Unit III

ECON 3406

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Differences of two proportions

Differences of Two Proportions

- Like with $\hat{p}$, the difference of two sample proportions $\hat{p}_1 - \hat{p}_2$ can be modeled using a normal distribution (when conditions are met).

$$SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

- This standard error comes from the fact that variances of independent variables **add**, even when subtracting.

Differences of Two Proportions: Standard Errors

- When we talk about the spread of an estimate, we're really talking about **variance** (the square of the standard error).
- If two random variables **A** and **B** are independent, then:

$$\mathrm{Var}(A - B) = \mathrm{Var}(A) + \mathrm{Var}(B)$$

- This might seem counterintuitive — but remember:
  - Even if you're subtracting two noisy measurements, the **uncertainty (noise)** from both still adds up.
  - Think of it like using two shaky rulers. Subtracting doesn't cancel the shakiness — it just combines it!

Differences of Two Proportions: Simulation Setup

We'll use simulation to understand the sampling distribution of sample proportions for two independent groups.

- Group 1 has a true proportion $p_1 = 0.5$
- Group 2 has a true proportion $p_2 = 0.4$
- Each group has $n = 500$ individuals per sample
- We'll repeat this sampling 1,000 times to observe variation in sample means

Differences of Two Proportions: Simulation

```
set.seed(1)        # ensures reproducibility
B <- 1000           # number of simulations
n <- 500            # sample size per group
p1 <- 0.5           # true proportion in group 1
p2 <- 0.4           # true proportion in group 2

# Create empty vectors to store simulated means and S
mean_x1 <- mean_x2 <- numeric(B)
sd_x1 <- sd_x2 <- numeric(B)

# Loop to simulate samples for both groups
for (i in 1:B) {
  # Generate random binary outcomes for group 1 (succ
  x1 <- rbinom(n, size = 1, prob = p1)
  mean_x1[i] <- mean(x1)  # sample proportion for gro
  sd_x1[i]   <- sd(x1)    # sample SD for group 1

  # Repeat for group 2
  x2 <- rbinom(n, size = 1, prob = p2)
```

Comparing Theoretical and Empirical Standard Errors

- We can compare:
  - the **theoretical** standard error (from the formula)
  - the **empirical** standard error (from our simulations)

Comparing Theoretical and Empirical Standard Errors

```r
# Theoretical standard error for a sample proportion
sqrt(p1*(1-p1)/n)
```

```
[1] 0.02236068
```

```r
# Empirical standard error from simulated data
mean(sd_x1) / sqrt(n)
```

```
[1] 0.0223613
```

- The first line gives the theoretical SE: $\sqrt{p(1-p)/n}$
- The second line gives the empirical SE, based on simulated SDs
- These values should be nearly identical, validating the normal approximation for large n

Sampling Distribution for One Group (p1 = 0.5)

- We can visualize the distribution of sample proportions across simulations, and overlay a 95% confidence interval around the true mean.

Sampling Distribution for One Group (p1 = 0.5)

Sampling Distribution for One Group (p1 = 0.5)

- Roughly 5% of simulated sample proportions should fall outside this interval — confirming the 95% confidence level's interpretation.

```
# Calculate the proportion of estimates that fall outs
mean(ifelse(mean_x1 >= ub_x1 | mean_x1 <= lb_x1, 1, 0)
```

```
[1] 0.052
```

Sampling Distribution for One Group (p2 = 0.4)

Sampling Distribution for One Group (p2 = 0.4)

- Again, around 5% of simulated estimates will fall outside the interval.
- The spread is slightly narrower than for Group 1 because the variance is smaller.

```
# Share of points outside the 95% CI
mean(ifelse(mean_x2 >= ub_x2 | mean_x2 <= lb_x2, 1, 0)
```

```
[1] 0.048
```

Combining Two Proportions

- Now that we understand the sampling variation of each group separately, we can combine them just as we would when estimating a **difference in proportions**:

$$SE_{\hat{p}_1-\hat{p}_2} = \sqrt{SE^2_{\hat{p}_1} + SE^2_{\hat{p}_2}}$$

- This formula reflects that variances add, even though we're subtracting proportions.

Simulated Differences Between Groups

```r
# Compute simulated differences
diff <- mean_x1 - mean_x2

# Theoretical SE for the difference in proportions
sqrt(p1*(1-p1)/n + p2*(1-p2)/n)
```

```
[1] 0.03130495
```

```r
# Empirical SE estimate (not exact but illustrative)
mean(sd_x1 + sd_x2) / sqrt(n + n)
```

```
[1] 0.03129505
```

Simulated Differences Between Groups

Simulated Differences Between Groups

```r
# Check coverage rates for both formulas
mean(ifelse(diff >= ub_diff_1 | diff <= lb_diff_1, 1,
```

```
[1] 0.037
```

```r
mean(ifelse(diff >= ub_diff_2 | diff <= lb_diff_2, 1,
```

```
[1] 0.793
```

Differences of two proportions: Example 1

- Consider an experiment involving patients who underwent cardiopulmonary resuscitation (CPR) following a heart attack and were subsequently admitted to a hospital.
  - Patients were randomly assigned to either a **treatment group** (received a blood thinner) or a **control group** (no blood thinner).
  - The outcome of interest was **survival for at least 24 hours**.

Differences of two proportions: Example 1

	Survived	Died	Total
Control	11	39	50
Treatment	14	26	40
Total	25	65	90

Differences of two proportions: Example 1

- Create and interpret a **90% confidence interval** of the difference for the survival rates in the CPR study.
  - $p_t - p_c = 0.35 - 0.2 = 0.13$
  - $SE = \sqrt{\frac{0.35(1-0.35)}{40} + \frac{0.22(1-0.22)}{50}} \approx 0.095$
  - $0.13 \pm 1.645 \times 0.095 = (-0.263, 0.286)$

Differences of two proportions: Computing in R

```
pt <- 0.35
pc <- 0.2
nt <- 40
nc <- 50

point_est <- pt - pc
se <- sqrt((pt * (1 - pt) / nt) + (pc * (1 - pc) / nc)

z <- data.frame(
  sig_level = c(0.01, 0.05, 0.1),
  z_score = c(2.45, 1.95, 1.645)
)

z$min <- point_est - z$z_score * se
z$max <- point_est + z$z_score * se
```

Differences of two proportions: Visualizing Confidence Intervals

Differences of two proportions: Interpretation

- We are **90% confident** that blood thinners change the 24-hour survival rate by between –2.6 and +28.6 percentage points for patients similar to those in the study.
- Because **0% is within this range**, the evidence is inconclusive — we cannot determine whether blood thinners help or harm heart attack patients who have undergone CPR.

Differences of Two Proportions: Example 2

- A 5-year clinical trial evaluated whether **fish oil supplements** reduce the risk of **heart attacks**.
- Each participant was randomly assigned to one of two groups:
  - **Fish Oil group**
  - **Placebo group**
- We'll examine heart attack outcomes across both groups.

Differences of Two Proportions: Example 2

Group	Heart Attack	No Event	Total
Fish Oil	145	12,788	12,933
Placebo	200	12,738	12,938

Differences of Two Proportions: Example 2

- Construct a **95% confidence interval** for the effect of fish oil on heart attack incidence among patients represented by this study.
- Interpret the interval in context:
  - What does the direction and width of the interval suggest?
  - Is there evidence that fish oil has a meaningful effect on heart attack risk?

Differences of two proportions: Computing in R

```
nt <- 12933
nc <- 12938

pt <- 145 / nt
pc <- 200 / nc

point_est <- pt - pc
se <- sqrt((pt * (1 - pt) / nt) + (pc * (1 - pc) / nc)

z <- data.frame(
  sig_level = c(0.01, 0.05, 0.1),
  z_score = c(2.45, 1.95, 1.645)
)

z$min <- point_est - z$z_score * se
z$max <- point_est + z$z_score * se
```

Differences of two proportions: Interpretation

- The **point estimate** for the effect of fish oil is approximately **–0.0043**, meaning heart attacks occurred **0.43 percentage points less often** in the fish-oil group than in the placebo group.
- We are **90% confident** that fish oil changes the heart-attack rate by between **–0.66 and –0.19 percentage points** for patients similar to those in the study.
- Because this interval **does not include 0**, the reduction in heart-attack risk is **statistically significant** at the 10% (and even 5% an 1%) level.

Practical vs. Statistical Significance

- While statistically significant, the **effect size is extremely small** — roughly **0.4 fewer heart attacks per 100 individuals**.
- In a large clinical sample, even minor effects can reach significance if variability is low.
- From a **practical** standpoint, such a small reduction may **not justify** the cost, side effects, or adherence burden of treatment.