1. (6 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & -4 \\ 2 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}.$$

(a) Compute the determinant of the matrix **A**. Does **A** have an inverse? Explain your solution.

$$\det(A) = 0 - 2\begin{vmatrix} 2 & -4 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & -4 \\ 2 & 0 \end{vmatrix} = -2(0+8) + 2(0+8) = 0$$

* A^{-1} = DNE because $\det(A) = 0$

(b) Let $\mathbf{x} = (x_1, x_2, x_3)$, we want to find a set of vectors whose linear combinations span the solution space of $\mathbf{Ax} = \mathbf{0}$.

i. Reduce the augmented coefficient matrix to its reduced row echelon form.

$$\begin{bmatrix} 0 & 2 & -4 \\ 2 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & -4 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 0 & 4 \\ 2 & 2 & 0 \\ 0 & 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

ii. Find a basis for the solution space of the matrix equation $\mathbf{Ax} = \mathbf{0}$.

$x_3 = t$, $x_1 = -2t$, $x_2 = 2t$

$$\therefore x = \begin{bmatrix} -2t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Basis = 1 dimension

$v_1 = (-2, 2, 1)$

(c) Let $\mathbf{b} = (b_1, b_2, b_3)$, we want to find a basis for the set of all vectors $\mathbf{b}$ such that the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a solution.

i. Reduce the augmented coefficient matrix for the nonhomogeneous linear system to its reduced row echelon form.

$$\left[\begin{array}{ccc|c} 0 & 2 & -4 & b_1 \\ 2 & 0 & 4 & b_2 \\ 2 & 2 & 0 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 4 & b_2 \\ 0 & 2 & -4 & b_1 \\ 2 & 2 & 0 & b_3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & b_2/2 \\ 0 & 1 & -2 & b_1/2 \\ 0 & 2 & -4 & b_3-b_2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & b_2/2 \\ 0 & 1 & -2 & b_1/2 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

$x_3 = t$

$x_2 = \frac{b_1}{2} + 2t$

$x_1 = \frac{b_2}{2} - 2t$

$$\therefore \mathbf{x} = \begin{bmatrix} \frac{b_2}{2} - 2t \\ \frac{b_1}{2} + 2t \\ t \end{bmatrix} = \begin{bmatrix} \frac{b_1}{2} \\ \frac{b_2}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

ii. What condition do we need to impose on the vector $\mathbf{b}$ so that the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a solution?

$b_1 - b_2 - b_3 = 0$

iii. Using part (ii) find a basis for the set of all vectors $\mathbf{b}$ such that the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b_2/2 \\ 0 & 1 & -2 & b_1/2 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right]$$

$b_3 = 0$

$b_2 = 0$

$b_1 = 0$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$x_2 = 0 + 2t$

$x_1 = 0 - 2t$

$$\mathbf{x} = \begin{bmatrix} -2t \\ 2t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}$$

2. (7 points) **Matrices as Linear Transformations & Determinants**

We can think of any $n \times n$ matrix $\mathbf{A}$ as representing a map or function, that transforms vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^n$. This type of map is called a **linear transformation**. In summary, a matrix times a vector can always be viewed as doing the following transformations of the vector: rotation, reflection, and stretching.

(a) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

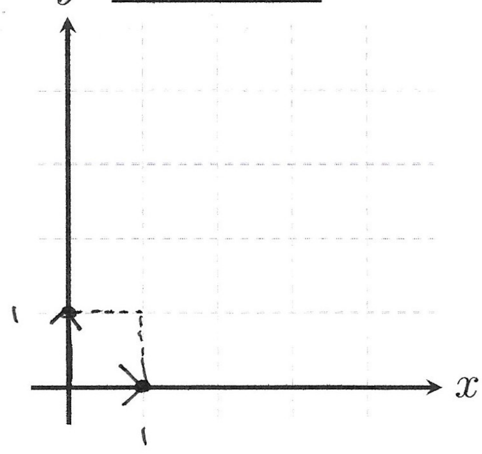
Compute the vectors $\mathbf{u}_1 = \mathbf{A}\mathbf{e}_1$ and $\mathbf{u}_2 = \mathbf{A}\mathbf{e}_2$.

$u_1)$ $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(1) + 1(0) \\ 1(1) + 3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

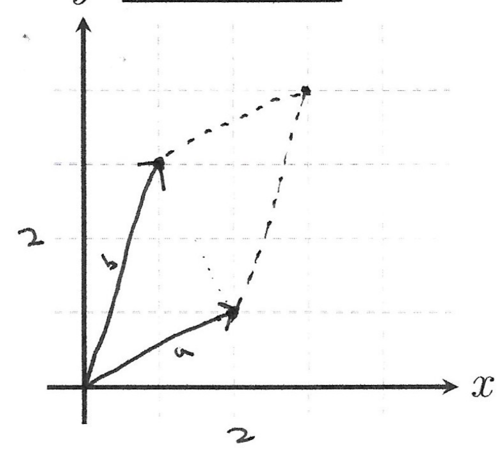
$u_2)$ $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 1 \\ 0 + 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(b) Sketch $\mathbf{e}_1$ and $\mathbf{e}_2$ along with the square they determine on the provided axes. Similarly, sketch $\mathbf{u}_1$ and $\mathbf{u}_2$ along with the parallelogram they determine on the separate axes. To determine the square and parallelogram consider the parallelogram law for vector addition.

Plot $\mathbf{e}_1$, $\mathbf{e}_2$



Plot $\mathbf{u}_1$, $\mathbf{u}_2$



(c) Compute the area of the parallelogram that you drew in part (b). How does this relate to the determinant of the matrix **A**?

$\cancel{A = bh =}$

$l_a = \sqrt{2^2+1} = \sqrt{5} = b$

$l_b = \sqrt{9+1} = \sqrt{10}$

$h =$

$A = |a||b| \sin\theta = \sqrt{5}\sqrt{10} \sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{\sqrt{10}}\right)\right) = \boxed{5}$

$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6-1 = 5$

$\boxed{\text{Area}_p = \det(A)}$

(d) Let

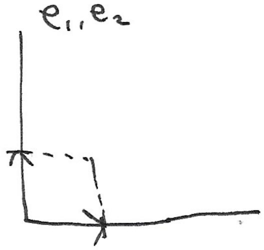
$$\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}.$$

REPEAT parts (a)—(c) with the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}_1 = \mathbf{B}\mathbf{e}_1$, and $\mathbf{v}_2 = \mathbf{B}\mathbf{e}_2$.

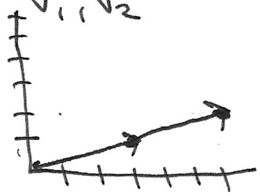
$V_1 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3(1) + 6(0) \\ 1(1) + 2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$V_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(0) + 6(1) \\ 1(0) + 2(1) \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

e_1, e_2



V_1, V_2



$\det(B) = \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 3(2) - 6(1) = 0$

$|\sqrt{10}||\sqrt{40}| \sin(0) = 0$

3. (7 points) Consider the following two sets of vectors in $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}.$$

(a) Verify that $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^3$.

$\underbrace{c_1v_1+c_2v_2+c_3v_3=0}$

$$\left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&1&1&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$c_1=c_2=c_3=0$

$$\det(B) = \begin{vmatrix}1&0&1\\0&1&0\\0&1&1\end{vmatrix} = 1\begin{vmatrix}1&0\\1&1\end{vmatrix} = 1-0=1\neq 0$$

✓

$$\det(C) = \begin{vmatrix}1&0&0\\-1&1&-1\\1&0&1\end{vmatrix} = \begin{vmatrix}1&-1\\0&1\end{vmatrix} = 1-0=1\neq 0$$

✓

(b) Write the vector $\mathbf{x} = (3,1,1)$ as a linear combination of the basis vectors in $\mathcal{B}$. Similarly, write the vector $\mathbf{x} = (3,1,1)$ as a linear combination of the basis vectors in $\mathcal{C}$.

$\underbrace{c_1V_1+c_2V_2+c_3V_3=X}$ $\rightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&1\\0&1&1&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&1\\0&0&1&0\end{array}\right]$ $\therefore c_3=0.$ $c_2=1$ $c_1=3$

$$\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

$$\therefore 3\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

$c_1u_1+c_2u_2+c_3u_3=X$ $\rightarrow \left[\begin{array}{ccc|c}1&0&0&3\\-1&1&-1&1\\1&0&1&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&0&3\\0&1&-1&4\\0&0&1&-2\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&0&3\\0&1&0&2\\0&0&1&-2\end{array}\right]$ $c_3=-2$ $c_2=2$ $c_1=3$

$$\therefore 3\begin{bmatrix}1\\-1\\1\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\end{bmatrix}$$

(c) We define $[\mathbf{x}]_{\mathcal{B}}$ to be "the representation of the vector $\mathbf{x}$ in the basis $\mathcal{B}$", this is what was found in part (b) with the example of $\mathbf{x} = (3, 1, 1)$. What is the matrix B such that $\mathbf{x} = B[\mathbf{x}]_{\mathcal{B}}$ for all $\mathbf{x} \in R^3$? Similarly, what is the matrix C such that $\mathbf{x} = C[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x} \in \mathbb{R}^3$?

$\begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}B_1\\B_2\\B_3\end{bmatrix}\begin{bmatrix}3\\1\\0\end{bmatrix}$ $\quad B[X]_B = X \therefore B[x_B][x_B]^{-1} = [x_B]^{-1}[X]$

$B =$

$[x_B]^{-1}: \begin{bmatrix}3&1\\1&1&0\\0&1&0\end{bmatrix} \rightarrow \begin{bmatrix}3&1\\0&1\\0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&1&1/3\\0&1&1\\0&1&0\end{bmatrix} \therefore (A_B)^{-1} = \begin{bmatrix}1/3\\1\\0\end{bmatrix} \therefore B =$

$X = \begin{bmatrix}B_1\\B_2\\B_3\end{bmatrix}\underbrace{\begin{bmatrix}3&0&0\\0&1&0\\0&1&0\end{bmatrix}}_{A} \rightarrow B = A^{-1}X$ $\quad A^{-1} = \left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\0&1&0&0&1&0\\0&1&1&0&0&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&-1&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}1&0&0&1/3&0&0\\0&1&0&0&1&0\\0&0&1&0&-1&1\end{array}\right]$

$B = \begin{bmatrix}1/3&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}1+0+0\\0+1+0\\0-1+1\end{bmatrix} = \boxed{\begin{bmatrix}1\\1\\0\end{bmatrix}}$

$A_2^{-1}: \left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\-3&2&2&0&1&0\\3&0&-2&0&0&1\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\0&2&2&1&1&0\\0&0&-2&-1&0&1\end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\0&2&0&0&1&1\\0&0&-2&-1&0&1\end{array}\right] \therefore A_2^{-1} = \begin{bmatrix}1/3&0&0\\0&1/2&1/2\\1/2&0&-1/2\end{bmatrix}$ $\quad C = A_2^{-1}X$

$\begin{bmatrix}\frac{1}{3}&0&0\\0&1/2&1/2\\1/2&0&-1/2\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}1+0+0\\0+1/2+1/2\\3/2+0-1/2\end{bmatrix} = \boxed{\begin{bmatrix}1\\1\\1\end{bmatrix}}$

(d) Find a matrix that represents a change of basis from $\mathcal{B}$ to $\mathcal{C}$. That is, given $[\mathbf{x}]_{\mathcal{B}}$ find a matrix $\mathbf{A}$ such that $[\mathbf{x}]_{\mathcal{C}} = \mathbf{A}\ [\mathbf{x}]_{\mathcal{B}}$.

$\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}A_1\\A_2\\A_3\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$