Snow Trends and a Lindley Random Walk

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Introduction to the Lindley random walk

Definition:

- The Lindley random walk models waiting times or storage-like processes where the current value is based on the previous state plus some random increment, but bounded at some barrier.

Applications:

- Common in queuing theory and storage problems; useful for phenomena with random inputs and boundaries like warehouse volume and snow pack accumulation.

Motivation for modeling snow depth

Importance of snow:

- Snow influences hydrology, climate, and ecosystem dynamics, making snow a crucial indicator of climate change.
- Snow plays a key role in Earth's energy balance due to its high albedo and insulating properties, which protect Arctic permafrost from solar radiation and help prevent the release of carbon and methane gas.
- Assessing snow trends is essential, as mountain snow melt sustains rivers and reservoirs that support human and ecological water needs worldwide.
- Although many studies focus on snow presence or absence of snow water equivalent (SWE), snow depth is important for implications for climate feedback and energy exchange.

Challenges:

- Daily snow depth data has seasonality, correlation, and a boundary of zero.

Storage model

Storage balance equation: The process $\{X_t\}$ for $t \geq 0$ is defined as

- $X_t = \max(X_{t-1} + C_t, 0)$, where X_t is the snow depth at time t and $\{C_t\}$ represents daily changes.
- $t = (n-1)P + \nu$ for $\nu = 1, 2, \ldots, P$. Here, $P = 365$, while $n = 1, 2, \ldots, N$ and N is the total number of years.
- $\{C_t\}$ is a periodic changes process where α_2 represents the trend.

$$C_t = \mu_\nu + \alpha_2 t + \epsilon_t.$$

  where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$, IID.
- The periodic mean μ_ν is modeled by

$$\mu_\nu = \alpha_0 + \alpha_1 \cos\left(\frac{2\pi(\nu - \tau_1)}{P}\right).$$

Daily Snow Depths in Napoleon

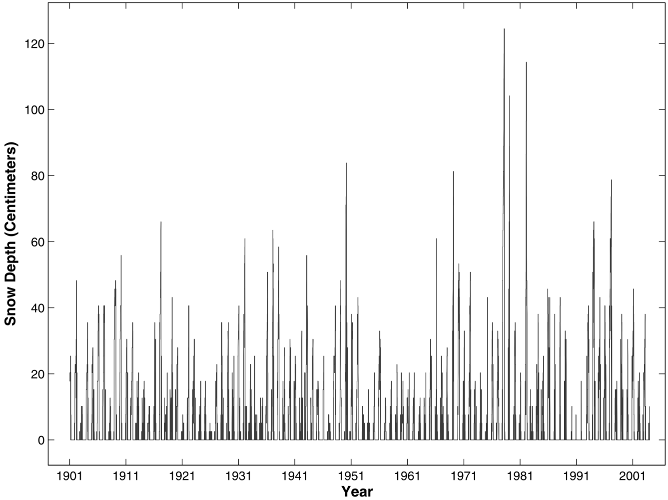


Figure 1. Daily snow depths at Napoleon, North Dakota, from 1 January 1901 to 6 December 2003.

Single Winter

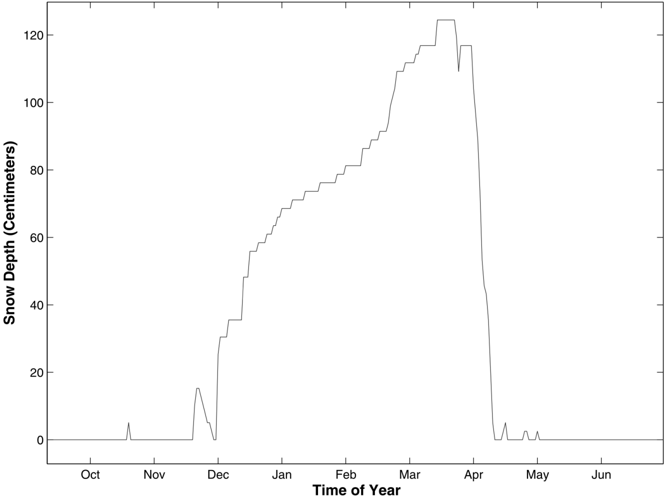


Figure 2. Daily Napoleon snow depths during the winter of 1977–1978.

Likelihood structure

- $L(\Theta|\mathbf{X}) = \Pi_{t=1}^{NP} \left[1_{\{X_t>0\}} f_{C_t}(C_t) + 1_{\{X_t=0\}} F_{C_t}(-X_{t-1})\right]$.

- Here, $1_{\{A\}}$ is an indicator for the event A. If A occurs, then $1_{\{A\}}$ is unity; otherwise, it is set to zero.
- Joint distribution for censored process C_t.

The Gaussian density is

$$f_{C_t}(c) = \frac{e^{-\frac{1}{2}\left(\frac{c-\mu_t}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}}, \qquad -\infty < c < \infty, \tag{1}$$

where

$$\mu_{\mathbf{t}} = \alpha_0 + \alpha_1 \cos\left(\frac{2\pi(\nu - \tau_1)}{P}\right) + \alpha_2 t$$

Here, $t = kP + \nu$ for some $k \in \{0, 1, 2, \ldots, d-1\}$ and $\nu \in \{1, 2, \ldots, P\}$. Likewise,

$$F_{C_t}(-X_{t-1}) = \int_{-\infty}^{-X_{t-1}} \frac{e^{-\frac{1}{2}\left(\frac{c-\mu_{\mathbf{t}}}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} dc.$$

Simulation study

$$C_t = \alpha_0 + \alpha_1 \cos\left(\frac{2\pi(\nu - \tau_1)}{P}\right) + \alpha_2 t + \epsilon_t.$$

α_0	α_1	τ_1	σ	α_2
-0.6	1	150	1	-0.000005

Table 1: Parameter values used in the simulation study.

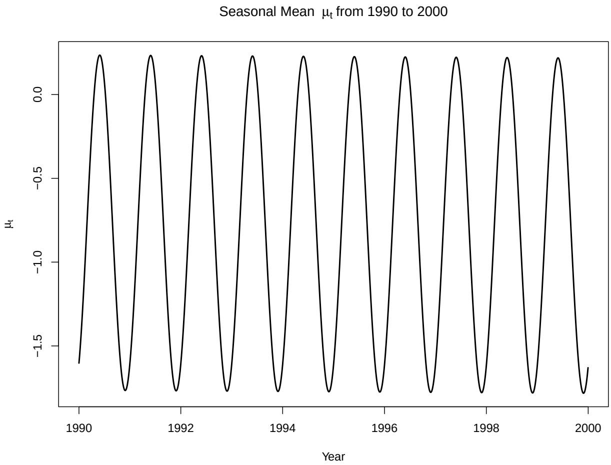


Figure 3: Seasonal Pattern of Snow from 1990 to 2000

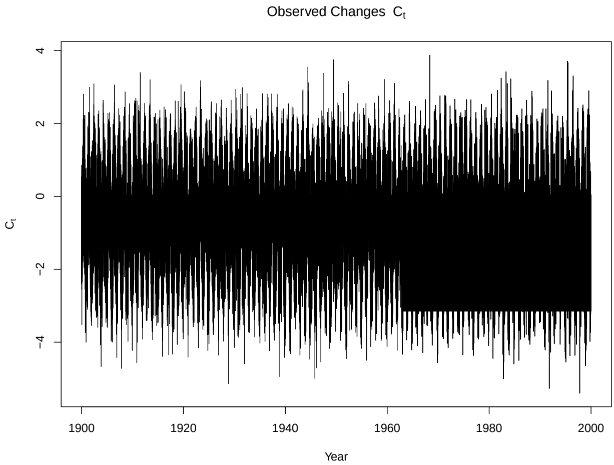
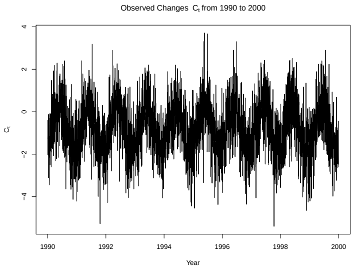
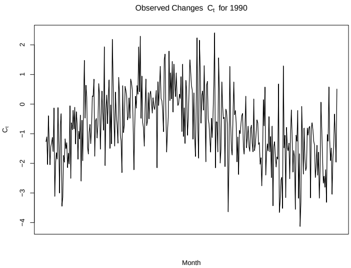


Figure 4: 100 Years of C_t Process

Plots



(a) 10 Years of C_t Process



(b) 1 Year of C_t Process

Plots continued

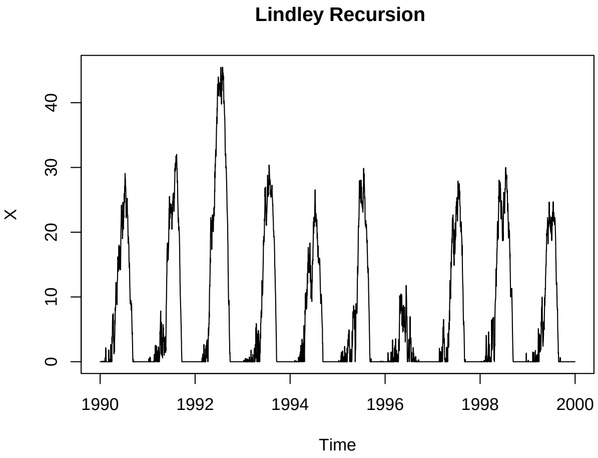


Figure 5: The Lindley Random Walk Model, X_t

Simulation overview

The simulation study:

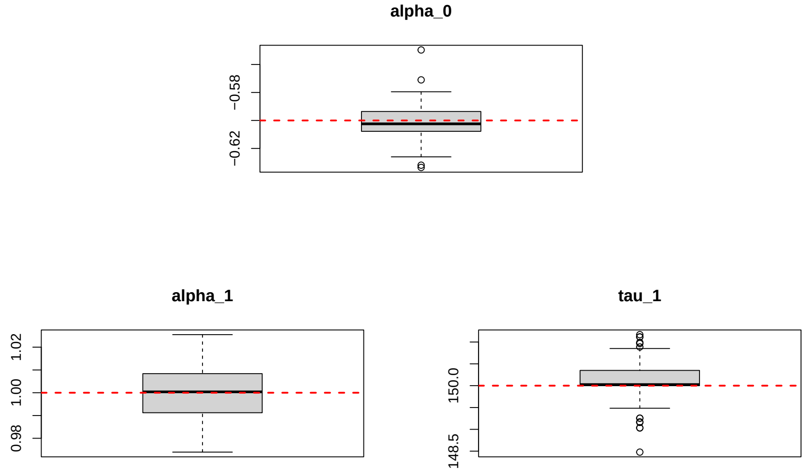
- Simulated $N = 100$ years of snow depths using a Lindley recursion.
- Obtained parameter estimates of each series of simulated snow depths.
- Repeated the process 100 times.
- Each iteration included:
  - Simulated a new dataset C_t and constructed X_t.
  - Obtained parameter estimates via maximum likelihood estimation via the optim function in R using the Nelder-Mead algorithm.
  - Stored the estimated parameters for later analysis.

Parameter estimates: summary statistics

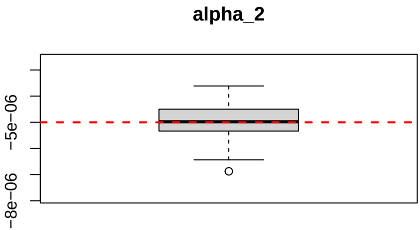
Parameters	α_0	α_1	τ_1	σ	α_2
True	-0.600	1.0000	150.00	1.00000	-0.00000500
Mean Estimate	-0.601	0.9987	150.12	0.99982	-0.00000493
Std Error	0.013	0.0111	0.4427	0.00472	0.00000062

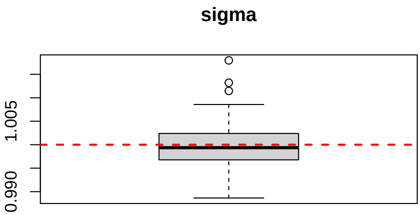
Table 2: Results of the simulation study: parameter estimates.

Plots: parameter estimates



Plots: parameter estimates





Summary of findings

Key Results:

- Likelihood approach allows us to increase complexity in the storage model while still actively obtaining parameter estimates.
- The Lindley recursion structure, combined with a flexible seasonal mean function and linear trend, allowed for a realistic simulation of snow depth dynamics.

Implications:

- Provides a framework for modeling and understanding seasonal snow depth evolution, particularly useful under scenarios of changing climate conditions.
- Enables more accurate simulations and predictions of snowpack behavior, which is vital for: water resource management and flood risk.

Future work

Extensions:

- Investigating seasonal trends.
- Adapting the model for other climatic variables such as rainfall or temperature.

Applications:

- Development of real-time prediction tools for snow-related disasters.
- Integration with geographic information systems (GIS) for enhanced spatial analysis.

Acknowledgments

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