Question 1

a)

In this part, I will generate 500 data points, with x on the interval $(0, 2\pi)$, and $y_i = 4sin(x_i) + Z_i$, where $Z_i \sim iid \ N(0, 1)$.

In [1]:
```
import numpy as np
import pandas as pd
import scipy.stats as st
import matplotlib.pyplot as plt
%matplotlib inline
x = np.linspace(start = 0, stop = 2 * np.pi, num = 500)
y = 4 * np.sin(x) + np.random.normal(loc = 0, scale = 1, size = len(x))
```

As instructed, I will use Gaussian Kernal in the model. The bandwidth will be evaluated based on cross-validated $L2$ norm.

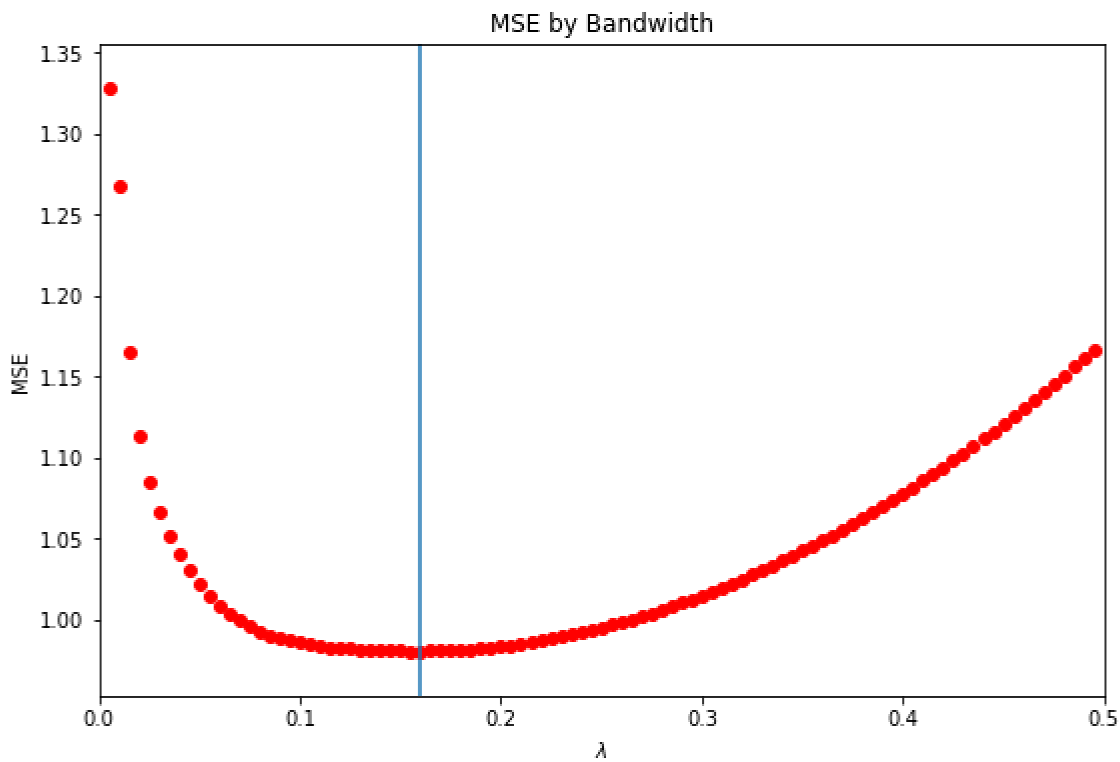
We can use function `norm.pdf` in the `scipy` package as the kernel function. Note that `norm.pdf`'s standard deviation argument works nicely as the bandwith parameter λ.

I will set λ ranges from .02 to 4, and pick the optimal λ value.

In [3]:
```
lambdas = np.arange(.005, .5, .005)
result = np.empty(len(lambdas) * len(x)).reshape(len(x), len(lambdas))
for i in range(len(x)):
    trainingx = np.delete(x, i)
    trainingy = np.delete(y, i)
    testx = x[i]
    testy = y[i]
    for j in range(len(lambdas)):
        densities = st.norm.pdf(x = trainingx, loc = testx, scale = lambdas[j
])
        predicted = sum(trainingy * densities) / sum(densities)
        result[i, j] = (testy - predicted) ** 2
```

In [4]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(lambdas, result.mean(0), color = "r", marker = "o")
ax.set_xlabel(r'$ \lambda $')
ax.set_ylabel("MSE")
ax.set_xlim(0, .5, 100)
ax.set_title("MSE by Bandwidth")
ax.axvline(x = lambdas[result.mean(0).argmin()])

plt.plot()
round(lambdas[result.mean(0).argmin()], 2)
```

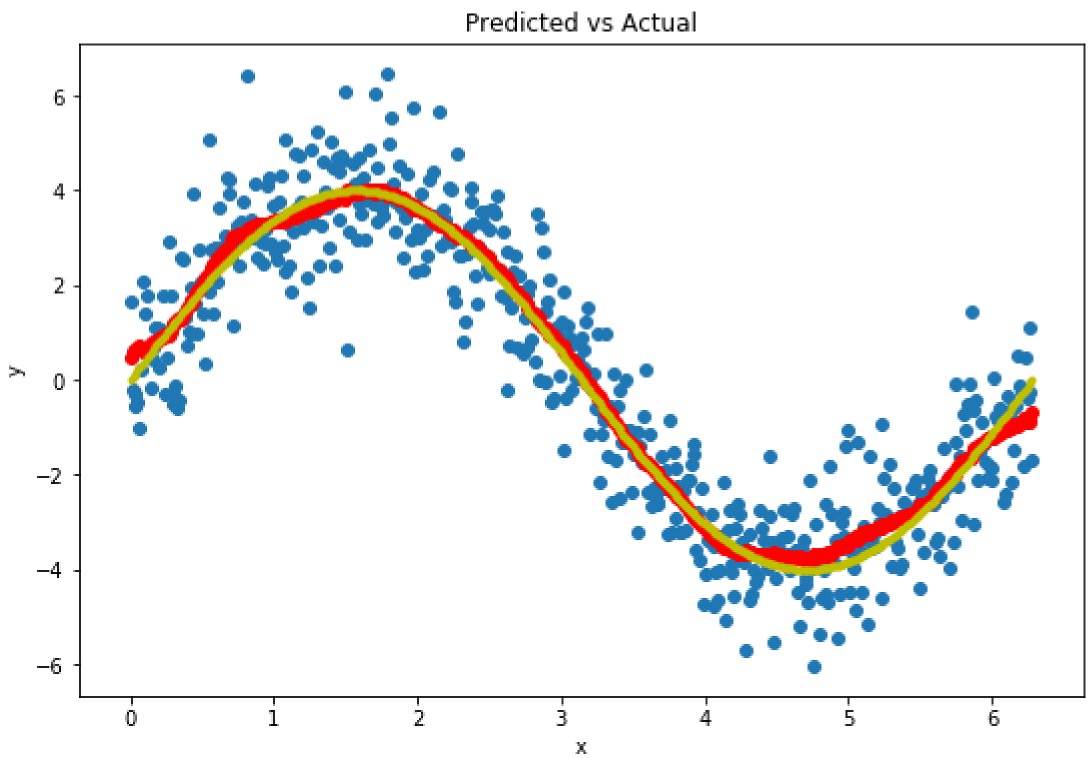
Out[4]: 0.16



The λ that achieved the minimum mean square error is 0.15. Now I am going to plot the predicted value as a line with the true value as points:

In [16]:
```
predicted = []
for i in range(len(x)):
    densities = st.norm.pdf(np.delete(x, i), loc = x[i], scale = lambdas[resul
t.mean(0).argmin()])
    predicted.append(sum(np.delete(y, i) * densities) / sum(densities))
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(x, y)
ax.scatter(x, predicted, color = "r")
ax.scatter(x, 4 * np.sin(x), marker = ".", color = "y")
ax.set_title("Predicted vs Actual")
ax.set_xlabel("x")
ax.set_ylabel("y")
plt.plot()
```

Out[16]: []



(b)

In this part, we are going to use a dataset specified by the homework requirement, and implement our algorithm on the real-world data.

In [184]:
```
videogames = pd.read_csv(r'D:\Zhao\Documents\fall_2017\542\hw4\videogames.csv'
)
videogames['Logged_Sales'] = np.log(1 + videogames.Global_Sales)
```

Because we are doing three kernal regression, each on one predictor variable, I will repeat the following procedure three times, each time on one predictor variable:

First, remove observations with missing values, for each of the predictor variable;

Second, find the best bandwidth λ with cross validation;

Third, with the best bandwidth, find the MSE.

After all the procedured repeated, I will compare the best MSE across the three models, and decide which predictor variable works the best.

Note that if we want to do leave-one-out-cross-validation, we need to compute density of each of the n points, based on all other $n-1$ data points, and take the average. Therefore, the time complexity is $O(n^2)$.

If we want to do k-fold cross validation, we still need to compute the density of each of the n points, based on all points that are not in the fold, of which there are $\frac{n(k-1)}{k}$. Therefore, the time complexity is $O(n^2\frac{k-1}{k})$.

With thousands of observations, it is still expensive to do even 10-fold cross validation.

Initially, I was using `R` to implement my algorithm. Due to `R`'s low speed, I had to implement cross-validation like this: I used k-fold cross validation. However, each fold will be the training data set, with all observations outside the fold being the testing dataset. That way, we can reduce time complexity to $O(\frac{n^2}{k})$. Note that this strategy is different from the usual cross-validation on many other analysis methods. For example, in linear regression, the cross validation is performed in such a way that each fold is a testing dataset, and all observations outside a fold are training dataseet.

But still the speed is quite slow. Therefore, I decided to use `Python` to implement the algorithm.

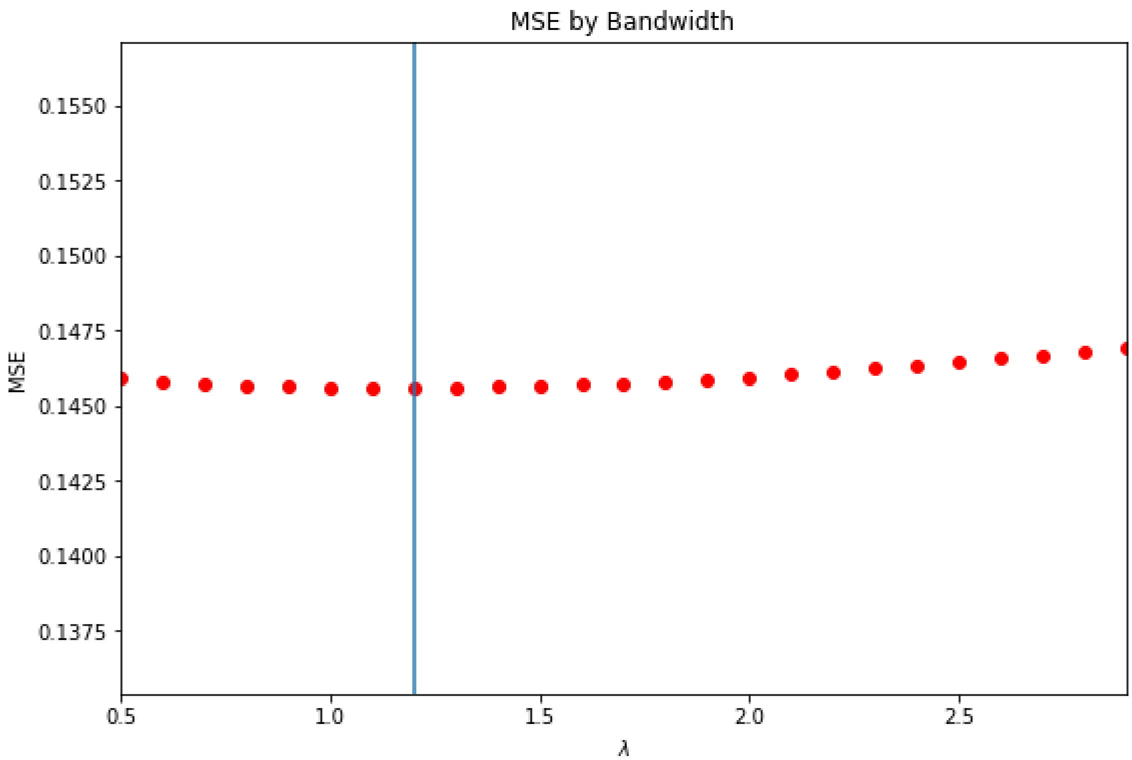
First, we use `Critic_Score` as predictor variable.

In [185]:
```
c_s = videogames[['Critic_Score', 'Logged_Sales']]
c_s = c_s.dropna()
x = np.array(c_s.Critic_Score)
y = np.array(c_s.Logged_Sales)
```

In [7]:
```
lambdas = np.arange(.5, 3, .1)
result = np.empty(len(lambdas) * len(x)).reshape(len(x), len(lambdas))
for i in range(len(x)):
    trainingx = np.delete(x, i)
    trainingy = np.delete(y, i)
    testx = x[i]
    testy = y[i]
    for j in range(len(lambdas)):
        densities = st.norm.pdf(x = trainingx, loc = testx, scale = lambdas[j
])
        predicted = sum(trainingy * densities) / sum(densities)
        result[i, j] = (testy - predicted) ** 2
```

In [8]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(lambdas, result.mean(0), color = "r", marker = "o")
ax.set_xlabel(r'$ \lambda $')
ax.set_ylabel("MSE")
ax.set_xlim(min(lambdas), max(lambdas))
ax.set_title("MSE by Bandwidth")
ax.axvline(x = lambdas[result.mean(0).argmin()])

plt.plot()
round(lambdas[result.mean(0).argmin()], 3)
```

Out[8]:
```
1.2
```

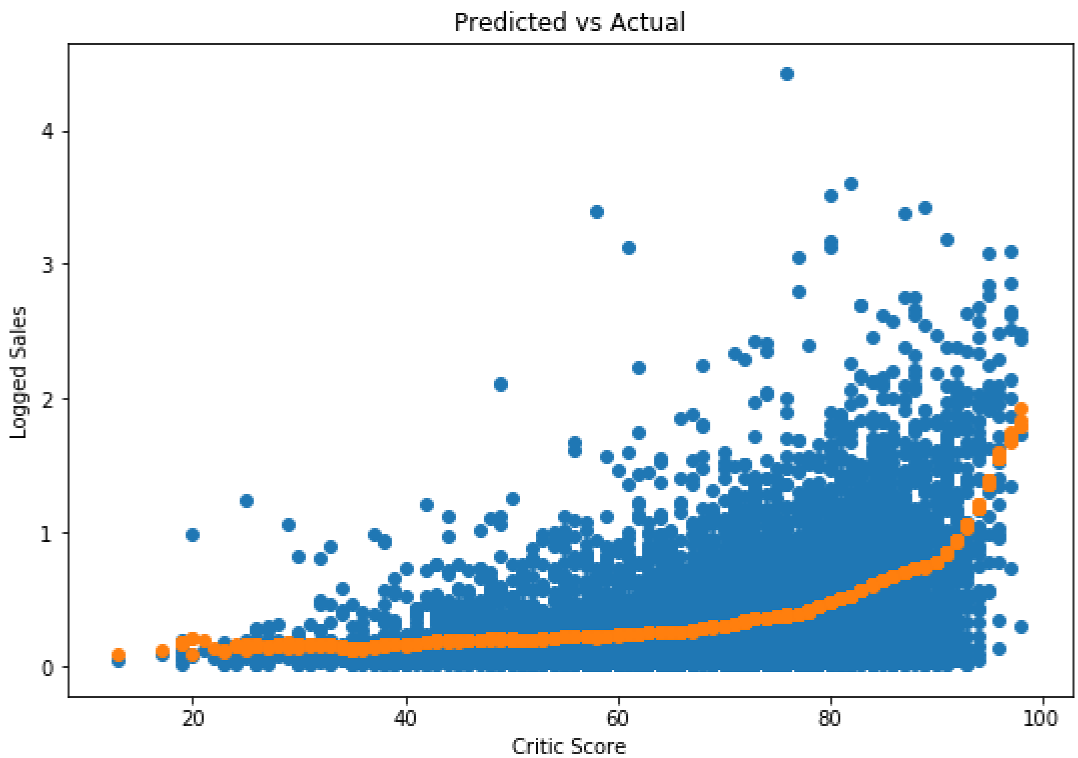


Now, I am going to use the obtained optimal bandwidth to fit kernal regression, and get the MSE.

In [9]:
```
predicted = []
for i in range(len(x)):
    densities = st.norm.pdf(np.delete(x, i), loc = x[i], scale = lambdas[resul
t.mean(0).argmin()])
    predicted.append(sum(np.delete(y, i) * densities) / sum(densities))
```

In [10]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(x, y)
ax.scatter(x, predicted)
ax.set_title("Predicted vs Actual")
ax.set_xlabel("Critic Score")
ax.set_ylabel("Logged Sales")
plt.plot()
round(np.mean((predicted - y) **2), 4)
```

Out[10]: 0.14560000000000001



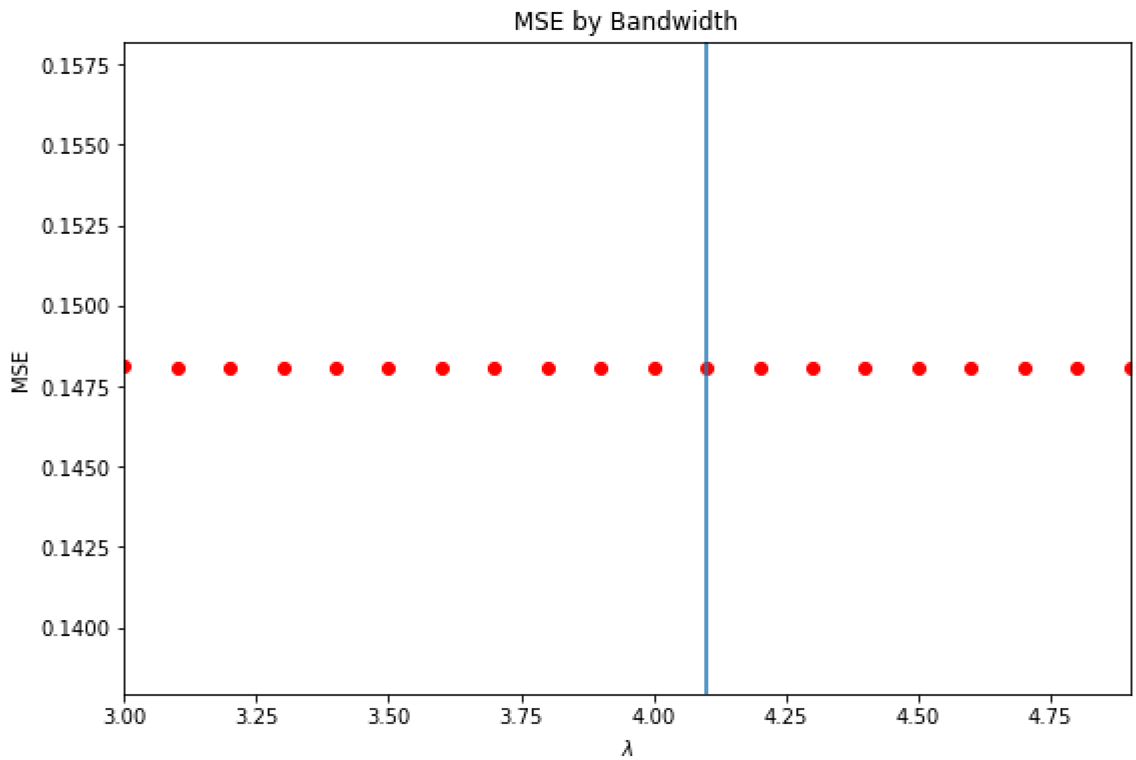
Second, I will use `Critic_Count` as predictor variable.

In [11]:
```
c_c = videogames[['Critic_Count', 'Logged_Sales']]
c_c = c_c.dropna()
x = np.array(c_c.Critic_Count)
y = np.array(c_c.Logged_Sales)
```

In [12]:
```
lambdas = np.arange(3, 5, .1)
result = np.empty(len(lambdas) * len(x)).reshape(len(x), len(lambdas))
for i in range(len(x)):
    trainingx = np.delete(x, i)
    trainingy = np.delete(y, i)
    testx = x[i]
    testy = y[i]
    for j in range(len(lambdas)):
        densities = st.norm.pdf(x = trainingx, loc = testx, scale = lambdas[j
])
        predicted = sum(trainingy * densities) / sum(densities)
        result[i, j] = (testy - predicted) ** 2
```

In [13]:
```python
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(lambdas, result.mean(0), color = "r", marker = "o")
ax.set_xlabel(r'$ \lambda $')
ax.set_ylabel("MSE")
ax.set_xlim(min(lambdas), max(lambdas))
ax.set_title("MSE by Bandwidth")
ax.axvline(x = lambdas[result.mean(0).argmin()])

plt.plot()
round(lambdas[result.mean(0).argmin()], 2)
```

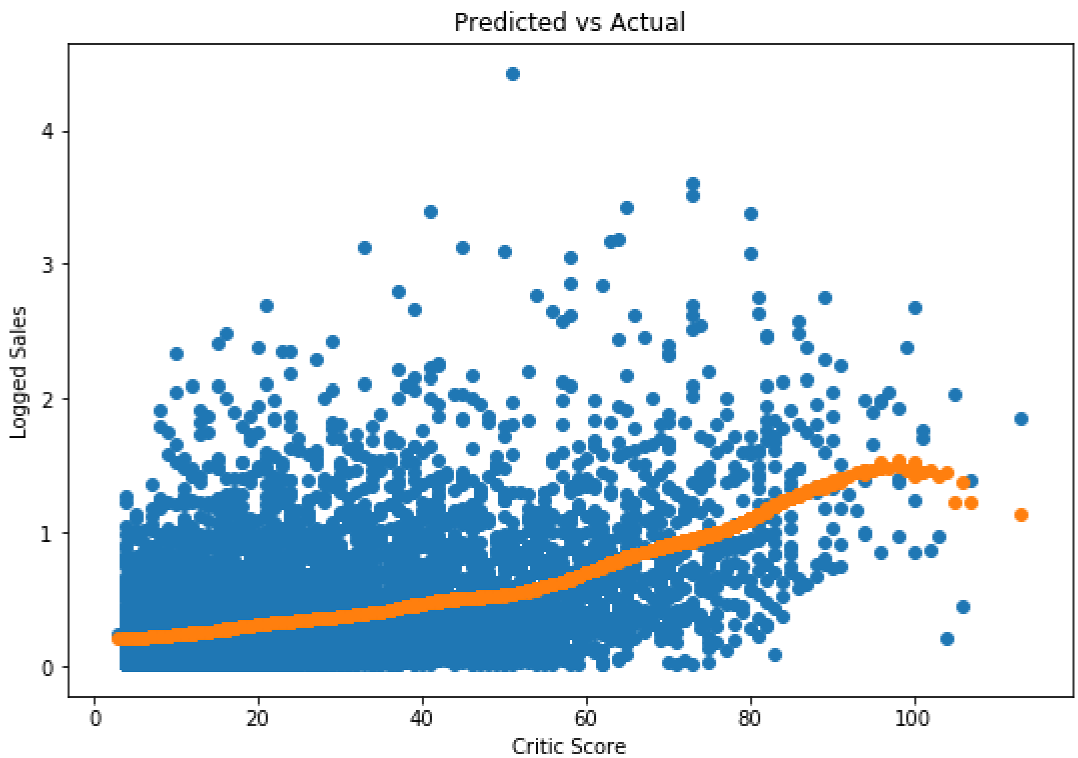
Out[13]: 4.0999999999999996



In [14]:
```python
predicted = []
for i in range(len(x)):
    densities = st.norm.pdf(np.delete(x, i), loc = x[i], scale = lambdas[resul
t.mean(0).argmin()])
    predicted.append(sum(np.delete(y, i) * densities) / sum(densities))
```

In [15]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(x, y)
ax.scatter(x, predicted)
ax.set_title("Predicted vs Actual")
ax.set_xlabel("Critic Score")
ax.set_ylabel("Logged Sales")
plt.plot()
round(np.mean((predicted - y) **2), 4)
```

Out[15]: 0.14810000000000001



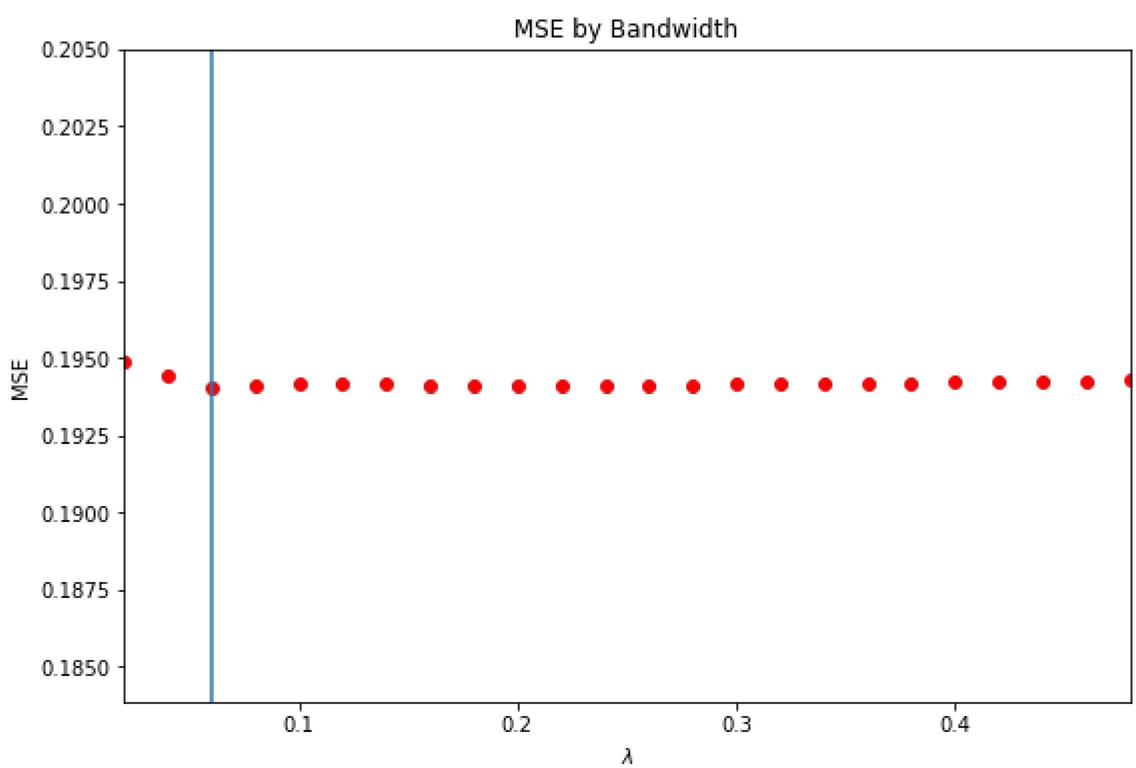
Last, I will use `User_Score` as predictor variable.

In [186]:
```
u_s = videogames[['User_Score', 'Logged_Sales']]
u_s = u_s.dropna()
u_s = u_s[u_s.User_Score != 'tbd']
x = np.array(u_s.User_Score)
x = x.astype(np.float)
y = np.array(u_s.Logged_Sales)
```

In [187]:
```
lambdas = np.arange(.02, .5, .02)
result = np.empty(len(lambdas) * len(x)).reshape(len(x), len(lambdas))
for i in range(len(x)):
    trainingx = np.delete(x, i)
    trainingy = np.delete(y, i)
    testx = x[i]
    testy = y[i]
    for j in range(len(lambdas)):
        densities = st.norm.pdf(x = trainingx, loc = testx, scale = lambdas[j
])
        predicted = sum(trainingy * densities) / sum(densities)
        result[i, j] = (testy - predicted) ** 2
```

In [188]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(lambdas, result.mean(0), color = "r", marker = "o")
ax.set_xlabel(r'$ \lambda $')
ax.set_ylabel("MSE")
ax.set_xlim(min(lambdas), max(lambdas))
ax.set_title("MSE by Bandwidth")
ax.axvline(x = lambdas[result.mean(0).argmin()])

plt.plot()
round(lambdas[result.mean(0).argmin()], 2)
```

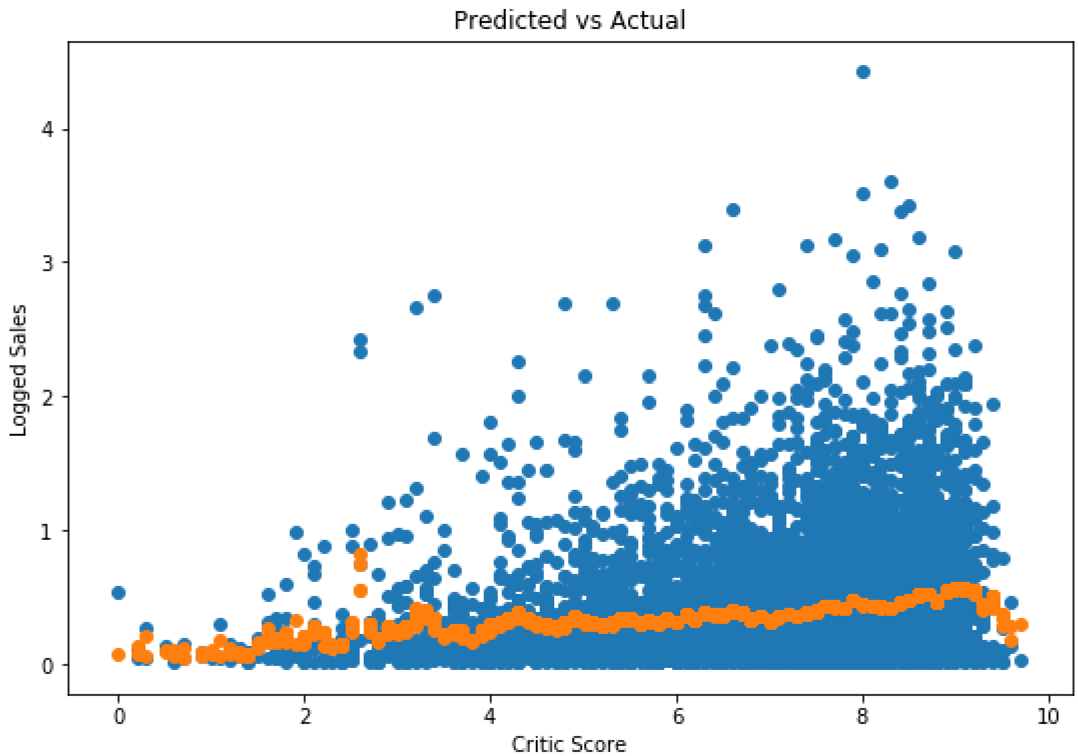
Out[188]: 0.059999999999999998



In [189]:
```
predicted = []
for i in range(len(x)):
    densities = st.norm.pdf(np.delete(x, i), loc = x[i], scale = lambdas[resul
t.mean(0).argmin()])
    predicted.append(sum(np.delete(y, i) * densities) / sum(densities))
```

In [190]:
```
fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(x, y)
ax.scatter(x, predicted)
ax.set_title("Predicted vs Actual")
ax.set_xlabel("Critic Score")
ax.set_ylabel("Logged Sales")
plt.plot()
round(np.mean((predicted - y) **2), 4)
```

Out[190]: 0.19400000000000001



Among the three models, the model with `Critic_Score` has the lowest mean square prediction error. Therefore, I draw the conclusion that expert critics' opinion is the best indicator of the sales of a game.

Question 3

(a)

In [4]:

```
def CART(x, y, w = []):
    '''
    Takes 3 numpy.array objects x, y, and w.

    Paramters
    ---------
    x: a numpy array containing x values;
    y: a numpy array containing y values, of equal length as x;
    w: a numpy array containing weights, of equal length as x;
        assumed to be uniformly 1/n if not supplied; normalized if not summing
```

```
to 1.


    Returns
    -------
    c: the cutting point.
    fL: prediction of points on the left-hand side of c.
    fR: prediction of points on the right-hand side of c.
    '''
    n = len(x)
    if len(y) != n:
        print('Error: x and y have different lengths.')
        return(None)
    if len(w) != n:
        w = np.ones(n)

    w = w / sum(w)

    y = [y for _,y in sorted(zip(x,y))]
    y = np.array(y)
    w = [w for _,w in sorted(zip(x,w))]
    w = np.array(w)
    x.sort()

    yis1 = (y==1) * 1

    scores = np.zeros(n)
    scores[0] = sum(w * yis1) / sum(w) * (1 - sum(w * yis1)) / sum(w) * -sum(w
) / sum(w)
    scores[-1] = scores[0]
    for i in range(1, n-1):

        pL = sum(w[:i + 1] * yis1[:i + 1]) / sum(w[:i + 1])
        pR = sum(w[i + 1:] * yis1[i + 1:]) / sum(w[i + 1:])

        giniL = pL * (1 - pL)
        giniR = pR * (1 - pR)
        score = - sum(w[:i]) / sum(w) * giniL - sum(w[i + 1:]) / sum(w) * gini
R
        scores[i] = score

    maxindex = np.argmax(scores)
    c = x[maxindex]
    pL = sum(w[:maxindex + 1] * yis1[:maxindex + 1]) / sum(w[:maxindex + 1])
    pR = sum(w[maxindex + 1:] * yis1[maxindex + 1:]) / sum(w[maxindex + 1:])

    if pL > .5:
        fL = 1
    else:
        fL = -1
    if pR > .5:
        fR = 1
    else:
        fR = -1
    return(c, fL, fR)
```

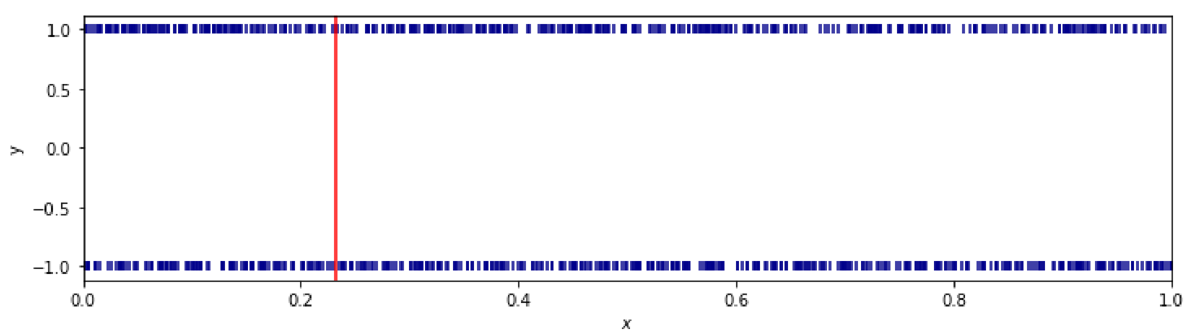
Then, I simulated a dataset to test run the algorithm.

In [24]:
```
### do not modify this cell.
n = 1000
x = np.random.uniform(low = 0, high = 1, size = n)
y = 2 * (np.random.binomial(n = 1, p = np.sin(4 * x * np.pi) / 2 + .5, size =
n) - .5)
w = np.ones(n) / n

c, fL, fR = CART(x, y)
print(c)
print(fL)
print(fR)
```

```
0.231896510466
1
-1
```

In [25]:
```
fig, ax = plt.subplots(figsize = (12, 3))
ax.scatter(x, y, color = "darkblue", marker = "|")
ax.set_xlabel(r'$x$')
ax.set_ylabel("y")
ax.set_xlim(0, 1)
ax.axvline(x = c, color = 'r')

plt.plot()
```

Out[25]: []



(b)

In this part, I will write an algorithm that performs adaboost.

In [26]:
```
def adaboost(x, y, T):
    '''
    Takes 2 numpy.array objects x, y, and an integer T.

    Paramters
    ---------
    x: a numpy array containing training x values;
    y: a numpy array containing training y values, of equal length as x;
    T: an integer specifying how many iterations are desired.
```

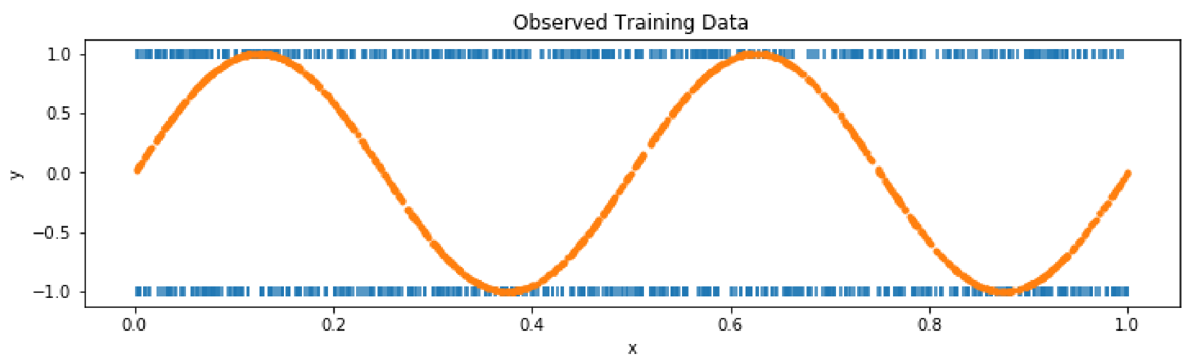
```
    Returns
    -------
    F: a function that predicts y value.
    Cuts: a list of floats, each being the cutting point.
    fLs: classification of point if falling on the left of the cutting point.
    fRs: classification of point if falling on the right of the cutting point.
    alpha: alpha value of each iteration.
    '''
    n = len(x)
    if len(y) != n:
        print('Error: x and y have different lengths.')
        return(None)
    y = [y for _,y in sorted(zip(x, y))]
    y = np.array(y)
    x.sort()
    alpha = np.empty(T)
    cut = np.empty(T)
    fL = np.empty(T)
    fR = np.empty(T)

    w = np.empty(n * (T + 1)).reshape(T + 1, n)
    w[0, :] = 1 / n
    fs = [] # to store the classification rule as functions.
    yis1 = (y==1) * 1
    for t in range(T):
        scores = np.zeros(n)
        scores[0] = - sum(w[t, ] * yis1) / sum(w[t, ]) * (1 - sum(w[t, ] * yis
1)) / sum(w[t, ])
        scores[-1] = scores[0]
        for i in range(n - 1):

            pL = sum(w[t, :i + 1] * yis1[:i + 1]) / sum(w[t, :i + 1])
            pR = sum(w[t, i + 1:] * yis1[i + 1:]) / sum(w[t, i + 1:])

            giniL = pL * (1 - pL)
            giniR = pR * (1 - pR)
            score = - sum(w[t, :i]) / sum(w[t, ]) * giniL - sum(w[t, i + 1:])
/ sum(w[t, ]) * giniR
            scores[i] = score

        maxindex = np.argmax(scores)

        cut[t] = x[maxindex]
        pL = sum(w[t, :maxindex + 1] * yis1[:maxindex + 1]) / sum(w[t, :maxind
ex + 1])
        pR = sum(w[t, maxindex + 1:] * yis1[maxindex + 1:]) / sum(w[t, maxinde
x + 1:])

        fL[t] = np.sign(sum(w[t, :maxindex + 1] * y[:maxindex + 1]))
        fR[t] = np.sign(sum(w[t, maxindex + 1:] * y[maxindex + 1:]))

        predicted_y = np.empty(n)
        predicted_y[:maxindex + 1] = fL[t]
        predicted_y[maxindex + 1:] = fR[t]

        indicator = (predicted_y != y) * 1
```

```
        epsilon = sum(w[t, ] * indicator)


        alpha[t] = .5 * np.log((1 - epsilon) / epsilon)


        w[t + 1, ] = w[t, ] * np.exp(-alpha[t] * y * predicted_y)
        w[t + 1, ] = w[t + 1, ] / sum(w[t + 1, ])

    # define F here

    def F(test_x):
        classifier = lambda x: np.sign(sum([fL[i] * alpha[i] if x <= cut[i] el
se fR[i] * alpha[i] for i in range(T)]))
        test_y = list(map(classifier, test_x))
        return(test_y)
    return(F, cut, fL, fR, alpha)
```

In [47]:
```
F, cut, fL, fR, alpha = adaboost(x, y, 30)
```

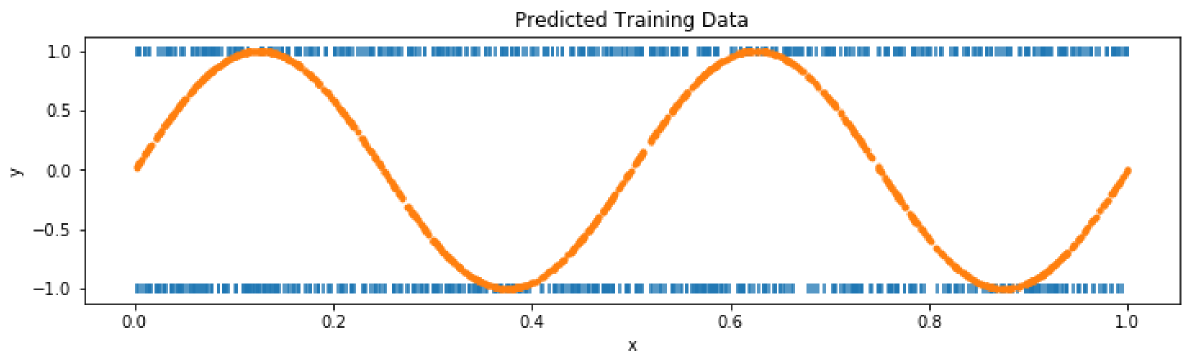
In [48]:
```
plt.figure(figsize=(12, 3))
plt.scatter(x, y, marker = "|")
plt.scatter(x, np.sin(4 * x * np.pi), marker = ".")
plt.xlabel("x")
plt.ylabel("y")
plt.title("Observed Training Data")
```

Out[48]: <matplotlib.text.Text at 0x26d5b1c7240>



```
In [44]: plt.figure(figsize=(12, 3))
         plt.scatter(x, F(y), marker = "|")
         plt.scatter(x, np.sin(4 * x * np.pi), marker = ".")
         plt.xlabel("x")
         plt.ylabel("y")
         plt.title("Predicted Training Data")
```

```
Out[44]: <matplotlib.text.Text at 0x26d5aff5b38>
```



And the prediction accuracy rate is:

```
In [30]: np.round(np.mean(y == F(x)), 2)
```

```
Out[30]: 0.54000000000000004
```

Now, I will generate an independent data set to test my model.

```
In [45]: test_y = 2 * (np.random.binomial(n = 1, p = np.sin(4 * x * np.pi) / 2 + .5, si
         ze = n) - .5)
         plt.figure(figsize=(12, 3))
         plt.scatter(x, test_y, marker = "|")
         plt.scatter(x, np.sin(4 * x * np.pi), marker = ".")
         plt.xlabel("x")
         plt.ylabel("y")
         plt.title("Observed Test Data")
```

```
Out[45]: <matplotlib.text.Text at 0x26d5b08c518>
```