Review:

1. Unit step function

$$\mathcal{U}_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

In particular

$$\mathcal{U}_0(t) = 1, \quad \mathcal{U}_\infty(t) = 0.$$

2. e.g.

$$f(t) = \begin{cases} f_1(t), & 0 \le t \le a \\ f_2(t) & a < t < b \\ f_3(t) & t \ge b \end{cases}$$

We can express $f(t)$ as

$$f(t) = f_1(t)\left(\cancelto{1}{u_0(t)} - u_a(t)\right) + f_2(t)\left(u_a(t) - u_b(t)\right) + f_3(t)\left(u_b(t) - \cancelto{0}{u_\infty(t)}\right)$$

3. The second shifting Theorem

Given $\mathcal{L}[f(t)] = F(s)$, we have

$$\mathcal{L}[f(t-a)\,u_a(t)] = e^{-as}F(s)$$

$$\Longleftrightarrow \quad \mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\,u_a(t)$$

Very Rough!

$$ay'' + by' + cy = f(t)$$

$\downarrow$

impulse function

$$f(t) = \delta(t - t_0)$$

$\boxed{\text{Definition, property}}$

Impulse Functions

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Definition

Introduce the Dirac **delta function**, denoted by $\delta(t)$ such that

$$\delta(t) = 0, \quad t \neq 0, \quad \int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{6}$$

There is no ordinary function of the kind studied in elementary calculus with such properties. Notice that for any t_0,

$$\delta(t - t_0) = 0, \quad t \neq t_0; \quad \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1. \tag{7}$$

The delta function can be viewed as the limit of usual functions. For example, consider

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau, \\ 0, & \text{otherwise}, \end{cases} \quad \tau > 0 \quad \Rightarrow \quad \lim_{\tau \to 0^+} d_\tau(t) = \delta(t) = \begin{cases} +\infty, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

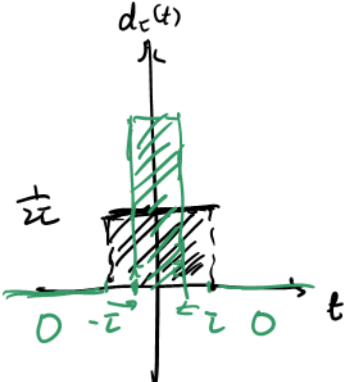
Based on this, we can show that for any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) f(t)dt = f(0), \qquad \int_{-\infty}^{\infty} \delta(t - t_0) f(t)dt = f(t_0). \tag{8}$$

Visualize $\delta(t)$

Define

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0 & |t| \geq \tau \end{cases}$$



$$\int_{-\infty}^{\infty} d_\tau(t)\,dt = 1.$$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$

Theorem (Laplace Transform of Delta function)

We have

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}, \quad t_0 > 0; \qquad \mathcal{L}[\delta(t)] = 1. \tag{9}$$

Example

Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \quad y(0) = y'(0) = 0.$$

Prove

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}$$

proof

Use definition of LT:

$$\mathcal{L}[\delta(t-t_0)] = \int_0^{\infty} e^{-st}\delta(t-t_0)\,dt$$

(Recall: $\int_{-\infty}^{+\infty}\delta(t-t_0)\, f(t)\,dt = f(t_0)$)

$$= \int_{-\infty}^{\infty} e^{-st}\delta(t-t_0)\,dt$$

$$= e^{-st}\Big|_{t=t_0}$$

$= e^{-st_0}$

$\neq$

Solution: Let $Y(s) = \mathcal{L}[y]$. Applying the Laplace transform leads to

$$(2s^2 + s + 2)Y(s) = e^{-5t}.$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2} \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}.$$

Recall that

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}\right\} = \frac{4}{\sqrt{15}} e^{-t/4} \sin \frac{\sqrt{15}}{4} t.$$

Therefore

$$y(t) = \mathcal{L}^{-t}[Y(s)] = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right). \quad \square$$

End of Chapter 4

Solve $\quad 2y'' + y' + 2y = \delta(t-5), \quad y(0) = y'(0) = 0$

<u>Soln</u>: Let $Y(s) = \mathcal{L}[y(t)]$. Apply LT on both sides.

$$\mathcal{L}[2y'' + y' + 2y] = \mathcal{L}[\delta(t-5)] = e^{-5s}$$

$$\Rightarrow 2\mathcal{L}[y''] + \mathcal{L}[y'] + 2\mathcal{L}[y] = e^{-5s}$$

$$2\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + \left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = e^{-5s}$$

$$\Rightarrow \underbrace{(2s^2 + s + 2)}_{\text{Characteristic poly.}} Y(s) = e^{-5s}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-5s}}{2s^2+s+2}.$$

Then
$$y(t) = \mathcal{L}^{-1}\left[\frac{e^{-5s}}{2s^2+s+2}\right].$$

(Use the 2nd shifting theorem:

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\,u_a(t).$$
)

Observe that
$$F(s) = \frac{1}{2s^2+s+2}.$$

$$= \frac{1}{2} \frac{1}{s^2 + \frac{s}{2} + 1} = \frac{1}{2} \frac{1}{(s+\frac{1}{4})^2 + 1 - \frac{1}{16}}$$

$$= \frac{1}{2} \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}} \qquad \frac{15}{16} = \left(\frac{\sqrt{15}}{4}\right)^2$$

$$= \frac{1}{2} \cdot \frac{4}{\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

Shifted from: $\frac{b}{s^2 + b^2}$

$\therefore$ Recall 1st-shifting theorem:

$$\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t)$$

$$= \frac{2}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}}{4} t\right).$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{e^{-5s}}{2s^2+s+2}\right] = \underbrace{\frac{2}{\sqrt{15}} e^{-\frac{t-5}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)}_{f(t-5)\ =\ g(t)} \times u_5(t)$$

Solution $\Uparrow$

#

Note: $\boxed{2y'' + y' + 2y = \delta(t-5),\ y(0) = y'(0) = 0}$

$g(t)u_5(t) =$

$$y(t) = \begin{cases} 0, & 0 \le t < 5 \\ g(t) & t \ge 5 \end{cases}$$

In Summary:

$$\begin{cases} ay'' + by' + cy = f_1(t) + \delta(t-5) = f(t) \end{cases}$$

piecewise continuous.

$$\left\{ y(0) = y'(0) = 0 \right.$$

Remark: Initial values Not at ~~t~~=0.

$$\begin{cases} 2y''(x) + y'(x) + 2y(x) = \delta(x-5) \\ y(1) = y'(1) = 0 \end{cases}$$

Firstly, we make a change of variable

$$t = x - 1 \qquad \text{So that } x = 1 \Leftrightarrow t = 0$$

$$y(t) = y(x-1)$$

$$\boxed{\begin{cases} 2y''(t) + y'(t) + 2y(t) = \delta(t-4). \\ y(0) = y'(0) = 0. \end{cases}}$$

Chapter 5 Systems of First-Order Linear ODEs (Part 1)

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MH3110 – Ordinary Differential Equations **March 15, 2023**

Outline

1. Matrix form for linear system
2. Linear theory of linear system
   - LI/LD & Wronskian
   - Abel's formula
   - Main results on solution structures

System of first-order linear ODEs

- Consider the system of first-order linear DE:

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t),\\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t),\\
&\;\;\vdots \quad\;\; \vdots \quad\;\; \vdots \qquad\quad \vdots \quad\;\; \vdots\\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t),
\end{aligned}
\tag{1}
$$

  where $x_1(t), \cdots, x_n(t)$ are unknown functions, $a_{ij}(t)$ and $f_i(t),\ 1 \le i, j \le n$, are given functions, which are continuous on some interval I.

- If $f_i(t) = 0$, for all $1 \le i \le n$, then the system (1) is said to be **homogeneous**, otherwise, it is **nonhomogeneous**.

2×2 System of 1st-order linear DEs

$$\begin{cases} x_1'(t) = a_{11}(t)\, x_1(t) + a_{12}(t)\, x_2(t) + f_1(t) \\ x_2'(t) = a_{21}(t)\, x_2(t) + a_{22}(t)\, x_2(t) + f_2(t) \end{cases} \qquad t \in I$$

given, continuous fuctions

1 Matrix form

$$\begin{bmatrix} x_1'(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \quad \downarrow$$

$$\boxed{\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t)}$$

Concepts: Homogeneous $\Longleftrightarrow$ $\vec{f}(t) \equiv \vec{0}$

$$\vec{x}'(t) = A\vec{x}(t)$$

Otherwise, if $\vec{f} \neq \vec{0}$, then it's nonhomogeneous

Matrix form

- Matrix form:
$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{F}(t), \tag{2}$$
or equivalently,
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \dots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

- **Note:** The differentiation and integration of vector and matrix functions are performed **component-wisely**.

Example

Let

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -2e^{5t} + 4e^{-t} \\ e^{5t} + e^{-t} \end{pmatrix}.$$

Then we have

$$\boldsymbol{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -10e^{5t} - 4e^{-t} \\ 5e^{5t} - e^{-t} \end{pmatrix},$$

and

$$\begin{aligned} \int \boldsymbol{x}(t)dt &= \begin{pmatrix} -\frac{2}{5}e^{5t} - \frac{1}{4}e^{-t} + c_1 \\ \frac{1}{5}e^{5t} - e^{-t} + c_2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix} e^{5t} - \begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned}$$

Example: Rewrite the following DEs as a system of first-order DEs:

i) $y'' + 2y' + 3y = e^t, \quad y(0) = 1, \;\; y'(0) = 0$

ii) $y''' + ty'' + e^t y' - y = \sin t$

iii) $x_1'' + 2x_2' - x_1' + x_1 - x_2 = 1, \quad x_1' - x_2' + 3x_1 - 2x_2 = 0$

Statement: Any nth linear DE in chapters 2 & 3, can be reformulated as a system of first-order DEs.

(i) Introduce

$$x_1 = y \Rightarrow x_1' = y' = x_2$$

$$x_2 = y' \Rightarrow x_2' = y''$$

$\Rightarrow$ i) $\Rightarrow$ $x_2' + 2x_2 + 3x_1 = e^t$

Standard form $\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -3x_1 - 2x_2 + e^t \end{cases}$

Initial values: $x_1(0) = 1$, $x_2(0) = 0$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'}_{\vec{x}'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ e^t \end{bmatrix}}_{\vec{F}}$$

$\vec{x}' = A\vec{x} + \vec{F}$

$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(iii) $y''' + t y'' - e^t y' - y = \sin t$

$x_1 = y$

$x_1' = x_2$
$x_2' = x_3$

$x_2 = y' \Rightarrow y'' = x_2'$

$x_3 = y'' \Rightarrow y''' = x_3'$

$y = x_1, \quad y' = x_2, \quad y'' = x_3, \quad y''' = x_3'$

$x_3' = -t x_3 + e^t x_2 + x_1 + \sin t$

$= x_1 + e^t x_2 - t x_3 + \sin t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & e^t & -t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ St \end{bmatrix}$$

Principle of superposition

Theorem
If the vector-valued functions $\{\boldsymbol{x}_j(t)\}_{j=1}^n$ *are solutions of the homogeneous linear system*

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t),$$

then the linear combination

$$\boldsymbol{x}(t) = c_1\boldsymbol{x}_1(t) + c_2\boldsymbol{x}_2(t) + \cdots + c_n\boldsymbol{x}_n(t)$$

is also a solution for any constants $c_1, c_2, \cdots, c_n$.

Extend the concepts for LA:

Vector Space

$$V_n(I) = \{\vec{x}(t) \in \mathbb{K}^n : \vec{x}'(t) = A\vec{x}(t)\}$$

$$\dim\left(V_n(I)\right) = n$$

Exercise: Verify that

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} \tag{3}$$

is a solution of

$$\boldsymbol{x}'(t) = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}(t).$$