A. Given,

$$u_{,xx} + f = 0 \qquad \bar{\Omega} =]0,1[$$

$$u(1) = g$$

$$-u_{,x}(0) = h$$

$$f = qx \qquad \text{and} \qquad g = h = 0$$

1) given equation

$$u_{,xx} + qx = 0$$

Integrate

$$u_{,x} + q\frac{x^2}{2} + c_1 = 0$$

using given B.C., $-u_{,x}(0) = 0$

$$c_1 = 0$$

$$\Rightarrow \quad u_{,x} + q\frac{x^2}{2} = 0$$

Integrate

$$u(x) + q\frac{x^3}{6} + c_2 = 0$$

using BC $u(1) = 0$

$$\Rightarrow \quad c_2 = -q/6$$

$$\Rightarrow \quad \boxed{u^h(x) = \frac{q}{6}(1 - x^3)} \qquad \text{for } \Omega = [0,1]$$

2) Galerkin - finite Element Equations,

we know the Galerkin-finite element Eq^n,

$$a(N_A, N_B)\, d_b = (N_A, f) + N_A(0)h - a(N_A, N_{AH})g$$

but since $h = g = 0$,

$$\Rightarrow \boxed{a(N_A, N_B)\, d_b = (N_A, f)}$$

for $n=1$ or $h=1$,

$$N_1 = 1 - x \qquad x \in]0,1[$$

$$k_{11} = \int_0^1 N_{1,x} N_{1,x}\, dx = 1$$

since $k_{AB} = a(N_A, N_B)$

stiffness matrix, $k = [1]$

also, $F = (N_A, f)$

$$F_1 = \int_0^1 N_1 f\, dx = \int_0^1 (1-x)\, qx\, dx$$

$$= q/6$$

vector, $F = [q/6]$

solving linear eq^n $kd = f$

$$\Rightarrow d = [q/6]$$

since $u(x) = d_i N_i$

$$\Rightarrow \boxed{u^h = \left(\frac{q}{6}\right)(1-x)}$$

for $n = 2$ or $h = 1/2$

$$N_1 = \begin{cases} -2x+1 & \text{for } x \in (0, 1/2] \\ 0 & \text{for } x \in (1/2, 1) \end{cases} \quad (\text{since } h = 1/2)$$

$$N_2 = \begin{cases} 2x & \text{for } x \in (0, 1/2] \\ -2x+2 & \text{for } x \in (1/2, 1) \end{cases}$$

since, $K_{AB} = a(N_A, N_B)$

$$\Rightarrow K_{11} = \int_0^1 N_{1,x} N_{1,x}\, dx = 2$$

similarly

$$K_{12} = K_{21} = -2 \quad \text{and} \quad K_{22} = 4$$

$\Rightarrow$ stiffness Matrix,

$$K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

and, $F = (N_A, f)$

$$F_1 = \int_0^1 N_1 f = \int_0^{0.5} (-2x+1)\, qx\, dx = q/24$$

$$F_2 = q/4$$

$\Rightarrow$ vector,

$$F = \begin{bmatrix} q/24 \\ q/4 \end{bmatrix}$$

solving $Kd = F$

$$\Rightarrow \quad d = \begin{bmatrix} q/6 \\ 7q/48 \end{bmatrix}$$

$$\Rightarrow \quad u^h = d_i N_i$$

$$u^h = \frac{q}{6} N_1 + \frac{7q}{48} N_2$$

for $n = 3$ or $h = 1/3$

$$N_1 = \begin{cases} -3x+1 & x \in (0, 1/3] \\ 0 & x \in (1/3, 1) \end{cases} \quad N_2 = \begin{cases} 3x & x \in (0, 1/3] \\ -3x+2 & x \in (1/3, 2/3) \\ 0 & x \in [2/3, 1) \end{cases} \quad N_3 = \begin{cases} 0 & x \in (0, 1/3] \\ 3x-1 & x \in (1/3, 2/3) \\ -3x+3 & x \in [2/3, 1) \end{cases}$$

since $K_{AB} = a(N_A, N_B)$

$$K_{11} = \int_0^1 N_{1,x} N_{1,x}\, dx = 3$$

similarly calculating rest of K_{mn}

we get,

stiffness Matrix
$$K = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix}$$

Also
$$f_\cdot = (N_A, f)$$

$$F_1 = \int_0^1 N_1 f\, dx = 9/54;$$ similarly $F_2 = 9/9$ and $F_3 = 29/9$

vector F,
$$f = \begin{bmatrix} 9/54 \\ 9/9 \\ 29/9 \end{bmatrix}$$

solving, $Kd = F$

$$\Rightarrow \quad d = \begin{bmatrix} 9/6 \\ 139/81 \\ 199/162 \end{bmatrix}$$

$$\Rightarrow \quad u^h = d_i N_i$$
$$= \frac{9}{6} N_1 + \frac{139}{81} N_2 + \frac{199}{162} N_3$$

for $n=4$ or $h=1/4$

$$N_1 = \begin{cases} -4x+1 & x\in(0,1/4] \\ 0 & x\in(1/4,1) \end{cases} \qquad N_2 = \begin{cases} 4x & x\in(0;1/4] \\ -4x+2 & x\in(1/4,2/4) \\ 0 & x\in[2/4,1) \end{cases}$$

$$N_3 = \begin{cases} 0 & x\in(0,1/4] \\ 4x-1 & x\in(1/4,2/4) \\ -4x+3 & x\in[2/4,3/4) \\ 0 & x\in[3/4,1) \end{cases} \qquad N_4 = \begin{cases} 0 & x\in(0,2/4] \\ 4x-2 & x\in(2/4,3/4) \\ -4x+4 & x\in[3/4,1) \end{cases}$$

stiffness Matrix $\quad K_{AB} = a(N_A, N_B)$

$$K = \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{bmatrix}$$

where, $K_{ij} = \int_0^1 N_{i,x} N_{j,x}\,dx$

vector $\quad F = (N_A, f)$

$$F_1 = \int_0^1 N_1 f\,dx = \int_0^{1/4} (-4x+1)\,q x\,dx = q/96$$

similarly,

$$F_2 = q/16 \;;\; F_3 = q/8 \text{ and } F_4 = 3q/16$$

$$F = \begin{bmatrix} q/96 \\ q/16 \\ q/8 \\ 3q/16 \end{bmatrix}$$

solving $\quad Kd = F$

$$\Rightarrow \quad d = \begin{bmatrix} q/6 \\ 21q/128 \\ 7q/48 \\ 37q/384 \end{bmatrix}$$

$$u^h = d_i N_i$$

$$= \frac{q}{6} N_1 + \frac{21q}{128} N_2 + \frac{7q}{128} N_3 + \frac{37q}{384} N_4$$

3) matrix is banded when $a_{i,j} = 0$ for $A = [a_{i,j}]$
$j < i - k_1$ or $j > i + k_2$
and $k_1, k_2 \geq 0$

so given defination, Yes stiffness matrix is banded with $k_1 = k_2 = 1$
(triadiagonal matrix)

Also, since stiffness matrix is made of shape functions N_i's, Boundry condition such as 'g' and 'h' will have no effect on bandedness of stiffness matrix.
(since $K_{AB} = a(N_A, N_B)$)

4) $$re_{,x} = \frac{|u^h_{,x} - u_{,x}|}{q/2}$$ relative error in $u_{,x}$ and compute all $re_{,x}$ at midpoint

$$u_{,x} = \frac{q}{6}(-3x^2) = -\frac{qx^2}{2}$$

for $n=1$, $$re_{,x} = \frac{\left|-q/6 - \left(-\frac{q(1/2)^2}{2}\right)\right|}{q/2} = \frac{1}{12}$$

similarly calculating,

for $n=2$,
$$u^h_{,x} = \begin{cases} -\frac{q}{24}x + \frac{q}{6} & x \in (0, 1/2] \\ \frac{7q}{24}(1-x) & x \in (1/2, 1) \end{cases}$$
$$re_{,x} = 1/48$$

for $n=3$
$$re_{,x} = 1/108$$

$$u^h_{,x} = \begin{cases} q\left(\frac{1}{6} - \frac{x}{54}\right) & x \in (0, 1/3] \\ q\left(\frac{11}{54} - \frac{7x}{54}\right) & x \in [1/3, 2/3) \\ \frac{19q}{54}(1-x) & x \in [2/3, 1) \end{cases}$$

for $n=4$

$$u^h_{,x} = \begin{cases} q\left(\frac{1}{6} - \frac{x}{96}\right) & x \in (0, 1/4] \\ q\left(\frac{35}{192} - \frac{7x}{96}\right) & x \in [1/4, 2/4) \\ q\left(\frac{47}{192} - \frac{19x}{96}\right) & x \in [2/4, 3/4) \\ \frac{37q}{96}(1-x) & x \in [3/4, 1) \end{cases}$$

computing them at midpoints of each domain

$$u^h_{,x}\Big|_{\text{at midpoints}} = \begin{cases} -q/96 & \text{at } x = 1/8 \\ -7q/96 & \text{at } x = 3/8 \\ -19q/96 & \text{at } x = 5/8 \\ -37q/96 & \text{at } x = 7/8 \end{cases}$$

for respective domain

$$re_{,x}(x = 1/8) = \frac{\left|-q/96 - \left(-\frac{q}{128}\right)\right|}{q/2} = 1/192$$

similarly

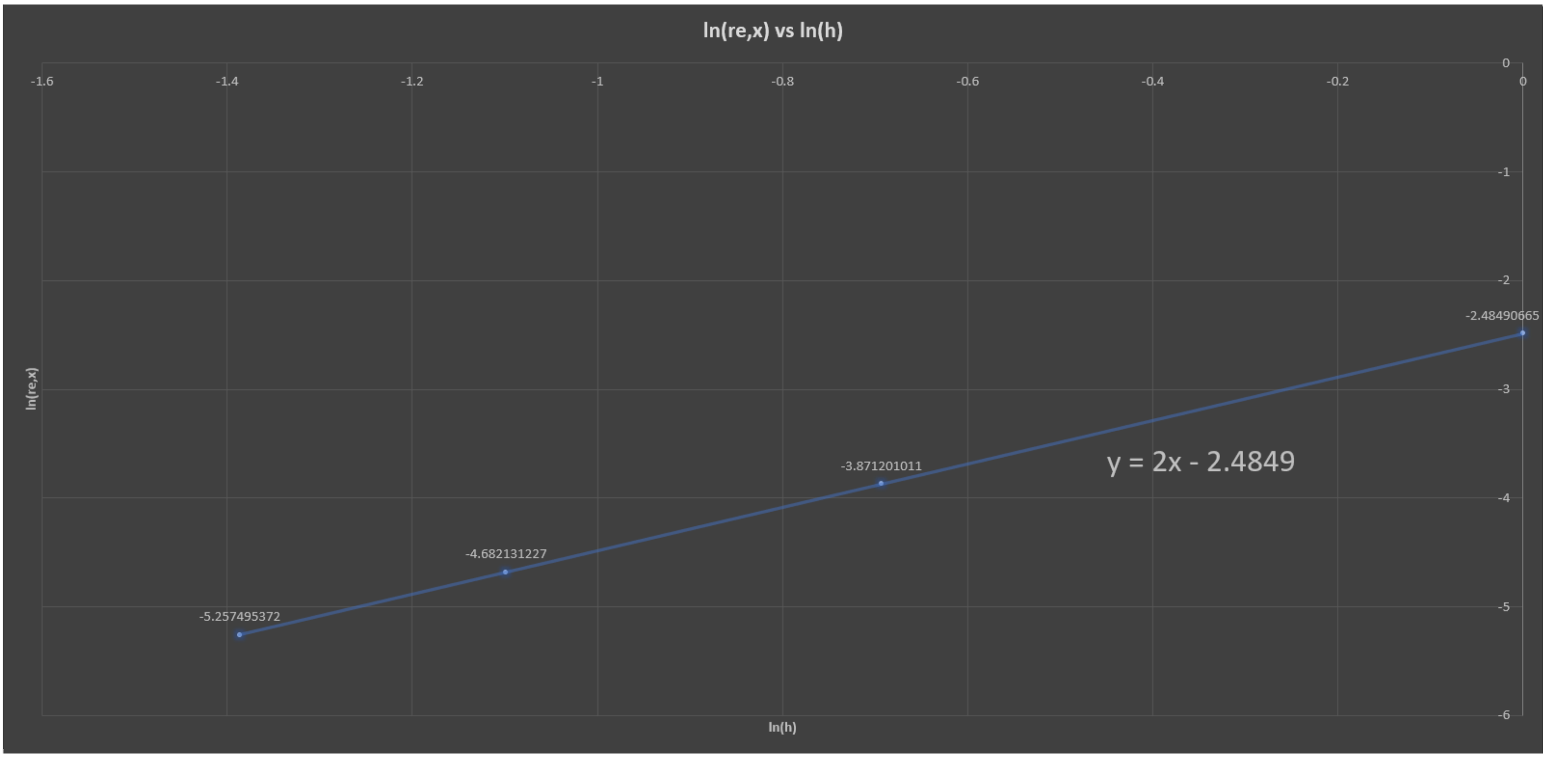
$$re_{,x}(x = 3/8) = \frac{1}{192} = re_{,x}\left(x = \frac{5}{8}\right) = re_{,x}\left(x = \frac{7}{8}\right)$$

we can see that $re_{,x}$ is same for all midpoints of each element

5) plot $\ln(re_{,x})$ vs $\ln(h)$

$\ln(1/2)$	$\ln(1)$
$\ln(1/48)$	$\ln(1/2)$
$\ln(1/108)$	$\ln(1/3)$
$\ln(1/192)$	$\ln(1/4)$

all values are calculated in subsection 4.



6) y intercept $= -2.4849$

$$= \ln(re_{,x\,max})$$

$$= \ln(1/12)$$

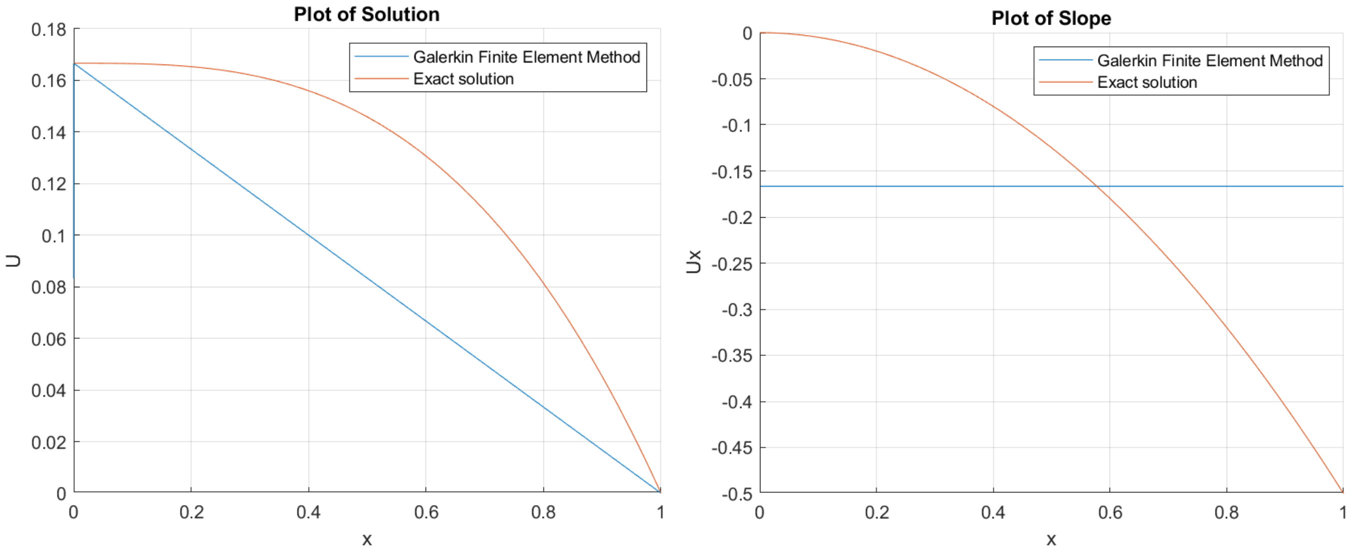
intercept is value of maximum relative error.

slope = 2 is rate at ~~wh~~ which relative error decreases as we increase number of element. ~~In this case~~

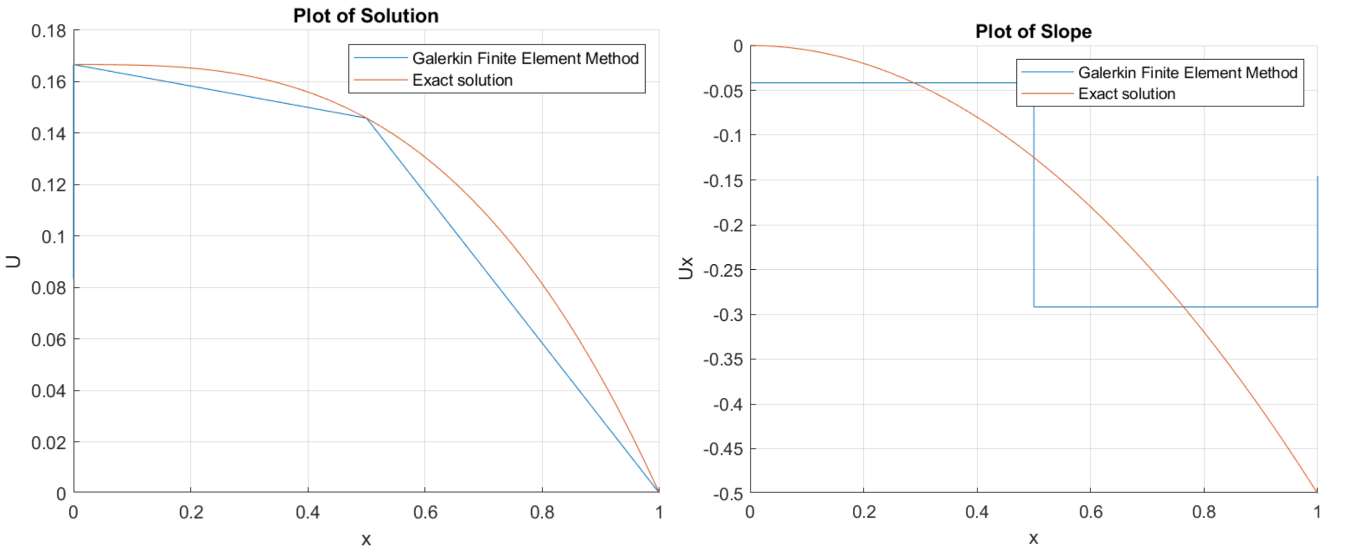
Part B.

```
function re = GFEM(n)

K = zeros(n, n);
for c = 1:n
    for r = 1:n
        if r == c && r == 1
            K(r, c) = n;
        elseif r == c
            K(r, c)= 2*n;
        elseif abs(r - c) == 1
            K(r, c) = -n;
        else
            K(r, c) = 0;
        end
    end
end

t = 0;
syms x
N = cell(1, n);
N{1} = triangularPulse(0, 0, 1/ n, x);

for i = 2:n
N{i} = triangularPulse(t, t+1/n, t+2/n, x);
t = t + 1/n;
end

F = zeros(n,1);

for i = 1:n
F(i) = int(N{i}.*x, x, 0,1);
end

db = K\F;

uh = 0;
for i = 1:n
    uh = uh + db(i)*N{i};
end

u = (1 - x^3)/6; du = diff(u, x); duh = diff(uh, x);
f1 = subs(du, x, 1/(2*n)); f2 = subs(duh, x, 1/(2*n));
re = abs(f1 - f2)*2;

end
```

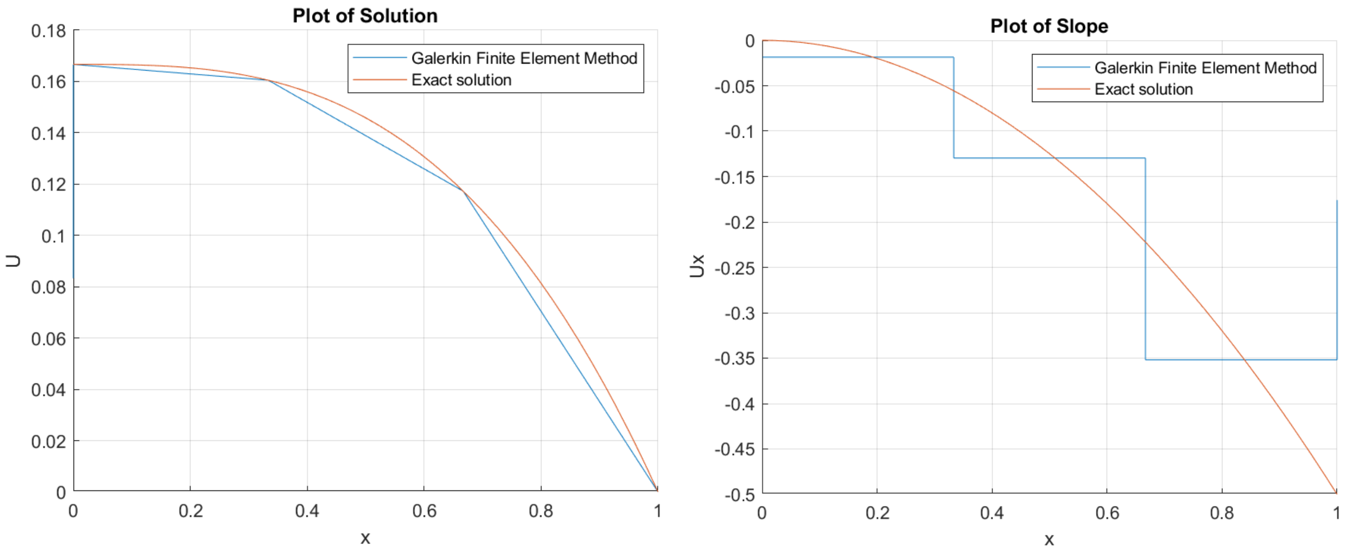
For n = 1



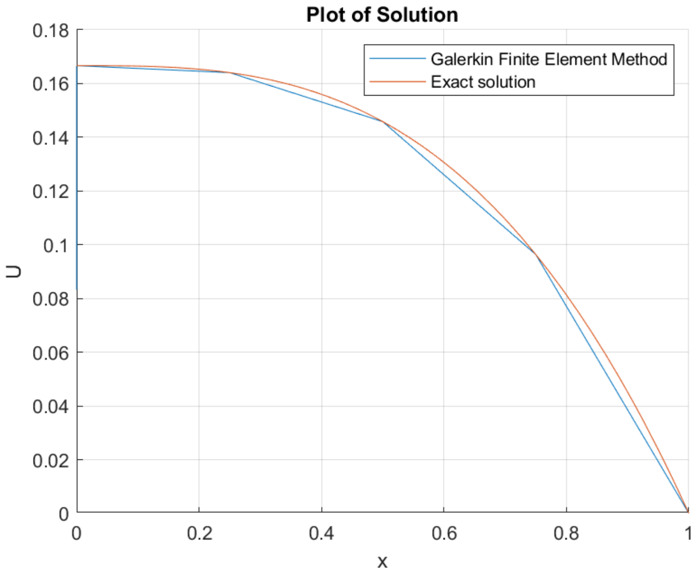
For n = 2

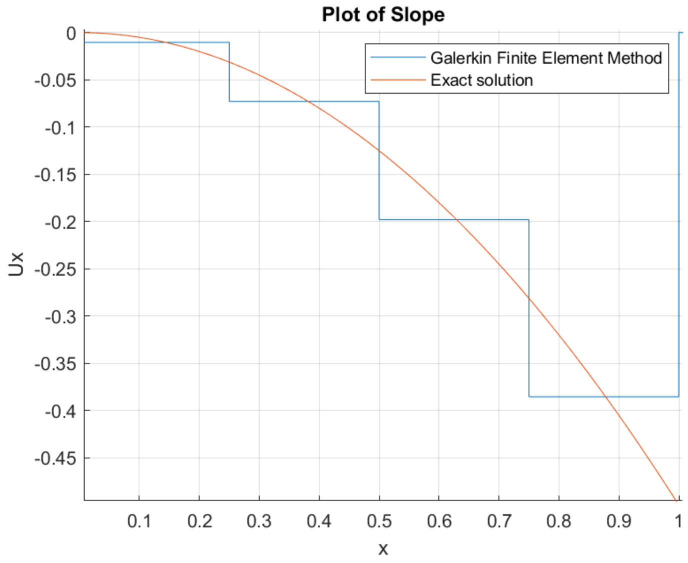


For n = 3

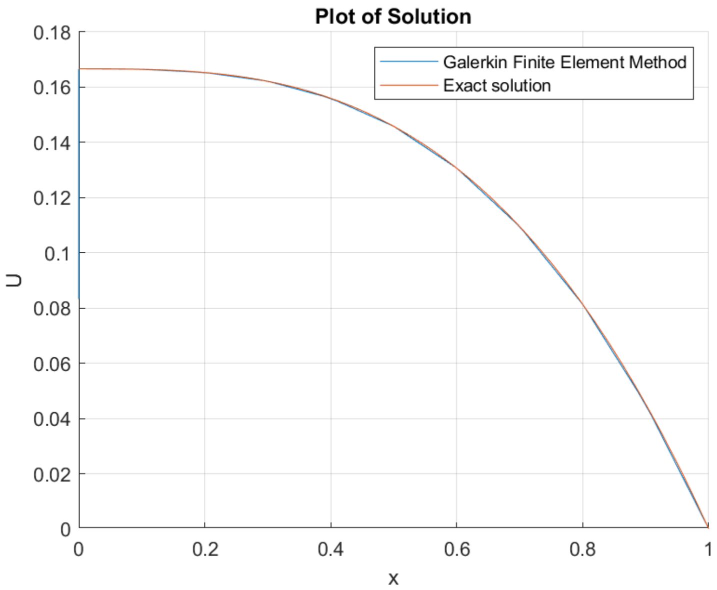


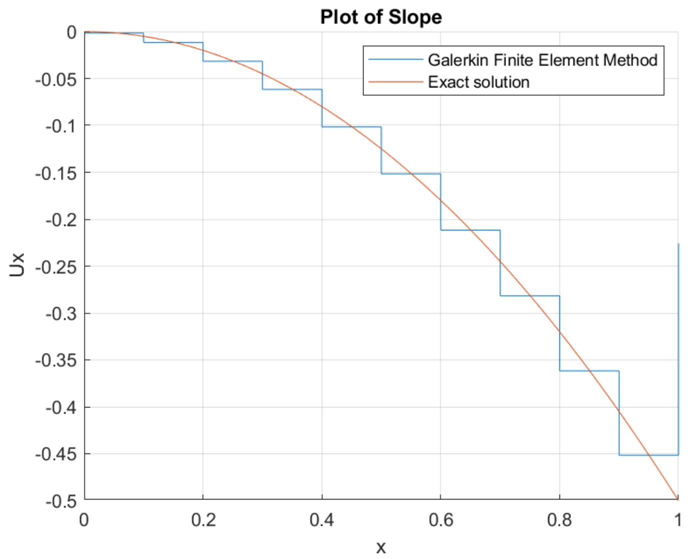
For n = 4



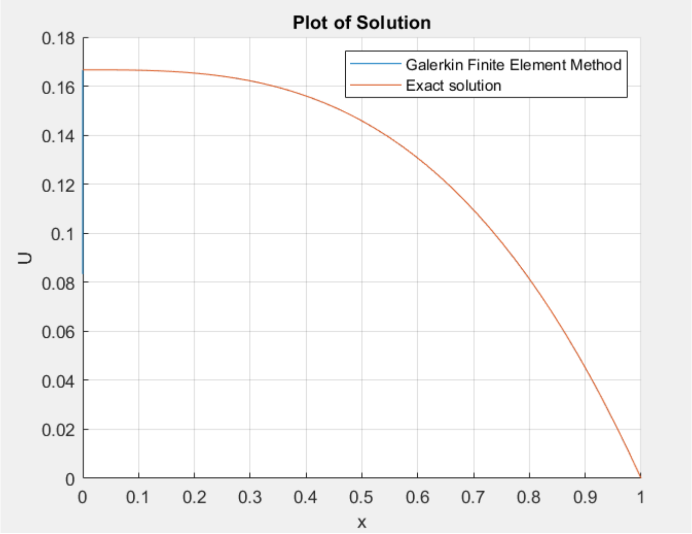


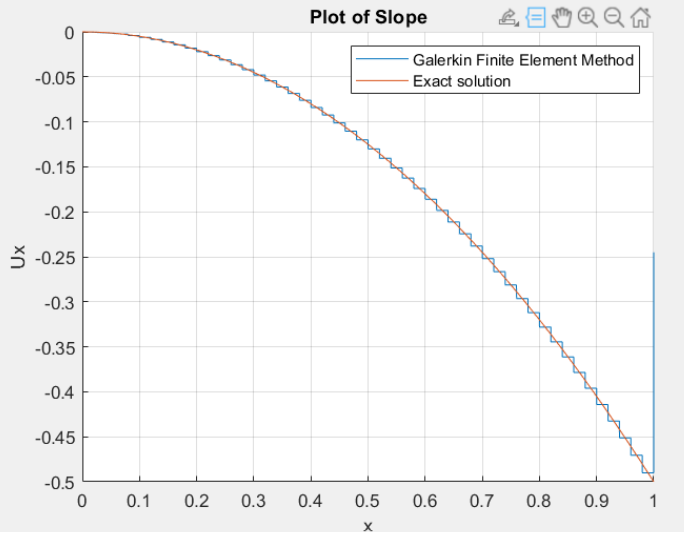
For n = 10



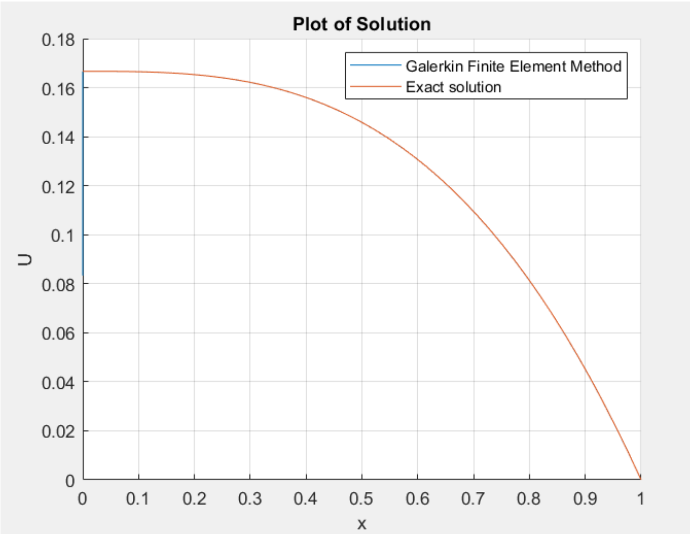


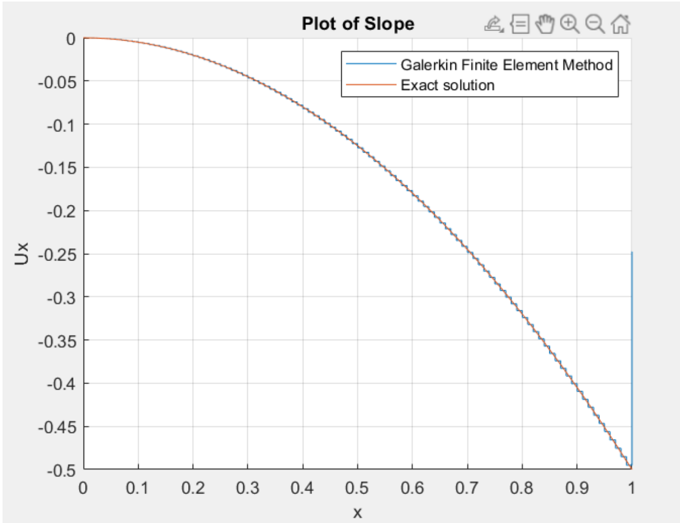
For n = 50

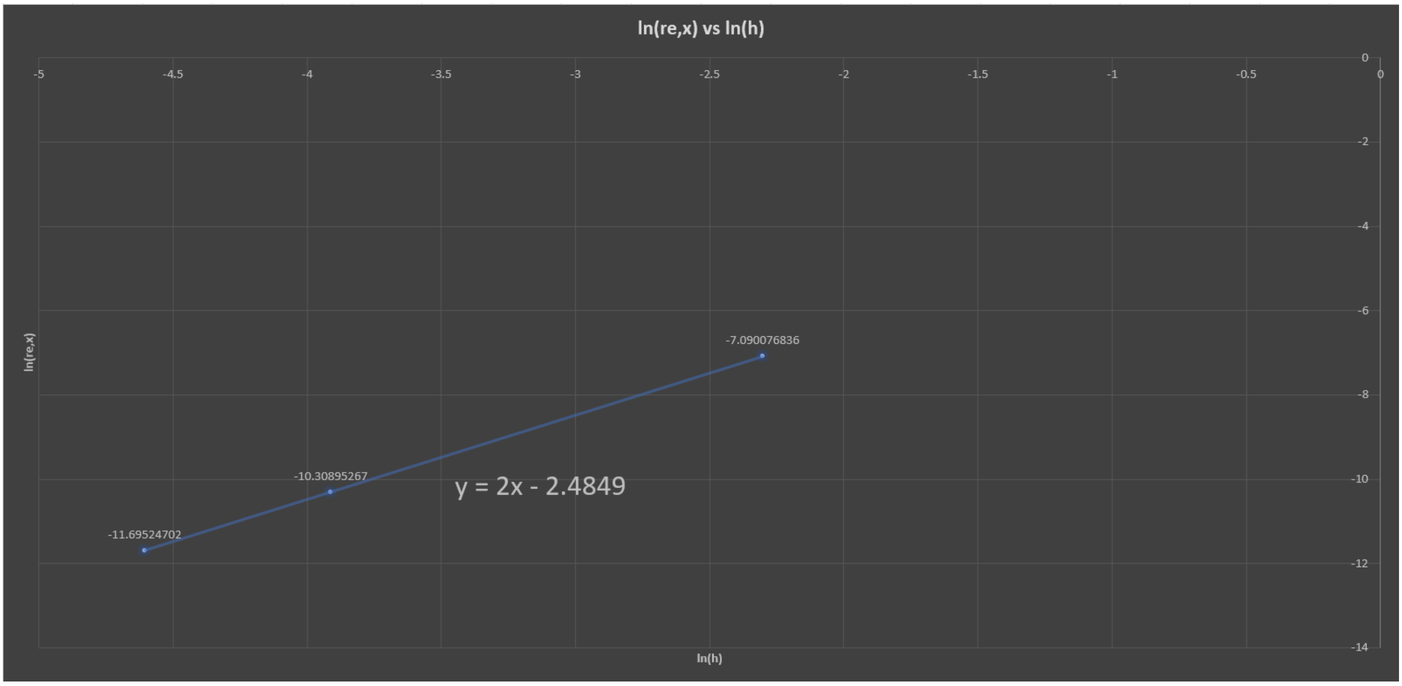




For n = 100







$re_{,x} = |u_x^h - u_x| / (q/2)$

for $n=10$, $re_{,x} = 1/1200 \simeq 8.33 \times 10^{-4}$

$n=50$, $re_{,x} = 3.33 \times 10^{-5}$

$n=100$, $re_{,x} = 8.33 \times 10^{-6}$

$\ln(re_x)$	vs	$\ln(h)$
-7.09008		$\ln(1/10)$
-10.3089		$\ln(1/50)$
-11.6952		$\ln(1/100)$

as we can see slope is same