The Mathematics of Decision Making I

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1 Introduction

The mathematics of decision making is very closely tied to the field of mathematical optimization. One of the primary ways mathematics is used to help guide decisions is by maximizing (or minimizing) specific outcomes subject to a list of constraints. Mathematical optimization provides the formal tools to model and solve such problems.

There are many kinds of mathematical optimization. There are two basic types depending on whether the variables to optimize or discrete or continuous. A few types of optimization are[1]

- Linear Programming,
- Integer Programming,
- Stochastic programming,
- Combinatorial optimization,
- Dynamic programming.

[1]"Program" is not a computer program but comes from the United States military's use of the word for training and logistics schedules.

Unsurprisingly there are many real-world applications; to list a few we have network optimization, pricing strategy, scheduling, supervised machine learning training, supply chain optimization, and transportation problems.

In this module, we will introduce the fundamentals of **linear programming**, also called *linear optimization* and *operations research*, such as the simplex method, polyhedral geometry, and the notion of duality. Depending on the time, we may also delve into **integer programming**.

1.1 History

Mathematical optimization has quite an interesting history. In the 17th century, combinatorial optimization problems were solved using game theory, combinatorics, and ad hoc methods. In the 19th century, transportation problems involving post and rail were studied and solved. And in the 20th century with the two World Wars and rise of the assembly line, operations research took off developing the mathematics for all kinds of optimization problems.

One of the most influential figures in mathematical optimization, and linear programming in particular, is George Dantzig. He was the recipient of the President's National Medal of Science in 1975 [3] and was credited for

> *inventing linear programming and discovering methods that led to wide-scale scientific and technical applications to important problems in logistics, scheduling, and network optimization, and to the use of computers in making efficient use of the mathematical theory.*

The proof of the simplex method, name coined by Motskin, was developed by Dantzig in the late 1940s [1]. I find it interesting that the "inductive proof of the simplex method" was published by the Mathematics Division of the RAND Corporation in 1960 (by Dantzig) and was made classified [2]. Now, of course, it is no longer classified.

After explaining the Simplex Method to John von Neumann at the Institute of Advanced Study in Princeton during 1948, von Neumann immediately conjectured the notion of duality because of his recent foray into game theory.

1.2 Five examples

We describe five example problems that touch on the tools we will develop in this module. For now, these problems are meant to introduce basic concepts and vocabulary.

1.2.1 A diet problem

Erin is planning her breakfast and wants to make oats with milk. (These numbers of simplified and not accurate to real life.)

	Milk (100ml)	Oats (100g)
fat	2g	3g
carbohydrates	1g	3g
protein	4g	3g

Erin wants the meal to provide at least 18g of fat, at least 12g of carbohydrates, and at least 24g of protein. If milk costs 20 cents per 100ml and oats 25 cents per 100g, what mixture minimizes the cost of the desired meal?

We could express this more mathematically. For example, let x and y be variables such that $x = 1$ means 100ml of milk and $y = 1$ means 100g of oats. Calculating the grams of fat relative to x and y is

$$2x + 3y.$$

For carbohydrates it is $x + 3y$, and for protein it is $4x + 3y$. Because we want *at least* 18g of fat, we express this via

$$2x + 3y \geqslant 18.$$

We can set up similar inequalities for the other two:

$$\begin{aligned} 2x + 3y &\geqslant 18, \\ x + 3y &\geqslant 12, \\ 4x + 3y &\geqslant 24. \end{aligned}$$

Since we cannot have negative amounts of milk or oats, we have $x \geqslant 0$ and $y \geqslant 0$. Since we want to minimize costs, we want to minimize

$$C = 0.2x + 0.25y.$$

Putting all of this together, we have the following optimization problem.

Determine values for x and y that minimize

$$C = 0.2x + 0.25y$$

subject to the constraints: $x \geqslant 0$, $y \geqslant 0$, and

$$\begin{aligned} 2x + 3y &\geqslant 18, \\ x + 3y &\geqslant 12, \\ 4x + 3y &\geqslant 24. \end{aligned}$$

1.2.2 A transportation problem

Javier has two production sites: one in Sligo and another in Kilkenny. There are three distributing warehouses in Dublin, Galway, and Cork. The Sligo site can supply 120 products per week, whereas the site in Kilkenny can supply 140 per week. The warehouses in Dublin, Galway, and Cork need 100, 60, and 80 products per week respectively to meet demand. The shipping costs are giving in the following table.

	Dublin	Galway	Cork
Sligo	5	7	9
Kilkenny	6	7	10

How many products should Javier ship from each production site to minimize total shipping costs while still meeting demand?

We need many variables, so let's define a variable for each shipment—for example, from Kilkenny to Dublin. Write them as

$$x_{kd}, x_{kg}, x_{kc}, x_{sd}, x_{sg}, x_{sc}.$$

Since Kilkenny and Sligo can only produce 140 and 120 products, respectively, we have

$$\begin{aligned} x_{kd} + x_{kg} + x_{kc} &\leqslant 140, \\ x_{sd} + x_{sg} + x_{sc} &\leqslant 120. \end{aligned}$$

We need to meet demands, so we have

$$\begin{aligned} x_{kd} + x_{sd} &\geqslant 100, \\ x_{kg} + x_{sg} &\geqslant 60, \\ x_{kc} + x_{sc} &\geqslant 80. \end{aligned}$$

Lastly, we want to minimize cost, so we want to minimize

$$C = 6x_{kd} + 7x_{kg} + 10x_{kc} + 5x_{sd} + 7x_{sg} + 9x_{sc}.$$

Altogether we have the following linear program.

Minimize

$$C = 6x_{kd} + 7x_{kg} + 10x_{kc} + 5x_{sd} + 7x_{sg} + 9x_{sc}$$

subject to the constraints: $x_{ij} \geqslant 0$ for all i and j and

$$\begin{aligned} x_{kd} + x_{kg} + x_{kc} &\leqslant 140, \\ x_{sd} + x_{sg} + x_{sc} &\leqslant 120, \\ x_{kd} + x_{sd} &\geqslant 100, \\ x_{kg} + x_{sg} &\geqslant 60, \\ x_{kc} + x_{sc} &\geqslant 80. \end{aligned}$$

1.2.3 The travelling salesperson problem

Kofi need to deliver n products in n different cities starting in Paris. He wants to do this by visiting each city exactly one time and then returning back to Paris at the end. Which path minimizes the distance traveled?

This problem is perhaps the most famous combinatorial optimization problem and is the core problem of many other more complex problems. We will not do much more with this, but note that different "distance functions" can allow for all kinds of slow-downs and speed-ups.

1.2.4 A financial problem

Julia runs an investment and must invest exactly €100,000 in two types of securities: bond A paying a dividend of 7% and stock B paying a dividend of 9%. Due to her incredible experience, she knows that

- no more than €40,000 can be invested in stock B and
- the amount invested in bond A must be at least twice that in stock B.

How much should Julia invest in each security to maximize her return?

2 General linear programming

Linear programs are the basis of what we consider throughout this module. In the example problems above, we sometimes wanted to maximize and sometimes we wanted to minimize. Although these are technically different, we can treat them as the same. Suppose f is some function we want to maximize. Then

$$\max(f) = \min(-f).$$

So maximizing f is the same as minimizing $-f$. Thus, we can use the two interchangeably—as long as we correctly compensate!

General linear program

Determine values for $x_1, x_2, \ldots, x_n$ that maximize

$$z = c_1x_1 + c_2x_2 + \cdots c_nx_n$$

subject to the constraints:

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \square & b_1, \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \square & b_2, \\
\vdots & & \vdots & & \vdots & & \vdots & \square & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \square & b_m,
\end{array}$$

where each of the $\square$ can be replaced with one of $\{=, \leqslant, \geqslant\}$.

Definition 2.1. A ***linear program (LP) problem*** is a problem of the form above. The function z is called the ***objective function***, and the m (in-)equalities are called the ***constraints***.

A key feature of LPs is that the objective function as well as each of the constraint (in-)equalities are *linear* in the $x_1, x_2, \ldots, x_n$.

2.1 Standard form

Can we play around with the constants a_{ij} and b_k to get all of the (in-)equalities into the same "shape"? For example,

$$4x_1 - 5x_2 - x3 \geqslant 1$$

is equivalent to

$$-4x_1 + 5x_2 + x3 \leqslant -1.$$

Thus, if we have an inequality, we can force it to use just $\leqslant$. Moreover, if we have an equality, we can use two inequalities to obtain the same solutions:

$$4x_1 - 5x_2 - x3 = 1 \qquad \text{is equivalent to} \qquad \begin{cases} 4x_1 - 5x_2 - x3 \geqslant 1 \text{ and} \\ 4x_1 - 5x_2 - x3 \leqslant 1 \end{cases}$$

So we can transform equalities to inequalities, but what about the other way around? We will look at this soon.

In some examples, variables only took on non-negative values. This actually has an advantage of constraining the possible values of the variables, and it is something we will come back to later on. But what about situations were variables are allowed to have negative values? Suppose x_i can be negative. We can introduce two new variables, say, x_i^+ and x_i^-, and we can rewrite x_i as follows:

$$x_i = x_i^+ - x_i^-.$$

In this way, x_i can be negative while both x_i^+ and x_i^- are non-negative. Thus, we can replace all instances of x_i with $x_i^+ - x_i^-$, so that all variables take non-negative values.

Now we can define the standard form for an LP.

Linear program standard form

Determine values for $x_1, x_2, \ldots, x_n$ that maximize

$$z = c_1x_1 + c_2x_2 + \cdots c_nx_n$$

subject to the constraints: for all $i \in \{1, \ldots, n\}$, $x_i \geqslant 0$ and

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leqslant b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leqslant b_2, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leqslant b_m. \end{aligned}$$

Example 2.2. The following LP is not in standard form.

Determine values for x and y that minimize

$$z = 3x + 2y$$

subject to the constraints: $x \geqslant 0$, $y \geqslant 0$, and

$$\begin{aligned} 2x + y &\leqslant 4 \\ 3x - 2y &\leqslant 6. \end{aligned}$$

We can put it into standard form as follows.

Determine values for x and y that maximize

$$z = -3x - 2y$$

subject to the constraints: $x \geqslant 0$, $y \geqslant 0$, and

$$\begin{aligned} 2x + y &\leqslant 4 \\ 3x - 2y &\leqslant 6. \end{aligned}$$

Example 2.3. Put the following LP into standard form.

Determine values for x and y that minimize

$$z = -4x + y$$

subject to the constraints:

$$\begin{aligned} x - 3y &= 2, \\ x + y &\leqslant 6. \end{aligned}$$

2.2 Canonical form

The canonical form is slightly different to that of the standard form of an LP.

Linear program canonical form

Determine values for $x_1, x_2, \ldots, x_s$ that maximize

$$z = c_1 x_1 + c_2 x_2 + \cdots c_n x_s$$

subject to the constraints: for all $i \in \{1, \ldots, s\}$, $x_i \geqslant 0$ and

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_s & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_s & = & b_2, \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{array}$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_s = b_m.$$

Proposition 2.4. *Every LP in standard form can be brought into canonical form. In other words, every LP has an associated LP in canonical form.*

References

[1] G. B. Dantzig. Reminiscences about the origins of linear programming. *Operations Research Letters*, 1(2):43–48, 1982.

[2] G. B. Danzig. Inductive proof of the simplex method. https://apps.dtic.mil/sti/tr/pdf/AD0224306.pdf, 1960.

[3] National Science Foundation. The President's National Medal of Science: Recipient Details. https://www.nsf.gov/od/nms/recip_details.jsp?recip_id=95. Accessed: 2024-09-08.