Boolean expressions

If B is the Boolean algebra with 2 elements $\{0, 1\}$, any Boolean function with n variables can be represented by the 'truth table' that lists for each of the 2^n possible assignments of values to the variables, the value of the output. Such a table can be easily converted into an expression. Consider a function that takes value 1 for exactly one possible input assignment and is 0 for all others. Such a function can be represented by the expression $y_1.y_2.\cdots.y_n$ where $y_i = x_i$ if x_i has value 1 in the assignment for which the function is 1 and $y_i = x_i^c$ otherwise. Any other function can be obtained from these by applying the $+$ operation. This is called the sum of products representation and though it is far from the smallest in general, it is unique for a function.

If the algebra has more than 2 elements, the expressions as defined are too weak to define all possible functions. One property that such expressions satisfy , but definitely not all functions, is that if each x_i has value 0 or 1, then the value of the expression is also 0 or 1. This can be proved easily by induction on the number of operations in the expression. Thus if the algebra has 4 elements $\{0, 1, a, a^c\}$, the function that maps 0 to a cannot be represented by such an expression.

We can allow more general expressions by allowing any constant $a \in B$ to be considered an expression. In the 2-element case, this is not needed because $0 = x.x^c$ and $1 = x + x^c$. This is equivalent to defining polynomials if $+$ and . are considered to be addition and multiplication of numbers. However, even this is not sufficient to represent all possible functions for Boolean algebras with more than 2 elements. In general, polynomials form a small subset of the set of all possible functions. Their value at a few inputs determines their value for all others. Try to list all functions with one variable that can be represented by polynomials, for a Boolean algebra with 4 elements.

However, expressions without constants are still useful for proving properties that hold for all Boolean algebras. Since they must hold for all Boolean algebras, they must hold for the algebra with 2 elements. Then any such expression can be reduced to the unique sum-of-products form by considering their values for 0 and 1. The two expressions are equal iff they have the same sum of products form. For any expression, the sum-of-products form can be derived using the axioms and De Morgan's laws, by induction on the number of operations used to define the expression. Thus if two such expressions are equal, we can derive one from the other using the axioms. Therefore, if they are equal for the 2-element Boolean algebra and represent the same function, then they are equal in any Boolean algebra. The same property can be used to show that if a statement is true in a Boolean algebra, then it can be proved to be true starting from the axioms.