1. (6 points) Consider the vector space $V = \text{span}\{u(t), v(t)\}$ where $u(t)$ and $v(t)$ are linearly independent functions.

   (a) Define two new functions $p(t) = a(u(t)+v(t))$ and $q(t) = b(u(t)-v(t))$. Show that for any $a, b \neq 0$ the set $\{p(t), q(t)\}$ is also a basis for V.

vectors $p(t), q(t)$ have to be lin independent
vectors $p(t), q(t)$ have to span V ✓

$p(t) = a\,u(t) + a\,v(t)$
$q(t) = b\,u(t) - b\,v(t)$

$a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} x_1 & x_2 & x \\ y_1 & y_2 & y \end{array}\right]$

$\left[\begin{array}{cc|c} x_1 & x_2 & x \\ 0 & y_2x_1 - x_2y_1 & yx_1 - xy_1 \end{array}\right]$

$x = x_1 + x_2$

$yx_1 = y_2x_1 \cancel{- y_1x_2} + x_1y_1 \cancel{+ x_2y_1}$

$y = y_2 + y_1$ ✓ ~~V span~~

   (b) Verify that $u(t) = e^{it}$ and $v(t) = e^{-it}$ are linearly independent solutions of $x'' + x = 0$.

$W(e^{it}, e^{-it}) = \begin{vmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it} \end{vmatrix}$ ~~$(ie^{it} \cdot e^{it}) - (ie^{it} \cdot e^{it}) = i - i = 0$~~

$(e^{it}(-i \cdot e^{-it})) - (i \cdot e^{it} \cdot e^{-it}) =$

$= -i(1) - i(1) = -2i \neq 0 \therefore$ independent

(c) Use Euler's identity ($e^{it} = \cos(t) + i\sin(t)$) along with part (a) to show that $\{\cos(t), \sin(t)\}$ is a basis for the solution space of $x'' + x = 0$.

$u(t) = \cos(t) + i\sin(t)$

2. (7 points) Let P be the following set:

$$P = \{\mathbf{p}_1(x),\ \mathbf{p}_2(x),\ \mathbf{p}_3(x),\ \mathbf{p}_4(x)\},$$

where $\mathbf{p}_1(x) = 1, \mathbf{p}_2(x) = x, \mathbf{p}_3(x) = 3x^2 - 1$, and $\mathbf{p}_4(x) = 5x^3 - 3x$.

(a) Determine whether the set P is linearly independent or linearly dependent.

$P = \{1, x, 3x^2-1, 5x^3-3x\}$

$$W(P) = \begin{vmatrix} 1 & x & 3x^2-1 & 5x^3-3x \\ 0 & 1 & 6x & 15x^2-3 \\ 0 & 0 & 6 & 30x \\ 0 & 0 & 0 & 30 \end{vmatrix} = 1\begin{vmatrix} 1 & 6x & 15x^2-3 \\ 0 & 6 & 30x \\ 0 & 0 & 30 \end{vmatrix} = 1\begin{vmatrix} 6 & 30x \\ 0 & 30 \end{vmatrix} = 6(30)-0 = 180$$

$\neq 0$

$\therefore$ $\boxed{\text{independent}}$

(b) Find a linear combination of elements of P that represents the polynomial $y(x) = 1 + x + x^2 + x^3$.

$1, x, 3x^2-1, 5x^3-3x$ $\quad c_1(1) + c_2(x) + c_3(3x^2-1) + c_4(5x^3-3x) = 1 + x + x^2 + 3x^3$

$1c_1 = 1 \therefore \boxed{c_1 = 1}$

$c_2 x = x \therefore \boxed{c_2 = 1}$

$c_3(3x^2-1) = x^2$

$c_3 = \dfrac{x^2}{3x^2-1} = \boxed{\dfrac{1}{3} - x^2}$

$c_4(5x^3-3x) = x^3 \therefore c_4 = \dfrac{x^3}{5x^3-3x} = \boxed{\dfrac{1}{5} - \dfrac{x^3}{3}}$

(c) Does P span the vector space V of all polynomials of third-degree or less? Justify your answer.

Yes. P is linearly independent and is 3° order