1. Recall that the definition of the Laplace transform is

$\int u\,dv = uv - \int v\,du$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Use the definition to determine $F(s) = \mathcal{L}\{\sin(at)\}$, where $a \in \mathbb{R}$ is an arbitrary constant.

$\int_0^\infty e^{-st} \sin(at)\,dt$ $\quad u = \sin(at)$ $\quad dv = e^{-st}$

$du = a\cos(at)$ $\quad v = \frac{-1}{s} e^{-st}$

$= \frac{-1}{s} e^{-st} \sin(at) + \frac{a}{s} \int_0^\infty e^{-st} \cos(at)\,dt$ $\quad u_2 = \cos(at)$ $\quad dv_2 = e^{-st}$

$du_2 = -a\sin(at)$ $\quad v_2 = \frac{-1}{s} e^{-st}$

$= \frac{-1}{s} e^{-st} \sin(at) + \frac{+a}{s}\left[\frac{-1}{s} e^{-st} \cos(at) \cancel{+} \frac{a}{s}\int_0^\infty e^{-st} \sin(at)\,dt\right]$

let $\int_0^\infty e^{-st} \sin(at)\,dt = G$

$\therefore G = \frac{-1}{s} e^{-st} \sin(at) + \frac{a}{s}\left[\frac{-1}{s} e^{-st} \cos(at) \cancel{+} \frac{a}{s} G\right]_0^\infty$

$G = \frac{-1}{s} e^{-st} \sin(at) - \frac{a}{s^2} e^{-st} \cos(at) + \frac{a^2}{s^2} G$

$G\left(1 + \frac{a^2}{s^2}\right) = \frac{-1}{s} e^{-st} \sin(at) - \frac{a}{s^2} e^{-st} \cos(at)$

$$G = \left.\frac{\frac{-1}{s} e^{-st} \sin(at) - \frac{a}{s^2} e^{-st} \cos(at)}{\frac{s^2+a^2}{s^2}}\right]_0^\infty = \frac{0 + \frac{a}{s^2}}{\frac{s^2+a^2}{s^2}} = \frac{+a}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2+a^2} = \boxed{\frac{+a}{s^2+a^2}}$$

2. Recall that the **shifted unit step function** with (time) shift $a > 0$ is defined as

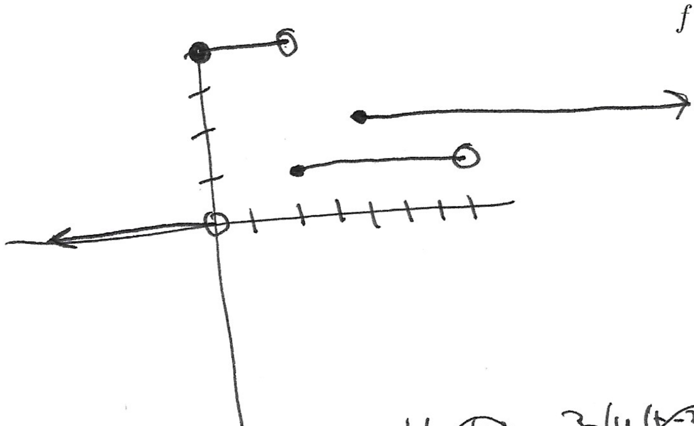
$$u_a(t) = u(t-a) = \begin{cases} 0 & \text{for} \quad t < a, \\ 1 & \text{for} \quad t \geq a. \end{cases}$$

Sketch each of the following functions and rewrite it as a linear combination of shifted unit step functions. For example,

$$f(t) = \begin{cases} 0 & t < 0, \\ 5 & 0 \leq t < 100, \\ -3 & 100 \leq t < \infty. \end{cases} \implies f(t) = 5u(t) - 8u(t-100).$$

(a)

$$f(t) = \begin{cases} 0 & t < 0, \\ 4 & 0 \leq t < 2, \\ 1 & 2 \leq t < 7, \\ 2 & 4 \leq t < \infty. \end{cases}$$



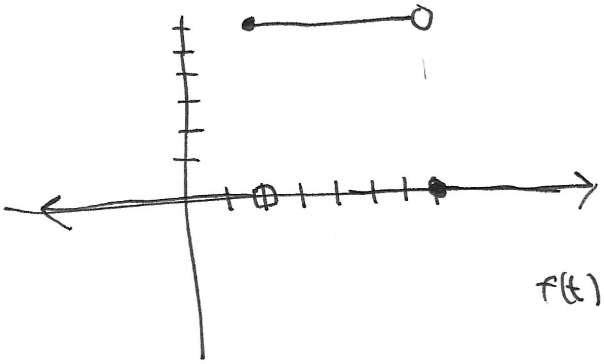
$\cancel{4u(t) - 3(u(t-2)) + u(t-4)}$

$f(t) = 4[(u(t-0)) - u(t-2)] + 1(u(t-2) - u(t-7)) + 2u(t-4)$

$= 4u(t) - 4u(t-2) + u(t-2) - u(t-7) + 2u(t-4)$

$= 4u(t) - 3u(t-2) - u(t-7) + 2u(t-4)$

(b)

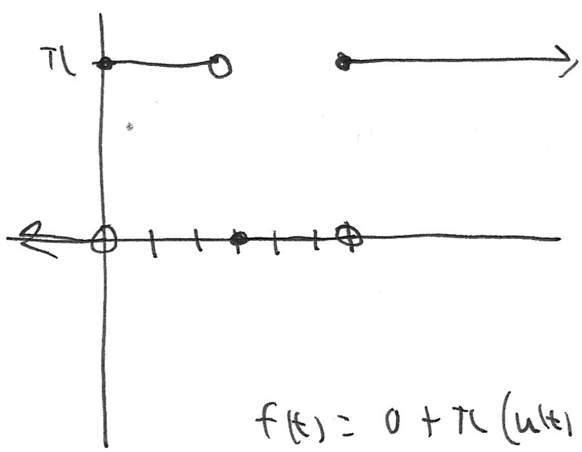


$$f(t) = \begin{cases} 0 & t < 2, \\ 6 & 2 \le t < 7, \\ 0 & 7 \le t < \infty. \end{cases}$$

$f(t) = 0\,(u(t) - u(t-2)) + 6\,(u(t-2) - u(t-7)) + 0$

$= 6u(t-2) - 6(u(t-7)$

(c)



$$f(t) = \begin{cases} 0 & t < 0, \\ \pi & 0 \le t < 3, \\ 0 & 3 \le t < 6, \\ \pi & 6 \le t < \infty. \end{cases}$$

$f(t) = 0 + \pi\,(u(t) - u(t-3)) + 0 + \pi\, u(t-6)$

$= \pi u(t) - \pi u(t-3) + \pi u(t-6)$

3. Excited for the new series *The Grand Tour*, you start watching reruns of *Top Gear*, and end up in a mathematical modeling frenzy. Everyone is given an identical Mazda Miata, which has a sprung mass of 1200 kg (the mass of the main body of the car that sits atop the suspension). They have to drive the car through a sinusoidal washboard-like road surface with an amplitude of 25 cm and a wavelength of 12 m. James May, a.k.a Captain Slow, states confidently that the best thing to do is to drive very slowly (30 kph). Hammond and Jeremy Clarkson say to drive as fast as the car can go (200 kph).

   The only one who takes his job seriously on the show is The Stig. The Stig assumes the car oscillates vertically as if it were a sprung mass $m = 1200/4$ kg $= 300$ kg on a single spring (with constant $k = 2.2 \times 10^4$ N/m) attached to a dashpot (with constant $c = 3500$ N $\cdot$ sec/m) and models the system accordingly.

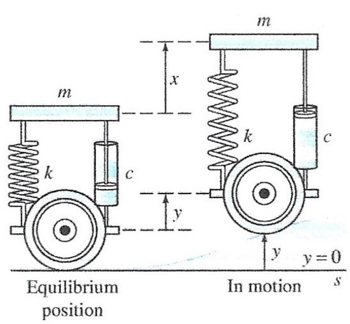


FIGURE 5.6.7. The "unicycle model" of a car.

(a) If the car's speed is given by s, determine the height $y(t)$ of the washboard under the car's wheel is at every time t.

$\cancel{f(t) = 300x'' + 3500x' + 2.2\times10^4 x}$

$$Y = a\cos\frac{2\pi v t}{L} \Rightarrow Y(t) = 0.25\cos\frac{2\pi \overset{s}{v} t}{12}$$

$$\boxed{= 0.25\cos\frac{\pi v t}{6}}$$

(b) Let $x(t)$ be the height of the sprung mass m, where $x = 0$ when the car is stationary and not going over any bumps. The spring is compressed/stretched when the relative position $(x - y)$ of the ground and the mass moves from the neutral position $(x - y = 0)$, while the dashpot exerts a force proportional to the relative velocity $\frac{d}{dt}(x - y)$. Using The Stig's assumptions, write down a differential equation expressing the position of the sprung mass $x(t)$.

$F = ma : mx'' - k(x-y) \rightarrow ky = mx'' + kx$

$mx'' = k(x-y) + x' - y'$

$$\boxed{mx'' - x' - kx = -ky - y'}$$

(c) At what car speed will practical resonance occur?

m.

(d) If the dashpot is disconnected, at what car speed will resonance occur?

$F = ma \rightarrow mx'' = -k(x-y) \rightarrow ky = mx'' + kx$

$\rightarrow mx'' + kx = k(0.25)\cos\frac{2\pi vt}{12}$

$v = \frac{L}{2\pi}\sqrt{k/m} = \frac{6}{\pi}\sqrt{\frac{2.2\times10^4}{340}} \boxed{= 16.36\ \frac{m}{s}}$

(e) Which driver, James May, or Jeremy Clarkson, will experience the rougher ride, as defined as the amplitude of the particular solution $x_p(t)$? Compute each amplitude and decide.