Øving 4 ✓

11.4 – ~~2~~, ~~3~~, ~~9~~, ~~13~~

2) $f(x) = x \quad (-\pi < x < \pi)$

Finner Fourierrekka til $f(x) = x$ som er en odde funksjon

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

delvis integrasjon

$$\frac{1}{\pi}\left(x\frac{-\cos(nx)}{n}\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right)$$

$$\frac{1}{\pi}\left(\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right)\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{-2\pi n(-1)^n}{n^2}\right)$$

$$= \underline{-\frac{2(-1)^n}{n}}$$

gir rekka: $\underline{\sum_{n=1}^{N} -\frac{2(-1)^n}{n}\sin(nx)}$

$$E^* = \int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = 20.671$$

$$20.671 - \pi\left(4\sum_{n=1}^{N}\frac{(-1)^{2n}}{n^2}\right)$$

$$\left.\begin{array}{l} N=1 : 8.1046 \\ N=2 : 4.9630 \\ N=3 : 3.5668 \\ N=4 : 2.7814 \\ N=5 : 2.2787 \end{array}\right\} E^*$$

3) $f(x) = |x| \quad (-\pi < x < \pi)$

Like funksjon gjør at vi kan doble det ene integralet:

$$a_0 = \frac{1}{2\pi}\cdot 2\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \underline{\frac{\pi}{2}}$$

$$a_1 = \frac{1}{\pi}2\cdot\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{\pi}\left(x\frac{\sin(nx)}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(x\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Big|_0^{\pi}$$

$$= \underline{\frac{2(-1)^n - 2}{\pi n^2}}$$ trekker ut $\frac{2}{\pi}$

gir Fourierrekka:

$$\underline{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx)}$$

Regner ut E^* på samme måte som sist:

$$\underbrace{\int_{-\pi}^{\pi} |x|^2\,dx}_{} - \pi\left(2\cdot\left(\frac{\pi^2}{4}\right) - \frac{4}{\pi}\sum_{n=1}^{N}\left(\frac{(-1)^n}{n^2}\right)^2\right)$$

$$\int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

$N=1$: 0.0747
$N=2$: 0.0747
$N=3$: 0.0119
$N=4$: 0.0119
$N=5$: 0.0037

9) komplekse Fourierrekka av $f(x) = x\ (-\pi < x < \pi)$, $f(x) = f(x+2\pi)$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx = \text{delvis integrasjon}$$

$u = x \quad dv = e^{-inx}$
$du = 1 \quad v = -\frac{e^{-inx}}{in} = \frac{ie^{-inx}}{n}$

$$c_n = \frac{1}{2\pi}\left[-\frac{xe^{-inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{xe^{-inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi i^2 n^2}\left[e^{-inx}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{2in}\underbrace{(e^{-in\pi} + e^{in\pi})}_{\cos(n\pi)} + \frac{1}{2\pi n^2}\underbrace{(e^{in\pi} - e^{-in\pi})}_{\sin(\pi n)=0}$$

$$c_n = -\frac{(-1)^n}{in} = i\frac{(-1)^n}{n} \text{ for en } n \neq 0$$

Den komplekse Fourierrekka blir da:

$$\sum_{n=-\infty}^{\infty} \frac{i}{n}(-1)^n e^{inx}$$

13) $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots = \frac{\pi^4}{96}$

Fra en tidligere øving har vi at, for $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$

blir $a_0 = \frac{\pi}{2}$, $a_n = \frac{4}{\pi n^2}\cdot\cos(nx)$

$b_n = 0$

Parseval:

$$\frac{\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4\cos(nx)}{\pi n^2}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2\,dx$$

Løser: $\left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2\,dx\right) - \frac{\pi^2}{2}\right)\cdot\left(\frac{\pi}{4}\right)^2 =$

$$\int_{-\pi}^{0}(\pi+x)^2\,dx = \int_{0}^{\pi} u^2\,du = \frac{u^3}{3}\Big|_0^{\pi} = \frac{\pi^3}{3}$$

$$\int_{0}^{\pi}(\pi-x)^2\,dx = \int_{\pi}^{0} u^2\,du = \frac{u^3}{3}\Big|_{\pi}^{0} = \frac{\pi^3}{3}$$

$$\left.\right\} = \frac{2\pi^3}{3}$$

$$\left(\left(\frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) - \frac{\pi^2}{2}\right)\cdot\left(\frac{\pi}{4}\right)\right)^2 = \underline{\underline{\frac{\pi^4}{96}}}$$

oppgaver 11.7)

1) $$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\,d\omega = \begin{cases} 0 & \text{om } x<0 \\ \pi/2 & \text{om } x=0 \\ \pi e^{-x} & \text{om } x>0 \end{cases}$$

$f(x) = \pi e^{-x}$ for $x>0$: bruker eksempel 3 i boka:

$A(\omega) = \int_0^\infty e^{-x}\cos(\omega x)\,dx$ = Bruker Laplace:

$$\int_0^\infty e^{-st} f(t) = \mathcal{L}(f(t))(s)$$

Laplace til $\cos(\omega t)$ m/ $s=1$

$= \frac{1}{1+\omega^2} = A(\omega)$ og $B(\omega)$ er det samme, men m/ $\sin(\omega x)$

$B(\omega) = \frac{\omega}{1+\omega^2}$ (Laplace til $\sin(\omega x)$)

Setter inn i Fourierintegralet:

$$\int_0^\infty \frac{\cos(\omega x)}{1+\omega^2} + \frac{\omega\sin(\omega x)}{1+\omega^2}$$

$$= \underline{\underline{\int_0^\infty \frac{\cos(\omega x)+\omega\sin(\omega x)}{1+\omega^2}}}$$

for $f(x) = \frac{\pi}{2}$ i $x=0$: $\frac{\pi e^{-0} + 0}{2} = \underline{\underline{\frac{\pi}{2}}}$

for $f(x) = 0$ for $x<0$ vil alle integralene for $A(\omega)$ og $B(\omega) = 0$.

Oppgaver 11.9 Fouriertransform $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$

5) $f(x) = \begin{cases} e^x & \text{om } -a < x < a \\ 0 & \text{ellers} \end{cases}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^x e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{x(1-i\omega)}\,dx = \quad u = x(1-i\omega), \quad du = 1-i\omega\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{(1-i\omega)}\int_{-a(1-i\omega)}^{a(1-i\omega)} e^u\,du = \frac{1}{(1-i\omega)\sqrt{2\pi}}\underbrace{\left[e^{a(1-i\omega)} - e^{-a(1-i\omega)}\right]}_{\sinh(a(1-i\omega))}$$

$$\hat{f}(\omega) = \frac{\sinh(a-ai\omega)}{(1-i\omega)\sqrt{2\pi}}$$

7) $f(x) = \begin{cases} x & \text{om } 0 < x < a \\ 0 & \text{ellers} \end{cases}$ $\quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^a x e^{-i\omega x}\,dx$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{ixe^{-i\omega x}}{\omega}\Big|_0^a - \frac{i}{\omega}\left[-\frac{ie^{-i\omega x}}{\omega}\right]_0^1\right)$$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{iae^{-i\omega a}}{\omega} - \frac{i}{\omega}\left(-\frac{ie^{-i\omega}}{\omega} + \frac{i}{\omega}\right)\right)$$

$$\frac{i}{\omega}ae^{-i\omega a} - \frac{e^{-i\omega}}{\omega^2} + \frac{1}{\omega^2}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{-e^{-i\omega}+1}{\omega^2} + \frac{i}{\omega}ae^{-i\omega a}\right)$$

9) $f(x) = \begin{cases} |x| & \text{om } -1 < x < 1 \\ 0 & \text{ellers} \end{cases}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\underbrace{\int_{-1}^{0} -x e^{-i\omega x}\,dx}_{(1)} + \underbrace{\int_0^1 x e^{-i\omega x}\,dx}_{(2)}\right)$$

(1) $-\int_{-1}^{0} x e^{-i\omega x}\,dx = \quad u = x, \quad du = 1dx, \quad v = e^{-i\omega x}, \quad dv = \frac{ie^{-i\omega x}}{\omega}$

$$\left(\frac{ixe^{-i\omega x}}{\omega}\Big|_{-1}^{0} + \frac{i}{\omega}\int_{-1}^{0} e^{-i\omega x}\,dx\right)$$

$$\left(-\frac{i}{\omega}e^{i\omega} + \frac{i}{\omega}\left[\frac{ie^{-i\omega x}}{\omega}\right]_{-1}^{0}\right)$$

$$\left(-\frac{i}{\omega}e^{i\omega} + \frac{i}{\omega}\left(\frac{i}{\omega} - \frac{ie^{i\omega}}{\omega}\right)\right)$$

$$-\frac{ie^{i\omega}}{\omega} - \frac{1}{\omega^2} + \frac{e^{i\omega}}{\omega^2}$$

$$= \frac{-1 + e^{i\omega}(1-i\omega)}{\omega^2}$$

(2) $\left(\frac{ixe^{-i\omega x}}{\omega}\Big|_0^1 + \frac{i}{\omega}\int_0^1 e^{-i\omega x}\,dx\right)$

$$= \left(\frac{ie^{-i\omega}}{\omega} + \frac{i}{\omega}\left[\frac{ie^{-i\omega x}}{\omega}\right]_0^1\right)$$

$$\frac{ie^{-i\omega}}{\omega} + \frac{i}{\omega}\left(\frac{ie^{-i\omega}}{\omega} - \frac{i}{\omega}\right)$$

$$= \frac{ie^{-i\omega}}{\omega} - \frac{e^{-i\omega}}{\omega^2} - \frac{1}{\omega^2}$$

$$= \frac{e^{-i\omega}(1+i\omega) - 1}{\omega^2}$$

$$(1)+(2)\cdot\frac{1}{\sqrt{2\pi}} = \frac{i\omega e^{-i\omega}+e^{-i\omega}-1-1+e^{i\omega}-i\omega e^{i\omega}}{\omega^2}$$

$$= \frac{i\omega(e^{-i\omega}-e^{-i\omega})+e^{-i\omega}+e^{i\omega}-2}{\sqrt{2\pi}\,\omega^2}$$