Question 4

The statement in the question as stated is actually not correct. If in a finite Boolean algebra B, the assumption about the existence of a complement is dropped, and replaced by the assumption $a + 1 = 1$ and $a.0 = 0$ for all a, we get more examples apart from the set of all multisubsets of a multiset. For example consider the algebra with 8 elements which are subsets of $\{a, b, c, d\}$. The subsets are $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, c, d\}$, with $+$ as union and . as intersection. This satisfies all the modified properties but cannot be obtained by considering all possible multisubsets of a multiset.

However, this does indicate the actual characterization. Every such algebra is obtained by considering some family of subsets of a set which is closed under union and intersection. This can be proved as follows. Again, define $I(a) = \{b | b+a = a\}$. However, 'atoms' are defined differently here and called irreducible elements. An element is irreducible if it cannot be written as the sum of two other elements. In other words, a is irreducible if $a = b + c$ implies $a = b$ or $a = c$. In the case of multisets, irreducible elements are exactly those that contain a single element any number of times. Every multisubset can be written uniquely as the union of these. The same property holds in general. So let A be the set of irreducible elements and let $S(a) = I(a) \cap A$ be the subset corresponding to the element a. In this case, we only need to show S is a one-to-one function (need not be onto) and $S(a.b) = S(a) \cap S(b)$ and $S(a+b) = S(a) \cup S(b)$. The proof is similar to that for Boolean algebra.

Even with the modified assumptions, it is true that $I(a.b) = I(a) \cap I(b)$. If $x \in I(a) \cap I(b)$, then $x+a = a$ and $x+b = b$, hence $a.b = (x+a).(x+b) = x+a.b$, hence $x \in I(a.b)$. Conversely, if $x+a.b = a.b$ then $x+a = x+a.(b+1) = x+a.b+a = a.b+a = a.(b+1) = a$, hence $x \in I(a)$. Similarly, $x \in I(b)$ and hence $x \in I(a) \cap I(b)$. This implies $S(a.b) = S(a) \cap S(b)$.

For the union, first note that $I(a) \cup I(b) \subseteq I(a+b)$. This is because if $x + a = a$ then $x + a + b = a + b$ and hence $x \in I(a + b)$. Similarly if $x \in I(b)$ then $x \in I(a+b)$. This implies $S(a) \cup S(b) \subseteq S(a+b)$. Conversely, suppose $S(a+b)$ contains an irreducible element c. Then $c \in I(a+b)$ hence $c+a+b = a+b$. If $c \notin S(a)$ and $c \notin S(b)$ then $c.a \neq c$ and $c.b \neq c$. However $c.a + c.b = c.(a+b) = c.(c+a+b) = (c+0).(c+a+b) = c+0.(a+b) = c$. This contradicts the assumption that c is irreducible.

The only thing remaining is to show that if $S(a) = S(b)$ then $a = b$. Suppose $S(a) = \{a_1, a_2, \ldots, a_k\}$. We show that $a = a_1 + a_2 + \cdots + a_k$ by induction on $|I(a)|$. Let $b = a_1 + a_2 + \cdots + a_k$. Then since $a_i + a = a$ for all i, $b + a = a$, and for any x such that $x + b = b$, we have $x + a = a$. Thus $I(b) \subseteq I(a)$. To show the other way, if a is irreducible, then $a = a_i$ for some i, which implies $I(a) \subseteq I(b)$. Otherwise $a = c + d$ for some two elements $c, d \neq a$ which implies $I(c) \subset I(a)$ and $I(d) \subset I(a)$. By induction, c is the sum of elements from $S(c) \subseteq S(a)$ and d is the some of elements from $S(d) \subseteq S(a)$. This implies a is the sum of some subset of elements from $S(a)$ and hence $I(a) \subseteq I(b)$. This implies $a = b$.

To capture the properties of the set of all multisubsets of a multiset, some additional assumptions are needed. To represent a multiset in which an element a can occur k times, we consider k distinct elements $a_1, a_2, \ldots, a_k$ but only allow subsets with the property that if a_i belongs to the subset then so does a_j for all $j < i$. Such a collection of subsets is closed under union and intersection but forms a particular case of the general structure.