

# Geometric Foundations of Data Analysis I

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## 1 Introduction

Data analysis and more broadly Data Science is a vast and important field within Computer Science and Mathematics. At the core, the goal is to make sense of data, which can be measurements, survey results, behavior patterns, etc. Often this data comes to us in a very “high dimension”. That is, there are so many variables that it is impossible to visualize, and even in low dimensions, it may not be clear what the best conclusion is based on the analysis.

A few references seem to agree that the *total data* on all computers is something like  $10^{23}$  bytes or about 100 zettabytes. While all of this data is not concentrated in one organization, we still require highly sophisticated tools to make sense of

a huge amount of data. My goal with this course is that you will have a solid foundation with some standard tools. From this bedrock one could explore more sophisticated methods of data analysis more easily.

We will consider four key topics:

1. Least Squares Fitting,
2. Principal Component Analysis,
3.  $k$ -Means and Hierarchical Clustering,
4. Nearest Neighbors and the Johnson–Lindenstrauss Theorem

We will be working with the assumption that **the data we care about is preserved by orthogonal and linear transformations**. This is not true with all data—for example, one should not take (proper) linear combinations of people. However, for data like grams of different kinds of food, this is completely plausible. This assumption will not always be necessary, but we will just keep this in mind.

Half of our time will be spent bringing these ideas to life and getting our hands dirty. We will be working with Jupyter Notebooks to build familiarity with the concepts we will discuss. This will be done using Python and standard data analysis packages like Pandas.

## 2 Least squares fitting

This method of analysis is both simple and powerful. Like with most things in mathematics, without some good guiding examples, we can get lost in the formulas.

### 2.1 Build up

Suppose we have the following data as seen in Figure 2.1. (Maybe a company produces parts once per month in lots that vary in size depending on demand. We write  $i$  for the production run,  $x_i$  for the lot size, and  $y_i$  for the hours of labor.)

It seems clear from the plot that the data fits a geometric pattern—there is a linear phenomenon. If we try to find the line that is somehow closest to all the data points, we might draw something like in Figure 2.2.

Although the line is not a perfect fit, it seems to tell us something about the relationship between the lots size and the amount of hours.

#### Questions:

- What makes this line “better” than alternatives?
- How are we quantifying “better”?
- Why are we using a line?

$i$	$x_i$	$y_i$
1	30	73
2	20	50
3	60	128
4	80	170
5	40	87
6	50	108
7	60	135
8	30	69
9	70	148
10	60	132

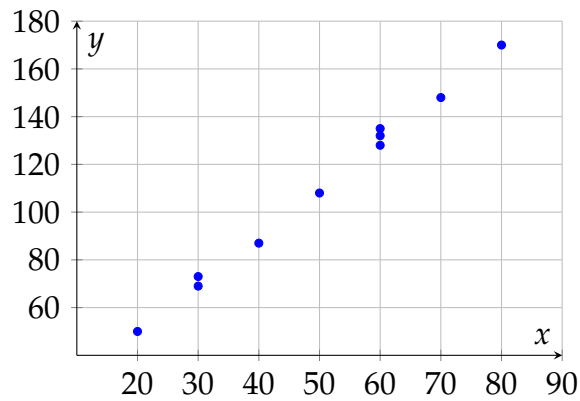


Figure 2.1: Data points exhibiting a linear relationship.

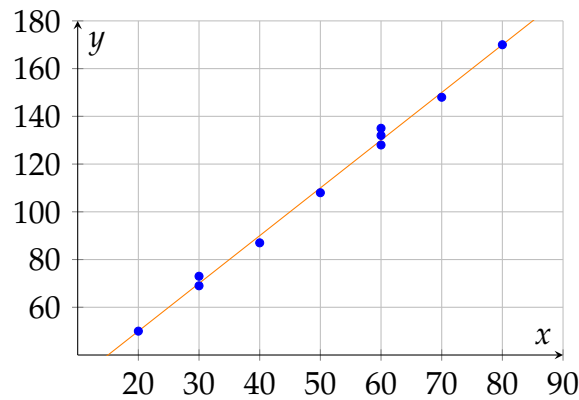


Figure 2.2: Line of best fit with data points.

We will answer the first question later, probably. Let us consider the question of quantifying “better”. Least squares fitting is all about minimizing the squares of differences between the line and the actual data points. We will make this precise very soon.

## 2.2 Line of best fit

We know that the equation of a non-vertical line has the form

$$y = b_0 + b_1x.$$

If we had  $n$  data points of the form  $(x_i, y_i)$ , we could choose  $b_0$  and  $b_1$  to minimize the following sum

$$S(b_0, b_1) = \sum_{i=1}^n (y_i - (b_0 + b_1x_i))^2.$$

We can even solve for these values. Since  $S$  is a function in terms of  $b_0$  and  $b_1$ , all possible minima occur when the partial derivatives of  $S$  are 0. In other words,

the minima arise as values  $(b_0, b_1)$  such that

$$\begin{aligned}\frac{\partial S}{\partial b_0} &= -2 \sum_{i=1}^n (y_i - (b_0 + b_1 x_i)) = 0, \\ \frac{\partial S}{\partial b_1} &= -2 \sum_{i=1}^n (x_i y_i - x_i (b_0 + b_1 x_i)) = 0.\end{aligned}$$

These equations are linear equations, so we can solve for these with techniques from linear algebra. This mean, we need to solve two equations in the unknown  $b_0$  and  $b_1$ :

$$\begin{aligned}nb_0 + b_1 \sum x_i &= \sum y_i, \\ b_0 \sum x_i + b_1 \sum x_i^2 &= \sum x_i y_i.\end{aligned}\tag{2.1}$$

Using the data from our example in Section 2.1, we have

$$\sum x_i = 500, \quad \sum y_i = 1100, \quad \sum x_i^2 = 28400, \quad \sum x_i y_i = 61800.$$

Thus, the equations we need to solve are

$$\begin{aligned}10b_0 + 500b_1 &= 1100, \\ 500b_0 + 28400b_1 &= 61800,\end{aligned}$$

which yield  $b_0 = 10$  and  $b_1 = 2$ . Going back to the context of the initial problem: this solution tells us that by increasing the lot size by one, we expect to increase the labor hours by two.

### 2.2.1 Written as matrices

Let us write the equations in (2.1) with matrices. This might seem like overkill at this stage, but it will set us up nicely to generalize. Let

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad B = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.\tag{2.2}$$

Therefore, the equations in (2.1) are equivalent to the single matrix equation:

$$X^t X B = X^t Y.\tag{2.3}$$

If  $X^t X$  is invertible, then  $B = (X^t X)^{-1} X^t Y$ .

## 2.3 In class exercises pt. I

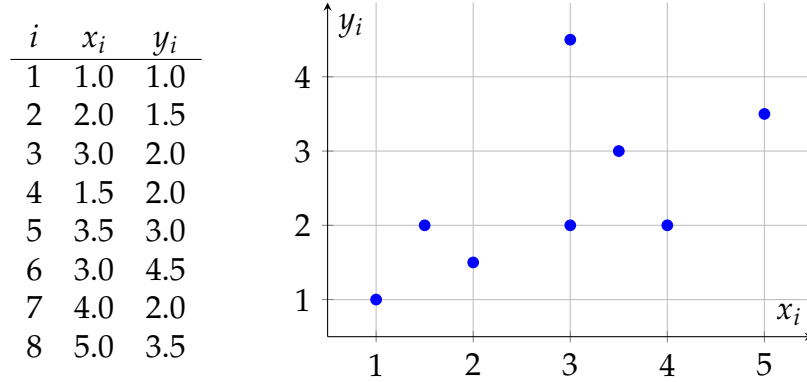
- (a) With  $X$ ,  $Y$ , and  $B$  as defined in Equation (2.2), show that

$$X^t X = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}, \quad X^t Y = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}.$$

(b) Show that  $X^tXB = X^tY$  is equivalent to Equation (2.1):

$$\begin{aligned} nb_0 + b_1 \sum x_i &= \sum y_i, \\ b_0 \sum x_i + b_1 \sum x_i^2 &= \sum x_i y_i. \end{aligned}$$

2. Find a least squares fitting line to the following data and draw in the line:



(Round  $b_0$  and  $b_1$  to the nearest half integer.)

Week 1

## 2.4 Plane of best fit

We consider two independent variables and one dependent variable now. Consider the following data points as given in Figure 2.3.

We can put some meaning to these data. For example, suppose a company is selling a product, and we have 16 populations of people labeled 0 through 15. The values  $x_{i1}$  are the population sizes in 100s of people; the values  $x_{i2}$  are the average yearly income in €1000 per capita; and the values  $y_i$  are the number of sales of the product. (There might be dependencies between population size and average income, but our model treats them as independent.)

It looks like though there is a plane of best fit for the data—thanks to the suggestive viewing angle. Our goal is to find a plane, given by

$$y = b_0 + b_1x_1 + b_2x_2.$$

We can just do what we did last time. That is, for

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad B = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix},$$

we need to solve for  $B$  in the equation

$$X^tXB = X^tY.$$

$i$	$x_{i1}$	$x_{i2}$	$y_i$
0	278	36	287
1	252	31	256
2	344	35	300
3	134	33	182
4	215	35	248
5	261	40	271
6	131	39	149
7	463	43	411
8	167	46	214
9	298	42	291
10	230	60	314
11	293	67	352
12	290	37	298
13	271	31	252
14	385	63	439
15	354	36	328

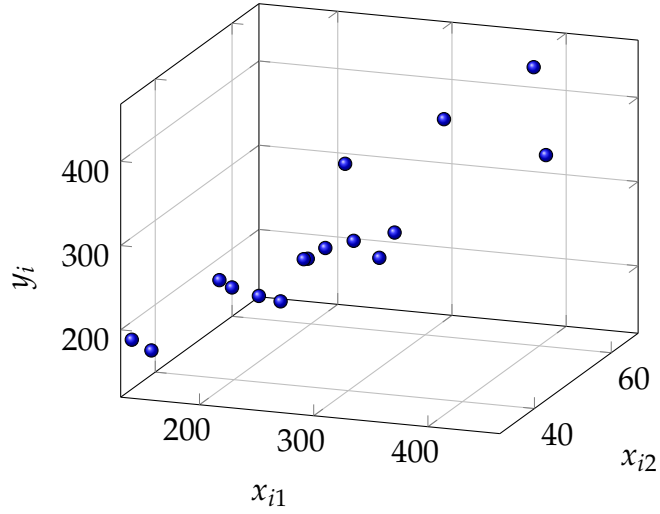


Figure 2.3: Data points in  $\mathbb{R}^3$ .

Thus, if  $X^t X$  is invertible, there is a unique  $B$ , which is equal to  $(X^t X)^{-1} X^t Y$ . For our example, we have

$$X^t X = \begin{pmatrix} 16 & 4366 & 674 \\ 4366 & 1309480 & 187024 \\ 674 & 187024 & 30330 \end{pmatrix}, \quad X^t Y = \begin{pmatrix} 4592 \\ 1343400 \\ 200571 \end{pmatrix}.$$

Therefore, the plane of best fit is approximately

$$y = -11.3 + 0.7x_1 + 2.6x_2.$$

Putting all the data together we have a plane of best fit as seen in Figure 2.4.

## 2.5 Hyperplane of best fit

Now we go to the general case. Suppose we have  $p - 1$  independent variables and 1 dependent variable, where  $p \geq 2$ . We assume we have  $n$  data points of the form

$$(x_{i1}, x_{i2}, \dots, x_{i,p-1}, y_i) \in \mathbb{R}^p.$$

The least squares fitting for these data is a hyperplane of the form

$$y = b_0 + b_1x_1 + b_2x_2 + \dots + b_{p-1}x_{p-1}.$$

To solve for the values  $b_i$ , we do as we did before. We define matrices

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1,p-1} \\ 1 & x_{21} & x_{22} & \cdots & x_{2,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{n,p-1} \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad B = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-1} \end{pmatrix}.$$

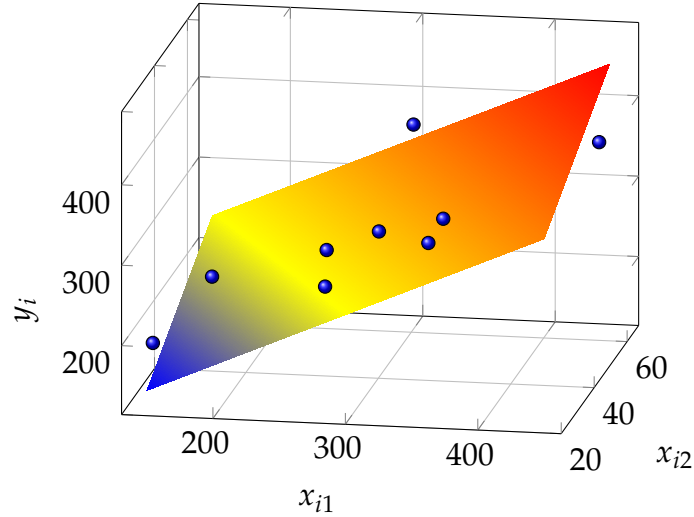


Figure 2.4: Data points together with the plane of best fit.

As before, the values we want are given by the equation

$$X^t X B = X^t Y. \quad (2.4)$$

## 2.6 Why Equation (2.4) works

The heart of least squares is (Euclidean) distance. The distance between two points  $x = (x_1, \dots, x_p)$  and  $y = (y_1, \dots, y_p)$  in  $\mathbb{R}^p$  is

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_p - y_p)^2}.$$

For a vector  $v = (v_1, \dots, v_p) \in \mathbb{R}^p$ , the **length** of  $v$  is

$$\|v\| = d(0, v) = \sqrt{v_1^2 + v_2^2 + \dots + v_p^2}.$$

Recall that the dot product of two (column) vectors  $u$  and  $v$  is

$$u \cdot v = u^t v = u_1 v_1 + u_2 v_2 + \dots + u_p v_p.$$

Thus, the length of  $v$  is  $\|v\| = \sqrt{v \cdot v}$ ; in other words  $\|v\|^2 = v \cdot v$ . In addition, if  $u \cdot v = 0$ , we say that  $u$  and  $v$  are **orthogonal** (or perpendicular).

The goal of least squares is to *minimize distance*; more specifically to minimize  $\|Y - XB\|$ . Note that the column vector  $Y$  has entries that are the *actual*  $y_i$  values, and the column vector

$$XB = \begin{pmatrix} B \cdot (1, x_{11}, x_{12}, \dots, x_{1,p-1}) \\ B \cdot (1, x_{21}, x_{22}, \dots, x_{2,p-1}) \\ \vdots \\ B \cdot (1, x_{n1}, x_{n2}, \dots, x_{n,p-1}) \end{pmatrix} = \begin{pmatrix} b_0 + b_1 x_{11} + b_2 x_{12} + \dots + b_{p-1} x_{1,p-1} \\ b_0 + b_1 x_{21} + b_2 x_{22} + \dots + b_{p-1} x_{2,p-1} \\ \vdots \\ b_0 + b_1 x_{n1} + b_2 x_{n2} + \dots + b_{p-1} x_{n,p-1} \end{pmatrix}$$

Therefore,  $\|Y - XB\|$  is the square root of a sum of squares of the form

$$y_i - b_0 + b_1x_{i1} + b_2x_{i2} + \cdots + b_{p-1}x_{i,p-1}.$$

Hence minimizing  $\|Y - XB\|$  is the same as minimizing  $\|Y - XB\|^2$ , which is a sum of *squares*.

**Proposition 2.1.** *The minimal distance  $\|Y - XB\|$  is achieved by solving for  $B$  in*

$$X^tXB = X^tY.$$

Week 2

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*Proof.* Consider the subspace  $U = \{Xu \mid u \in \mathbb{R}^p\}$  of  $\mathbb{R}^n$ , and observe that our desired solution  $XB$  is contained in  $U$ . Since  $\|Y - XB\|$  is minimal, we must have that the vector  $Y - XB$  is orthogonal to all vectors contained in  $U$ .<sup>1</sup> That is,  $(Xu) \cdot (Y - XB) = 0$  for all  $u \in \mathbb{R}^p$ . In other words, we have for all  $u \in \mathbb{R}^p$ ,

$$\begin{aligned} 0 &= (Xu)^t(Y - XB) = u^tX^t(Y - XB) \\ &= u^t(X^tY - X^tXB). \end{aligned}$$

Because  $u^t(X^tY - X^tXB) = 0$  for all  $u \in \mathbb{R}^p$ , it follows that  $X^tY - X^tXB = 0$ .  $\square$

## 2.7 In class exercises pt. II

1. Determine  $X^tX$  and  $X^tY$  with

$$X = \begin{pmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1,p-1} \\ 1 & x_{21} & x_{22} & \cdots & x_{2,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{n,p-1} \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

2. Using (1) and by taking partial derivatives of

$$S(b_0, \dots, b_{p-1}) = \sum_{i=1}^n (y_i - (b_0 + b_1x_{i1} + b_2x_{i2} + \cdots + b_{p-1}x_{i,p-1}))^2, \quad (2.5)$$

show that the hyperplane of best fit is obtained by solving  $X^tXB = X^tY$ . (You could try this for  $p = 3$  first.)

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<sup>1</sup>To see why this is true, see Section 6.3.1 of [3], which is all about orthogonal decompositions.



## 2.8 Nonlinear fittings

Although all of our examples so far have been linear fittings, we will demonstrate that least squares fittings works in the nonlinear case. What is important is that we have a candidate equation to fit. In the linear cases, we tried to fit

$$y = b_0 + b_1x_1 + b_2x_2 + \cdots + b_{p-1}x_{p-1}.$$

Suppose we have the following data as given in Figure 2.5. Instead of trying to fit the line  $y = b_0 + b_1x$ , we could try to fit the parabola:

$$y = b_0 + b_1x + b_2x^2.$$

We can treat this the same way as before. Of course the quantities  $x$  and  $x^2$  are *not* independent, but we can ignore this. Set

$$x_{i1} = x_i, \quad x_{i2} = x_i^2.$$

Therefore, the hyperplane of best fit for the data  $(x_{i1}, x_{i2}, y_i)$  will give us the parabola of best fit. *Try this on your own!*

$x_i$	$y_i$
2.27	2.50
5.06	-16.13
1.45	4.23
5.89	-22.46
0.48	1.37
-0.22	0.86
1.44	11.85
-1.77	-14.71
2.45	9.42
-1.54	-14.07
7.55	-55.62
1.76	4.45
5.16	-19.56
3.26	-2.79
3.23	5.20
0.85	8.09

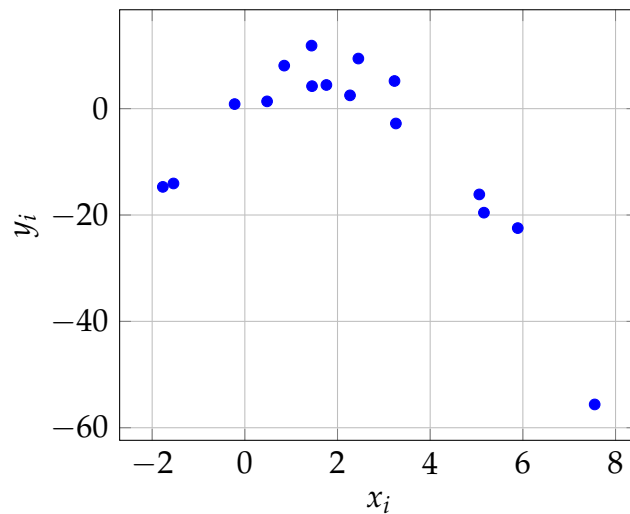


Figure 2.5: Data points demonstrating a nonlinear relationship.

So one can fit any hypersurface  $y = f(x_1, \dots, x_{p-1})$  to the given data. The function  $f$  in this case is called the **regression function**. This general method of analysis is known as **regression analysis**. A few questions arise:

- Which surface is “best”?
- How can we quantify “best”?
- Even in the line case ( $p = 2$ ), how can we quantify how well data fits our line?

## 2.9 Coefficient of determination ( $R^2$ values)

We are going to make precise how well our hyperplane fits our data. Recall that hyperplanes can be replaced by hypersurfaces; see Section 2.8. First we establish some notation. Suppose we have  $n$  data points  $(x_{i1}, x_{i2}, \dots, x_{i,p-1}, y_i) \in \mathbb{R}^p$ . Then we define

$$\begin{aligned} \text{(Fitted value)} \quad & \hat{y}_i = b_0 + b_1 x_{i1} + b_2 x_{i2} + \dots + b_{p-1} x_{i,p-1}, \\ \text{(Residual)} \quad & e_i = y_i - \hat{y}_i, \\ \text{(Sample mean)} \quad & \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i. \end{aligned}$$

These yield vectors in  $\mathbb{R}^n$  as follows

$$\hat{Y} = \begin{pmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{pmatrix} = XB, \quad E = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = Y - \hat{Y}, \quad \bar{Y} = \begin{pmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix} = \bar{y} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

From our  $n$  data points, we have three points in  $\mathbb{R}^n$  given by  $Y$ ,  $\hat{Y}$ , and  $\bar{Y}$ . Three points always lie on a plane, so the three points determine a triangle on such a plane. What does this triangle look like? If it is a triangle (and not a line or a single point), then the next lemma proves it must be a right triangle.

**Lemma 2.2.** *The vectors  $E = Y - \hat{Y}$  and  $\hat{Y} - \bar{Y}$  are orthogonal.*

*Proof.* Suppose  $X^t X B = X^t Y$ . We need to prove two equations. For the first,

$$0 = X^t(Y - XB) = X^t(Y - \hat{Y}) = X^t E.$$

Hence,  $X^t E = 0$ . For the second,

$$\begin{aligned} \bar{Y}^t E &= \bar{y} \sum_{i=1}^n (y_i - \hat{y}_i) \\ &= \bar{y} \sum_{i=1}^n (y_i - (b_0 + b_1 x_{i1} + b_2 x_{i2} + \dots + b_{p-1} x_{i,p-1})) \\ &= -\frac{\bar{y}}{2} \cdot \frac{\partial S}{\partial b_0} = 0, \end{aligned}$$

where  $S$  is defined in Equation (2.5), so  $\bar{Y}^t E = 0$ . Thus, we have

$$(\hat{Y} - \bar{Y}) \cdot E = (XB)^t E - \bar{Y}^t E = 0. \quad \square$$

**Remark 2.3.** One can simplify the proof for Lemma 2.2 by applying an isometry to the data, so that  $\bar{y} = 0$ . That is, one only needs to prove that  $E$  and  $\hat{Y}$  are orthogonal.

The lengths of the differences of the vectors are important and have names:

$$\begin{aligned} (\text{Sums of Squares Total} - SST) &: \|Y - \bar{Y}\|^2, \\ (\text{Sums of Squares Error} - SSE) &: \|Y - \hat{Y}\|^2, \\ (\text{Sums of Squares Regression} - SSR) &: \|\hat{Y} - \bar{Y}\|^2. \end{aligned}$$

The value  $SST$  measures the *total variability* of the data set. For example,  $\sqrt{SST} = \|Y - \bar{Y}\|$  is the distance from the actual data  $Y$  to the sample mean  $\bar{Y}$ . Using the same ideas, we can see that  $SSE$  measures the error of our regression and that  $SSR$  measures the distance from our regression to the sample mean.

**Proposition 2.4.**

$$SST = SSE + SSR.$$

*Proof.* Apply Lemma 2.2 and the Pythagorean Theorem:

$$\|Y - \bar{Y}\|^2 = \|Y - \hat{Y}\|^2 + \|\hat{Y} - \bar{Y}\|^2. \quad \square$$

Now we can describe a quantity that measures how good our regression fits the given data.

**Definition 2.5.** The **coefficient of determination** (also known as the  $R^2$ -value) is

$$R^2 = \frac{SSR}{SST} = \frac{\|\hat{Y} - \bar{Y}\|^2}{\|Y - \bar{Y}\|^2}.$$

**Proposition 2.6.**  $0 \leq R^2 \leq 1$ .

*Proof.* Since each  $SST$  and  $SSR$  are squares, they are nonnegative. By Proposition 2.4, we have  $0 \leq SSR \leq SST$ .  $\square$

### 2.9.1 What do the extremes means?

The one case where  $R^2$  is meaningless is when  $SST = 0$ . This implies both  $SSR = SSE = 0$ . Moreover,  $Y = \bar{Y} = \bar{y}\mathbb{1}$ , where  $\mathbb{1}$  is the all ones column vector. Hence, every data point  $y_i$  is the same and, therefore, equal to the mean. Let's never return to this case.

We can have  $SSR = 0$ , which is equivalent to  $R^2 = 0$ . This implies that  $\|\hat{Y} - \bar{Y}\|^2 = 0$ , so that  $\hat{Y} = \bar{Y}$ . In other words, our prediction  $\hat{y}_i$  is just simply the mean. This means we have not found any relationship between the independent variables and the dependent variables.

In the other extreme we have  $SSR = SST$ , which is equivalent to  $R^2 = 1$ . This implies that  $Y = \hat{Y}$ , so the given data lies (exactly) on the surface given by  $y = f(x_1, \dots, x_{p-1})$ . That is, the regression function exactly predicts the data.

To summarize, when  $R^2 = 0$ , we cannot deduce any relationship between the independent and dependent variables, and when  $R^2 = 1$ , we understand completely the relationship between the independent and dependent variables. Very roughly speaking, the  $R^2$  can be thought of as the ratio of how well the regression fits the data.

## 2.10 In class exercises pt. III

1. Prove the following.

(a)  $\|\bar{Y}\|^2 = n\bar{y}$ .

(b)  $Y \cdot \bar{Y} = \hat{Y} \cdot \bar{Y} = \|\bar{Y}\|^2$ .

(c)  $Y \cdot \hat{Y} = \|\hat{Y}\|^2$ .

2. Use (1) to show that

(a)  $SST = \|Y\|^2 - \|\bar{Y}\|^2$ ,

(b)  $SSE = \|Y\|^2 - \|\hat{Y}\|^2$ ,

(c)  $SSR = \|\hat{Y}\|^2 - \|\bar{Y}\|^2$ .

3. What are the  $R^2$  values for the examples above?

## 3 Principal component analysis

Principal component analysis (PCA) is a power method of analysis that comes standard in all data science tool kits. With little effort, one can reduce a complex data set to data that we can more easily see structure. More specifically, the goal of PCA is to find the “best” basis to express the data. In other words, our initial reference frame may not be the one that best expresses the structure of our data—PCA is a method to find the “best” reference frame.

### 3.1 Introducing PCA

We will start with a toy example, where the analysis is quite simple. Suppose we have many many data points in  $\mathbb{R}^2$  as seen in Figure 3.1.

Let’s assume that our data points live in  $\mathbb{R}^m$ , so that  $m = 2$  in Figure 3.1. Suppose we write those data points in an  $m \times n$  matrix  $X$ . Our current (and default) basis is  $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$ , and we want a basis  $\{p_1, p_2, \dots, p_m\}$  that better reflects the structure of our data. That is, we want an  $m \times m$  matrix  $P$ , whose rows are the  $p_i$ , that provides us a better reference frame. Therefore, we want to transform our data  $X$  into a new data set  $Y$  such that

$$PX = Y.$$

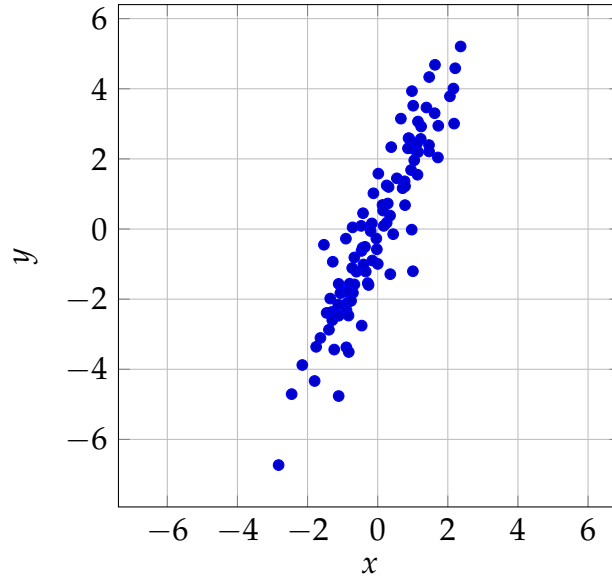


Figure 3.1: Some data in  $\mathbb{R}^2$ .

The columns of  $X$  are the “old” data, and the columns of  $Y$  are the “new” data. If the  $p_i$  are row vectors and the  $x_i$  column vectors, then we want

$$PX = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} (x_1 \ x_2 \ \cdots x_n) = \begin{pmatrix} p_1 \cdot x_1 & p_1 \cdot x_2 & \cdots & p_1 \cdot x_n \\ p_2 \cdot x_1 & p_2 \cdot x_2 & \cdots & p_2 \cdot x_n \\ \vdots & \vdots & \ddots & \vdots \\ p_m \cdot x_1 & p_m \cdot x_2 & \cdots & p_m \cdot x_n \end{pmatrix}.$$

Thus, if the columns of  $Y$  are written  $y_i$ , we have

$$y_i = \begin{pmatrix} p_1 \cdot x_i \\ p_2 \cdot x_i \\ \vdots \\ p_m \cdot x_i \end{pmatrix}. \quad (3.1)$$

Back to our example from Figure 3.1. We want data with a high *signal-to-noise* (SNR) ratio as to minimize noise. Assuming our data in Figure 3.1 was collected reasonably well, the direction of largest variance is the direction of most interesting dynamics. Therefore, the variance of signal,  $\sigma_s^2$ , would correspond to the length of the orange vector in fig. 3.2 pointing to the top right, and the variance of the noise,  $\sigma_n^2$ , would correspond to the length of the orange vector pointing to the top left.

Note also that in Figure 3.2 knowing the  $x$  value gives one a good approximation for the  $y$  value and vice versa. In this case, we might say that the data has a moderate amount of redundancy, whereas if the data points had a much higher  $R^2$  value to its line of best fit, we would say the data have high redundancy. (And if the data had a much lower  $R^2$  value, we would say the data have low redundancy.) One of the aims of PCA is to lower redundancy. For the 2-dimensional

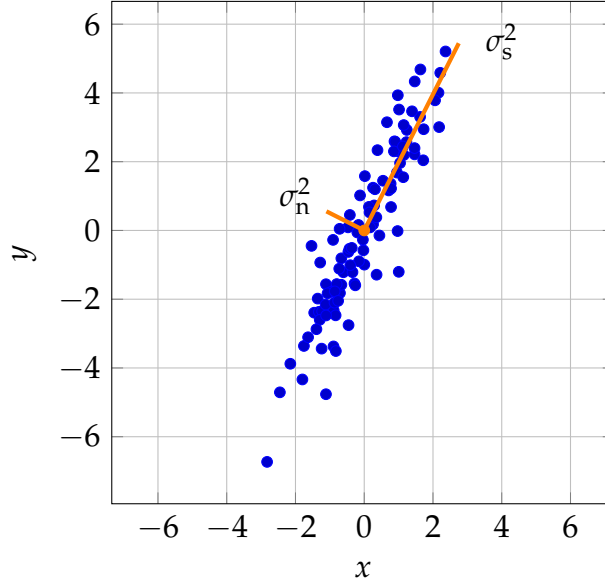


Figure 3.2: Signal and noise variances represented graphically.

case, this is simple—take the line of best fit, but for arbitrarily higher dimensions, this is not obvious.

Suppose we have  $n$  measurements

$$U = \{u_1, u_2, \dots, u_n\} \quad \text{and} \quad V = \{v_1, v_2, \dots, v_n\}$$

**with mean equal to 0.** The *variances* are equal to

$$\sigma_U^2 = \frac{1}{n} \sum_{i=1}^n u_i^2, \quad \sigma_V^2 = \frac{1}{n} \sum_{i=1}^n v_i^2.$$

The *covariance* between the data sets  $U$  and  $V$  is

$$\sigma_{UV}^2 = \frac{1}{n} \sum_{i=1}^n u_i v_i.$$

The covariance measures the degree of the linear relationship between the two variables. Thus, a large positive value would imply that the data are positively correlated, and a large negative value would imply negatively correlated. And  $\sigma_{UV}^2 = 0$  if and only if the data  $U$  and  $V$  are uncorrelated. Moreover the absolute magnitude of the covariance measures the degree of redundancy.

If instead we wrote  $u = (u_1, u_2, \dots, u_n)$  and  $v = (v_1, v_2, \dots, v_n)$  as row vectors, then

$$\sigma_{uv}^2 = \frac{1}{n} u v^t. \quad (3.2)$$

We generalize from two vectors to  $m$  vectors. Write

$$X = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{pmatrix}$$

where  $x'_i$  is a row vector. Note that the rows of  $X$  correspond to measurements of a particular type, and the columns of  $X$  correspond to all measurements of a particular trail. Using Equation (3.2), we define the **covariance matrix** of  $X$  to be

$$C_X = \frac{1}{n} X X^t.$$

**Lemma 3.1.** 1. The matrix  $C_X$  is symmetric. That is,  $C_X = C_X^t$ .

2. The diagonal entries of  $C_X$  are variances.

3. The off-diagonal entries of  $C_X$  are covariances.

Recall our goal is to find a new (and better) basis  $\{p_1, p_2, \dots, p_m\}$ . Namely, we want an invertible matrix  $P$  to turn our data  $X$  into a data set  $Y$  where we can better understand the structure. If we could do this on the level of covariance matrices, then we could pick an ideal covariance matrix, one where the diagonal entries are large in absolute magnitude and where the off-diagonal entries are small in absolute magnitude. (Variances being high in magnitude suggest interesting dynamics, and covariances low in magnitude suggest low redundancy.) In other words, we would like to find a matrix  $P$  such that

$$C_Y = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}, \quad (3.3)$$

where  $Y = PX$ . Typically we have  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ .

The main work of PCA is finding such a matrix  $P$ . We will describe how to construct such a  $P$  later, but let us return to our running example.

The first principal component is in the direction of the largest variance, and the second principal component is in the orthogonal direction. So in  $\mathbb{R}^2$ , this is quite simple. Although we have not given all the data point explicitly, the covariance matrix is

$$C_X = \begin{pmatrix} 1.27 & 2.52 \\ 2.52 & 5.95 \end{pmatrix}.$$

The slope of the line in the direction of the highest variance is 1.98, and it passes through the point  $(0,0)$ . By rotating and permuting, we get

$$P = \begin{pmatrix} 0.40 & 0.92 \\ 0.92 & -0.40 \end{pmatrix},$$

and the graph of the data  $Y = PX$  is seen in Figure 3.3.

**Remark 3.2.** There is a whole art of scaling data in a pre-processing stage that we will not explore in this course. Basically, if one variable ranges between  $\pm 10$  and another  $\pm 10^3$ , the second variable will bias the process simply by its scale. There are many different methods to rescale the data as not to lose (too much) information. Throughout we will assume our data has roughly the same scale and not worry about rescaling, but in practice this is an important issue.

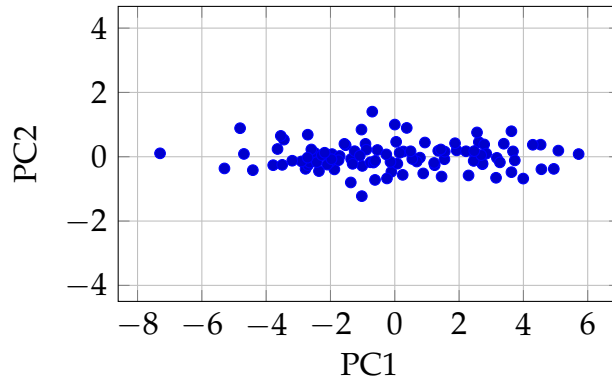


Figure 3.3: A new basis for our data.

### 3.1.1 Assumptions of PCA

Before we close this introduction to PCA, let us come back to some of the assumptions we have made along the way. There are three key assumptions we have made in introducing PCA. We will not focus much on these, but users of PCA should know that assumptions have been made. These statements may not necessarily hold for a particular data set.

1. *Linearity.*

This allows us to reframe the problem as a change of basis problem.

2. *Large variance is important (and  $SNR > 1$ ).*

This can be a very strong assumption and really needs to take into account how the data was collected.

3. *Principal components are orthogonal.*

This is not always the case, but orthogonality allows us to use linear algebra.

## 3.2 In class exercises pt. IV

1. (a) Show that  $C_X$  is symmetric.  
(b) Prove that the diagonal entries of  $C_X$  are variances and the off-diagonal entries are covariances.
2. Suppose  $X$  is an  $m \times n$  matrix with sample mean  $\bar{x} \in \mathbb{R}^m$ . Let  $X'$  be the shifted data of  $X$ , so that its sample mean is 0. What is  $C_{X'}$  in terms of  $X$  and  $\bar{x}$ ?

## 3.3 Performing PCA

At the heart of performing PCA in practice is the following question. Recall that  $C_Y$  is a diagonal matrix; see the discussion around Equation (3.3).

**Question 3.3.** What is the relationship between  $C_X$  and  $C_Y$  if  $Y = PX$ ?



*Proof.* From above, we have

$$C_Y = \frac{1}{n} Y Y^t = \frac{1}{n} (P X) (P X)^t = P C_X P^t. \quad \square$$

In order to perform a PCA, we need to find a matrix  $P$  such that  $P C_X P^t$  is diagonal. Importantly, we do not want to change the variances, so we want  $P$  to be *distance preserving*; that is, we want  $P$  to satisfy

$$\|Pv\| = \|v\| \quad (3.4)$$

for all vectors  $v \in \mathbb{R}^m$ . We say such a matrix  $P$  is an **isometry**. We can take Equation (3.4) and massage it, so that  $P$  must satisfy

$$v^t P^t P v = (Pv) \cdot (Pv) = v \cdot v = v^t v$$

for all  $v \in \mathbb{R}^m$ . Since this needs to hold for all vectors, it must hold for all pairs of basis vectors  $(e_i, e_j)$  for all  $i, j \in \{1, \dots, m\}$ . Thus, distance preserving is equivalent to  $P^t P = I_m$ , but these matrices are called **orthogonal**. (Note: pairs of distinct columns of such a matrix  $P$  are pairwise orthogonal. Can you prove this?!)

Let us bring this back to the equation we established. We want  $P$  to be an orthogonal matrix such that

$$C_Y = P C_X P^t. \quad (3.5)$$

Moreover, we want  $C_Y$  to be a diagonal matrix. Since  $P^t P = I_m$ , it follows that  $P^{-1} = P^t$ , so using this identity we have

$$C_Y = P C_X P^{-1}.$$

Since  $C_Y$  is diagonal, this is accomplished through *eigendecomposition*. Therefore, the rows of  $P$  are eigenvectors, and the diagonal entries of  $C_Y$  are eigenvalues.

All the entries of  $C_Y$  are *real*, and we know that some matrices have complex eigenvalues. For example, the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues  $i$  and  $-i$ , for  $i = \sqrt{-1}$ .

**Question 3.4.** Is it always possible to find a real matrix  $P$  such that Equation (3.5) holds?

The answer to Question 3.4 is “Yes,” and we will prove it later. For now, let us assume it is always possible.

We give a recipe for cooking up the principal components.

**Given:**  $n$  data points  $x_i \in \mathbb{R}^m$  (of roughly the same scale),

**Return:**  $m$  principal components.

1. Compute the mean of each coordinate:  $\mu_j = \sum_i x_{ij}$ ,
2. Organize the normalised data into a matrix  $X = (x_{ij} - \mu_j)$ ,
3. Compute the covariance matrix  $C_X$  of  $X$ ,
4. Compute the eigenvectors of  $C_X$ , and sort them based on their eigenvalues: largest is first and smallest is last.
5. Return the (ordered) orthonormal basis of eigenvectors.

PCA is a phrase used for this algorithm together with its analysis, and one of the main ways PCA is performed is by taking only the first  $k$  principal components (rather than all  $m$ ). Here,  $k$  is usually determined by the *eigenvalues*.

Both  $C_X$  and  $C_Y$  have the same eigenvalues:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ . Since the trace of matrix is the sum of its eigenvalues, both  $C_X$  and  $C_Y$  have the same trace. In other words,

$$\text{tr}(C_X) = \sum_{i=1}^m \lambda_i,$$

and by Lemma 3.1, the sum of the variances is the sum of the eigenvalues. Because we view variability as an important measurement to keep track of, we can choose a  $k$  that both maximizes the amount of variability “seen” and while minimizing the value of  $k$ . This is a bit more of an art than a science, but one general rule could be to choose the smallest  $k$  such that

$$\frac{\sum_{i=1}^k \lambda_i}{\sum_{i=1}^m \lambda_i} \geq 0.95. \quad (3.6)$$

For such a  $k$ , one might say that the first  $k$  principal components capture 95% of the total variability.

**Question 3.5.** Are the principal components  $k + 1$  through  $m$  useless?

### 3.4 Projections

The answer to Question 3.5 is essentially “Yes”, and it can be helpful to consider an idealised example.

Recall that  $X$  is our  $m \times n$  matrix whose columns are our  $n$  data points in  $\mathbb{R}^m$ . The matrix  $P$  is obtained from the eigenvectors of  $C_X$ , which are the rows of  $P$ , and  $Y = PX$ . Moreover,  $P$  is an orthogonal matrix. Suppose  $k < m$  and

$$\lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_m = 0.$$

Therefore, we have

$$C_Y = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}.$$

**Remark 3.6.** In this situation,  $y_{rj} = 0$  for all  $k + 1 \leq r \leq m$  and all  $1 \leq j \leq n$ . (Can you show this?!) In other words, the “new” data point  $y_j$  has a tail of zeroes.

**Lemma 3.7.** Let  $Q$  be the first  $k$  rows of  $P$ , so that  $Q$  is  $k \times m$ . Then for all columns  $x_i, x_j \in \mathbb{R}^m$  of  $X$ ,

$$d_{\mathbb{R}^m}(x_i, x_j) = d_{\mathbb{R}^k}(Qx_i, Qx_j).$$

*Proof.* For each column  $x_i$  of  $X$ , we have  $Px_i = y_i$ , where  $y_i$  is the  $i$ th column of  $Y$ . Since  $P$  is orthogonal,  $d_{\mathbb{R}^m}(Px_i, Px_j) = d_{\mathbb{R}^m}(x_i, x_j)$ . Thus, by Remark 3.6

$$\begin{aligned} d_{\mathbb{R}^m}(Px_i, Px_j)^2 &= d_{\mathbb{R}^m}(y_i, y_j)^2 \\ &= \sum_{r=1}^m (y_{ri} - y_{rj})^2 \\ &= \sum_{r=1}^k (y_{ri} - y_{rj})^2 \\ &= d_{\mathbb{R}^k}(Qx_i, Qx_j)^2. \end{aligned} \quad \square$$

The key application of Lemma 3.7 is that ignoring rows  $k + 1$  through  $m$  in the matrix  $P$  does not change the geometry! That is, we can project the data to a smaller dimension and distances between the data points remain unchanged!

In practice the  $\lambda_i$  are strictly greater than 0, so this idealised situation does not occur. Using our rule in Equation (3.6), then the projected data would approximate the original geometry quite well. The larger the ratio, the better the approximation, so there is indeed a trade off. Thus, after constructing all of the principal components, one can take the first  $k$ , and project the original data (i.e. the matrix  $X$ ) into a smaller dimension by constructing the matrix  $Q$  from the first  $k$  rows of  $P$ .

### 3.5 PCA is always possible—the Spectral Theorem

The real power of PCA is that we can *always* perform it. One does not need to input parameters; just the data. Note there is a pre-processing stage of normalising and rescaling, but this can be applied to all data. We address Question 3.4 which, at the time, we just assumed was true. In order to do this, we prove the Spectral Theorem.

**Theorem 3.8** (Spectral Theorem). Let  $M$  be a real symmetric  $n \times n$  matrix. Then  $\mathbb{R}^n$  has an orthonormal basis coming from eigenvectors of  $M$ .

**Corollary 3.9.** Every eigenvalue and eigenvector of a real symmetric matrix are real.

In particular, the Spectral Theorem makes principal component analysis possible since the covariance matrix is always a real symmetric matrix.

**Remark 3.10.** Covariance matrices are examples of *positive semi-definite matrices*, which are real matrices whose eigenvalues are all real and nonnegative. We will not need this much, nor will we prove this, but it might be useful to know that covariance matrices are particularly nice.

Let's unpack what the [Spectral Theorem](#) says exactly. Recall that an orthonormal basis  $\{b_1, \dots, b_n\}$  for a vector space  $V$  satisfies

$$b_i \cdot b_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

There are two components to being an orthonormal basis: the basis vectors are pairwise orthogonal and each basis vector has unit length. The latter condition is not so particularly important (though useful); really, the magic is in the pairwise orthogonal condition.

We already saw in Section 3.3 that the [Spectral Theorem](#) does not hold if we drop 'symmetric'. Now we will build our way to prove the [Spectral Theorem](#).

**Lemma 3.11.** *Let  $M$  be a real symmetric matrix. Then eigenvectors corresponding to distinct eigenvalues are orthogonal.*

*Proof.* Let  $u$  and  $v$  be eigenvectors of  $M$  corresponding to  $\lambda$  and  $\mu$  with  $\lambda \neq \mu$ . Then

$$\mu u^t v = u^t (Mv) = (u^t M)v = (Mu)^t v = \lambda u^t v.$$

Suppose via contradiction that  $u^t v \neq 0$ , but then  $\mu = \lambda$ , which is a contradiction, so we must have  $u^t v = 0$ . Hence,  $u$  and  $v$  are orthogonal.  $\square$

From Lemma 3.11, we can already prove that all eigenvalues (and therefore) all eigenvectors are real. Before we dive into that proof, recall the operation of *complex conjugation*. If  $a, b \in \mathbb{R}$ , then  $a + bi$  is a complex number, and the complex conjugate is

$$\overline{a + bi} = a - bi.$$

If  $z \in \mathbb{C}$ , then  $\bar{\bar{z}} = z$  if and only if  $z \in \mathbb{R}$ .

**Proposition 3.12.** *Let  $M$  be a real symmetric matrix. Then all eigenvalues of  $M$  are real.*

*Proof.* Suppose  $Mv = \lambda v$ , and assume, via contradiction, that  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . If we apply complex conjugation to  $Mv = \lambda v$ , we have  $\overline{Mv} = \overline{\lambda v}$ . Since  $M$  is a real matrix,  $\overline{M} = M$ . Thus, we have a new eigenvector and eigenvalue of  $M$  since

$$M\bar{v} = \bar{\lambda}\bar{v}.$$

The dot product of  $v$  and  $\bar{v}$  is a sum of squares: suppose  $M$  is written with respect to some basis  $\{b_1, \dots, b_m\}$ ; then

$$v \cdot \bar{v} = \sum_{i=1}^m v_i \bar{v}_i > 0.$$

Thus,  $v$  and  $\bar{v}$  are not orthogonal, but  $\lambda \neq \bar{\lambda}$ , so we get a contradiction thanks to Lemma 3.11. Therefore,  $\lambda \in \mathbb{R}$ , and hence all eigenvalues of  $M$  are real.  $\square$

It follows from Proposition 3.12 that all eigenvectors of a real symmetric matrix are real, but we still want more—namely orthogonality.

**Definition 3.13.** Let  $M$  be an  $n \times n$  matrix with entries in a field  $K$ . A subspace  $U$  of  $K^n$  is  **$M$ -invariant** if  $Mu \in U$  for all  $u \in U$ .

**Definition 3.14.** Let  $U$  be a subspace of a vector space  $V$ . Define  $U^\perp$ , sometimes called the **perp-space** (or orthogonal space) of  $U$ , to be the set of all  $v \in V$  such that  $v \cdot u = 0$  for all  $u \in U$ .

One thing to try on your own: Show that  $U^\perp$  is a subspace of  $V$  if  $U$  is a subspace of  $V$ .

**Lemma 3.15.** Let  $M$  be an  $n \times n$  real symmetric matrix. If  $U$  is an  $M$ -invariant subspace of  $\mathbb{R}^n$ , then  $U^\perp$  is  $M$ -invariant.

*Proof.* Exercise. □

**Lemma 3.16.** Let  $M$  be an  $n \times n$  real symmetric matrix. If  $U$  is a nonzero  $M$ -invariant subspace of  $\mathbb{R}^n$ , then  $U$  contains an eigenvector of  $M$ .

*Proof.* Let  $Q$  be a matrix whose columns define an orthonormal basis for  $U$ , and assume  $U$  is  $m$ -dimensional. Thus,  $Q^t Q = I_m$ . Since  $U$  is  $M$ -invariant, each basis vector is mapped to some linear combination of all of the basis vectors. Therefore, there exists an  $m \times m$  real matrix  $N$  such that

$$MQ = QN.$$

Since  $QQ^t = I_m$ , we have  $Q^t MQ = N$ . Because  $M$  is symmetric, so is  $N$ . By Proposition 3.12, all the eigenvalues of  $N$  are real, and since  $m \geq 1$ , let  $0 \neq v \in \mathbb{R}^m$  such that  $Nv = \lambda v$  for some  $\lambda \in \mathbb{R}$ . Hence,

$$MQv = QNv = \lambda Qv.$$

Since the columns of  $Q$  are linearly independent,  $Qv \neq 0$  and is, therefore, an eigenvector of  $M$ . But  $Qv \in U$ , so  $U$  contains an eigenvector of  $M$ . □

*Proof of the Spectral Theorem.* We will prove this by induction on  $n$ . For  $n = 1$ , there is nothing to prove, so we assume that the statement of the theorem holds for  $n$ . We will show that it holds for  $n + 1$ .

Let  $v$  be an eigenvector of  $M$ , which is real by Proposition 3.12. Let  $U$  be the subspace of  $\mathbb{R}^{n+1}$  that is orthogonal to  $v$ , so  $\dim(U) = n$ . Since the span of  $v$  is  $M$ -invariant, so is  $U$  by Lemma 3.15. By Lemma 3.16,  $U$  contains a real eigenvector of  $M$ . By iterating this argument,  $U$  contains all other eigenvectors of  $M$ . Hence, by induction, the statement holds. □

## References

- [1] Christos Sotiriou, Soek-Ying Neo, Lisa M. McShane, Edward L. Korn, Philip M. Long, Amir Jazaeri, Philippe Martiat, Steve B. Fox, Adrian L. Harris, and Edison T. Liu, *Breast cancer classification and prognosis based on gene expression profiles from a population-based study*, Proceedings of the National Academy of Sciences **100** (2003), no. 18, 10393–10398.
- [2] Google, *What is Clustering?* (2023), date accessed: 16 Oct. 2023. <https://developers.google.com/machine-learning/clustering/overview>.
- [3] Dan Margalit and Joseph Rabinoff, *Interactive Linear Algebra*, 2019, <https://textbooks.math.gatech.edu/ila/>.
- [4] Jonathon Shlens, *A tutorial on principal component analysis*, 2014, [arXiv:1404.1100](https://arxiv.org/abs/1404.1100).