Additional Problems

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These problems are taken essentially directly from our main text: Bernard Kolman and Robert E. Beck's textbook titled *Elementary linear programming with applications*.

Problem 1

Show that the following hold.

(i) A hyperplane in $\mathbb{R}^n$ is a convex set.

(ii) A close half-space in $\mathbb{R}^n$ is a convex set.

(iii) An intersection of a finite collection of convex sets in $\mathbb{R}^n$ is convex.

(iv) Suppose $f : \mathbb{R}^a \to \mathbb{R}^b$ is a linear transformation. If $S \subseteq \mathbb{R}^a$ is a convex set, show that the following set is convex:

$$\{f(v) \mid v \in S\} \subseteq \mathbb{R}^b.$$

Problem 2

Consider the following LP

Maximize

$$z = 3x + 2y$$

subject to $x, y \geqslant 0$ and

$$\begin{aligned} 2x - y &\leqslant 6, \\ 2x + y &\leqslant 10. \end{aligned}$$

(i) Transform this problem to a problem in canonical form.

(ii) For each extreme point of the new problem, identify the basic variables.

(iii) Solve the given LP.

Problem 3

Consider the following simplex tableau that is missing its objective row and column.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0	0	2	1	5/2	0	0	6/7
x_1	1	0	5	0	-3	0	-2	2/7
x_6	0	0	3	0	4	1	-4	5/7
x_2	0	1	0	0	3/2	0	0	1/7

Determine the departing variable if the entering variable is

(i) x_5;

(ii) x_3;

(iii) x_7.

Problem 4

Suppose $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Consider the following LP.

Maximize

$$z = c^\top x$$

subject to $x \geqslant 0$ and

$$Ax \leqslant b.$$

Show that if u and v are feasible solutions, then the following point is a feasible solution:

$$w = \frac{1}{3}u + \frac{2}{3}v.$$

Problem 5

Use one iteration of the Simplex Algorithm to obtain the next tableau from the following.

	x_1	x_2	x_3	x_4	x_5	
x_3	2/3	0	1	3/5	0	3/2
x_2	3/2	1	0	1	0	5/2
x_5	5	0	0	2/9	1	2/3
	4	0	0	-5	0	7/3

Problem 6

Consider the system of equations $Ax = b$ where

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 6 & 1 & 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

Determine (with justification) whether each of the following points is a basic solution to the system:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 2 \\ -5 \\ 0 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 7

Consider the following tableau that arose from solving an LP by the simplex method.

x_1	x_2	x_3	s_1	s_2	s_3	
1	5	2	0	0	3	20
0	2	4	1	0	-4	6
0	2	-1	0	1	3	12
0	-5	-3	0	0	3	12

(i) identify the basic feasible solution and basic variables in this tableau.

(ii) Compute the next tableau using the simplex method.

(iii) identify the basic feasible solution and basic variables in the new tableau from (ii).

Problem 8

Use the Two-Phase Method to solve the following LP.

Maximize

$$z = x_1 + 2x_2 + x_3$$

subject to $x \geqslant 0$ and

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 15, \\ 8x_1 + 4x_2 - x_3 &= 50, \\ 2x_1 + 2x_2 + x_3 &= 20. \end{aligned}$$