Øving 3

11.1 – ~~2~~, ~~15~~, ~~17~~, ~~21~~

Oppgaver 11.1

2) Fundamentalperioden til:

$\cos nx$ og $\sin nx$ er $\frac{2\pi}{n}$, fordi:

La P være perioden: $f(x) = f(x+P)$

$$\cos(nx) = \cos n(x+P)$$
$$\cos(nx) = \cos(nx + nP)$$

Siden vi vet at cos/sin(x) begge har periode 2π:

$$\cancel{nx} + 2\pi = \cancel{nx} + nP \rightarrow P = \underline{\underline{\frac{2\pi}{n}}}$$

Fra dette kan vi se at fundamentalperioden til:

$\cos\left(\frac{2\pi x}{k}\right)$ blir $\frac{2\pi}{\left(\frac{2\pi}{k}\right)} = \underline{\underline{k}}$ (samme for $\sin\left(\frac{2\pi x}{k}\right)$)

For cos og $\sin\left(\frac{2\pi n x}{k}\right)$: $\frac{2\pi}{\left(\frac{2\pi n}{k}\right)} = \frac{\cancel{2\pi}}{\cancel{2\pi}\cdot n}\cdot k = \underline{\underline{\frac{k}{n}}}$

15) Fourierrekken til $f(x) = x^2$ $(0 < x < 2\pi)$:

transformerer integralet fra:

$$\int_0^{2\pi} x^2\,dx \quad \text{til} \quad \int_{-\pi}^{\pi} (x+\pi)^2\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+\pi)^2\,dx \qquad u = x+\pi, \quad u(\pi) = 2\pi,\ u(-\pi) = 0$$

$$\frac{1}{\pi}\int_0^{2\pi} u^2\,du = \frac{1}{\pi}\left[\frac{u^3}{3}\right]_0^{2\pi} = \frac{(2\pi)^3}{3}\cdot\frac{1}{\pi} = \frac{8\pi^3}{3}\cdot\frac{1}{\pi} = \underline{\frac{8\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)^2\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx) + 2\pi x\cos(nx) + \pi^2\cos(nx)\,dx$$

$$= x^2\frac{\sin(nx)}{\pi n}\Big|_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi} x\sin(nx)\,dx + \underbrace{2\pi\int_{-\pi}^{\pi} x\cos(nx)\,dx}_{=0} + \pi^2\int_{-\pi}^{\pi}\cos(nx)\,dx$$

$$= \underbrace{\cancel{\frac{2\pi\sin(\pi n)}{n}}}_{=0} - \frac{2}{\pi n}\int_{-\pi}^{\pi} x\sin(nx)\,dx + \pi^2\left(\frac{\sin(nx)}{n}\right)_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -\frac{2x\cos(nx)}{\pi n^2}\Big|_{-\pi}^{\pi} + \frac{2}{\pi n^2}\int_{-\pi}^{\pi}\cos(nx)\,dx$$

$$= \frac{4\cos(\pi n)}{\pi n^2} + \frac{2}{\pi n^2}\int_{-\pi n}^{\pi n}\cos(u)\,du$$

$$= \frac{4\cos(\pi n)}{\pi n^2} + \pi^2\cancel{\left(\frac{\sin(nx)}{n}\right)_{-\pi}^{\pi}} = 0 \qquad + \cancel{\frac{4\sin(\pi n)}{\pi n^3}} = 0$$

$$\underline{a_n = \frac{4(-1)^n}{\pi n^2}} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx) + 2\pi x\sin(nx) + \pi^2\sin(nx)\,dx$$

deler opp integralet

$$\int_{-\pi}^{\pi} x^2\sin(nx) = 0$$ odde funksjon over symmetriske intervall

$$2\pi\int_{-\pi}^{\pi} x\sin(nx)dx = 2x\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \frac{2\pi}{n}\int_{-\pi}^{\pi}\cos(nx)dx = -\frac{4\pi^2(-1)^n}{n} + \cancel{\frac{4\pi\sin(\pi n)}{n^2}}^{=0}$$

$$\pi^2\int_{-\pi}^{\pi}\sin(nx)dx = \left(\frac{\cos(nx)}{n}\right)_{-\pi n}^{\pi n} = 0 \qquad \underline{b_n = -\frac{4\pi^2(-1)^n}{n\pi}}$$

Fourier:

$$\frac{8\pi^3}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi n^2}\cos(nx) - \frac{4\pi^2(-1)^n}{n}\sin(nx)$$

17) Stykkevis funksjon: $f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ -x+\pi, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} x+\pi\,dx + \frac{1}{2\pi}\int_{0}^{\pi} -x+\pi\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\Big(\underbrace{\int_{-\pi}^{0} x\cos(nx) + \pi\cos(nx)\,dx}_{} + \int_{0}^{\pi} -x\cos(nx) + \pi\cos(nx)\,dx$$

$$\frac{1}{\pi}\left(\frac{x\sin(nx)}{\pi n}\right)\Big|_{-\pi}^{0} - \frac{1}{\pi n}\int_{-\pi}^{0}\sin(x)\,dx + \int_{-\pi}^{0}\cos(nx)\,dx$$

$$\left(\frac{1-\cos(n\pi)}{\pi n^2}\right)$$ og det neste integralet blir det samme, men m/ motsatt fortegn.

$$a_n = \frac{2(1-\cos(n\pi))}{\pi n^2}, \quad \cos(n\pi) = (-1)^n$$

$$a_n = \frac{2(1-(-1)^n)}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0}(x+\pi)\sin(nx)\,dx + \int_{0}^{\pi}(-x+\pi)\sin(nx)\,dx\right) = 0$$

Odde funksjon over et symmetrisk intervall $= 0$.

Fourierrekka: $$\underline{\underline{\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{\pi n^2}\cos(nx)}}$$

21) $f(x) = \begin{cases} -x-\pi, & -\pi < x < 0 \\ -x+\pi, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$ ser det fra grafen

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\cos(nx)\,dx$$

Liknende integraler som i forrige oppgave, men forskjellige fortegn gir:

$$= \frac{1-\cos(n\pi)}{n^2\pi} - \frac{\cos(n\pi)-1}{n^2\pi} = \underline{0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin(nx)\,dx + \int_{0}^{\pi}(-x+\pi)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}(-x+\pi)\sin(nx)\,dx$$

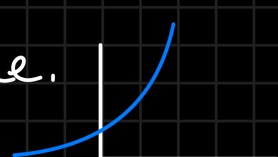
$$= \frac{2}{\pi}\left(\frac{x\cos(nx)}{nx}\Big|_0^{\pi} - \frac{1}{n^2}(\sin(nx))_0^{\pi} - \frac{\pi}{n}(\cos(nx))_0^{\pi} \right)$$

$$= \frac{2}{\pi}\left(\frac{\pi}{n}\cos(n\pi) - 0 - \frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\right)$$

$$= \underline{\frac{2}{n}}$$ Fourierrekka blir $\underline{\underline{2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}}}$

<u>Oppgaver 11.2</u> ~~1~~, ~~10~~, ~~17~~, ~~24~~, 29

1) e^x er verken odde eller like.
(ingen symmetri)

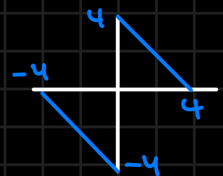


$e^{-|x|}$ er symmetrisk om y-aksen, og er derfor like.

$x^3\cos(nx) \rightarrow$ produktet av en odde funksjon x^3 og en like funksjon $\cos(nx)$ = en odde funksjon.

$x^2\tan(\pi x) \rightarrow \tan(\pi x)$ er en odde funksjon, x^2 er like
= odde funksjon

$\sinh(x) - \cosh(x) \rightarrow$ differansen mellom en like ($\cosh$) og en odde ($\sinh$) funksjon er verken like eller odde.

10) 

$$f(x) = \begin{cases} -x+4 & , x \in (0,4) \\ -x-4 & , x \in (-4,0) \end{cases}$$

Odde funksjon, $a_n = 0$, antar periode $2L$

Består bare av sinusledd.

$$b_n = \frac{1}{L}\int_{-4}^{0} -x\sin(\pi x) - 4\sin(nx)\,dx + \int_0^4 -x\sin(nx) + 4\sin(nx)\,dx$$

(1) (2)

Integral 1:

$$\frac{1}{L}\int_{-4}^{0} -x\sin(nx)\,dx - 4\int_{-4}^{0}\sin(nx)\,dx$$

$$\frac{1}{L}\left(\frac{x\cos(nx)}{n}\Big|_{-4}^{0} - \frac{1}{n}\int_{-4}^{0}\cos(nx)\,dx - \frac{4}{n}\int_{-4}^{0}\sin(nx)\,dx \right)$$

$$\frac{1}{L}\left(\frac{4\cos(4n)}{n} - \frac{\sin(4n)}{n^2} - \frac{8\sin^2(2n)}{n} \right) = \frac{4n - \sin(4n)}{Ln^2}$$

Integral 2 blir det samme:

Fourierrekka blir: $\underline{\underline{\sum_{n=1}^{\infty} \frac{2(4n - \sin(4n))}{Ln^2} \cdot \sin(nx)}}$

17) Ser at $f(x)$ er like (symmetri om y-aksen). Dvs. $b_n = 0$ og rekka består bare av cosinusledd.

$$a_0 = \frac{1}{2L}\int_{-1}^{0} x+1\,dx + \frac{1}{2L}\int_0^1 1-x\,dx$$

$$\frac{1}{2L}\left(\frac{x^2}{2}+x\right)_{-1}^{0} + \frac{1}{2L}\left(x - \frac{x^2}{2}\right)_0^1$$

$$\frac{1}{2L}\left(-\left(\frac{1}{2}-1\right)\right) + \frac{1}{2L}\left(\frac{1}{2}\right) = \frac{1}{2L}$$

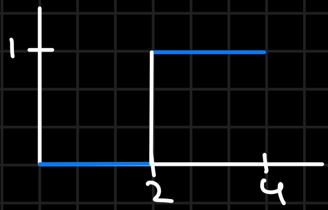
$$a_n = \frac{1}{L}\left(\int_{-1}^{0}(x+1)\cos(nx)\,dx + \int_0^1 (1-x)\cos(nx)\,dx\right)$$

$$= \frac{1}{L}\left(\int_{-1}^{0} x\cos(nx)\,dx + \int_{-1}^{0}\cos(nx)\,dx\right)$$

$$\frac{1}{L}\left(\frac{x\sin(nx)}{n} - \frac{1}{n}\int_{-1}^{0}\sin(nx)\,dx + \int_{-1}^{0}\cos(nx)\,dx \right)$$

$$= \frac{1}{L}\left(\frac{1-\cos(n)}{n^2}\right)\cdot 2$$ ganger m/2 pga neste integral bare har snudd om alle fortegn. verdiene er like

Fourierrekka blir $\underline{\underline{\frac{1}{2L} + \frac{2}{L}\sum_{n=1}^{\infty} \frac{1-\cos(n)}{n^2} \cdot \cos(nx)}}$

24) 

a) Fourier-cosinusrekka:

$$f(x) = \begin{cases} 0, & 0 < x < 2 \\ 1, & 2 < x < 4 \end{cases}$$

4-periodiske

$$a_0 = \frac{1}{4}\int_0^2 0\,dx + \frac{1}{2}\int_2^4 1\,dx = 1$$

$$a_n = \frac{1}{4}\int_2^4 \cos\left(\frac{n\pi x}{4}\right) dx = \begin{matrix} u = \frac{\pi}{4}nx & u(4) = \pi n \\ du = \frac{\pi}{4}n & u(2) = \pi n/2 \end{matrix} \quad \frac{1}{\pi n}\int_{\pi n/2}^{\pi n} \cos(u)\,du =$$

$$\frac{\sin(\pi n) - \sin(\frac{\pi}{2}n)}{\pi n} = \frac{-\sin(\frac{\pi}{2}n)}{\pi n} =$$

teller varierer om n er odde eller partall

0 om n er 0, 2, 4...
1 om n er 1, 5, 9... $4k-3$
−1 om n er 3, 7, 11. $4k-1$

Fourier-cosinusrekka:

$$1 + \sum_{k=1}^{\infty} \frac{-\sin(\frac{\pi}{2}(4k-3))}{(4k-3)\pi}\cos((4k-3)x) + \sum_{k=1}^{\infty} \frac{-\sin(\frac{\pi}{2}(4k-1))}{(4k-1)\pi}\cos((4k-1)x)$$

Fourier-sinusrekka:

$$b_n = \frac{1}{2}\int_2^4 \sin\left(\frac{n\pi x}{4}\right) dx = \begin{matrix} u = \frac{\pi}{4}nx \\ du = \frac{\pi}{4}n \end{matrix} \quad \int_{\pi n/2}^{\pi n} \sin(u)\,du = \left[-\cos(u)\right]_{\pi n/2}^{\pi n} \cdot \frac{1}{2}$$

$$= \frac{\cos(\frac{\pi}{2}n) - \cos(\pi n)}{2}$$

hvor $\cos(\pi n) = (-1)^n$
$\cos(\frac{\pi}{2}n) =$ 0 om $n =$ oddetall
1 om $n = 2, 4, 8...$
−1 om $n = 6, 10, 14...$

Fourier-sinusrekka:

$$\sum_{n=1}^{\infty} \frac{\cos(\frac{\pi}{2}n) - (-1)^n}{2\pi n}\sin(nx)$$

29) $f(x) = \sin(x)\ (0 < x < \pi)$

Fourier-sinusrekka til sin(x) er bare $\underline{\underline{\sin(x)}}$.

Fourier-cosinusrekka til sin(x):

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} \sin(x)\,dx = \left[-\cos(x)\right]_0^{\pi} = 2 \cdot \frac{1}{2\pi} = \underline{\underline{\frac{1}{\pi}}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cdot\cos(nx)\,dx =$$

bruker identiteten $\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$

$$a_n = \frac{1}{2\pi}\int_0^{\pi} \sin(x - nx) + \sin(x + nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi n - n} \sin(u)\,du + \frac{1}{2\pi}\int_0^{\pi n + n} \sin(u)\,du =$$

$$\frac{1}{2\pi}\left(-\cos(u)\right)_0^{\pi n - n} + \frac{1}{2\pi}\left(-\cos(u)\right)_0^{\pi n - n}$$

$$= \frac{\cos(\pi n)+1}{\pi - \pi n^2} = \underline{\frac{(-1)^n+1}{\pi - \pi n^2}}$$

Fourier-cosinusrekka til $\sin(x)$:

$$\underline{\underline{\frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n+1}{\pi - \pi n^2}\cos(nx)}}$$

Oppgaver 11.3

15) steady-state oscillation av $y'' + cy' + y' = r(t)$ m/ $c > 0$, $k = 1$

$r(t) = \pi^2 t - t^3$, $-\pi < t < \pi$ og $r(t)$ er 2π-periodisk.

finner Fourierrekka til $r(t)$:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 t - t^3 \, dt = \frac{1}{2\pi}\left(\frac{\pi^2 t^2}{2} - \frac{t^4}{4}\right)_{-\pi}^{\pi} = 0$$

$r(t)$ er en odde funksjon, dvs. det holder å finne b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \pi^2 t \sin(nt) - t^3 \sin(nt) \, dt$$

deler opp integralet: $\pi^2 \int_{-\pi}^{\pi} t \sin(nt) \, dt = -\frac{t}{n}\cos(nt)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} -\frac{1}{n}\cos(nt) \, dt$

$u = t$ $du = 1$

$dv = \sin(nt)$ $v = -\frac{1}{n}\cos(nt)$

$$\pi^2\left(\frac{\sin(nt) - nt\cos(nt)}{n^2}\right)\Big|_{-\pi}^{\pi} = \pi^2\left(\frac{2\sin(\pi n) - 2\pi n(\cos(\pi n))}{n^2}\right)$$

$\cos(\pi n) = (-1)^n$

$\sin(\pi n) = 0$

$$= \pi^2\left(\frac{2\pi(-1)^n}{n}\right) = \frac{2\pi^3(-1)^n}{n}$$

neste integral: $\int_{-\pi}^{\pi} t^3 \sin(nt) \, dt$ delvis integrasjon $u = t^3$ $du = 3t^2$

$dv = \sin(nt)$

$v = -\frac{1}{n}\cos(nt)$

$$-\frac{t^3\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \frac{3}{n}\int_{-\pi}^{\pi} t^2 \cos(nt) \, dt$$

$$= -\frac{2\pi^3(-1)^n}{n} + \frac{6t\cos(nt)}{n^3}\Big|_{-\pi}^{\pi} - \frac{6}{n^3}\int_{-\pi}^{\pi} \cos(nt) \, dt$$

$$= \frac{12\pi(-1)^n}{n^3} - \frac{2\pi^3(-1)^n}{n}$$

Summerer integralene for en $b_n = \frac{12(-1)^n}{n^3}$

Fourierrekka til $r(t) = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3}\sin(nx)$

$$= -12\sin(x) + \frac{3}{2}\sin(2x) - \frac{4}{9}\sin(3x) + \frac{3}{16}\sin(4x) - \dots$$

Innsatt i diff.likning:

$y'' + cy' + y' = \frac{12(-1)^n}{n^3}\sin(nx)$, og vi vet: $y_n = A_n\cos(nt) + B_n\sin(nt)$

$$-n^2(A_n\cos(nt)+B_n\sin(nt))+cn(b_n\cos(nt-a\sin(nt))$$
$$+a_n\cos(nt)+b_n\sin(nt)=\frac{12(-1)^n}{n^3}\sin(nt)=y_p$$

$$-n^2a\cos nt-n^2b\sin nt+cnb\cos(nt)-cna\sin(nt)$$
$$+a\cos(nt)+b\sin(nt)$$

$$a(-n^2\cos(nt)-cn\sin(nt)+\cos(nt))+b(-n^2\sin(nt)$$
$$+cn\cos(nt)+\sin(nt))=\frac{12(-1)^n}{n^3}\sin(nt)$$

forsvinner om vi løser for 0

$$a=\frac{\left(\frac{12(-1)^n}{n^3}\sin(nt)\right)-b(-n^2\sin(nt)+cn\cos(nt)+\sin(nt))}{-n^2\cos(nt)-cn\sin(nt)+\cos(nt)}$$