**Exercise 11.5** Let $x_1, \ldots, x_n \in \mathbb{N}$. For each of the following problems, either (a) design and analyze a polynomial time algorithm (the faster the better), or (b) prove that a polynomial time algorithm would imply a polynomial time algorithm for SAT.[2]

[2]You can use the solution of one subproblem to solve another, as long as there's no circular dependencies overall.

**Exercise 11.5.1.** The *partition problem* asks if one can partition $x_1, \ldots, x_n$ into two parts such that the sums of each part are equal.

*Solution.* We claim that a polynomial time solution for the partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from subset sum, a problem known to be hard, to the partition problem.

**Notation.** Given some set $S = \{s_1, \ldots, s_n\}$, we denote the sum $\sum\limits_{s \in S} s$ with $\Sigma S$.

Consider an arbitrary instance of subset sum. That is, suppose we have a set of positive integers

$$A := \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{N}$$

and a positive integer target value $T \in \mathbb{N}$.

Now, let $x := 2T - \Sigma A$ and define a new set $\overline{A} := A \cup \{x\}$.

Note that if $T < \Sigma A/2$, the value of $x$ becomes less than 0. However, if there exists some set $B = \{\beta_1, \ldots, \beta_k\} \subseteq A$ such that $\Sigma B = T$, then $A \setminus B$ sums to $\Sigma A - T \leq \Sigma A/2$. Thus, WLOG we can simply rephrase the problem to use $\Sigma A - T$ as the target value instead.

Then, notice that

$$\begin{aligned} \Sigma\overline{A} &= \Sigma A + 2T - \Sigma A \\ &= 2T. \end{aligned}$$

*Correctness.* (SS $\Longrightarrow$ PP) Suppose there exists some $B \subseteq A$ such that $\Sigma B = T$. Consider the partition of $\overline{A}$ defined as $\overline{B} = B \cup \{x\}$. Then,

$$\Sigma\overline{B} = T + 2T - \Sigma A.$$

The remaining partition is then $C := \overline{A} \setminus \overline{B}$, and

$$\begin{aligned} \Sigma C &= 2T - T + 2T - \Sigma A \\ &= T + 2T - \Sigma A, \end{aligned}$$

and we can see that these two sums are equal. Hence, the partition problem is solved.

(SS $\Longleftarrow$ PP) Suppose the set $\overline{A}$ has a valid partition such that each of the two subsets sum to $T$. Recall that $\overline{A}$ is defined as the union of $A$ and the singleton set $\{x\}$. By the pigeonhole principle, we know that one of these subsets of $\overline{A}$ is a subset of $A$, whence the subset sum problem is solved. ■

Since each step in the reduction process takes only $O(1)$ or $O(n)$ time, the entire reduction can be done in polynomial time relative to the size of $A$. Thus, a polynomial time solution for the partition problem implies a polynomial time solution for SAT. □

**Exercise 11.5.2.** The *3-partition problem* asks if one can partition $x_1, \ldots, x_n$ into 3 parts such that the sums of each part are all equal.

*Solution.* We claim that a polynomial time solution for the 3-partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, which we showed to be hard in 11.5.1.

**Notation.** Given sets $A$ and $B$, we denote the *disjoint union* of $A$ and $B$ by $A \sqcup B$.

Consider an arbitrary instance of the partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \ldots, \alpha_m\} \subseteq \mathbb{N}.$$

The partition problem seeks two disjoint subsets $B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C$.

Note that if $\Sigma A$ is odd or the cardinality of $A$ is less than 2, then the problem becomes impossible. Thus, WLOG we may assume that $\Sigma A = 2n$ for some $n \in \mathbb{N}$ and that $A$ contains at least 2 elements.

Now, let $x := n$ and define a new set $\overline{A} := A \cup \{x\} \implies \Sigma\overline{A} = 3n$.

*Correctness.* (PP $\implies$ 3P) Assume that $\exists B \subseteq A$ such that $\Sigma B = n$. Let $C := A \setminus B$ and notice that $\Sigma C = 2n - n = n$. By construction,

$$\overline{A} = C \sqcup B \sqcup \{x\} \text{ and } \Sigma C = \Sigma B = \Sigma\{x\}.$$

Hence, the 3-partition problem is solved.

(PP $\impliedby$ 3P) Assume that there exists a valid 3-partition of $\overline{A}$. That is, assume that there exist $A_1, A_2, A_3 \subseteq \overline{A}$ such that

$$\Sigma A_1 = \Sigma A_2 = \Sigma A_3 = n \text{ and } A_1 \sqcup A_2 \sqcup A_3 = \overline{A}.$$

We already know $\{x\} \subseteq \overline{A}$ and $x = n$, so WLOG we can set $A_1 := \{x\}$. Then, we have that $A_2 \sqcup A_3 = \overline{A} \setminus A_1 = A$, and we know $\Sigma A_2 = \Sigma A_3 = n$, whence the partition problem is solved. ■

This reduction can obviously be done in polynomial time relative to the size of $A$. Thus a polynomial time solution for the 3-partition problem would imply a polynomial time solution for the partition problem, which we have already shown would imply a polynomial time solution for SAT. □

**Exercise 11.5.3.** The *any-$k$-partition problem* asks if one can partition $x_1, \ldots, x_n$ into $k$ parts, for any integer $k \geq 2$, such that the sums of each part are all equal.

*Solution.* We claim that a polynomial time solution for the $k$-partition problem would imply a polynomial time solution for SAT. To see this, we present an inductive proof of a polynomial time reduction from the 3-partition problem to the $k$-partition problem.

As stated, our base case will be the 3-partition problem, which we showed to be hard in 11.5.2. Assume that we have shown that the $\ell$-partition problem is hard for all $3 \leq \ell < k$.

Consider an arbitrary instance of the $(k-1)$-partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \ldots, \alpha_m\} \subseteq \mathbb{N}.$$

The $(k-1)$-partition problem seeks $k-1$ pairwise disjoint subsets $A_1, A_2, \ldots, A_{k-1} \subseteq A$ such that

$$\bigsqcup_{1 \leq i \leq k-1} A_i = A \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \cdots = \Sigma A_{k-1}.$$

Note that if $\Sigma A$ is not divisible by $k-1$ or if the cardinality of $A$ is less than $k-1$, then the problem is rendered impossible. Thus, WLOG we may assume that $\Sigma A = (k-1)n$ for some $n \in \mathbb{N}$ and that $A$ contains at least $k-1$ elements.

By our inductive hypothesis, we have that the existence of a polynomial time solution for the $(k-1)$-partition problem implies the existence of a polynomial time solution for SAT.

Now, let $x := n$ and define a new set $\overline{A} := A \cup \{x\} \implies \Sigma\overline{A} = kn$.

*Correctness.* ($(k-1)$P $\implies k$P) Assume that there exists a valid $(k-1)$-partition for $\overline{A}$. That is, assume that there exist $k-1$ pairwise disjoint subsets $A_1, \ldots, A_{k-1} \subseteq A$ such that

$$\bigsqcup_{1 \leq i \leq k-1} A_i = A \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \cdots = \Sigma A_{k-1}.$$

By construction, we have that

$$\overline{A} = \left[ \bigsqcup_{1 \leq i \leq k-1} A_i \right] \sqcup \{x\} \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \cdots = \Sigma A_{k-1} = \Sigma\{x\} = n.$$

Hence, the $k$-partition problem is solved.

($(k-1)$P $\impliedby k$P) Assume that there exists a valid $k$-partition of $\overline{A}$. That is, assume there exist $k$ pairwise disjoint subsets $A_1, A_2, \ldots, A_k \subseteq \overline{A}$ such that

$$\overline{A} = \bigsqcup_{1 \leq i \leq k} A_i \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \cdots = \Sigma A_k = n.$$

We already know $\{x\} \subseteq \overline{A}$ and $x = n$, so WLOG we can set $A_1 := \{x\}$. Then, we have that

$$\bigsqcup_{2 \leq i \leq k} A_i = \overline{A} \setminus A_1 = A \quad \text{and} \quad \Sigma A_2 = \Sigma A_3 = \cdots = \Sigma A_k = n,$$

whence the $(k-1)$-partition problem is solved. ■

This reduction can obviously be done in polynomial time relative to the size of $A$. Thus, a polynomial time solution for the $k$-partition problem implies a polynomial time solution for the $(k-1)$-partition problem, and by induction does so for the 3-partition problem (and equivalently for SAT). □

**Exercise 11.5.4.** The *almost-partition problem* asks if one can partition $x_1, \ldots, x_n$ into two parts such that the two sums of each part differ by at most 1.

*Solution.* We claim that a polynomial time solution for the almost-partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, which we proved in 11.5.1 to be hard.

Suppose we want to solve the partition problem on a set of positive integers $A := \{a_1, ..., a_n\} \subseteq \mathbb{N}$, given a solution to the almost-partition problem as a blackbox. We transform $A$ into the the set

$$A' := \{2a_i : a_i \in A\} = \{2a_1, ..., 2a_n\}$$

and then apply the almost-partition solution to $A'$. Since all elements in $A'$ are even, it is impossible for partitions to differ by exactly 1. Hence, we claim $A$ has a partition if and only if $A'$ has an almost-partition.

*Correctness.* To prove correctness, let us first assume that $A'$ has an almost-partition; that is, there exists some $B' \subseteq A'$ for which

$$\sum B' = \sum (A' \setminus B') \quad \text{or} \quad \sum B' = \sum (A' \setminus B') \pm 1$$

$B'$ and $A' \setminus B'$ are both subsets of $A'$, so we have $2 \mid \sum B'$ and $2 \mid \sum(A' \setminus B')$.

Since $\sum B' = \sum (A' \setminus B') \pm 1$ cannot be true, we must have $\sum B' = \sum (A' \setminus B')$, which can be rewritten $2 \sum B' = \sum A'$.

Let $B := \{a_i : 2a_i \in B'\} \subseteq A$. Then $2 \sum B = \sum B' = \frac{1}{2} \sum A' = \sum A$; hence, $A$ has an exact partition.

Conversely, we now assume that $A$ has an exact partition given by $\sum B = \sum(A \setminus B)$ for some $B \subseteq A$. If we define $B' := \{2a_i : a_i \in B\} \subseteq A$, then $A'$ also has an exact partition given by $\sum B' = \sum(A' \setminus B')$ which is, by definition, an almost-partition of $A'$. ■

This reduction can clearly be performed in polynomial time relative to the input size of $A$ and expression size of the integers in $A$. Since we proved above that almost-partition can be used to solve the exact partition problem, which is known to be hard, we can conclude that a polynomial time solution for the almost-partition problem would also imply a polynomial-time solution for SAT. □

**Exercise 11.5.5.** [3]Let $n$ be even. The *perfect partition problem* asks if one can partition $x_1, \ldots, x_n$ into two parts such that

(a) Each part has the same sum.

(b) Each part contains the same number of $x_i$'s.

[3]IMO, this one is the trickiest.

*Solution.* We claim that a polynomial time solution for the partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, a problem known we showed to be hard in 11.5.1, to the perfect partition problem.

Consider an arbitrary instance of the partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \ldots, \alpha_m\} \subseteq \mathbb{N}.$$

The partition problem seeks two disjoint subsets $B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C = \Sigma A/2$.

Note that if $\Sigma A$ is odd or the cardinality of $A$ is less than 2, then the problem becomes impossible. Thus, WLOG we may assume that $\Sigma A = 2T$ for some $T \in \mathbb{N}$ and that $A$ contains at least 2 elements.

In any valid partition of A, the two parts may have different cardinalities. Suppose that in a given partition we have $|B| = p$ and $|C| = q$. WLOG we can assume $p \geq q$, and let the difference between the cardinalities be $x = p - q$.

Note that if $x = 0$, then the partition is already 'perfect'. However if $x > 0$, although A is partitionable, it is not perfectly partitionable.

We can't yet know the value of $x$, but it is easy to see that it is bounded above by $m$. Then for each possible $x \in \{1, \ldots, m\}$, define $A_x = A \cup S_x \cup Y_x$ where $S_x := \{s_1, \ldots, s_{x+1}\}$ such that $s = 1$ for each $s \in S_x$ and $Y_x := \{x + 1\}$.

*Correctness.* (PP $\implies$ PPP) Assume that $\exists B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C = T$. Let $x = |B| - |C|$

*Case 1 ($x = 0$).* As above, this case is trivial. The partition is already perfect and hence the perfect partiton problem is solved. ■

*Case 2 ($x \geq 1$).* Construct $A_x$ as above. Now, consider the partition of $A_x$ in which $B_x := B \cup Y$ and $C_x := C \cup S_x$. Then $\Sigma B_x = T + (x+1)$ and $\Sigma C_x = T + (x+1)$. Also, $|B_x| = |B| + 1$ and $|C_x| = |C| + (x+1)$. However notice that by definition of $x$, $|C_x| = |C| + (|B| - |C| + 1) = |B| + 1 = |B_x|$ and thus the sets have equal cardinality. Hence the perfect partition problem is solved. ■

(PP $\impliedby$ PPP) Assume that there exist disjoint $B, C \subseteq A$ such that

$$B \sqcup C = A, \qquad \Sigma B = \Sigma C = n, \qquad \text{and} \qquad |B| = |C|.$$

Obviously, the partition problem is solved. ■

Each step of the reduction process can be done in polynomial time relative to the size of $A$, so it follows that the a polynomial time solution to the perfect partition problem implies a polynomial time solution to the partition problem, whence a polynomial time solution to SAT. □