Math 2250-001: Differential Equations and Linear Algebra

December 9, 7:55 AM - 10:15 AM, Fall 2020, FINAL EXAM

Instructions and Honor Code

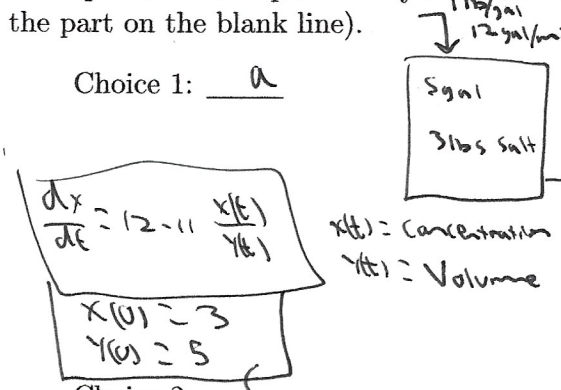
1. You have 140 minutes to complete the exam (120 minutes is the designed length for the exam, 15 minutes are added for uploading ONE SINGLE FILE for all your work, another additional 5 minutes are added for reading the instructions in detail and reviewing your submission in detail). **Time management is your responsibility.**
2. Please make sure that the uploaded file contains ALL THE WORK for ALL THE EXAM QUESTIONS.
3. Uploaded files should be .pdf, .png, .jpg, or .jpeg. One image per page is fine, you do not need to take a picture of individual questions/parts. Similar to HW assignments or lab worksheets, you will upload ONE SINGLE FILE and match your pages with exam questions. If you can't see the image when you submit then I won't be able to either.
4. Check the orientation to ensure it is upright.
5. Whenever asked to explain work all work needs to be shown or explained to receive credit.
6. **Grading will primarily be based on work shown and not the final answer.**
7. This exam is open book, notes, homework, labs, or other written class material. However, given the time restraint it is not a good strategy to depend on these resources.
8. You will not need a calculator on the exam.
9. **You should not google, WolframAlpha, use Matlab, etc... on the exam.**
10. Aside from using Gradescope and accessing written course material mentioned above **you should not use the internet in any way during the exam.**
11. **You must work alone, no collaboration of any kind is allowed.**
12. You should not discuss the exam in any way to any classmates. The exam will lock at 10:15 AM. Late work will be accepted until 10:35 with a severe penalty (5 point deduction per minute late, **time management is your responsibility.**).
13. I recommend you read through the entire exam before starting.
14. I recommend you set a separate timer for 120 minutes. At the end of that time upload all completed work.
15. You have only ONE SINGLE FILE to upload containing all your work for the entire exam.

I agree that I will not receive or provide assistance on this exam. I understand I am allowed to use my written class material (notes, textbook, homework, personal notes, etc). I will not communicate with other students about this exam or use the internet to receive help. By agreeing to this statement I am stating that I am aware of the consequences for breaking this policy. Write your name below to confirm that you have read and understood the above instructions and agree to the above statement.

Josh Whitehead

1. This question has 4 parts but you choose which 3 to answer. Clearly label which you are answering (write the part on the blank line).

Choice 1: a



$\frac{dx}{dt} = r_i c_i - r_o c_o = 12\frac{\text{gal}}{\text{min}}\left(\cdot 1\frac{\text{lb}}{\text{gal}}\right) - r \cdot x(t)$

$\frac{dx}{dt} = 12 - r\cdot\frac{x(t)}{y} = 12 - 11\frac{x(t)}{y(t)}$

Choice 2: (

$\frac{dP}{dt}$ = birth - death = ~~$1130 - 10P - 30 = 1100 - 10P$~~

$\frac{dP}{dt} = P(t)(1130 - 10P - 30) = P(1100 - 10P) = 10P(110 - P) = \frac{dP}{dt}$ $\quad kP(M-P)$

Carrying Cap. = 110

Choice 3: ~~b~~

~~$x'' - 3x' = \frac{1}{s^2}$~~

$\frac{dT}{dt} = k(T(t) - 64) \rightarrow -20 = k(90 - 64) \rightarrow k = \frac{-20}{34} = -\frac{10}{17}$

$\therefore \frac{dT}{dt} = -\frac{10}{17}(T(t) - 64)$

$T(0) = 150°$

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2. $y' = 3y + te^{3t}$ $\quad y(1) = 0$ $\quad \frac{dy}{dt} + P(t)y = Q(t)$

$\frac{dy}{dt} - 3y = te^{3t}$

$P(t)$ $\quad Q(t)$

$\mu(x) = e^{\int P(t)dt} \Rightarrow \int -3\,dt = -3t \rightarrow \mu(t) = e^{-3t}$

$e^{-3t}\left(\frac{dy}{dt} - 3y\right) = te^{3t}\cdot e^{-3t} \rightarrow \int\frac{d}{dt}\left(e^{-3t}y\right) = \int t\,dt \rightarrow e^{-3t}y = \frac{t^2}{2} + C$

$y = \frac{t^2 e^{3t}}{2} + Ce^{3t}$

$y(1) = 0 = \frac{1\cdot e^3}{2} + Ce^3 = e^3\left(\frac{1}{2} + C\right) = 0 \rightarrow C = -\frac{1}{2}$

$\therefore y = \frac{t^2 e^{3t}}{2} - \frac{e^{3t}}{2}$

$\frac{dy}{dx} = y^4 - 2y^3 - 8y^2 = 0$

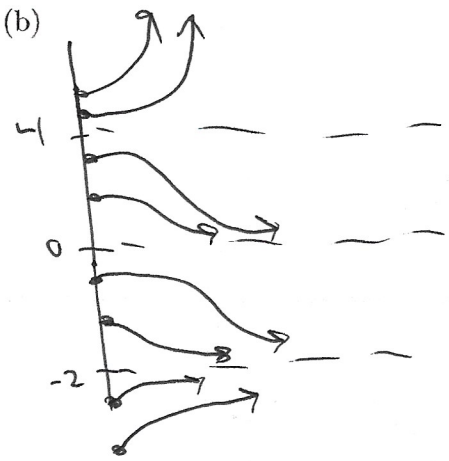
3. (a)

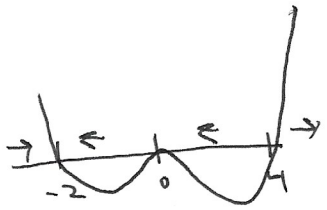
$y^2(y^2 - 2y - 8) = 0$

$(y-4)(y+2) = 0$

$y = 0, 4, -2$

(b)





-2: stable

0, 4: unstable

(c)

4.

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 3 & 4 & 0 \end{bmatrix} \xrightarrow{-R_1+R_3} \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$ ∴ lin independent

5. $\begin{bmatrix} 6 & -3 & 7 \end{bmatrix}$ $6\begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix} - 3\begin{bmatrix} x_2 \\ \vdots \\ \vdots \end{bmatrix} - 7\begin{bmatrix} x_3 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\begin{bmatrix} 6 \\ -3 \\ 7 \end{bmatrix} x = 0$ (closed under vector addition & multiplication

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6. (a) $4x'' + Cx' + 5x = 0$ $\quad C \geq C_{crit}$

$C_{crit} = 2\sqrt{5 \cdot 4} = 2 \cdot 2\sqrt{5} = \boxed{4\sqrt{5}}$

when $C < C_{crit}$, oscilations occure

(b) $4x'' + 8x' + 5x = 0$

$4r^2 + 8r + 5 = 0$

$\frac{-8 \pm \sqrt{64 - 16 \cdot 5}}{8} = \frac{-8 \pm \sqrt{64 - 80}}{8}$

$= \frac{-8 \pm \sqrt{-16}}{8}$

$= -1 \pm 4i$

$f(x) = e^{-x}\left(C_1 \cos(4x) + C_2 \sin(4x)\right)$

7. This question has 2 parts but you choose which 1 to answer. Clearly label which you are answering (write the part on the blank line).

Your choice? a

$$f(t) = \begin{cases} 0 & \text{if } t<0 \\ -3t & \text{if } 0<t<1 \\ 3t & \text{if } 1<t<2 \\ 0 & \text{if } t>2 \end{cases}$$

$$0\left(u(t-0)\right) + -3t\left(u(t-0)-u(t-1)\right) + 3t\left(u(t-1)-u(t-2)\right) + 0$$

$$= -3t\,u(t) + 3(t)\left(u(t-1)\right) + 3t\,(ut-1) - 3t(u(t-2))$$

$$= -3t\,u(t) + 6t\left(u(t-1)\right) - 3t\,(u\,t-2)$$

$$\mathcal{L}\{f(t)\} = e^{0s}\mathcal{L}\{-3t\} + e^{-s}\mathcal{L}\{6t\} + e^{-2s}\mathcal{L}\{-3t\}$$

$$= \frac{-3}{s^2} + \frac{6e^{-s}}{s^2} + \frac{-3e^{-2s}}{s^2}$$

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8. $x'' - x' - 12x = 12 \qquad x(0) = 0 \qquad x'(0) = 2$

$$\bar{x} = \mathcal{L}\{x(t)\}$$

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 12\mathcal{L}\{x\} = 12$$

$$s^2\bar{x} - sx(0) - x'(0) - s\bar{x} + x(0) - 12\bar{x} = 12$$

$$s^2\bar{x} - 2 - s\bar{x} - 12\bar{x} = 12$$

$$s^2\bar{x} - s\bar{x} - 12\bar{x} = 14$$

$$\bar{x}(s^2 - s - 12) = 14$$

$$\bar{x} = \frac{14}{s^2-s-12} = \frac{14}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3}$$

$$A(s+3) + B(s-4) = 14$$

$$B(-7) = 14$$

$$B = -2$$

$$A = +2$$

$$\therefore \bar{x} = \frac{2}{s-4} + \frac{-2}{s+3}$$

$$\mathcal{L}^{-1}\{\bar{x}\} = 2e^{4t} - 2e^{-3t}$$

9. (a) ~~Invertible if independent~~

False $\begin{bmatrix} 3-\lambda & 5 \\ 4 & 8-\lambda \end{bmatrix}\begin{bmatrix} A^{-1} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix}$ $A^{-1} = \frac{1}{24-20}\begin{bmatrix} 8 & -5 \\ -4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -5/4 \\ -1 & 3/4 \end{bmatrix}$

$A \cdot A^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 2 & -5/4 \\ -1 & 3/5 \end{bmatrix} = \begin{bmatrix} 6-5 & \\ -15/4-15/4 & 5-24/5 \end{bmatrix}$

$\lambda \neq 0$

(b) $\begin{bmatrix} 6 & -5 \\ 10 & -9 \end{bmatrix}$ $\begin{bmatrix} 6-\lambda & -5 \\ 10 & -9-\lambda \end{bmatrix}$

$= -54 + 3\lambda + \lambda^2 + 50$

$= \lambda^2 + 3\lambda - 4$

$\lambda = 4, -1$

$\lambda_1 = -1$: $\begin{bmatrix} 7 & -5 & 0 \\ 10 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 7 & -5 & 0 \\ 0 & -6 & 0 \end{bmatrix}$ $x_1 = 0$, $x_2 = 0$

$\lambda_2 = 4$: $\begin{bmatrix} -1 & 5 & 0 \\ 4 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 5 & 0 \\ 0 & 24 & 0 \end{bmatrix}$ $x_1 = x_2 = 0$

$\therefore x(t) = c_1\begin{bmatrix} 0 \\ 0 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 0 \\ 0 \end{bmatrix}e^{4t}$

10. (a)

(b) $m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2$

~~$m_2 x_2'' = k_2 x_1 - (k_2 + k_3) x_2 + k_3 x_3$~~

$m_2 x_2'' = k_2 x_1 - (k_2 + k_3) x_2$

$$M = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \qquad K = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix}$$

$x'' = Ax \quad A = M^{-1}K \qquad M^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}(k_1 + k_2) & \frac{1}{4}k_2 \\ \frac{1}{2}k_2 & -\frac{1}{2}(k_2 + k_3) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4}\cdot 12 & 2 \\ 4 & -\frac{1}{2}\cdot 10 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix} \rightarrow \begin{vmatrix} -3-\lambda & 2 \\ 4 & -5-\lambda \end{vmatrix}$$

$= 15 + 8\lambda + \lambda^2 - 8 \rightarrow \lambda^2 + 8\lambda + 7 \rightarrow \lambda = 7, 1$

$$\begin{bmatrix} -10 & 2 \\ 4 & -12 \end{bmatrix}$$

(c)