MATH 318 Homework 1

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Problem 1

Part A

The sample space of this scenario is:

$$S = \left\{ \begin{array}{c} (T,T),(H,T,T),(T,H,T,T),(H,H,T,T),(T,H,H,T,T),(H,H,H,T,T), \\ (H,T,H,T,T),(H,H,H,H,H),(H,H,T,H,H),(T,H,H,H,H),(T,H,T,H,H), \\ (H,T,H,H,H),(T,H,H,H,T),(H,H,H,H,T),(H,H,T,H,T),(H,T,H,H,T), \\ (T,H,T,H,T),(T,H,H,T,H),(H,H,H,T,H),(H,T,H,T,H) \end{array} \right\}$$

Next, the probability of each outcome is $(\frac{1}{2})^n$ where n is the number of coin tosses:

$$P(T,T) = (\frac{1}{2})^2 = \frac{1}{4}$$
$$P(H,T,T) = (\frac{1}{2})^3 = \frac{1}{8}$$
$$P(T,H,T,T) = (\frac{1}{2})^4 = \frac{1}{16}$$
$$P(H,H,T,T) = (\frac{1}{2})^4 = \frac{1}{16}$$
$$P(T,H,H,T,T) = (\frac{1}{2})^5 = \frac{1}{32}$$
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$$P(H,H,H,T,H) = (\frac{1}{2})^5 = \frac{1}{32}$$
$$P(H,T,H,T,H) = (\frac{1}{2})^5 = \frac{1}{32}$$

Part B

Then, we can find the $P(E_i)$ for each $i \in \{2,3,4,5\}$ by counting the number of outcomes for each subset of number of tosses:

$$P(E_2) = \frac{1}{4}$$
$$P(E_3) = \frac{1}{8}$$
$$P(E_4) = \frac{1}{8}$$
$$P(E_5) = \frac{1}{2}$$

Problem 2

To start, let's find the probability of capturing 14 marked frogs in a row in terms of n:

$$P(M_{14}) = \frac{40}{n} \cdot \frac{39}{n-1} \cdot \frac{38}{n-2} \cdots \frac{27}{n-13} = \frac{40!}{26!} \cdot \frac{(n-14)!}{n!}$$

After that event, let's find the probability of capturing 36 unmarked frogs in a row in terms of n:

$$P(U_{36}) = \frac{n-40}{n-14} \cdot \frac{n-41}{n-15} \cdot \frac{n-42}{n-16} \cdots \frac{n-40-35}{n-14-35} = \frac{(n-40)!}{(n-76)!} \cdot \frac{(n-50)!}{(n-14)!}$$

Then, we can calculate the product of both $P(M_{14})$ and $P(U_{36})$ to find the probability of selecting 14 marked frogs and 36 unmarked frogs in a row:

$$P(M_{14} \cap U_{36}) = \frac{40!}{26!}\frac{(n-14)!}{n!} \cdot \frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{(n-14)!} = \frac{40!}{26!}\frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{n!}$$

Since getting 14 marked frogs and 36 unmarked frogs in a row is only one possible arrangement, we should multiply $P(M_{14} \cap U_{36})$ by $\binom{50}{14}$ to consider all possible arrangements::

$$L(n) = \binom{50}{14}\frac{40!}{26!}\frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{n!} = \frac{50!}{14! \cdot 36!}\frac{40!}{26!}\frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{n!}$$

Onto the main question, to show that the function L(n) is increasing up to some n and decreasing afterwards, we should find $\frac{L(n)}{L(n-1)}$ to make it clear:

$$\frac{L(n)}{L(n-1)} = \frac{50!}{14! \cdot 36!}\frac{40!}{26!}\frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{n!} \cdot \frac{14! \cdot 36!}{50!}\frac{26!}{40!}\frac{(n-77)!}{(n-41)!}\frac{(n-1)!}{n-51!}$$

$$\frac{L(n)}{L(n-1)} = \frac{(n-40)!}{(n-76)!}\frac{(n-50)!}{n!}\frac{(n-77)!}{(n-41)!}\frac{(n-1)!}{(n-51)!}$$

$$\frac{L(n)}{L(n-1)} = \frac{(n-40)(n-50)}{(n-76)n}$$

Finally, we can find the value of n that brings the maximum probability of grabbing 14 marked frogs by doing the following:

$$\frac{L(n)}{L(n-1)} = 1$$

$$\frac{(n-40)(n-50)}{(n-76)n} = 1$$

$$(n-40)(n-50) = (n-76)n$$

$$n^2 - 90n + 2000 = n^2 - 76n$$

$$14n = 2000$$

n**=142.857**

Rounding down to meet our inequality of $\frac{L(n)}{L(n-1)} \leq 1$, we have found that the maximum is reached when n **= 142**.

Problem 3

Part A

i.) To calculate $P(OnePair)$, we need to consider the probability of getting the next 4 cards and keep in mind that the order doesn't matter.

$$P(OnePair) = \binom{5}{2} \cdot \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{44}{49} \cdot \frac{40}{48}$$

$$P(OnePair) = \frac{352}{833} = \mathbf{0.42}$$

ii.) To calculate $P(TwoPairs)$, we can do a different strategy with face sets instead:

$$P(TwoPairs) = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}(\binom{52}{2})^{-}1$$

$$P(TwoPairs) = \frac{198}{4165} = \mathbf{0.047539}$$

Part B

i.) To calculate $P(OnePair)$, we can first choose the number will be a pair and which dice will be the pair. Also, we can choose which die are not the pair. We can calculate it like so:

$$P(OnePair) = \frac{\binom{6}{1}\binom{5}{2}\binom{5}{3}3!}{6^5}$$

$$P(OnePair) = \frac{3600}{7776} = \mathbf{0.46296}$$

ii.) To calculate $P(TwoPairs)$, we can first choose which two numbers will be a pair and which die will be pairs. We also need to choose which number out of the other 4 that the last remaining dice will be. We can calculate it like so:

$$P(TwoPairs) = \frac{\binom{6}{2}\binom{5}{2}\binom{3}{2}\binom{4}{1}}{6^5}$$

$$P(TwoPairs) = \frac{1800}{7776} = \mathbf{0.23148}$$

Problem 4

Part A

For $2n$ tosses, there needs to be n heads. Using the binomial coefficient, we can calculate it like this:

$$P_n = \frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{n!n!2^(2n)}$$

Part B

$$\frac{P_{n+1}}{P_n} = \frac{(2n+2)!}{(n+1)!(n+1)! \cdot\ 4 \cdot\ 2^{2n}} \frac{n!n!2^{2n}}{(2n)!}$$

$$\frac{P_{n+1}}{P_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)(n+1) \cdot\ 4}$$

$$\frac{P_{n+1}}{P_n} = \frac{4n^2+6n+2}{4n^2+8n+4}$$

Looking at the numerator and denominator, we can observe that $P_{n+1} < P_n$. Therefore, P_n must be decreasing.

Part C

We can calculate P_n using Sterling's Approximation:

$$P_n \approx \frac{\sqrt{4\pi n}(\frac{2n}{3})^{2n}}{\sqrt{2\pi n}(\frac{n}{e})^n \sqrt{2\pi n}(\frac{n}{e})^n (2)^{2n}} = \frac{\sqrt{4\pi n}}{\sqrt{4\pi n}\sqrt{\pi n}} \frac{(2)^{2n}(\frac{n}{e})^{2n}}{(2)^{2n}(\frac{n}{e})^{2n}} = \frac{1}{\sqrt{\pi n}}$$

From $P_n \approx \frac{\alpha}{\sqrt{n}}$, we conclude that α is $\frac{1}{\sqrt{\pi}}$.

Problem 5

Part A

If we have n balls and $m-1$ dividers, then we have $n+m-1$ total objects. So we can calculate the general case if we calculate $(n+m-1)$ choose n:

$$\binom{n+m-1}{n} = \binom{n+m-1}{m-1} = \frac{(n+m-1)!}{n!(m-1)!}$$

Part B

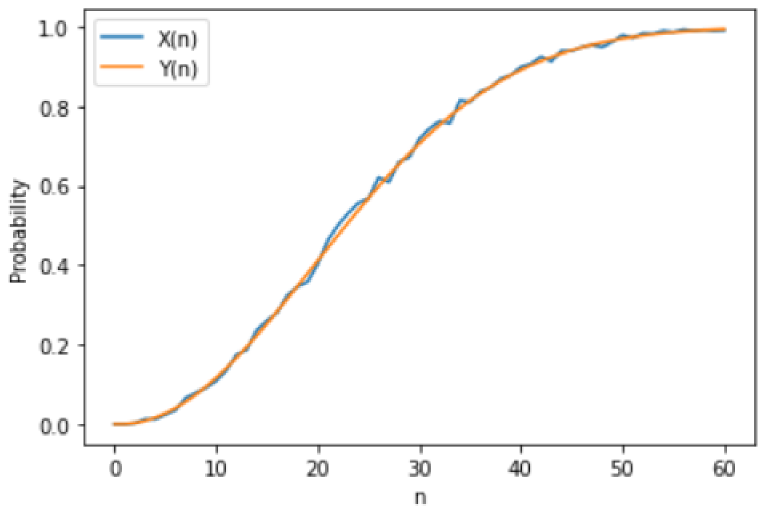
If an urn can only have at most 1 ball, then we can calculate m choose n to find how many ways we can place n balls into m urns:

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

Problem 6

Parts A-C

```
import numpy as np
import random
import matplotlib.pyplot as plt

def birthday(n):
    birthdays = np.random.randint(1,365,n)
    unique_birthdays = set(birthdays)

    if(len(unique_birthdays) == len(birthdays)):
        return 0
    else:
        return 1

Xn = [0] * 61
Yn = [0] * 61

total = 1000

for n in range(2,61):
    matches = 0
    for i in range(total):
        if birthday(n):
            matches+=1

    Xn[n] = matches/total
    Yn[n] = 1-(np.math.factorial(365)/(np.math.factorial(365-n)*365**n))
```



Part D

```
import numpy as np
import random
import matplotlib.pyplot as plt

def martian_birthday(n):
    birthdays = np.random.randint(1,670,n)
    unique_birthdays = set(birthdays)

    if(len(unique_birthdays) == len(birthdays)):
        return 0
    else:
        return 1

Xn = [0] * 61
Yn = [0] * 61

total = 1000

for n in range(2,61):
    matches = 0
    for i in range(total):
        if martian_birthday(n):
            matches+=1
```

```
    Xn[n] = matches/total
    Yn[n] = 1-(np.math.factorial(669)/(np.math.factorial(669-n)*669**n))

n = range(0,61)
plt.plot(n, Xn, label = 'X(n)')
plt.plot(n, Yn, label = 'Y(n)')
plt.xlabel('n')
plt.ylabel('Probability')
plt.legend()
plt.show()
```