MATH 318 Homework 2

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Problem 1

In order to find the proportion of the population that fills out fraudulent returns, we can first find out the population that fills out truthful returns.

$$P(Yes) = 0.45$$
$$P(Heads) = 0.5$$
$$P(Yes|Heads) = \frac{0.45}{0.5} = 0.90$$
$$P(No|Heads) = 1 - 0.90 = \mathbf{0.10}$$

There are **10%** of the honest population who cheat on their taxes.

Problem 2

Let's consider that A, B are independent events.

$$P(A \cap B) = P(A) \cdot P(B)$$

We can show that A, B^C are independent events.

$$P(A \cap B^C) = P(B) - P(A \cap B)$$
$$P(A \cap B^C) = P(B) - P(A) \cdot P(B)$$
$$P(A \cap B^C) = P(B) \cdot (1 - P(A))$$
$$\mathbf{P(A \cap B^C) = P(B) \cdot P(A^C)}$$

Additionally, we can show that A^C, B are independent events.

$$P(A^C \cap B) = P(A) - P(A \cap B)$$
$$P(A^C \cap B) = P(A) - P(A) \cdot P(B)$$
$$P(A^C \cap B) = P(A) \cdot (1 - P(B))$$
$$\mathbf{P(A^C \cap B) = P(A) \cdot P(B^C)}$$

Finally, we can prove that A^C, B^C are independent events as well.

$$P(A^C \cap B^C) = 1 - P(A \cup B)$$
$$P(A^C \cap B^C) = 1 - [P(A) + P(B) - P(A \cap B)]$$
$$P(A^C \cap B^C) = 1 - P(A) - P(B) + P(A \cap B)$$
$$P(A^C \cap B^C) = 1 - P(A) - P(B) + P(A) \cdot P(B)$$
$$\mathbf{P(A^C \cap B^C) = (1 - P(A)) \cdot (1 - P(B))}$$

Problem 3

Let's consider that A, B, C are independent events.

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = P(ABC)$$

We can prove the following by making the definition equal to the complement of the complement:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(A \cap B \cap C)$$
$$P(A \cup B \cup C) = 1 - (1 - P(A) - P(B) - P(C) + P(AB) + P(BC) + P(AC) - P(ABC)$$
$$\mathbf{P(A \cup B \cup C) = 1 - (1 - P(A)) \cdot (1 - P(B)) \cdot (1 - P(C))}$$

Alternatively, we could've also solved for the definition of P($A \cup B \cup C$) by foiling out $1 - (1 - P(A)) \cdot (1 - P(B)) \cdot (1 - P(C))$ as another solution.

Problem 4

For the condition of a 9-game series to be true, both teams each need to win 4 games out of the first 8. This means that we are solving for $P(A_4 \cap B_4)$. But we also have to keep in mind that order does not matter, so we must also multiply by a factor of $\binom{8}{4}$.

$$P(A_4 \cap B_4) = \binom{8}{4} \cdot p \cdot p \cdot p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p)$$
$$P(A_4 \cap B_4) = \binom{8}{4} \cdot p^4 \cdot (1-p)^4$$

Next, if we want to find the maximum, we should note that this function is continuous for p. Let f(p) represent $P(A_4 \cap B_4)$ in this case.

$$f(p) = \binom{8}{4} \cdot p^4 \cdot (1-p)^4$$
$$f'(p) = \binom{8}{4} \cdot (4p^3 \cdot (1-p)^4 + (-4 \cdot (1-p)^3 \cdot p^4))$$

$$f''(p) = \binom{8}{4} \cdot (12p^2 \cdot (1-p)^4 - 24p^3 \cdot (1-p)^3 + 12p^4 \cdot (1-p)^3)$$

Looking at f'(p) and f"(p), we can confirm that p = $\frac{1}{2}$ is a **local maximum** at -$\frac{1}{8}$.

Problem 5

Part A

The probability is the same for both strategies. For the first strategy, we can calculate the probability for each team member specifically to get P(S_1).

$$P(S_1) = 0.5p + 0.5p$$
$$P(S_1) = p$$

For the second strategy, we have to consider 3 cases:
(i) Both members having the correct answer = p^2
(ii) Both members having incorrect answers = $(1-p)^2$
(iii) One member has the correct answer while the other member has an incorrect answer. We also have to consider that order doesn't matter = $\frac{1}{2}\binom{2}{1}p(1-p) = \frac{1}{2}(2p(1-p)) = p(1-p)$

$$P(S_2) = p^2 + p(1-p)$$
$$P(S_2) = p^2 + p - p^2$$
$$P(S_2) = p$$

We can confirm that $P(S_1)$ **equals** $P(S_2)$ so that means the probability for both strategies is the same and there is no such thing as a better strategy.

Part B

Let's suppose $p = 0.6$ and the team chooses to adapt strategy (ii). Using Bayes' Theorem, we can calculate the conditional probability that the team gives the correct answer given they both agree.

$$P(Win|Agree) = \frac{P(Agree \cap Win) \cdot P(Win)}{P(Agree)}$$
$$P(Win|Agree) = \frac{p^2}{p^2 + (1-p)^2} = \frac{p^2}{p^2 + 1 - 2p + p^2} = \frac{p^2}{2p^2 - 2p + 1} \approx \mathbf{0.692}$$

We can also calculate the conditional probability that the team gives the correct answer given they disagree with their answers instead.

$$P(Win|Disagree) = \frac{P(Disagree \cap Win) \cdot P(Win)}{P(Disagree)}$$

$$P(Win|Disagree) = \frac{p(1-p)}{2p(1-p)} = \mathbf{\frac{1}{2}}$$

Problem 6

Based on the wording of the problem, we are given the following information.

$$P(Fraud) = \frac{1}{1000}$$
$$P(Legit) = 1 - P(Fraud) = \frac{999}{1000}$$
$$P(Pass|Legit) = 0.995$$
$$P(Fail|Fraud) = 0.99$$
$$P(Fail|Legit) = 1 - P(Pass|Legit) = 0.005$$

Part A

We want to calculate $P(Fraud|Fail)$ using Bayes' Theorem.

$$P(Fraud|Fail) = \frac{P(Fail|Fraud) \cdot P(Fraud)}{P(Fail)}$$
$$P(Fraud|Fail) = \frac{P(Fail|Fraud) \cdot P(Fraud)}{P(Fail|Fraud)P(Fraud) + P(Fail|Legit)P(Legit)}$$
$$P(Fraud|Fail) = \frac{0.99 \cdot \frac{1}{1000}}{0.99 \cdot \frac{1}{1000} + 0.005 \cdot \frac{999}{1000}}$$
$$P(Fraud|Fail) = \frac{0.99}{0.99 + 0.005 \cdot 999}$$
$$P(Fraud|Fail) \approx \mathbf{0.1654}$$

Part B

Let A denote the event that a transaction originates in Region A, and let B denote the event that a transaction originates in Region B. Then $P(Fraud|A) = 0.0005$ and $P(Fraud|B) = 0.002$. We can calculate $P(A)$ by using $P(Fraud)$.

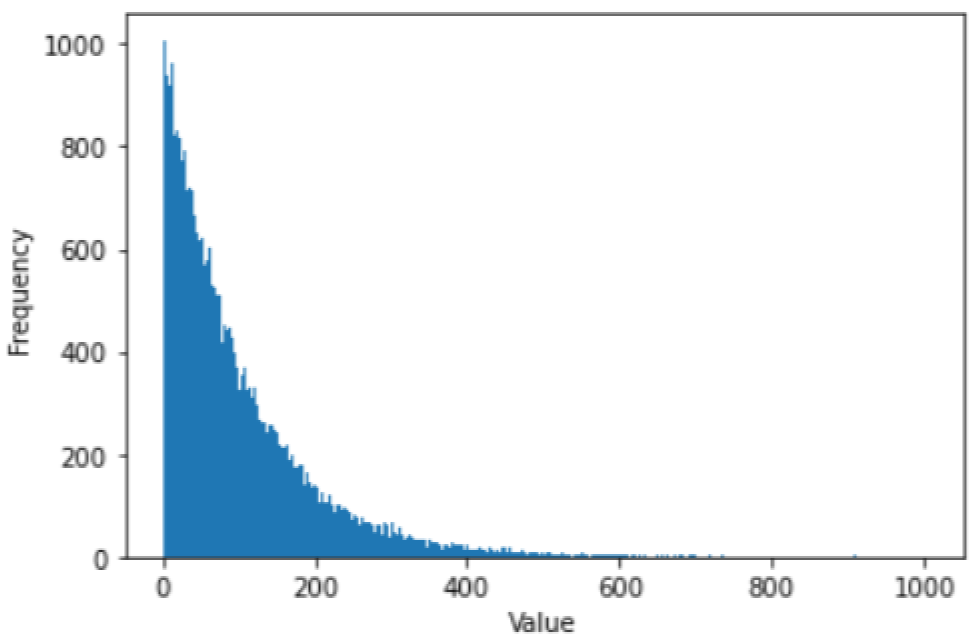
$$P(Fraud) = P(Fraud|A) \cdot P(A) + P(Fraud|B) \cdot P(B) = 0.001$$
$$0.001 = (0.0005) \cdot P(A) + (0.002) \cdot (1 - P(A))$$

There are a total of $\mathbf{\frac{2}{3}}$ of transactions in Region A.

Problem 7

Part A

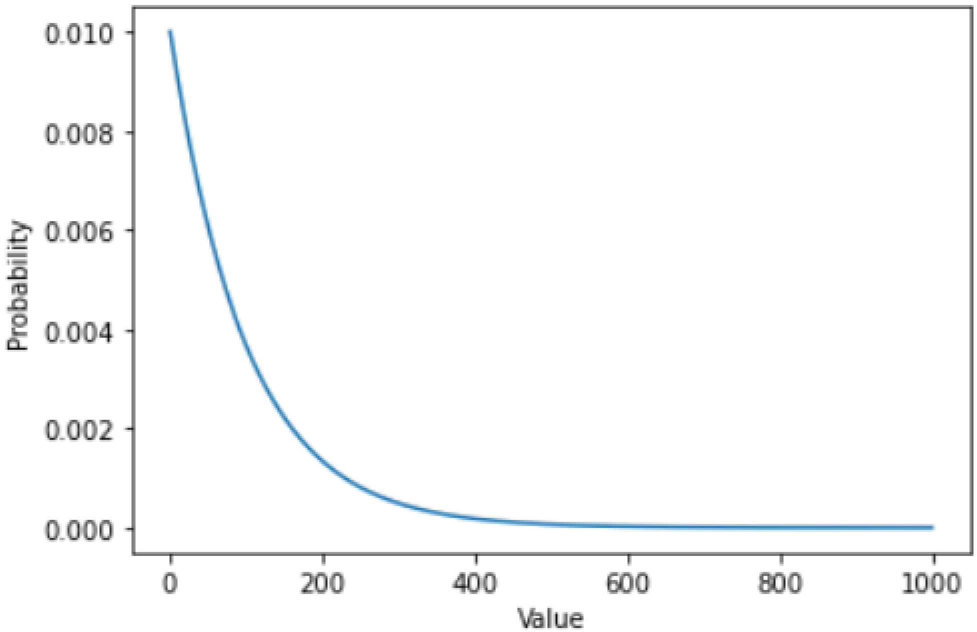
```
import numpy as np
import matplotlib.pyplot as plt

# Sample 100,000 geometric random variables with parameter
p = 0.01
size = 100000
data = np.random.geometric(p, size)

# Create a histogram of the resulting values, with buckets for each of the values 1 to 1000.
plt.hist(data, bins=range(1,1001))
plt.xlabel("Value")
plt.ylabel("Frequency")
plt.show()
```



Part B

```
# Probability mass function of a geometric random variable
p = 0.01
x = np.arange(1, 1001)
pmf = np.power(1 - p, x - 1) * p
```

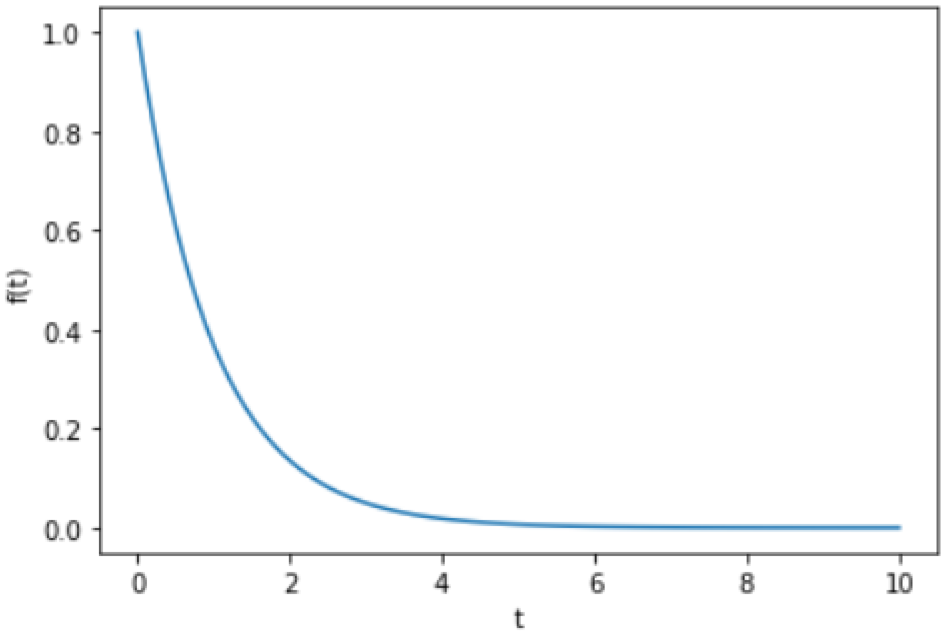
```
# Plot probability mass function
plt.plot(x, pmf)
plt.xlabel("Value")
plt.ylabel("Probability")
plt.show()
```



Unlike the histogram, this plot no longer has any noise or randomness. We would get this shape if we could average out an infinite number of times to approach the exact 'underlying' probability distribution and eliminate random variance.

Part C

```
# Plot function
plt.plot(np.linspace(0,10,100), np.exp(-x))
plt.xlabel("t")
plt.ylabel("f(t)")
plt.show()
```



The shape of this graph is identical to the earlier one, meaning that the continuous version of a geometric probability function is a smooth decaying exponential function.