1. (6 points) For each of the following differential equations, match it with its corresponding slope field and sketch the solution to the differential equation that passes through the given initial condition. Justify your answer by verifying specific points. Justify your answer by verifying specific points.

(a) $\dfrac{dy}{dx} = 1 - yx;\ y(1) = 1$

(b) $\dfrac{dy}{dx} = x^2 + y^2 - 1;\ y(1) = 0$

(c) $\dfrac{dy}{dx} = \dfrac{x}{1-y};\ y(0) = -3$

(d) $\dfrac{dy}{dx} = y - \sin(x);\ y(0) = 1$

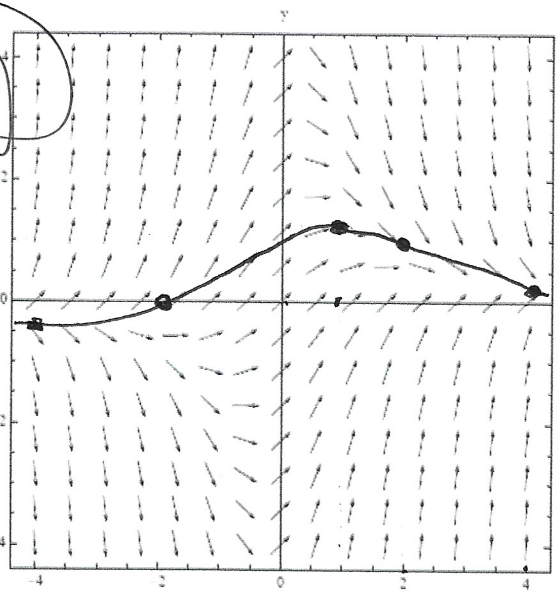
a.

$y'(-4,-\frac{1}{2}) = -1$

$y'(-2,0) = 1$

$y'(1,1) = 0$

$y'(2,1) = -1$

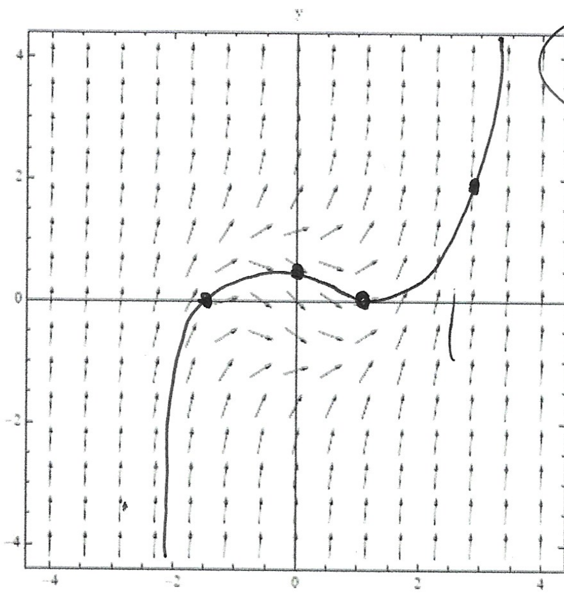


b

$y'(0,\frac{1}{2}) = -\frac{3}{4}$

$y'(-\frac{3}{2},0) = \frac{5}{4}$

$y'(3,2) = 12$



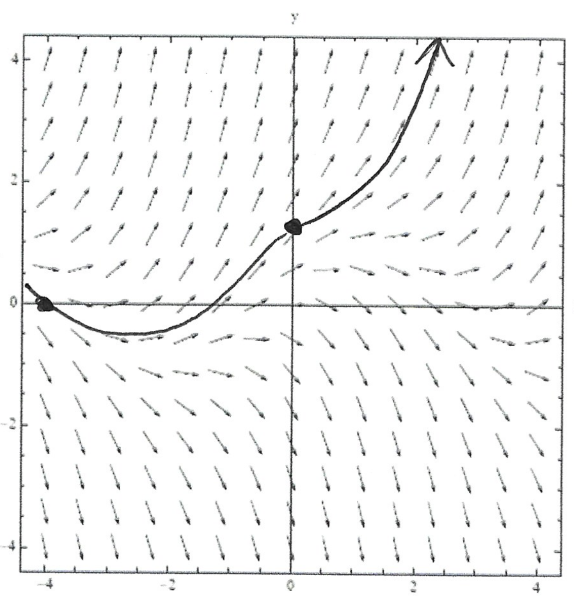
d

$y'(-4,0) =$

$0 - \sin(-4) < 0$

$y'(0,1) = y - \sin(0)$

$= 1$ ✓

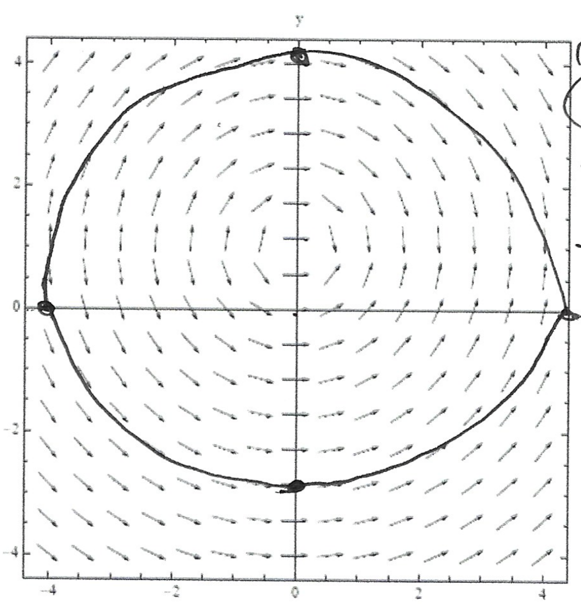


c

$y'(-4,0) = \frac{-4}{1}$

$y'(0,4) = 0$

$y'(4,0) = \frac{4}{1}$



2. (7 points) For each of the following problems, write the differential equation describing the given scenario. **DO NOT SOLVE IT.**

   (a) Let $v(t)$ be the velocity of a coasting motorboat at any time t. The motorboat's acceleration is proportional to the square of its velocity.

   $\cancel{\frac{dx}{dt}} = \boxed{\frac{dv}{dt} = k v^2}$

   (b) Let the population of Zootopia be $P(t)$ for any year t. Animals are born at a rate proportional to the population size, and the death rate is proportional to the population size as well. Animals move to Zootopia at a rate of 40,000 animals per year and away from Zootopia at a rate of 25,000 per year.

   $B = kP$ $\quad \cancel{\frac{dP}{dt}} =$ $\quad \Delta P = (B + 40000) - (D + 25000)\,\Delta t$

   $D = hP$

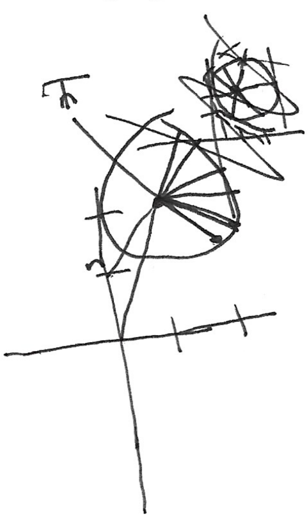
   $\therefore \frac{\Delta P}{\Delta t} = B - D + 15000 \Rightarrow \boxed{\frac{dP}{dt} = P(k-h) + 15000}$

   $\therefore \cancel{\frac{dB}{dt}} =$

   (c) A function $g(x)$ is described by some geometric property of its graph: At each point (x, y) on the graph $y = g(x)$, the normal line to the graph passes through the point $(1, 2)$. Use this information to write a differential equation of the form

   $$\frac{dy}{dx} = f(x, y),$$

   having the function $g(x)$ as its solution. **Hint:** If two lines, with slope m_1, m_2, are perpendicular then $m_1 m_2 = -1$.



@ $m_1 = 1$

$m_2 = -1$

$(y-2)^2 = \frac{r^2}{(x-1)^2}$

$g(x) = \text{circle} = (x-1)^2 + (y-2)^2 = r^2 \to g(x) = \frac{r}{x-1} + 2$

$\frac{dy}{dx} = g'(x) = \frac{d}{dx}\, r(x-1)^{-1} + 2 = -r(x-1)^{-2} \cdot 1 + 0 =$

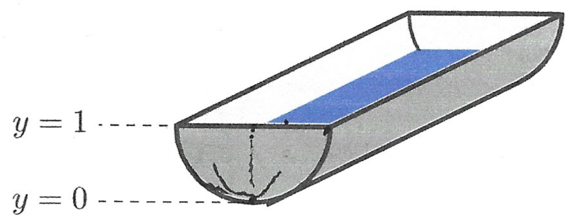
$\boxed{\frac{dy}{dx} = \frac{-r}{(x-1)^2}}$

3. (7 points) **Torricelli's Law**: Suppose we have a water tank with a hole at the bottom with area a. Denote the height of the water at time t (measured in seconds) by $y(t)$, where $y = 0$ is the bottom of the tank. Let $A(y)$ denote the cross-sectional area of the tank. Then

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity.

Consider a water trough (as pictured below) of length 2 meter that is made from cutting a cylindrical barrel (of radius 1 meter) in half. Assume there is a hole at the bottom of the tank that is 2/7 meter in radius.



$y^{-1/2}$

To simplify your calculation in this problem you may wish to use the approximation $\pi \approx \sqrt{10}$.

(a) Determine the cross-sectional area $A = A(y)$ as a function of y.

$A(y)\frac{dy}{dt} = -a\sqrt{2gy} \rightarrow \int \frac{dy}{dt} = \int \frac{-a}{A}\sqrt{2gy}$

$\int \frac{dy}{-a\sqrt{2gy}} = \int \frac{dt}{A} \rightarrow \frac{1}{-a\sqrt{2g}} \int \frac{dy}{y^{1/2}} = \frac{1}{A}\int dt \rightarrow \frac{2\sqrt{y}}{-a\sqrt{2g}} = \frac{t}{A} + C$

$A(y) = \frac{t}{\frac{2\sqrt{y}}{-a\sqrt{2g}}} + C$

$A = \frac{\pi r^2}{2}$, $A(0) = 0$ $\quad V = A \cdot l \rightarrow \frac{dV}{dt} = -a\sqrt{2gy}$ $\quad A = \frac{\pi y^2}{2}$

$A(y) = \frac{\pi}{2}$ $\quad V = \frac{1}{2}\pi r^2 l$

$A(1) = \frac{\pi r^2}{2}$ $\quad V(1) = \pi$ $\quad A(1) = \frac{\pi}{2}$ $\quad \boxed{A(y) = \frac{\pi y^2}{2}}$

$A(0) = 0$ $\quad V(0) = 0$ $\quad A(0) = 0$

(b) What is the ODE that describes the height of the water at time t?

$$\cancel{\frac{dy}{dt} = \frac{-a}{A}\sqrt{2gy}} = \frac{-a}{\frac{\pi L y^2}{2}}\sqrt{2gy} \to \frac{-2a\sqrt{2g}\, y^{1/2}}{\pi L y^2} \to \frac{-2a\sqrt{2g}}{\pi L \sqrt{y}^3}$$

$$\therefore \boxed{\frac{dy}{dt} = \frac{-2a\sqrt{2g}}{\pi L\sqrt{y}^3} \quad \text{where } a = \pi L\left(\frac{4}{49}\right)}$$

(c) Assume the trough starts completely full, how long does it take for the tank to completely empty?

$u^{-1/2}$

$$\cancel{\int \frac{dy}{\sqrt{2g}\sqrt{y}} = \int \frac{-a}{A}\,dt \to \frac{1}{\sqrt{2g}}\int y^{-1/2}\,dy = \frac{-a}{A}t + C}$$

$$\cancel{\to \frac{1}{\sqrt{2g}}\cdot 2\sqrt{y} = \frac{-a}{A}t + C \to \sqrt{y} = \frac{\sqrt{2g}}{2}\left(\left(\frac{-a}{A}\right)t + C\right) \to y(t) = \sqrt{\frac{\sqrt{2g}}{2}\left(\frac{-at}{A} + C\right)}}$$

$$\cancel{y(0) = 0 = \sqrt{\frac{\sqrt{2g}}{2}(+AC} \to \frac{\sqrt{2g}}{2}C = 0 \to C = 0 \to y(t) = \sqrt{\frac{\sqrt{2g}}{2}\left(\frac{-at}{A}\right)}}$$

$a = \pi L\left(\frac{2}{7}\right)^2$

$$\cancel{\int \pi\sqrt{y}^3\,dy = \int -2a\sqrt{2g}\,dt \to \frac{2\pi}{5}y^{5/2} = -2a\sqrt{2g}\,t + C \quad @(0,0),\ C = 0}$$

$$\cancel{\therefore \frac{2\pi}{5}y^{5/2} = -2a\sqrt{2g}\,t \to y^{5/2} = \frac{-10a\sqrt{2g}\,t}{2\pi} \to y = \left(\frac{-5\pi\frac{4}{49}\sqrt{2g}\,t}{\pi}\right)^{2/5}}$$

$$\cancel{y = \left(\frac{-20}{49}\sqrt{2g}\,t\right)^{2/5}}$$

NEXT PAGE

Lab 2

3 c) $\frac{dy}{dt} = \frac{-2a\sqrt{2g}}{\pi\sqrt{y}^3} \Rightarrow \frac{-2\cdot\pi(\frac{4}{49})\sqrt{2g}}{\pi\sqrt{y}^3} = \frac{\frac{-8}{49}\sqrt{2g}}{\sqrt{y}^3}$

$\int \sqrt{y}^3\,dy = \int \frac{-8\sqrt{2g}}{49}\,dt \rightarrow \frac{2}{5}y^{5/2} = \frac{-8\sqrt{2g}}{49}t + C$

$y(0) = 1 = C - \frac{8\sqrt{2g}}{49}(0) \rightarrow C = 1 \quad \therefore \quad y^{5/2} = \frac{-40\sqrt{2g}}{5\cdot 49}t + 1$

@ $y = 0$, $0 = 1 - \frac{40\sqrt{2g}t}{5\cdot 49} \rightarrow 1 = \frac{40\sqrt{2g}t}{5\cdot 49} \rightarrow \frac{5\cdot 49}{40\sqrt{2g}} = \boxed{t = 1.38 \text{ sec}}$