Additional Problems

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Solution 1

(i) Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, so that

$$H = \{v \in \mathbb{R}^n \mid a^\top v = b\}$$

define a hyperplane in $\mathbb{R}^n$. Let $u, v \in H$. Then for all $t \in [0, 1]$,

$$\begin{aligned} a^\top(tv + (1-t)u) &= ta^\top v + (1-t)a^\top u \\ &= tb + (1-t)b = b. \end{aligned}$$

Hence, $tv + (1-t)u \in H$, and therefore the whole line segment between u and v is in H. □

(ii) Apply the same argument as (i) except some equalities will be weak inequalities. □

(iii) We'll prove it for infinitely many. Let I be an indexing set and suppose $S_i \subseteq \mathbb{R}^n$ is convex for all $i \in I$. Set $S = \bigcap_{i\in I} S_i \subseteq \mathbb{R}^n$. Let $u, v \in S$. Then $u, v \in S_i$ for all $i \in I$. Since S_i is convex, the line segment connecting u and v is in S_i. Since this holds for all $i \in I$, the line segment connecting u and v is in S. Hence, S is convex. □

(iv) Write

$$f(S) = \{f(v) \mid v \in S\} \subseteq \mathbb{R}^b.$$

Let $u, v \in f(S)$. Then there exist $x, y \in S$ such that $u = f(x)$ and $v = f(y)$. Since S is convex, we have that $ty + (1-t)x \in S$ for all $t \in [0, 1]$. Thus, $f(ty + (1-t)x) \in f(S)$. Since f is a linear transformation,

$$f(ty + (1-t)x) = tf(y) + (1-t)f(x) = tv + (1-t)u$$

is contained in $f(S)$. Hence, $f(S)$ is convex. □

Solution 2

(i) The LP in CF is

Maximize

$$z = 3x + 2y$$

subject to $x, y, s_1, s_2 \geqslant 0$ and

$$\begin{aligned} 2x - y + s_1 &= 6, \\ 2x + y + s_2 &= 10. \end{aligned}$$

(ii) From a theorem from class, the extreme points are the basic feasible solutions. From another theorem, the basic solutions are obtained by taking all pairs of columns of

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

that form a basis. Every 2-element subset yields a basis, so we have:

$$\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 10 \\ 16 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -6 \\ 0 \\ 16 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 6 \\ 10 \end{bmatrix}.$$

We have exactly four of these points that are feasible:

$$\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 10 \\ 16 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 6 \\ 10 \end{bmatrix}.$$

(iii) The optimal point occurs at $(0, 10)^\top$.

Solution 3

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0	0	2	1	5/2	0	0	6/7
x_1	1	0	5	0	−3	0	−2	2/7
x_6	0	0	3	0	4	1	−4	5/7
x_2	0	1	0	0	3/2	0	0	1/7

(i) If x_5 is the entering variable, then θ-ratios are $12/35$, $-2/21$, $5/28$, and $2/21$ for x_4, x_1, x_6, and x_2 respectively. Therefore, x_6 is the departing variable.

(ii) If x_3 is the entering variable, then θ-ratios are $3/7, 2/35, 5/21$, and DNE for x_4, x_1, x_6, and x_2 respectively. Therefore, x_1 is the departing variable.

(iii) If x_7 is the entering variable, then there is no departing variable since the pivotal column is non-positive.

Solution 4

Suppose $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Consider the following LP.

Maximize

$$z = c^\top x$$

subject to $x \geqslant 0$ and

$$Ax \leqslant b.$$

The point

$$w = \frac{1}{3}u + \frac{2}{3}v$$

is on the line segment between u and v, namely, $w = L_{u,v}(2/3)$. We know that a finite intersection of closed half-spaces is convex, and we know that the set of feasible solutions is a finite intersection of close half-spaces. Therefore, w is a feasible solution. □

Solution 5

Original:

	x_1	x_2	x_3	x_4	x_5	
x_3	$2/3$	0	1	$3/5$	0	$3/2$
x_2	$3/2$	1	0	1	0	$5/2$
x_5	5	0	0	$2/9$	1	$2/3$
	4	0	0	-5	0	$7/3$

Next:

	x_1	x_2	x_3	x_4	x_5	
x_3	$-7/30$	$-3/5$	1	0	0	0
x_4	$3/2$	1	0	1	0	$5/2$
x_5	$14/3$	$-2/9$	0	0	1	$1/9$
	$23/2$	5	0	0	0	$89/6$

Solution 6

Let

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 6 & 1 & 0 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

And set

$$x = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad y = \begin{bmatrix} 0 \\ 2 \\ -5 \\ 0 \\ -1 \end{bmatrix}, \qquad z = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since only 2 entries of z are nonzero, z is not basic. Since

$$Ax = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

the vector x cannot be basic. Since $Ay = b$, it follows that y is a basic solution. $\square$

Solution 7

x_1	x_2	x_3	s_1	s_2	s_3	
1	5	2	0	0	3	20
0	2	4	1	0	-4	6
0	2	-1	0	1	3	12
0	-5	-3	0	0	3	12

(i) The basic variables are given by where the pivots are located: x_1, s_1, and s_2. Therefore, the corresponding basic feasible solution is $(20, 0, 0, 6, 12, 0)^\top$.

(ii) The next table is

	x_1	x_2	x_3	s_1	s_2	s_3	
x_1	1	0	-8	$-5/2$	0	13	5
x_2	0	1	2	$1/2$	0	-2	3
s_2	0	0	-5	-1	1	7	6
	0	0	7	$5/2$	0	-7	27

(iii) The corresponding basic feasible solution is $(5, 3, 0, 0, 6, 0)^\top$ with basic variables x_1, x_2, and s_2.

Solution 8

Maximize

$$z = x_1 + 2x_2 + x_3$$

subject to $x \geqslant 0$ and

$$3x_1 + x_2 - x_3 = 15,$$
$$8x_1 + 4x_2 - x_3 = 50,$$
$$2x_1 + 2x_2 + x_3 = 20.$$

We need to add in three artificial variables. For the first phase, we need to solve

Maximize

$$z = -85 - 13x_1 - 7x_2 + x_3$$

subject to $x \geqslant 0$, $y \geqslant 0$, and

$$3x_1 + x_2 - x_3 + y_1 = 15,$$
$$8x_1 + 4x_2 - x_3 + y_2 = 50,$$
$$2x_1 + 2x_2 + x_3 + y_3 = 20.$$

The corresponding tableau is

	x_1	x_2	x_3	y_1	y_2	y_3	
y_1	3	1	-1	1	0	0	15
y_2	8	4	-1	0	1	0	50
y_3	2	2	1	0	0	1	20
	13	7	-1	0	0	0	-85

The next tableau is

	x_1	x_2	x_3	y_1	y_2	y_3	
y_1	5	3	0	1	0	1	35
y_2	10	6	0	0	1	1	70
x_3	2	2	1	0	0	1	20
	15	9	0	0	0	0	-85

Therefore, there is no feasible solution.