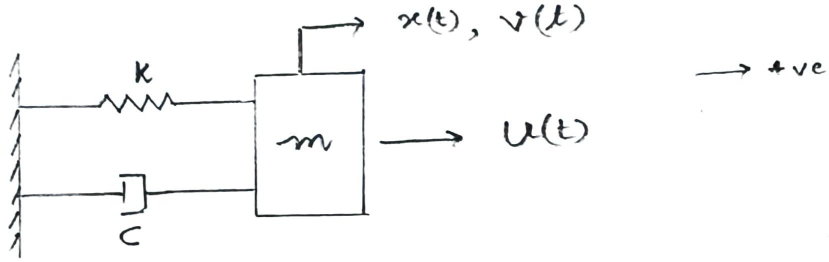



$u(t) \rightarrow$ Applied force

$x(t) \rightarrow$ Displacement

$v(t) \rightarrow$ velocity

Given values

$m = 0.003 N$

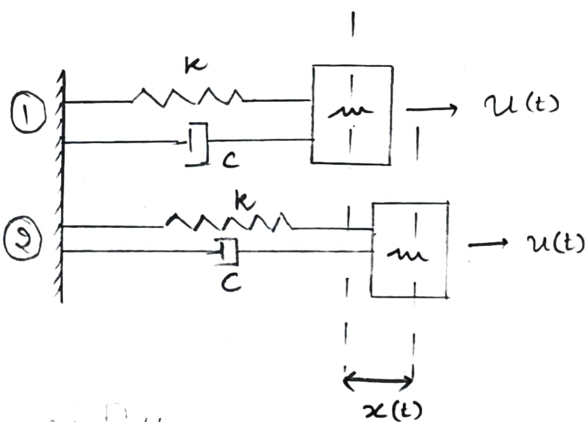
$m = \frac{0.003}{9.81}$ kg

$k = 9000 \frac{N}{m}$

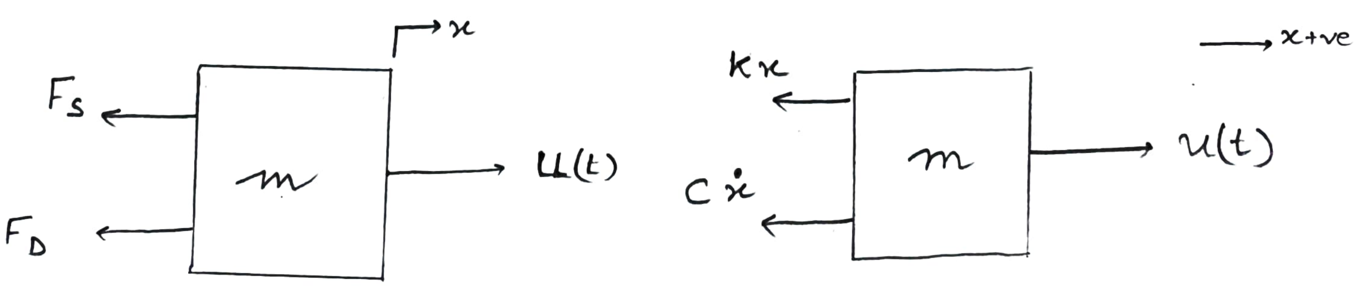
$C = 0.045 \frac{Ns}{m}$

Q 1.

Ans The free body diagram with the applied force $u(t)$.




∴ Displacement from mean position




∴ Free body diagram.

The force due to spring $F_S = kx$ (in -ve x direction)
The force due to damper $F_D = c\dot{x}$ (in -ve x direction)

Q 2

Ans

The equation of motion of mass spring damper can be given by the help of newtons 2nd law, that the net force acting on the body is equal to the product of mass and acceleration of the body.

$$\Sigma F_x = m a_x \quad \text{(in x direction)}$$

$$\Rightarrow m\dot{v}(t) = \Sigma F_x$$

$$\Rightarrow m\ddot{x}(t) = \Sigma F_x \quad \text{--- (1)}$$

By the free body diagram we can determine the net force acting on the body.

$$\Sigma F_x = \underbrace{u(t)}_{\text{Applied force}} - \underbrace{kx}_{\text{spring force}} - \underbrace{c\dot{x}}_{\text{damper force}}$$

Putting back to eqn (1).

$$\Rightarrow m\ddot{x} = u(t) - kx - c\dot{x}$$

where $x, \dot{x}, \ddot{x}$ are also function of $(t)$.

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = u$$

$$\Rightarrow m\ddot{x}(t) + c\dot{x}(t) + kx(t) = u(t) \quad \text{--- (11)}$$

where m is mass
c is damping coeff.
k is spring constant
u(t) is applied force
x(t) is displacement

$\dot{x}(t)$ is velocity $(dx/dt)$
$\ddot{x}(t)$ is accen. $(d^2x/dt^2)$

For the given set of values we can write the equation (II) as.

$$\Rightarrow \frac{0.003}{9.81}\ddot{x} + 0.045\dot{x} + 9\times10^{3}x = u(t)$$

The above equation is non homogeneous equation, where $u(t)$ is a fuction of time 't' and $u(t) \neq 0$.

Q(3).

Ans The order of the differential equation (II) is 2.

$$m\ddot{x} + c\dot{x} + kx = u$$

$$\Rightarrow \boxed{\text{order} = 2.}$$

The degree of the differential equation is 1.

$$\Rightarrow \boxed{\text{degree} = 1}$$

Q(4)

Ans It is given that $u(t) = 0$, by considering this in our diff equation, we can write as:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{——— (III)}$$

The above equation, (III) is a homogeneous equation, no applied force.

Lets solve the ODE by the power series.

We will use taylor series,

$$x(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + \dots \text{ HOT}$$

$$x(t) = \sum_{n=0}^{\infty} C_n t^n$$

on diff the above expression, we will get the expression for velocity.

$$\dot{x}(t) = \sum_{n=1}^{\infty} n C_n t^{n-1}$$

On again diff the above equation, we will get $\frac{d^2x}{dt^2}$, acceleration

$$\ddot{x}(t) = \sum_{n=2}^{\infty} n(n-1) C_n t^{n-2}$$

Putting back these expressions to equation (iii).

$$\Rightarrow m\left[\sum_{n=2}^{\infty} n(n-1) C_n t^{n-2}\right] + c\left[\sum_{n=1}^{\infty} n C_n t^{n-1}\right] + k\sum_{n=0}^{\infty} C_n t^n = 0$$

$$\Rightarrow m\left[\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2}\, t^n\right] + c\left[\sum_{n=0}^{\infty} (n+1) C_{n+1}\, t^n\right] + k\sum_{n=0}^{\infty} C_n t^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[m\left((n+2)(n+1)C_{n+2}\right) + c\,(n+1)\,C_{n+1} + kC_n\right] t^n = 0$$

$$\Rightarrow m(n+2)(n+1)\, C_{n+2} + c(n+1)\, C_{n+1} + kC_n = 0$$

$$C_{n+2} = \frac{1}{(n+1)(n+2)m}\left[-c\,(n+1)\,C_{n+1} - kC_n\right]$$

For $n=0$

$$C_2 = \frac{1}{2m}(-cC_1 - kC_0) \Rightarrow -\frac{cC_1}{2m} - \frac{kC_0}{2m} \quad \ldots\ldots (1)$$

For $n=1$

$$C_3 = \frac{1}{3\times 2\, m}\left(-c\,(2)\,C_2 - kC_1\right) \Rightarrow -\frac{2cC_2}{3!\,m} - \frac{kC_1}{3!\,m} \Rightarrow \frac{-\cancel{2}c}{3!\,m}\left[\frac{-cC_1}{\cancel{2}m} - \frac{kC_0}{\cancel{2}m}\right] - \frac{kC_1}{3!\,m}$$

$$\Rightarrow \frac{c^2C_1}{3!\,m^2} + \frac{kcC_0}{3!\,m^2} - \frac{kC_1}{3!\,m} \Rightarrow \frac{C_1}{3!}\left[\frac{c^2}{m^2} - \frac{k}{m}\right] + \frac{kcC_0}{3!\,m^2}$$

For $n=2$

$$C_4 = \frac{1}{4\times 3m}\left(-c\,(3)\,C_3 - kC_2\right)$$

$$\Rightarrow \frac{-3cC_3}{4\times 3m} - \frac{kC_2}{4\times 3m}$$

$$\Rightarrow \frac{-c}{4m}\left[\frac{c^2C_1}{3!\,m^2} - \frac{kC_1}{3!\,m} + \frac{kcC_0}{3!\,m^2}\right] + \frac{cC_1k}{4!\,m^2} + \frac{k^2C_0}{4!\,m^2}$$

$$\Rightarrow \frac{-c^3C_1}{4!\,m^3} + \frac{kcC_1}{4!\,m^2} - \frac{kc^2C_0}{4!\,m^3} + \frac{cC_1k}{4!\,m^2} + \frac{k^2C_0}{4!\,m^2}$$

$$C_4 \Rightarrow \frac{C_1}{4!}\left(\frac{-c^3}{m^3} + \frac{2kc}{m^2}\right) + \frac{C_0}{4!}\left(\frac{-kc^2}{m^3} + \frac{k^2}{m^2}\right)$$

For $n=3$

$$C_5 = \frac{C_1}{5!}\left[\frac{C^4}{m^4} - \frac{kC^2}{m^3} - \frac{kC^2}{m^3} - \frac{kC^3}{m^3} + \frac{k^2}{m^2}\right] - \frac{C_0}{5!}\left[\frac{kC^3}{m^4} - \frac{k^2C}{m^3} - \frac{k^2C}{m^3}\right]$$

$$C_5 = \frac{C_1}{5!}\left[\frac{C^4}{m^4} - \frac{3kC^2}{m^3} + \frac{k^2}{m^2}\right] - \frac{C_0}{5!}\left[\frac{kC^3}{m^4} - \frac{2k^2C}{m^3}\right]$$

$$\Rightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n t^n$$

$$\Rightarrow \quad x(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4$$

$$\Rightarrow \quad x(t) \Rightarrow C_0 + C_1 t + -\frac{CC_1}{2m} t^2 - \frac{C_0 k}{2m} t^2 + \frac{C_1}{3!}\left(\frac{c^2}{m^2} - \frac{k}{m}\right) t^3 + \frac{kCC_0}{3!m^2} t^3 + \dots$$

$$\Rightarrow x(t) = C_0\left[1 - \frac{k}{2m} t^2 + \frac{kC}{3!m^2} t^3 + \dots\right] + C_1\left[t - \frac{Ct^2}{2m} + \left(\frac{C^2}{m^3} - \frac{k}{m}\right)\frac{t^3}{3!} + \dots\right]$$

If given, initial value such as $x(0)$ = constant

this means at $t \to 0$

By the boundary condition $\quad \boxed{\dot{x}(0) \longrightarrow 0}$

$\underline{C_1 = 0}$

$$\Rightarrow x(t) = C_0\left[1 - \frac{k}{2m} t^2 + \frac{kC}{3!\,m^2} t^3 + \dots\right]$$

Let us solve the equation (III) by the exponential function

let $x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

again diff. with respect to $t$.

acceleration $= \frac{d^2x}{dt^2} = \ddot{x}(t) = \lambda^2 e^{\lambda t}$

Putting back these values to equation (3) we will get.

$$m(\lambda^2 e^{\lambda t}) + C(\lambda e^{\lambda t}) + k(e^{\lambda t}) = 0$$

Taking common $e^{\lambda t}$ from every element.

$$e^{\lambda t}\left[m\lambda^2 + C\lambda + k\right] = 0 \quad , \quad e^{\lambda t} \neq 0.$$

Thus this means,

$$\Rightarrow \boxed{m\lambda^2 + C\lambda + k = 0}$$

we will get a quadratic equation in terms of $\lambda$.

By solving the quadratic equation

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda_1, \lambda_2 \Rightarrow \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\Rightarrow -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

At given values, $m = 3.0581 \times 10^{-4}$ kg

$c = 0.045 \ \frac{Ns}{m}$

$k = 9000 \ \frac{N}{m}$

$$\lambda_1, \lambda_2 = \frac{-0.045}{2[3.0581 \times 10^{-4}]} \pm \sqrt{\left(\frac{0.045}{2 \times 3.0581 \times 10^{-4}}\right)^2 - \left(\frac{9000}{3.0581 \times 10^{-4}}\right)}$$

$$\lambda_1, \lambda_2 = -73.57509 \pm \sqrt{5413.2 - 2.943 \times 10^{7}}$$

$$\Rightarrow -73.575 \pm 5424.4469\, i \longrightarrow \alpha \pm i\beta$$

where $\alpha = -73.575$ real

$\beta = 5424.4469$ imagery

The sol$^n$ of diff eq$^n$ are

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$x(t) = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$

$$x(t) = c_1 e^{\alpha t} \cdot e^{i\beta t} + c_2 e^{\alpha t} \cdot e^{-i\beta t}$$

$$x(t) = e^{\alpha t}\left[(c_1 + c_2)\cos\beta t + (c_1 - c_2)\, i \sin\beta t\right]$$

$$x(t) = e^{-73.575t}\left[(c_1 + c_2)\cos(5424.4469\, t) + (c_1 - c_2)\sin(5424.4469\, t)\right]$$

Q 5.

Ans The equation (III) can be represented in a matrix form.

$$m\ddot{x} + c\dot{x} + kx = 0$$

let
$$\left.\begin{aligned} x_1 &= x \\ \dot{x}_1 &= x_2 = \dot{x} \\ \dot{x}_2 &= \ddot{x}_1 = \ddot{x} \end{aligned}\right\} \Rightarrow \text{putting back these values to eq}^n.$$

The homogenous equation of mass spring damper

$$m\dot{x}_2 + c\dot{x}_1 + kx_1 = 0 \qquad \Rightarrow \quad \dot{x}_1 = x_2$$

$$\Rightarrow \quad \dot{x}_2 = \frac{-c\dot{x}_1}{m} - \frac{kx_1}{m}$$

By these two above expression

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{X} = AX$$

where $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}$

we can represent as state space representation

$$\dot{X} = AX$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q6

Ans The root of system matrix in the state space formed in quation (5).

$$\dot{X} = AX$$

$$X(t) = \phi(t)\, x(0)$$

We can write $\phi(t)$ as.

$$\phi(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

$$\dot{X}(t) = \frac{d}{dt}[X(t)] = \frac{d}{dt}[\phi(t) \cdot x(0)] = AX$$

$$(a_1 + 2a_2 t + 3a_3 t^2 + \dots)x(0) = [A + Aa_1 t + Aa_2 t^2 + Aa_3 t^3 + \dots]$$

on comparing both the sides

$$\boxed{a_1 = A} \qquad \boxed{a_3 = \frac{A^3}{3!}}$$

$$2a_2 = Aa_1$$

$$\boxed{a_2 = \frac{A^2}{2!}} \qquad \boxed{a_k = \frac{A^k}{k!}}$$

$$\phi(t) = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$$

$$\boxed{\phi(t) = e^{At}}$$

$$\Rightarrow \boxed{x(t) = e^{At} \cdot x(0)}$$

The root of any matrix can be given by its eigen values, of the matrix.

$\Rightarrow$ $|A - \lambda I| = 0$ char. equation
where the $I$ is identity matrix.

$$\Rightarrow \begin{vmatrix} 0-\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = 0 \qquad \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda\left(\frac{c}{m} + \lambda\right) + \frac{k}{m} = 0$$

$$\Rightarrow c\lambda + m\lambda^2 + k = 0$$

$$\Rightarrow m\lambda^2 + c\lambda + k = 0$$

By solving the quadratic eq^n we can calculate the value of eigenvalues, roots of the matrix.

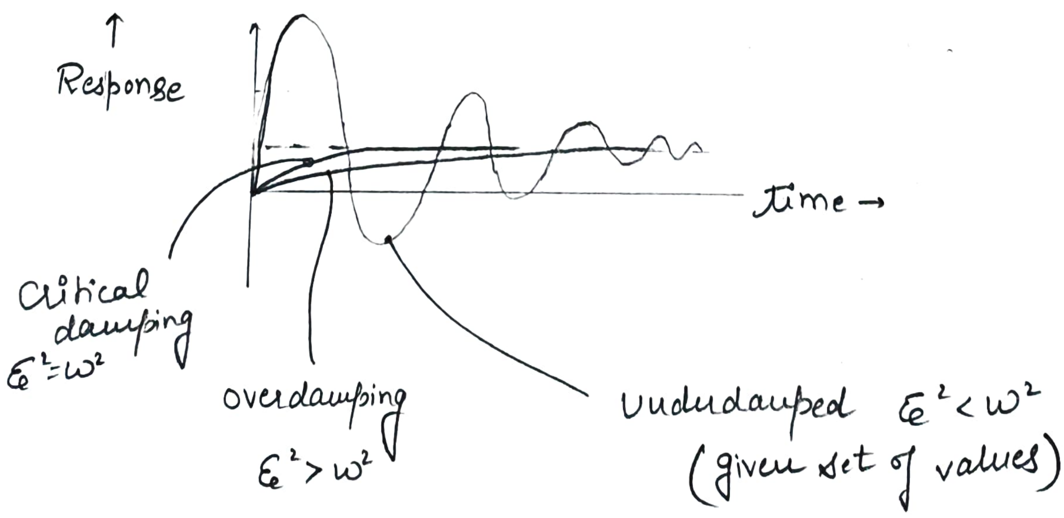
$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\lambda_1, \lambda_2 = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$\lambda_1, \lambda_2 = -\varepsilon_e \pm \sqrt{\varepsilon_e^2 - \omega^2}$$

$$\Rightarrow \lambda_1, \lambda_2 = -73.57509 \pm 5424.4469\, i$$

$\therefore$ on putting the values of $m, c, k$



# MATLAB

mass spring damper system

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$ , homogenous equation where applied force is zero.

$u(t) = 0$

boundary condition → –

Initial condition → $x(0) \rightarrow 1$

$\dot{x}(0) \rightarrow 0 = v(0)$

$$\frac{dx}{dt} = v$$

$$\Rightarrow m\frac{dv}{dt} + cv + kx = 0$$

$$y = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\frac{d}{dt}(y) = \begin{bmatrix} v \\ \frac{-(cv + kx)}{m} \end{bmatrix}_{2\times 1}$$

Q7. Plot the time response of x(t) for a step input in MATLAB.

ANS. MATLAB code:

```
%Name:- ANUBHAV JOSHI, Roll no:-21101008
%mass spring damper system
y0=[1;0]; % considering the initial value x(0)=1,v(0)=0
tspan=[0 0.1];
[tsol,ysol]=ode45(@(t,y) MSDfun(t,y), tspan, y0);
plot(tsol,ysol(:,1),'k');
xlabel('time t');
ylabel('response x(t)');
hold on;

%% Function Mass spring damper
function fval=MSDfun(t,y)
x=y(1);
v=y(2);
c=0.045;
k=9000;
m=0.003/9.81;
fval(1,1)=v;
fval(2,1)=-(c*v+k*x)/m;
end
```

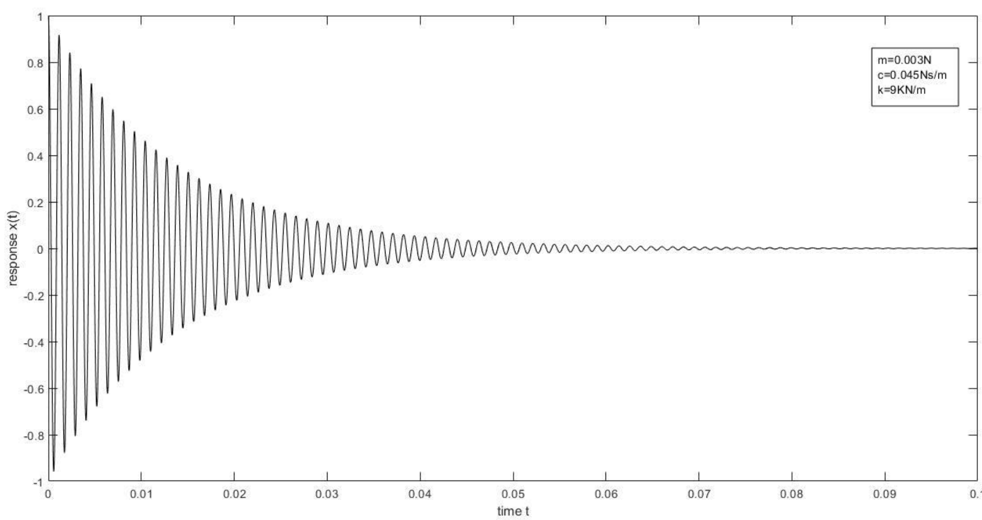



FIG.1 Response of displacement ‘x(t)’ with time ‘t’ of states

Q8. Taking c=0, plot the time response of x(t) and v(t) for a step input in MATLAB

ANS. MATLAB code:

```
y0=[1;0]; % considering the initial value x(0)=1,v(0)=0
tspan=[0 0.1];
[tsol,ysol]=ode45(@(t,y) MSDfun(t,y), tspan, y0);
plot(tsol,ysol(:,1),'k');   %  for the velocity plot change ysol(:,2)
xlabel('time t');
ylabel('response x(t)');  % for the velocity plot change ‘response v(t)’
hold on;

%% Function Mass spring damper
function fval=MSDfun(t,y)
   x=y(1);
   v=y(2);
   k=9000;
   m=0.003/9.81;
   fval(1,1)=v;
   fval(2,1)=-(k*x)/m;
end
```

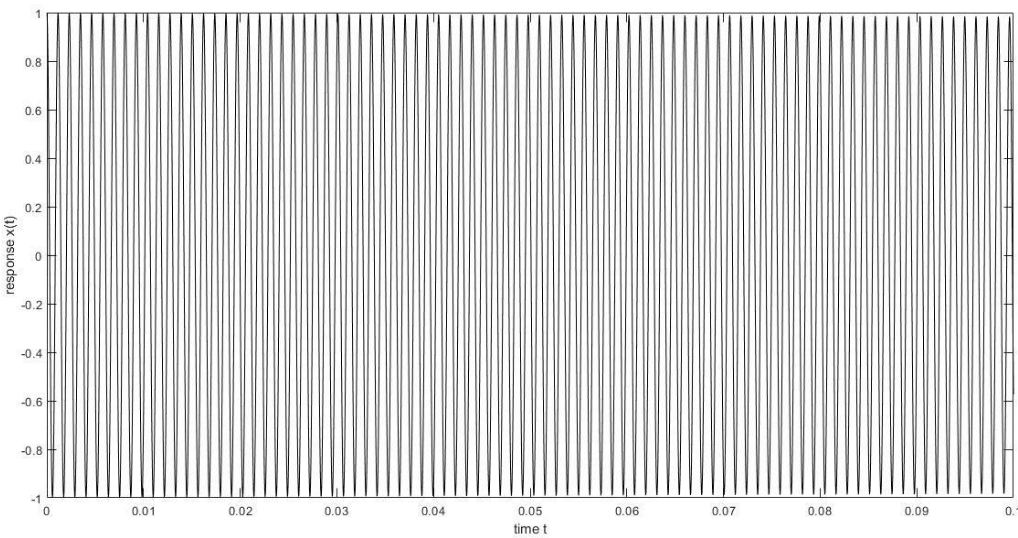



FIG.2.1 Response of displacement ‘x(t)’ with time ‘t’ of states

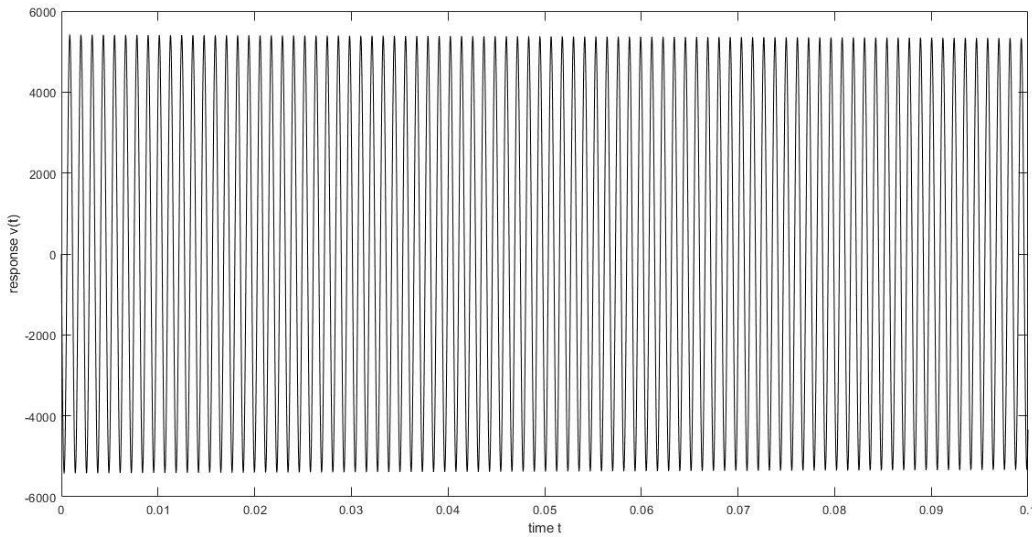



FIG.2.2 Response of velocity ‘v(t)’ with time ‘t’ of states

Q9. Taking k=0, plot the time response of x(t) and v(t) for a step input in MATLAB.

ANS. MATLAB code:

```
 y0=[1;0]; % considering the initial value x(0)=1,v(0)=0
tspan=[0 0.1];
[tsol,ysol]=ode45(@(t,y) MSDfun(t,y), tspan, y0);
plot(tsol,ysol(:,1),'k');
xlabel('time t');
ylabel('response x(t)');
hold on;

%% Function Mass spring damper
function fval=MSDfun(t,y)
   x=y(1);
   v=y(2);
   c=0.045;
   m=0.003/9.81;
   fval(1,1)=v;
   fval(2,1)=-(c*v)/m;
end
```

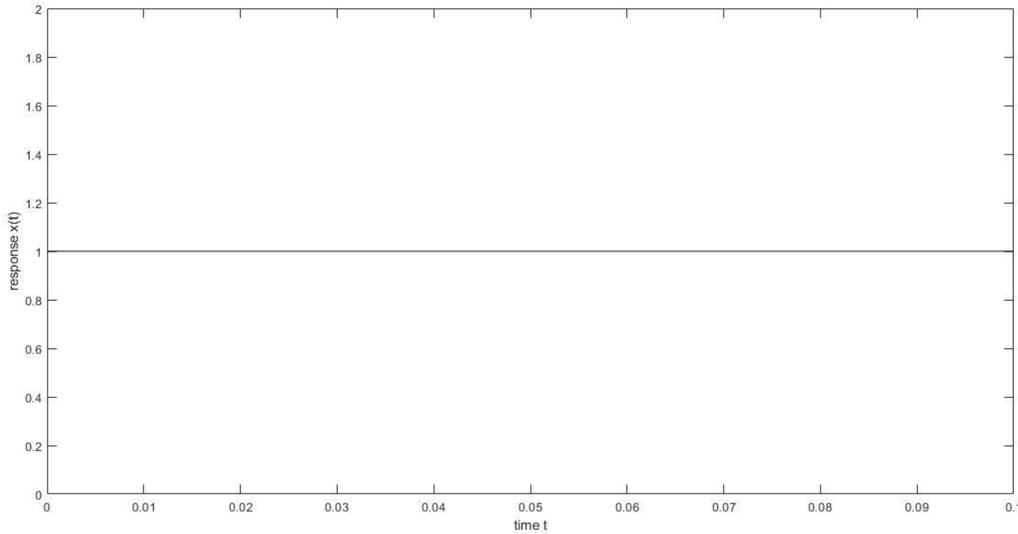



FIG.3.1 Response of displacement ‘x(t)’ with time ‘t’ of states

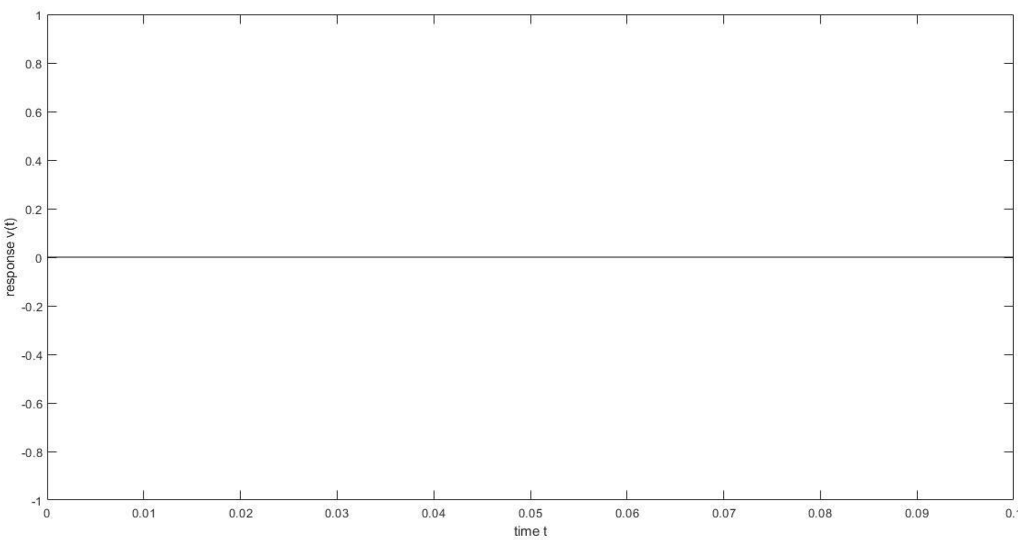



FIG.3.2 Response of velocity v(t) with time ‘t’ of states

Q10. Taking u(t)=0.1N plot the time response of x(t) and v(t) for a step input in MATLAB.

ANS. MATLAB code:

```
y0=[1;0]; % considering the initial value x(0)=1,v(0)=0
tspan=[0 0.1];
[tsol,ysol]=ode45(@(t,y) MSDfun(t,y), tspan, y0);
plot(tsol,ysol(:,1),'k');
xlabel('time t');
ylabel('response x(t)');
hold on;

%% Function mass spring damper
function fval=MSDfun(t,y)
   x=y(1);
   v=y(2);
   u=0.1;
   c=0.045;
   k=9000;
   m=0.003/9.81;
   fval(1,1)=v;
   fval(2,1)=u/m-(c*v+k*x)/m;
end
```

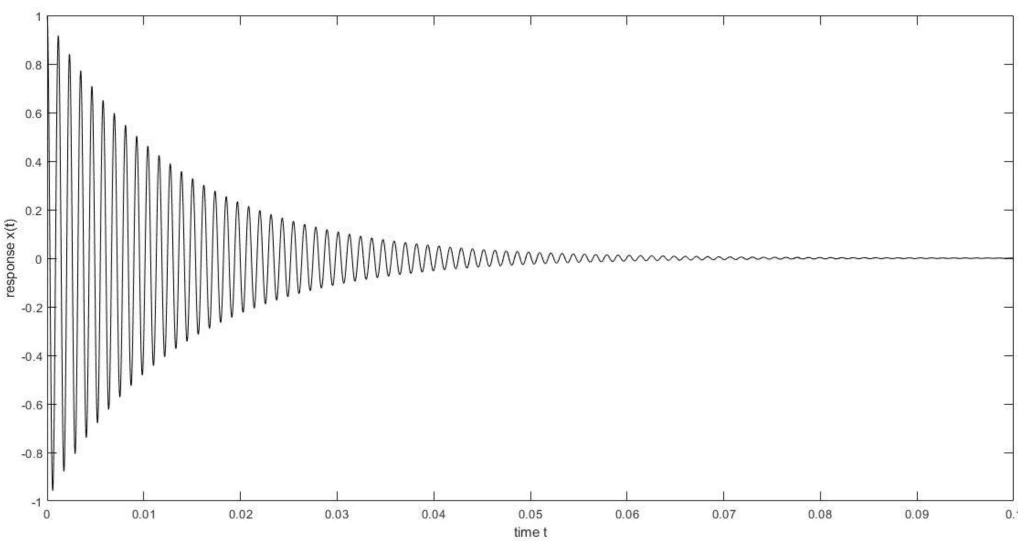



FIG.4.1 Response of displacement ‘x(t)’ with time ‘t’ of states

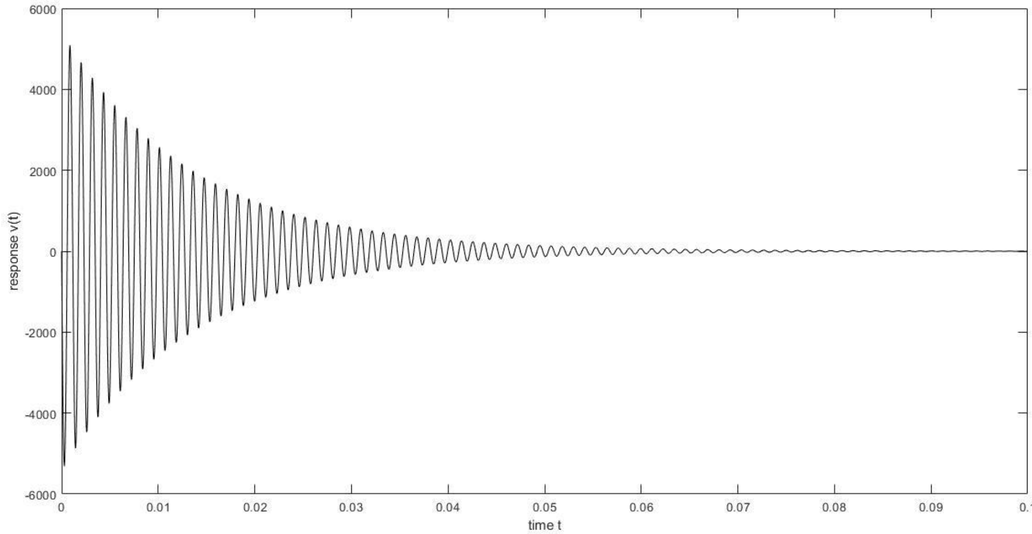



FIG.4.2 Response of velocity v(t) with time ‘t’ of states