$$(IP)_y = \frac{xf'(xy)[xf(xy) - xg(xy)] - f(xy)x^2(f'(xy) - g'(xy))}{x^2(f(xy) - g(xy))} = \frac{-2f'(xy)g(xy)}{f(xy) - g(xy)}$$

Exercise 1: Use the above method to show the integrating factor of the first-order linear equation

$$P_y = \frac{\partial}{\partial y}(p(x)y - q(x)) = p(x)$$

$$y' + p(x)y = q(x)$$

$$Q_x = 0$$

is $I(x) = e^{\int p(x)dx}$.

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$\frac{P_y - Q_x}{Q} = \frac{p(x)}{1}$$

$$\underline{1 \cdot dy + [p(x)y - q(x)]dx = 0}$$

$$I(x,y)dy + I(x,y)[p(x)y - q(x)]dx \overset{?}{=} 0$$

Exercise 2: Show that if

$$P(x,y)dx + Q(x,y)dy = 0$$

$$P(x, y) = yf(xy), \quad Q(x, y) = xg(xy),$$

$$\frac{\partial I(x,y)}{\partial x} = \frac{\partial[I(x,y)[p(x)y - q(x)]]}{\partial y}$$

then the integrating factor is

$$(IP)_y = (IQ)_x$$

$$I(x, y) = \frac{1}{xP - yQ}.$$

$$IP = \frac{yf(xy)}{xyf(xy) - xyg(xy)} = \frac{f(xy)}{xf(xy) - xg(xy)}$$

$$\Rightarrow I_x = I_y[p(x)y - q(x)] + Ip(x)$$

$$IQ = \frac{g(xy)}{xf(xy) - xg(xy)}$$

If $I(x,y) = I(x)$, then $I_x = Ip(x)$

Summary of Solvable First-order DE

$2xy\,dx + (x^2+2y)\,dy = 0$

Rule of thumb:

1. **Identify the type**
2. **Use the corresponding solution technique**

$P_y = 2x \quad Q_x = 2x$ Exact

$\frac{\partial u(x,y)}{\partial x} = 2xy$

$u(x,y) = x^2 y + f(y)$

$u_y = x^2 + f'(y)$

$x^2 + f'(y) = x^2 + 2y$

$f(y) = y^2 + c$

Exercise: Determine which of the five types of DEs we have studied the given equation falls into, and use an appropriate technique to find the general solution.

1. $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2+2y}$. Key: $y^2 + x^2 y + c = 0$.
2. $y' - x^{-1}y = x^{-1}\sqrt{x^2 - y^2}$. Key: $y = x\sin(\ln cx)$.
3. $\dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{25x^2 \ln x}{2y}$. Key: $y^2 = x^{-2}[x^5(5\ln x - 1) + c]$.

$2y\frac{dy}{dx} + \frac{2}{x}y^2 = 25x^2 \ln x$

Let $v = y^2$, we have

$\frac{dv}{dx} + \frac{2}{x}v = 25x^2 \ln x$

$I(x) = e^{\int \frac{2}{x}dx} = x^2$

Outline

1. Type-IV: First-order Exact ODE
2. Type-V: Non-Exact DE with IF
3. Type-VI: Homogeneous DE
4. **Type-VI: Reducible Second-order DE**

Reducible Second-order DE

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- Consider the second-order DE:

$$\frac{d^2y}{dx^2} = F\Big(x, y, \frac{dy}{dx}\Big), \tag{13}$$

where F is a known function.

- Introduce

$$v = dy/dx \Longrightarrow d^2y/dx^2 = dv/dx.$$

We can rewrite it as an equivalent system of 1st-order DEs:

$$\frac{d^2y}{dx^2} = F\Big(x, y, \frac{dy}{dx}\Big) \Rightarrow \begin{cases} \dfrac{dy}{dx} = v \\ \dfrac{dv}{dx} = F(x, y, v) \end{cases} \tag{14}$$

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- **Exercise:** Write the second-order DE as a system:

$$y'' = x \sin y' + e^x y + 1.$$

- In general, the second DE can not be solved directly, since the system involves three variables, namely, x, y and v.
- We shall explore the possibility to solve the DE, if it only involves two variables.

Let $v = y'$

$y'' = \frac{dv}{dx}$

$$\begin{cases} \frac{dy}{dx} = v \\ \frac{dv}{dx} = x \sin v + e^x y + 1 \end{cases}$$

Case I. y is missing in F

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- If y does not occur explicitly in the function F, then we have

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right), \tag{15}$$

which is equivalent to the system

$$\begin{aligned} \frac{dy}{dx} &= v, \qquad v = v(x) \\ \frac{dv}{dx} &= F(x, v). \\ &= 0 \end{aligned} \tag{16}$$

- If the second equation is solvable, then find its solution v.
- Substitute v into the first equation to obtain y.

Example: Find the general solution to

$$\frac{d^2y}{dx^2} = \frac{1}{x}\Big(\frac{dy}{dx} + x^2 \cos x\Big), \qquad x > 0. \tag{17}$$

Solution: The dependent variable y is missing (as the RHS only involves x, y'). We rewrite it as

$$\begin{aligned} \frac{dy}{dx} &= v, \\ \frac{dv}{dx} &= \frac{1}{x}(v + x^2 \cos x). \end{aligned} \tag{18}$$

$$\frac{dv}{dx} - \frac{1}{x} v = x \cos x$$

Question: What is the type of the second equation?

Answer: First-order linear equation with the standard form

$$\frac{dv}{dx} - \frac{1}{x} v = x \cos x, \tag{19}$$

with

$$p(x) = -\frac{1}{x}, \quad q(x) = x \cos x.$$

Hence,

$$I(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = x^{-1},$$

and the solution is

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2} v = \cos x$$

$$v(x) = x\left(\int \cos x dx + c\right) = x(\sin x + c_1). \qquad \frac{d}{dx}\left(\frac{1}{x} v\right) = \cos x$$

$$\frac{1}{x} v = \sin x + c_1$$

Substitute v into the first equation gives

$$\frac{dy}{dx} = \underline{x \sin x} + c_1 x,$$

which we can integrate to obtain

$$y(x) = -x \cos x + \sin x + c_1 x^2 + c_2,$$

where we have absorbed a factor $1/2$ into c_1. $\square$

Case II. x is missing in F

- Consider the independent variable x missing:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \Rightarrow \frac{d^2y}{dx^2} = F\left(y, \frac{dy}{dx}\right). \qquad y = y(x) \qquad (20)$$

- We still let

$$v = \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = v,$$

but use the chain rule to express d^2y/dx^2 in terms of dv/dy :

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} \overset{\text{chain rule}}{=} \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}.$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} v$$

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- Rewrite (20) as the first-order system

$$\begin{aligned} \frac{dy}{dx} &= v, \\ v\frac{dv}{dy} &= F(y,v). \end{aligned} \tag{21}$$

$v = v(y) = v(y(x))$

- If the second equation is solvable, then find its solution $v = v(y)$ as a function of y.
- Substitute v into the first equation and solve for y.

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Example: Find the general solution to

$$\frac{d^2y}{dx^2} = -\frac{2}{1-y}\left(\frac{dy}{dx}\right)^2. \tag{22}$$

Solution: In this DE, the independent variable does not appear explicitly. Therefore, we let $v = dy/dx$ and use the chain rule to obtain

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} \overset{\text{chain rule}}{=} \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}.$$

Substituting into (22) results in the equivalent system

$$\begin{aligned} \frac{dy}{dx} &= v, \\ v\frac{dv}{dy} &= -\frac{2}{1-y}v^2. \end{aligned} \tag{23}$$

$v = C(y-1)^2$

$\uparrow C = \pm e^{C_1}$

$\frac{1}{v}\,dv = -\frac{2}{1-y}\,dy = \frac{2}{y-1}\,dy$

$\ln|v| = \ln(y-1)^2 + C_1 \qquad \ln\frac{|v|}{(y-1)^2} = C_1 \qquad \frac{|v|}{(y-1)^2} = e^{C_1}$

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Separating the variables in the second equation gives

$$\frac{1}{v}dv = -\frac{2}{1-y}dy,$$

which can be integrated to obtain

$c_1 \neq 0$

$$\ln|v| = 2\ln|1-y| + c \quad \Rightarrow \quad v(y) = c_1(1-y)^2,$$

where denoted $c_1 = \pm e^c$. Substituting v into the first equation of (23) yields

$$\frac{dy}{dx} = c_1(1-y)^2. \quad \Rightarrow \quad \frac{dy}{(1-y)^2} = c_1\,dx$$

Separating the variables and integrating we obtain

$$(1-y)^{-1} = c_1 x + c_2.$$

That is

$$y = 1 - \frac{1}{c_1 x + c_2}. \qquad \square$$

$c_1 \in \mathbb{R}, \ c_2 \in \mathbb{R}$

Exercises

$\frac{du}{dx} - \tan x \, u = -1 \leftarrow \frac{1}{v^2}\frac{dv}{dx} + \tan x \frac{1}{v} = 1 \leftarrow \frac{dv}{dx} + \tan x \cdot v = v^2$

$u = \frac{1}{v}$, $\frac{du}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

Solve the following second-order DEs.

$y'' = (y')^2 - y'\tan x$ (y is missing)

$$\begin{cases} \frac{dy}{dx} = v \\ \frac{dv}{dx} = -v\tan x + v^2 \end{cases}$$

1. $y'' + y'\tan x = (y')^2$. Key: $y(x) = c_2 - \ln|c_1 - \sin x|$.

$y'' = (y')^2 + y'$

2. $y'' - (y')^2 - y' = 0$. Key: $y(x) = -\ln|c_1 + c_2 e^x|$.

$$\begin{cases} \frac{dy}{dx} = v \\ \frac{dv}{dx} = v^2 + v \end{cases}$$

$\frac{dv}{v^2+v} = dx$

$\left(\frac{1}{v} - \frac{1}{v+1}\right) dv = dx$

$\ln\left|\frac{v}{v+1}\right| = x + C_0$

$\frac{v}{v+1} = C_3 e^x$

$1 - \frac{1}{v+1} = C_3 e^x$

$v = \frac{1}{1 - C_3 e^x} - 1$

$v = \frac{C_3 e^x}{1 - C_3 e^x}$

$dy/dx = \frac{C_3 e^x}{1 - C_3 e^x} \Rightarrow dy = \frac{C_3 e^x}{1 - C_3 e^x} dx$

$d(1 - C_3 e^x) = -C_3 e^x dx$

$y = -\ln|1 - C_3 e^x| + C_4$

3. $yy'' = 3(y')^2$, Key: $y(x) = \frac{c_1}{\sqrt{1+c_2 x}}$

$$\begin{cases} \frac{dy}{dx} = v \rightarrow \frac{dy}{C_3 y^3} = dx \quad -\frac{1}{2C_3} y^{-2} = x + C_4 \\ \frac{dv}{dx} = \frac{3v^2}{y} \overset{v \neq 0}{\Rightarrow} \frac{dv}{dy} = \frac{3v}{y} \end{cases}$$

$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$

$\ln|v| = 3\ln|y| + C_0$

$v = \pm e^{C_0}|y^3|$

$v = C_3 y^3$

End of Chapter One

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Tutorial 2: First-Order ODEs (Part II)

[Attempt all questions before tutorial session]

A) Bernoulli's Equation

Solve

$$2x(\ln x)y' - y = -9x^3y^3 \ln x. \qquad (x>0)$$

Solution: This equation is nonlinear since it involves the term y^3. We rewrite it as the standard form

$$y' - \frac{1}{2x(\ln x)}y = -\frac{9}{2x^2}y^3,$$

which is a Bernoulli equation of order $n = 3$. ~~Diving~~ Dividing by y^3 yields the equivalent equation:

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x(\ln x)}y^{-2} = -\frac{9}{2}x^2. \tag{1}$$

We next make the change of variables

$$u = y^{-2} \implies \frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \implies y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}.$$

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = -2y^{-3}\frac{dy}{dx}$$

Substituting into (1) gives

$$-\frac{1}{2}\frac{du}{dx} - \frac{1}{2x(\ln x)}u = -\frac{9}{2}x^2. \quad (\text{linear equation})$$

Rewriting it as the standard form gives

$$\frac{du}{dx} + \frac{1}{x(\ln x)}u = 9x^2,$$

$$I(x) = e^{\int \frac{1}{x\ln x}dx} = e^{\int \frac{1}{\ln x}d\ln x} = e^{\ln|\ln x| + C_0} = e^{C_0}|\ln x| = \pm e^{C_0}\ln x = C_1 \ln x$$

For simplicity, we choose $C_1 = 1$.

which is a linear equation. The solution is

$$\begin{aligned} u(x) &= e^{-\int \frac{1}{x(\ln x)}dx}\Big[\int 9x^2 e^{\int \frac{1}{x(\ln x)}dx}dx + C\Big] \\ &= e^{-\ln(\ln x)}\Big[\int 9x^2 e^{\ln(\ln x)}dx + C\Big] \\ &= \frac{1}{\ln x}\Big[9\int x^2 \ln x dx + C\Big] = \frac{1}{\ln x}\Big[3x^3 \ln x - x^3 + C\Big]. \end{aligned}$$

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Thus, the general solution is

$$y^{-2} = \frac{1}{\ln x}\Big[3x^3 \ln x - x^3 + C\Big] \iff y^2 = \frac{\ln x}{x^3(3\ln x - 1) + C}.$$

B) **Exact DE**

Determine if the following equations are exact. If so, find its general solution

a) $\left(3x^2y - 2y^2\right) dx + \left(x^3 - 4xy + 6y^2\right) dy = 0,$

b) $(\sin(xy) + xy\cos(xy) + 2x)dx + (x^2\cos(xy) + 2y)dy = 0,$

c) $x^2ydx - (xy^2 + y^3)dy = 0.$

$M(x,y)\,dx + N(x,y)\,dy = 0$

Solution: (a) The equation is already in differential form with

$M_y = N_x$

$$M(x,y) = 3x^2y - 2y^2, \quad N(x,y) = x^3 - 4xy + 6y^2.$$

We test if it is exact by verifying

$$\frac{\partial M}{\partial y} = 3x^2 - 4y = \frac{\partial N}{\partial x}.$$

Therefore, this DE is exact! Then we look for a potential function u satisfying

$$\begin{cases} \dfrac{\partial u}{\partial x} = M = 3x^2y - 2y^2, \\ \dfrac{\partial u}{\partial y} = N = x^3 - 4xy + 6y^2. \end{cases}$$

From the first equation, integrating with respect to x keeping y constant, we have

$$u(x,y) = x^3y - 2xy^2 + g(y)$$

where $g(y)$ is the "constant" of integration. Substituting this into the second equation yields

$du(x,y) = M(x,y)\,dx + N(x,y)\,dy = 0$

$$x^3 - 4xy + g'(y) = x^3 - 4xy + 6y^2,$$

from which, we find $g'(y) = 6y^2$, i.e., $g(y) = 2y^3 + c$. We can take $c = 0$, and obtain the potential function: $u = x^3y - 2xy^2 + 2y^3$. Then the general solution is

$$u(x,y) = x^3y - 2xy^2 + 2y^3 = C.$$

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Alternative solution by inspection:

$$\begin{aligned}\left(3x^2y-2y^2\right)dx+\left(x^3-4xy+6y^2\right)dy &= \left(3x^2ydx+x^3dy\right)-\left(2y^2dx+4xydy\right)+6y^2dy\\ &= d\left(x^3y\right)-d\left(2xy^2\right)+d\left(2y^3\right)=d\left(x^3y-2xy^2+2y^3\right)\\ &= d\left(x^3y-2xy^2+2y^3+C\right)\end{aligned}$$

Then the required function is $x^3y-2xy^2+2y^3+c$. This method, called the grouping method, is based on ones ability to recognize exact differential combinations.

b) We have

$$M(x,y)=\sin(xy)+xy\cos(xy)+2x \implies \frac{\partial M}{\partial y}=2x\cos(xy)-x^2y\sin(xy),$$

$$N(x,y)=x^2\cos(xy)+2y \implies \frac{\partial N}{\partial x}=2x\cos(xy)-x^2y\sin(xy)=\frac{\partial M}{\partial y},$$

and so the equation is exact. Hence, there exists a potential function $u(x,y)$ such that

2 PDEs

$$\begin{aligned}\frac{\partial u}{\partial x}&=\sin(xy)+xy\cos(xy)+2x,\\ \frac{\partial u}{\partial y}&=x^2\cos(xy)+2y.\end{aligned} \tag{2}$$

In this case, the second equation is the simpler equation, and so we integrate it with respect to y, holding x fixed, to obtain

$$u(x,y)=x\sin(xy)+y^2+h(x),$$

where $h(x)$ is an arbitrary function of x. We now determine $h(x)$. Differentiating the resulting $u(x,y)$ partially with respect to x yields

$$\frac{\partial u}{\partial x}=\sin(xy)+xy\cos(xy)+h'(x).$$

Hence, we have

$$h'(x)=2x \quad \Rightarrow \quad h(x)=x^2.$$

$h(x)=x^2+C_0$

Here, we set the integration constant to zero, since we only need one potential function

$$u(x,y)=x\sin(xy)+x^2+y^2.$$

$u(x,y)=x\sin(xy)+x^2+y^2+C_0=C$

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The original equation can be written as

$$d(x\sin(xy) + x^2 + y^2) = 0,$$

and hence the general solution is

$$x\sin(xy) + x^2 + y^2 = C.$$

(c) We have that

$$M(x,y) = x^2y \quad \Longrightarrow \quad \frac{\partial M}{\partial y} = x^2,$$

while

$$N(x,y) = -(xy^2 + y^3) \quad \Longrightarrow \quad \frac{\partial N}{\partial x} = -y^2.$$

Since $M_y \neq N_x$, the equation is non-exact.

C) Determine an integrating factor for the equation

$$ydx - (2x + y^4)dy = 0,$$

and hence find the general solution.

Solution: Assume that the function $I(x,y)$ is an integrating factor of this equation. Multiplying the equation by $I(x,y)$ yields

$$\underbrace{Iy}_{P(x,y)}\,dx + \underbrace{(-(2x+y^4))I}_{Q(x,y)}dy = 0. \tag{3}$$

It is exact if and only if

$I_x = 0$

$$P_y = yI_y + I = Q_x = -2I - (2x + y^4)I_x. \quad \Longrightarrow yI_y + I = -2I$$

Observe that if I is only a function of y, i.e., $I_x = 0$, then we can solve

$I = I(y)$

$$yI_y = -3I \quad \Longrightarrow \quad I(y) = \frac{1}{y^3}.$$

Plugging it into the equation (3) gives $\quad y\frac{dI}{dy} = -3I \Rightarrow \frac{dI}{I} = -3\frac{dy}{y}$

$$\frac{1}{y^2}dx - \frac{2x + y^4}{y^3}dy = 0,$$

$\ln|I| = -3\ln|y| + C_0$

$|I| = e^{C_0}\frac{1}{|y^3|}$

$I = \pm e^{C_0}/y^3$

$I = C/y^3$

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which is exact. We now solve the potential function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \frac{1}{y^2}, \qquad \frac{\partial \phi}{\partial y} = -\frac{2x + y^4}{y^3}.$$

From the first equation, we get that

$$\phi(x, y) = \frac{x}{y^2} + h(y) \implies \frac{\partial \phi}{\partial y} = -\frac{2x}{y^3} + h'(y).$$

By the second equation,

$$h'(y) = -y \implies h(y) = -\frac{y^2}{2}.$$

Accordingly, the general solution is

$$\frac{x}{y^2} - \frac{y^2}{2} = c \implies 2x - y^4 = Cy^2.$$

D) Determine whether the given function $I(x, y) = \cos(xy)$, is an integrating factor for the DE

$$\left[\tan(xy) + xy\right] dx + x^2 dy = 0.$$

If so, find its general solution.

Solution: Multiplying the equation by $I(x, y)$ gives

$$\underbrace{\left[\sin(xy) + xy\cos(xy)\right]}_{P(x,y)} dx + \underbrace{x^2\cos(xy)}_{Q(x,y)} dy = 0.$$

Check that

$$P_y = 2x\cos(xy) - x^2 y \sin(xy) = Q_x.$$

Therefore, $\cos(xy)$ is an integrating factor of the given equation.

Note that by inspection, we find

$$d(x\sin(xy)) = \big(\sin(xy) + xy\cos(xy)\big)dx + x^2\cos(xy)dy,$$

so the general solution is $x\sin(xy) = C$.

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E) **Homogeneous DE** $f(tx, ty) = t^n f(x, y)$ $t > 0$

Solve

$$y' - x^{-1}y = x^{-1}\sqrt{x^2 - y^2}, \quad x > 0.$$

Solution: By inspection, we know that this equation could not be separable, linear or Bernoulli. We try to check if it is homogenous, and hence rewrite it as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2 - y^2}}{x} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}.$$

$V(x) = \frac{y}{x}$

We recognize it as a homogenous DE, since the RHS only depends on y/x. Change the variables

$$y = xV(x) \implies \frac{dy}{dx} = \underline{x\frac{dV}{dx} + V},$$

and substitute them into the equation to eliminate y, y' :

$$x\frac{dV}{dx} + V = V + \sqrt{1 - V^2} \implies \frac{dV}{\sqrt{1 - V^2}} = \frac{dx}{x}.$$

Direct integration leads to

$$\int \frac{dV}{\sqrt{1 - V^2}} = \int \frac{dx}{x} + C \implies \sin^{-1} V = C + \ln|x|.$$

$V = \sin\left[C + \ln|x|\right]$

Hence, the general solution is

$$y = xV = x\sin(C + \ln x), \quad \underline{x > 0}.$$

F) Solve the following equations by (i) Bernoulli's technique or (ii) finding a suitable integrating factor and then converting it to an exact equation:

$$\frac{dy}{dx} = -\frac{8x^5 + 3y^4}{4xy^3}.$$

Solution: We first rewrite it as

$$\frac{dy}{dx} + \frac{3}{4x}y = -2x^4y^{-3},$$

which we recognize as a Bernoulli equation of order $\underline{n = -3}$. We can solve it by using the corresponding solution formula.

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Here, we solve it by using the second method. We check for exactness and rewrite the given DE in a differential form $P(x,y)\,dx + Q(x,y)\,dy = 0$

$$(8x^5 + 3y^4)dx + 4xy^3dy = 0.$$

We have

$$P_y = 12y^3, \quad Q_x = 4y^3,$$

$\dfrac{P_y - Q_x}{P}$ $I(y)$

so that the equation is not exact. However, we see that

$$\frac{P_y - Q_x}{Q} = 2x^{-1}.$$

or $\dfrac{P_y - Q_x}{Q}$ $I(x)$

Therefore, $I(x) = x^2$ is an integrating factor, which is obtained by solving

$$\frac{dI}{dx} = \frac{P_y - Q_x}{Q} I = \frac{2I}{x}.$$

Multiplying both sides by x^2 yields

$$x^2(8x^5 + 3y^4)dx + 4x^3y^3dy = 0.$$

$\dfrac{\partial u}{\partial x} = x^2(8x^5+3y^4)$

From inspection, we find

$$x^2(8x^5 + 3y^4)dx + 4x^3y^3dy = d(x^8 + x^3y^4),$$

$\dfrac{\partial u}{\partial y} = 4x^3y^3$

so the general solution is

$$x^8 + x^3y^4 = C.$$