Exercise 16.6. Let $G = (V, E)$ be an undirected graph with distinct nonnegative edge weights $w : E \to \mathbb{R}$. For a spanning tree T, we say that the *bottleneck weight of* T is the maximum weight edge in T, $\max_{e \in T} w(e)$.

Exercise 16.6.1. Prove that the MST is also a minimum bottleneck weight spanning tree of G.

Solution. Let α be the MST of G. Assume ad absurdum that there exists an minimum bottleneck weight spanning tree (MBWST) β of G such that $\alpha \neq \beta$. Then by definition of bottleneck weight, we have that

$$\max_{e \in \alpha}\{w(e)\} > \max_{e \in \beta}\{w(e)\}.$$

That is to say, there exists some edge $e = (v_1, v_2) \in \alpha$ such that $w(e)$ is greater than $w(e')$ for every $e' \in \beta$. Obviously $e \notin \beta$, but by definition of spanning tree β spans e. Since α does not contain any cycles by definition of tree, WLOG there exists some $f = (v_1, v_3) \in \beta$ such that $f \notin \alpha$. Then we have that

$$w(e) > w(f).$$

By the spanning property of α, we know there exists an edge $(v_3, v_k) \in \alpha$ for some arbitrary vertex v_k. Thus by removing e and adding f to α, we can preserve the spanning property of α while reducing its total weight, contradicting that α is the MST of G. Thus the minimum spanning tree of G must also be a minimum bottleneck spanning tree of G. □

Exercise 16.6.2. Design and analyze a $O(m+n)$-time algorithm for computing a minimum bottleneck weight spanning tree of G. (This is faster than any of our algorithms for MST.)[4]

[4] Here's step 1: compute the median edge weight in $O(m)$ time.

Solution. □