Innlevering 7
TMA4115, våren 2022

Oppgaver til kapittel 13

1a) må finne egenverdiene og egenvektorene

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{bmatrix}$$

$$(2-\lambda)(-2-\lambda) + 3$$

$$-4 - 2\lambda + 2\lambda + \lambda^2 + 3 = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda = 1, -1$$

egenvektorer : $\lambda = 1$

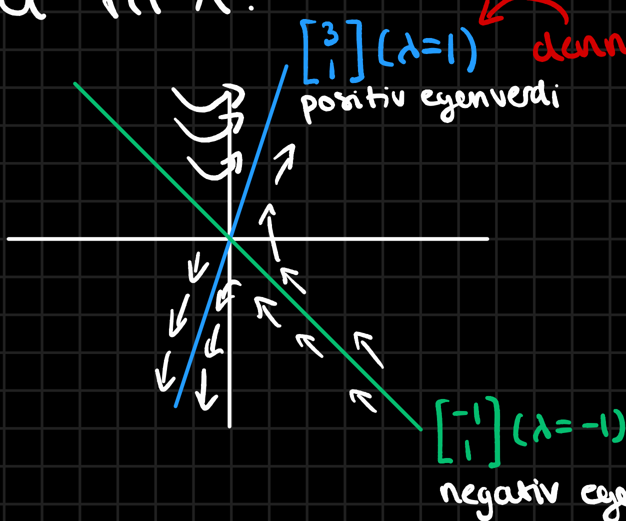
$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$ x_2 er fri variabel

$$\left.\begin{aligned} x_1 &= -3x_2 \\ x_2 &= x_2 \end{aligned}\right\} \underline{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$$

$\lambda = -1$: $$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$ x_2 er fri variabel

$$\left.\begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \end{aligned}\right\} \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Faseplottet til A:



denne er tegnet feil, skal egt være $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ men så den feilen litt sent.

Pilene beveger seg mot uendelig langs egenvektoren m/ positiv egenverdi, og mot origo langs den m/ negativ egenverdi.

b) $\begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$

$(7-\lambda)(3-\lambda)+3$

$21-7\lambda-3\lambda+\lambda^2+3$
$21-10\lambda+\lambda^2+3$

$\lambda^2-10\lambda+24=0$

$\dfrac{-(-10)\pm\sqrt{100-4\cdot 24\cdot 1}}{2} \rightarrow \begin{matrix}\lambda_1=6 \\ \lambda_2=4\end{matrix}$

Egenvektor for $\lambda=6$:

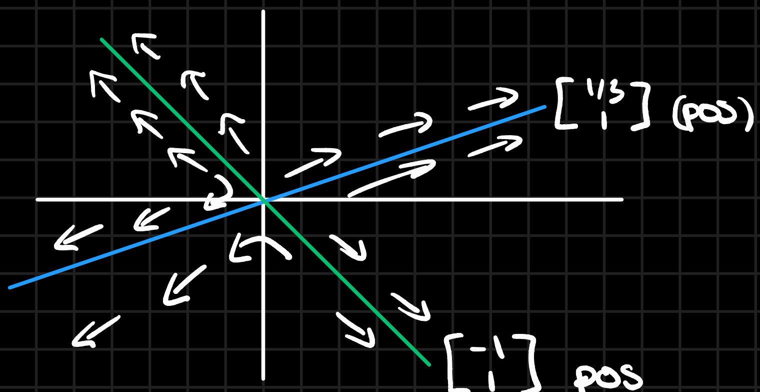
$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ x_2 er fri variabel

$\left.\begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix}\right\} \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

Egenvektor for $\lambda=4$:

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ x_2 er fri variabel

$\begin{matrix} 3x_1 = x_2 \\ x_2 = x_2 \end{matrix} \rightarrow \left.\begin{matrix} x_1 = 1/3\, x_2 \\ x_2 = x_2 \end{matrix}\right\} \underline{\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}}$



Siden begge egenverdiene er positive, peker alle pilene bort fra origo.

c) $\begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

$(-3-\lambda)(-1-\lambda)+2$

$3+3\lambda+\lambda+\lambda^2+2$

$\lambda^2+4\lambda+5=0 \rightarrow \lambda_1 = -2+i$
$\lambda_2 = -2-i$

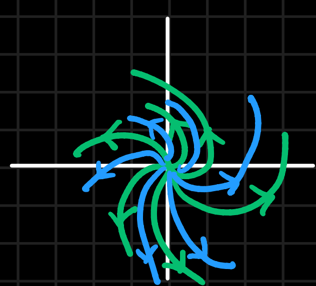
$\frac{-4 \pm 2i}{2}$

kan ikke finne en 2x1 egenvektor for en kompleks egenverdi, siden komplekse egenvektorer roterer planet.

For en kompleks egenverdi $\alpha \pm bi$ vil:

$\alpha > 0$: utadgående spiraler
$\alpha = 0$: sirkuler
$\alpha < 0$: innadgående spiraler

Her er $\alpha = -2 < 0$ i begge egenverdiene:



to utadgående spiraler mot klokka

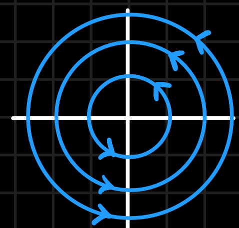
d) $\begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$

$(-3-\lambda)(3-\lambda)-(-18)$

$-9+3\lambda-3\lambda+\lambda^2+18$

$\lambda^2+9=0$
$\lambda^2=-9$
$\lambda^2=\pm(3i)$

Her er $\alpha = 0$: dvs det blir sirkulære baner rundt origo.



2a) $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} = A$ Basis for løsningsrommet til $Ay = y'$

Trappeform: egenverdiene ligger på diagonalen og er 2, 3 og 5

Egenvektor for $\lambda = 2$:

$$\begin{bmatrix} 0 & 3 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix} \sim \tfrac{1}{3}R_1 \sim \begin{bmatrix} 0 & 1 & 5/3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \sim -5R_3 + R_2 \sim \begin{bmatrix} 0 & 1 & 5/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim -\tfrac{5}{3}R_3 + R_1 \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 er fri variabel:

$$\left.\begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\} = \underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Egenvektor for $\lambda = 3$:

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 er fri:

$$\begin{aligned} -x_1 &= -3x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned} \quad \to \quad \left.\begin{aligned} x_1 &= 3x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}\right\} = \underline{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}$$

Egenvektor for $\lambda = 5$:

$$\begin{bmatrix} -3 & 3 & 5 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -5/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -25/6 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 er fri variabel:

$$\left.\begin{aligned} x_1 &= 25/6\, x_3 \\ x_2 &= 5/2\, x_3 \\ x_3 &= x_3 \end{aligned}\right\} = \underline{\begin{bmatrix} 25/6 \\ 5/2 \\ 1 \end{bmatrix}}$$

Sjekker at alle er lineært uavhengige:

$$\det\left(\begin{bmatrix} 1 & 3 & 25/6 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1(1) - 3(0) + 0 = 1$$

det(matrise) $\neq 0 \rightarrow$ kolonnene er lineært uavhengige.

Basisen for løsningsrommet blir:

$$\underline{\underline{\left\{ c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t},\ c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} e^{3t},\ c_3 \begin{bmatrix} 25/6 \\ 5/2 \\ 1 \end{bmatrix} e^{5t} \right\}}}$$

Generell løsning blir:

$$\underline{\underline{c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 25/6 \\ 5/2 \\ 1 \end{bmatrix} e^{5t}}}$$

b) $\begin{bmatrix} 3-\lambda & -1 & 2 \\ 3 & -1-\lambda & 6 \\ -2 & 2 & -2-\lambda \end{bmatrix}$

$(3-\lambda)((-1-\lambda)(-2-\lambda)-12))$

$-(-1)(3(-2-\lambda)+12) + 2(6+2(-1-\lambda)) = 0$

For $\lambda_1 = 2$ og $\lambda_2 = -4$

Egenvektor for $\lambda_1 = 2$:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 og x_3 er frie variabler:

$$\left.\begin{aligned} x_1 &= x_2 - 2x_3 \\ x_2 &= x_2 + 0x_3 \\ x_3 &= 0x_2 + x_3 \end{aligned}\right\} = \underline{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t,\ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} s}$$

Disse vektorene utgjør egenrommet til $\lambda = 2$.

For $\lambda = -4$:

$$\begin{bmatrix} 7 & -1 & 2 \\ 3 & 3 & 6 \\ -2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 7 & -1 & 2 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 7 & -1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 7 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} -7R_1 + R_2 \\ -2R_1 + R_3 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -8 & -12 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 er fri variabel:

$$\left.\begin{array}{l} x_1 = \frac{1}{2}x_3 \\ x_2 = -3/2\,x_3 \\ x_3 = x_3 \end{array}\right\} = \underline{\begin{bmatrix} -1/2 \\ -3/2 \\ 1 \end{bmatrix}} \sim \underline{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}$$

Basis: $\underline{\underline{\left\{ c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{2t},\ c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{2t},\ c_3\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}e^{-4t} \right\}}}$

Den generelle løsningen er summen av elementene i basisen.

3) a) $y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2\cdot 0} + c_2\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}e^{3\cdot 0} + c_3\begin{bmatrix} 25/6 \\ 5/2 \\ 1 \end{bmatrix}e^{5\cdot 0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ tok denne radreduksjonen i hodet

som betyr at $c_1 = 1,\ c_2 = 0,\ c_3 = 0$

$$= \underline{\underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t}}}$$

b) $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim -R_1 + R_2 \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & -2 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \sim -R_2 + R_3 \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$

$\sim -R_3 + R_2 \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \sim 2R_2 + R_1 \sim$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \sim -R_3 + R_1 \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right]$

$c_1 = 1,\ c_2 = -\frac{1}{3},\ c_3 = -\frac{2}{3}$

$$y = \underline{\underline{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}e^{2t} - \frac{1}{3}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}e^{2t} - \frac{2}{3}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}e^{-4t}}}$$

4) Teorem 13.14 (?) skjønte ikke

5) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix}$

a) $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ $y' = Ay$

Finner egenvektorer + egenverdier til A:

$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = (-\lambda)(1-\lambda) - 2 = 0$

$-\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$ for $\lambda_1 = -1$, $\lambda_2 = 2$

Egenvektor for $\lambda = -1$:

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \left.\begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix}\right\} = \underline{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

Egenvektor for $\lambda = 2$:

$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \to \left.\begin{matrix} -2x_1 = -x_2 \\ x_2 = x_2 \end{matrix}\right\} = \underline{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}$

Basis: $\underline{\underline{\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}}$ $y = c_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

b) $\begin{bmatrix} 0 \\ \frac{1}{2}e^{-2t} \end{bmatrix} \to y' = \begin{bmatrix} 0 \\ -e^{-2t} \end{bmatrix}$

$\begin{bmatrix} 0 \\ -e^{-2t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2e^{-2t} \end{bmatrix}} + \frac{1}{2} \begin{bmatrix} 3e^{-2t} \\ e^{-2t} \end{bmatrix}$

$\begin{bmatrix} 0 \\ -e^{-2t} \end{bmatrix} \neq \begin{bmatrix} 1/2e^{-2t} \\ 1/2e^{-2t} \end{bmatrix} + \begin{bmatrix} 3/2e^{-2t} \\ 1/2e^{-2t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}$

<u>Dette er ikke en løsning.</u>

$y = \begin{bmatrix} -e^{-2t} \\ 1/2e^{-2t} \end{bmatrix} \to y' = \begin{bmatrix} 2e^{-2t} \\ -e^{-2t} \end{bmatrix}$

$\begin{bmatrix} 2e^{-2t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -e^{-2t} \\ 1/2e^{-2t} \end{bmatrix} = \begin{bmatrix} 1/2e^{-2t} \\ -2e^{-2t} + \frac{1}{2}e^{-2t} \end{bmatrix} + \begin{bmatrix} 3/2e^{-2t} \\ 1/2e^{-2t} \end{bmatrix}$

$= (2e^{-2t}, -e^{-2t})$ <u>Dette er en løsning.</u>

c) generell løsning:

$$= \underbrace{c_1\begin{bmatrix}1/2\\1\end{bmatrix}e^{2t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-t}}_{y_h} + \underbrace{\begin{bmatrix}-e^{2t}\\1/2\,e^{2t}\end{bmatrix}}_{y_p}$$

Oppgaver til kapittel 14

1 a) $y'' - y = 0$ om til system

generelt: $y''(t) + a_1 y'(t) + a_0 y(t) = 0$

hvor $x_1 = y,\ x_2 = y' \rightarrow \begin{bmatrix}0 & 1\\-a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}'$

$a_1 = 0,\ a_0 = -1$

$$\underline{\underline{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}'}}$$

b) $y''(t) + 2y' + 3y = 0$

$$\underline{\underline{\begin{bmatrix}0 & 1\\-3 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}'}}$$

c) $y'' + y' = 0$

$a_1 = 1,\ a_0 = 0$

$$\underline{\underline{\begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}'}}$$

2) Finn generell løsning:

a) $y'' - y' - 2y = 0$, $a_1 = -1$, $a_0 = -2$

$\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$ egenverdier og egenvektorer

$\begin{bmatrix}-\lambda & 1\\2 & 1-\lambda\end{bmatrix}$ $\quad -\lambda(1-\lambda) - 2$

$-\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$ for $\lambda_1 = -1$, $\lambda_2 = 2$

Egenvektor for $\lambda = -1$

$\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \sim$ x_2 er fri variabel

$\left.\begin{matrix}x_1 = -x_2\\x_2 = x_2\end{matrix}\right\} = \underline{\begin{bmatrix}-1\\1\end{bmatrix}}$

Egenvektor for 2

$\begin{bmatrix}-2 & 1\\2 & -1\end{bmatrix} \sim \begin{bmatrix}-2 & 1\\0 & 0\end{bmatrix}$

x_2 er fri

$\left.\begin{matrix}-2x_1 = -x_2\\x_2 = x_2\end{matrix}\right\} = \begin{bmatrix}1/2\\1\end{bmatrix} \sim \underline{\begin{bmatrix}1\\2\end{bmatrix}}$

$y_h(t)$ for $a = \underline{\underline{c_1 e^{-t} + c_2 e^{2t}}}$

Husk den karakteristiske likningen:

$\lambda^2 + a_1\lambda + a_0 = 0$

$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

b) $y'' + y = 0$

$a_1 = 0$
$a_0 = 1$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1 \rightarrow \lambda_1 = i$
$\lambda_2 = -i$

For komplekse egenverdier $\lambda = a + bi$ m/ $b \neq 0$ er generell løsning på formen:

$y_h(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$

$y_h(t) = \underline{\underline{c_1\cos(t) + c_2\sin(t)}}$

c) $y'' - 4y' + 4y = 0$

$a_1 = -4$ $\quad \lambda^2 - 4\lambda + 4 = 0$
$a_0 = 4$ $\quad$ for $\lambda = 2$

$y(t) = \underline{\underline{tc_1 e^{2t} + c_0 e^{2t}}}$

For én reell egenverdi:
$y(t) = e^{\lambda t}$ om $\lambda = \frac{-a_1}{2} = 2$
som stemmer i dette tilfellet.
Generell:
$y_h(t) = c_1 t e^{\lambda t} + c_0 e^{\lambda t}$

3) Initialverdiproblemer

a) $y(0) = 0,\ y'(0) = 1$ for 2a)

generell løsning: $c_1 e^{-t} + c_2 e^{2t}$

$y(0) = 0 = c_1 e^{-0} + c_2 e^{2\cdot 0} = c_1 + c_2$

$y'(0) = -c_1 e^{-0} + 2c_2 e^{2\cdot 0} = -c_1 + 2c_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 2 & 1 \end{array}\right] \sim R_1 + R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 3 & 1 \end{array}\right] \sim \tfrac{1}{3}R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1/3 \end{array}\right]$$

$$\sim -R_2 + R_1 \sim \left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \end{array}\right] \quad \begin{array}{l} c_1 = -1/3 \\ c_2 = 1/3 \end{array}$$

$y_p(t) = \underline{\underline{-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}}}$

b) $y(\frac{\pi}{2}) = 1,\ y'(\frac{\pi}{2}) = 0$ for 2b)

generell: $c_1\cos(t) + c_2\sin(t)$
derivert: $-c_1\sin(t) + c_2\cos(t)$

$y(\frac{\pi}{2}) = 1 = c_1\cos(\frac{\pi}{2}) + c_2\sin(\frac{\pi}{2}) = c_2$

$y'(\frac{\pi}{2}) = 0 = -c_1$

$y_p(t) = \underline{\underline{\sin(t)}}$

c) $y(1)=0,\ y'(0)=-e^{-2}$ for 2c)

generell: $tc_1e^{2t}+c_2e^{2t}$

derivert: $2tc_1e^{2t}+2c_2e^{2t}$

$y(1)=0=c_1e^2+c_2e^2=e^2(c_1+c_2)$

$y'(0)=-e^{-2}=2c_2$

$$\left[\begin{array}{cc|c} e^2 & e^2 & 0 \\ 0 & 2 & -e^{-2} \end{array}\right] \sim \tfrac{1}{e^2}R_1 \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & -e^{-2} \end{array}\right]$$

$$\sim \tfrac{1}{2}R_2 \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -\frac{e^{-2}}{2} \end{array}\right] \sim -R_2+R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{e^{-2}}{2} \\ 0 & 1 & -\frac{e^{-2}}{2} \end{array}\right] \quad c_1=\frac{e^{-2}}{2},\ c_2=-\frac{e^{-2}}{2}$$

$$y_p(t)=\underline{\underline{\frac{e^{-2}}{2}te^{2t}-\frac{e^{-2}}{2}e^{2t}}}$$

Alle løsninger til den innhomogene likningen er på formen:

$y(t)=y_p(t)+y_h(t)$

y_p: partikulær, y_h: homogen

4) Finn generell løsning:

a) $y''-y'-2y=e^{-2t}$

$y_h(t)=c_1e^{-t}+c_2e^{2t}$

$f(t)=e^{-2t} \to y_p(t)=\underline{ce^{-2t}}$

ofte er det et mønster, $y_p(t)$ har samme form som $f(t)$

$(ce^{-2t})'=\underline{-2ce^{2t}}$

$(ce^{-2t})''=\underline{4ce^{2t}}$

setter alle disse inn i difflikningen

$4ce^{2t}+2ce^{2t}-2ce^{2t}=ce^{2t}$

$4ce^{2t}=ce^{2t}$

$4c=1$

$c=1/4 \to y_p(t)=\frac{1}{4}e^{-2t}$

$$y(t)=\underline{\underline{c_1e^{-t}+c_2e^{2t}+\frac{1}{4}e^{-2t}}}$$

b) Bruker formel for $y_p(t)$: $f(t) = e^{2t}$

$$e^{-t}\int_0^t \frac{e^{2s}e^{2s}}{\underbrace{-e^{2s}e^{-s}-e^{-s}2e^{2s}}} ds - e^{2t}\int_0^t \frac{e^{-s}e^{2s}}{\underbrace{-e^{2s}e^{-s}-e^{-s}2e^{2s}}_{=\frac{e^{s}}{-3e^{s}}}}$$

$$= \frac{e^{4s}}{-e^{2s-s}-2e^{-s+2s}} = \frac{e^{4s}}{-e^{s}-2e^{s}} = \frac{e^{4s}}{-3e^{s}}$$

$$= e^{-t}\int_0^t \frac{e^{4s}}{-3e^{s}} ds - e^{2t}\int_0^t \frac{e^{s}}{-3e^{s}} ds$$

$$= -\frac{1}{3}e^{-t}\int_0^t e^{s}\, ds + \frac{1}{3}e^{2t}\int_0^t 1\, ds$$

$$y_p(t) = -\frac{1}{3}e^{-t}(e^{t}-1) + \left(\frac{1}{3}te^{2t}\right) \leftarrow \text{bruker denne}$$

$$y(t) = c_1e^{-t} + c_2e^{2t} + \frac{1}{3}te^{2t}$$

c) $y'' + y = t$

$$y_h(t) = c_1\underbrace{\cos(t)}_{y_1(t)} + c_2\underbrace{\sin(t)}_{y_2(t)}$$

$$y_p(t) = y_2(t)\int_0^t \frac{y_1(t)f(t)}{y_1(t)y_2'(t)-y_2(t)y_1'(t)} dt - y_1(t)\int_0^t \frac{y_2(t)f(t)}{y_1(t)y_2'(t)-y_2(t)y_1'(t)} dt$$

$$= \sin(t)\int_0^t \frac{\cos(t)\cdot t}{\cos^2(t)+\sin^2(t)} dt - \cos(t)\int_0^t \frac{\sin(t)\cdot t}{\cos^2(t)+\sin^2(t)} dt$$

$$= \sin(t)\int_0^t t\cdot\cos(t)\, dt - \cos(t)\int_0^t t\cdot\sin(t)\, dt$$

delvis integrasjon

$u\,dv = uv - \int v\cdot du$

$u = t \qquad du = 1$

$dv = \cos(t) \qquad v = \sin(t)$

$\downarrow$ samme

$$\sin(t)\left(t\sin(t) - \int \sin(t)\, dt\right) - \cos(t)\left(t\sin(t) - \int\sin(t)\, dt\right)$$

$$\sin(t)\,(t\sin(t)+\cos(t)) \qquad - \cos(t)\,(\sin(t) - t\cos(t))$$

$$t\sin^2(t) + \sin(t)\cos(t) - \cos(t)\sin(t) + t\cos^2(t)$$

$$= \underline{t} = y_p(t)$$

$$y(t) = c_1\cos(t) + c_2\sin(t) + t$$

d) homogen løsning: $tc_1e^{2t} + c_2e^{2t}$

$y_1(t) = te^{2t}$
$y_2(t) = e^{2t}$

$f(t) = 4t \rightarrow y_p(t) = x + yt$ (et polynom av samme grad)

$(y_p(t))' = y$

$(y_p(t))'' = 0$

$-4y + 4(x+yt) = 4t$

$\underbrace{4x - 4y}_{=0} + \underbrace{4yt = 4t}_{= y=1}$

som betyr at $x=1$

$\rightarrow y_p(t) = 1 + t$

$y(t) = \underline{\underline{tc_1e^{2t} + c_2e^{2t} + t + 1}}$

5) For likning a): $a_1 = 0$, $a_0 = -1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'$$

karakteristiske polynom for matrisen:

$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \underline{\lambda^2 - 1}$

karakteristiske polynom for diff:

$y'' - y = 0$

om $y(t) = e^{xt} \rightarrow y'(t) = xe^{xt} \rightarrow y''(t) = x^2e^{xt}$

Innsetting av $y(t)$ i diff gir oss:

$\underline{(x^2 - 1)}e^{xt} = 0$

↑ Dette blir polynomet.

<u>Sammenhengen er at de er like.</u>

b) $\det\left(\begin{bmatrix} 0-\lambda & 1 \\ -3 & -2-\lambda \end{bmatrix}\right)$

$-\lambda(-2-\lambda) + 3$

$\underline{\lambda^2 + 2\lambda + 3}$ blir polynomet for matrisa

Om $y(t) = e^{xt} \rightarrow y'(t) = xe^{xt} \rightarrow y''(t) = x^2e^{xt}$
Innsetting av $y(t)$ i diff gir oss:

$$x^2e^{xt} + 2(xe^{xt}) + 3(e^{xt})$$

$$\underline{(x^2 + 2x + 3)}\, e^{xt}$$

polynomet for diff

$\underline{\underline{\text{De er like}}}$

c) $\begin{bmatrix} 0-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix}$

$-\lambda(-1-\lambda)$
$\underline{\lambda + \lambda^2}$ blir polynomet for matrisa

Om $y(t) = e^{xt} \rightarrow y'(t) = xe^{xt} \rightarrow y''(t) = x^2e^{xt}$
Innsetting av $y(t)$ i diff gir oss:

$$x^2e^{xt} + xe^{xt} = \underbrace{(x^2 + x)}\, e^{xt}$$

polynom for diff

$\underline{\underline{\text{De er like}}}$