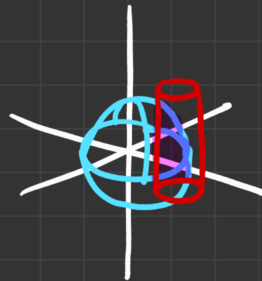
Innlevering 3
TMA4105, våren 2022



* 1) $x^2 + y^2 + z^2 = 4$ (kule)

$x^2 + (y-1)^2 = 1$ (sylinder)

Sylinderkoordinater: r, θ, z

Uttrykker z fra kuleflate:

$$z = \sqrt{4 - \underbrace{x^2 - y^2}_{r}} = \sqrt{4 - r^2}$$

Setter inn x og y for sylinderkoordinater:

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

målet er å finne θ for trippelintegralet.

Sylinder: $r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$

$$r^2\cos^2\theta + \underbrace{r^2\sin^2\theta - 2r\sin\theta + 1}_{\text{andre kvadratsetning}} = 1$$

løser for θ:

$$r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} - 2r\sin\theta = 0$$

$$r^2 - 2r\sin\theta = 0$$

$$r^2 = 2r\sin\theta$$

$$r = 2\sin\theta$$

Dvs. r går mellom 0 og 2 sin θ

Løser videre for θ:

Fra uttrykket til sirkelen får vi at

$r = 2 = 2\sin\theta$

$\rightarrow \sin\theta = 1$ når $\theta = \frac{\pi}{2}$

Integralet blir:

$$4\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$[rz]_0^{\sqrt{4-r^2}}$$

$$\int_0^{2\sin\theta} r\sqrt{4-r^2} \, dr$$

$u = 4-r^2$
$du = -2r \, dr$

$$-\frac{1}{2}\int_0^{2\sin\theta}\sqrt{u} \, du = -\frac{1}{\cancel{2}}\cdot 4\left[-\frac{\cancel{2}}{3}(4-r^2)^{3/2}\right]_0^{2\sin\theta}$$

$$-\frac{1}{3}\left(\underbrace{(4-4\sin^2\theta)}_{=4\cos^2\theta}{}^{3/2} - (4)^{3/2}\right)$$

$$= -\frac{4}{3}\int_0^{\pi/2} 8\cos^3\theta - 8 \, d\theta$$

$$= -\frac{32}{3}\int_0^{\pi/2}\cos^3\theta - 1 \, d\theta$$

$$= -\frac{32}{3}\int_0^{\pi/2} \cos\theta(1-\sin^2\theta)$$

$u = \sin\theta$
$du = \cos\theta$

$$-\frac{32}{3}\int_0^1 1-u^2\,du$$

$$-\frac{32}{3}\left[u-\frac{u^3}{3}-\frac{\pi}{2}\right]_0^1$$

$$= -\frac{32}{3}\left(\frac{2}{3}-\frac{\pi}{2}\right) = \underline{\underline{\frac{32\pi}{6}-\frac{64}{9}}}$$

Oppgave 2)

T er $(x, y, z) \in \mathbb{R}^3$ som oppfyller

$$0 < \sqrt{x^2+y^2+z^2} < 1 - |z|$$

Og massetetthetsfunksjon $\delta(x, y, z) = (x^2+y^2+z^2)^{-\frac{3}{4}}$

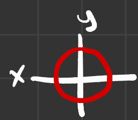
gjør om til kulekoordinater:

$$0 < \rho < 1 - |\cos\varphi \cdot \rho|$$

Integralet skal bli $\iiint \delta(\rho, \varphi, \theta)\, dV$, må

finne grensene:

$$0 \leq \theta \leq 2\pi$$

I xy-planet: 

sirkel m/radius 1

Setter $x=0$: $\sqrt{y^2+z^2} = 1 - |z|$

$$y^2 + z^2 = (1-z)^2$$

$$y^2 + \cancel{z^2} = 1 - 2z + \cancel{z^2}$$

$$y^2 = 1 - 2z \text{, i } y=0:$$

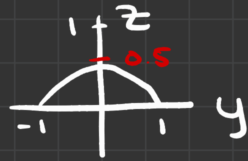
$$0 = 1 - 2z$$

$$z = 0.5$$

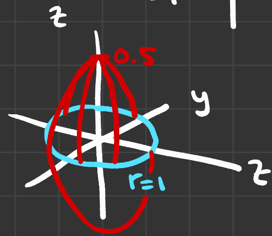
i $y=1$: $1 = 1 - 2z$
$\rightarrow z = 0$

z går opp til høyden 0.5:

I yz-planet:



Hele T: ca.



Integrasjonsgrenser:

$0 \leq \rho \leq 1 - \cos\varphi\, \rho$

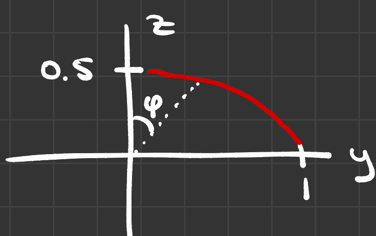
$0 \leq \rho + \cos\varphi\, \rho \leq 1$

$0 \leq \rho(1+\cos\varphi) \leq 1 \quad 1 \div (1-\cos\varphi)$

$0 \leq \rho \leq \frac{1}{1+\cos\varphi}$ som er integrasjonsgrensene for ρ

Må finne grensene for φ:

yz-planet ser ca. sånn ut:



som viser at φ blir max. $\pi/2$

$0 \leq \varphi \leq \pi/2$

Hele integralet:

$$2\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\frac{1}{1+\cos\varphi}} \underbrace{(\rho)^{2\cdot\left(-\frac{3}{4}\right)}}_{\text{massetetthetsfunksjonen, skrevet om siden } x^2+y^2+z^2=\rho^2} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

for å få m/ alt under xy-planet.

Kan skrive "$2\int_0^{2\pi}$" som 4π:

$$4\pi\int_0^{\pi/2}\int_0^{\frac{1}{1+\cos\varphi}} \rho^{1/2}\sin\varphi\, d\rho\, d\varphi$$

$$4\pi\int_0^{\pi/2}\left[\frac{2}{3}\rho^{3/2}\cdot\sin\varphi\right]_0^{\frac{1}{1+\cos\varphi}} d\varphi$$

$$= 4\pi\int_0^{\pi/2}\frac{2}{3(1+\cos\varphi)^{3/2}}\cdot\sin\varphi\, d\varphi$$

$u = 1+\cos\varphi$
$du = -\sin\varphi\, d\varphi$

$$-4\pi\int_0^{\pi/2}\frac{2}{3}\cdot\frac{1}{u^{3/2}}\cdot du$$

$$-\frac{8\pi}{3}\int_0^{\pi/2}\frac{1}{u^{3/2}}\, du = -\frac{8}{3}\pi\int_0^{\pi/2}\left(\frac{1}{u}\right)^{3/2} du$$

$$= -\frac{8}{3}\pi\cdot\left[\frac{u^{-\frac{3}{2}+\frac{2}{2}}}{-\frac{3}{2}+\frac{2}{2}}\right] = -\frac{8}{3}\pi\left(-\frac{2}{\sqrt{1+\cos\varphi}}\right)_0^{\pi/2}$$

$$= -\frac{8}{3}\pi\left(-2-\left(-\frac{2}{\sqrt{2}}\right)\right) \approx 4.9074$$

oppgave 3)

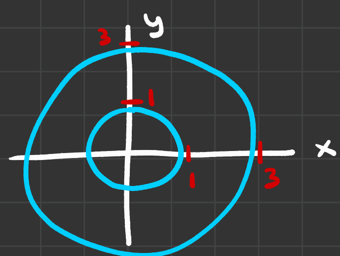
$S \in (r-2)^2 + z^2 \leq 1$

$z^2 \leq 1-(r-2)^2$

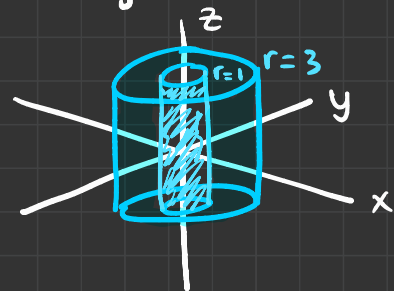
$z \leq \sqrt{1-(r-2)^2}$ dette blir grensene til z i det endelige integralet

Setter $z=0$:

$0 \leq 1-(r-2)^2$
$0 \leq 1-(r^2-4r+4)$
$0 \leq -r^2+4r-3$
ulikheten gjelder for $r=1$ og $r=3$.
Siden r sine grenser starter fra et tall høyere enn null, vil xy-planet se ca. sånn ut:



DVS formen vår er en hul sylinder:



$0 \leq \theta \leq 2\pi$
$1 \leq r \leq 3$
$0 \leq z \leq \sqrt{1-(r-2)^2}$
Fullstendig integral blir:

$$\int_0^{2\pi}\int_1^3\int_0^{\sqrt{1-(r-2)^2}} r \, dz \, dr \, d\theta$$

$$\int_0^{2\pi}\int_1^3 [rz]_0^{\sqrt{1-(r-2)^2}}\, dr\, d\theta$$

$$= 2\pi\int_1^3 r\sqrt{1-(r-2)^2}\, dr$$

$u = r-2$
$du = dr$

$$\rightarrow 2\pi\int_{-1}^{1} (u+2)\sqrt{1-u^2}\, du$$

nye grenser
(3−2) og (1−2)

bruker trig sub:
$u = \sin(x)$
$du = \cos(x)\, dx$

$$= 2\pi\int_{-\pi/2}^{\pi/2} (\sin(x)+2)\underbrace{\sqrt{1-\sin^2(x)}}_{=\cos^2(x)}\cos(x)\, dx$$

$$= 2\pi\int_{-\pi/2}^{\pi/2} (\sin(x)+2)\underbrace{\sqrt{\cos^2(x)}\;\cos(x)}_{=\cos^2(x)}\, dx$$

$$= 2\pi\int_{-\pi/2}^{\pi/2} (\sin(x)+2)\cos^2(x)\, dx$$

$$= 2\pi\int_{-\pi/2}^{\pi/2} \cos^2(x)\sin(x) + 2\cos^2(x)\, dx$$

substituerer igjen, og deler opp integralet:

$s = \cos(x)$
$ds = -\sin(x)\, dx$
$dx = \frac{1}{-\sin(x)}\, ds$

$$2\pi\int_{-\pi/2}^{\pi/2} -s^2\cdot \cancel{\sin(x)}\,\frac{1}{\cancel{\sin(x)}}\, ds$$

$$2\pi\left[-\frac{\cos^3(x)}{3}\right]_{-\pi/2}^{\pi/2}$$ første delen

Andre delen:

$$4\pi \int \cos^2(x)\,dx = 4\pi \int \frac{\cos(2x)+1}{2}\,dx$$

$$= \underbrace{2\pi \int \cos(2x)\,dx}_{= \frac{\sin(2x)}{2}} + \underbrace{2\pi \int 1\,dx}_{+\ x}$$

hele integralet: $\left[-\frac{\cos^3(x)}{3} + \frac{\sin(2x)}{2} + x \right]_{-\pi/2}^{\pi/2} \cdot 2\pi$

$= \pi$, ganget med 2π og med 2

$$= \underline{\underline{4\pi^2}}$$

Oppgave 4)

Konservativt om: $F_a(x,y,z) = \nabla\varphi(x,y,z)$

DVS:

$$\frac{\partial\varphi}{\partial x} = y^2\cos(x) - xy^2\sin(x)$$

$$\frac{\partial\varphi}{\partial y} = 2xy\cos(x) + z^2$$

$$\frac{\partial\varphi}{\partial z} = ayz + 1$$

Tar integralet mtp x av den første:

$$y^2\int \cos(x) - x\sin(x)\,dx$$

$$= y^2\int\cos(x) - y^2\int x\sin(x)\,dx$$

$$= \cancel{\sin(x)} + x\cos(x) - \cancel{\sin(x)}$$

$$\Rightarrow y^2x\cos(x) + c_1(y,z)$$

$$\frac{\partial}{\partial y}(y^2x\cos(x) + c_1(y,z)) = 2xy\cos(x) + z^2$$

$$\Rightarrow \cancel{2yx\cos(x)} + \frac{\partial c_1(y,z)}{\partial y} = \cancel{2xy\cos(x)} + z^2$$

$$\int\frac{\partial c_1(y,z)}{\partial y} = \int z^2 \text{ mtp } y$$

$$c_1(y,z) = yz^2 + c_2(z)$$

φ så langt: $y^2 x\cos(x) + yz^2 + C_2(z)$

$$ayz + 1 = \frac{\partial}{\partial z}(yz^2 + C_2(z))$$

$$= 2yz + \underbrace{C_2'(z)}_{=\text{konstant}}$$

Ser at $\underline{\underline{a = 2}}$ og $C_2'(z) = 1 \rightarrow C_2(z) = z$

$\varphi(x, y, z) = \underline{\underline{y^2 x \cos(x) + yz^2 + z}}$ for F

b) $\int_C F_a \cdot d\vec{r}$, $\vec{r}(t) = (\frac{1}{2}t, \cos(t), \cos(2t))$

for $0 \leq t \leq \pi$

Siden F er konservativ, kan vi bruke φ og $\vec{r}(t)$ med endepunktene til t sitt intervall:

$\varphi(\vec{r}(\pi)) - \varphi(\vec{r}(0))$ blir likt integralet vårt:

$\vec{r}(\pi) = (\frac{\pi}{2}, -1, 1)$ og $\vec{r}(0) = (0, 1, 1)$

$\varphi((\frac{\pi}{2}, -1, 1)) = 0$ og $\varphi(0, 1, 1) = 1 + 1 = 2$

Verdien til integralet blir $\underline{\underline{-2}}$.