Exercise 13.4. Let $G = (V, E)$ be a directed graph. We say that a set of vertices is *almost independent* if each $v \in S$ has at most one neighbor in S.[5] Consider the problem of computing the maximum cardinality of any almost independent set of vertices. For this problem, either (a) design and analyze a polynomial time algorithm (the faster the better), or (b) prove that a polynomial time algorithm would imply a polynomial time algorithm for SAT.

[5] Two vertices u and v are neighbors if they are connected by an edge.

Solution. We claim that the maximum cardinality almost independent set (MCAIS) problem is NP-complete. To see this, we present a polynomial time reduction from the maximum cardinality independent set (MCIS) problem, a problem we saw to be NP-complete in class, to MCAIS.

Note. Since the condition for vertices $v, w \in V$ to be neighbors only requires that there be a connection between the two, the directional property of the edges in E can be ignored. Hence, we will not discriminate between two edges (v, w) and (w, v).

Consider an arbitrary instance of independent set. That is, suppose we have a graph $G = (V, E)$. The MCAIS problem seeks the largest $S \subset V$ such that $(v, w) \notin E$ for all $v, w \in S$.

Create an auxiliary graph $G' = (V', E')$, where $V' = \{v_i' \mid v_i \in V\} \cup V$ and $E' = \{(v_i, v_i') \mid v_i \in V\} \cup E$. Then, V' contains duplicates of each vertex in V and E' contains each edge in E, as well as an additional edge connecting each pair of vertices v_j and v_j'.

Correctness. (MCIS $\Longrightarrow$ MCAIS) Suppose we have a set $S \subset V$ that solves MCIS for G. Create a new set $S' = \{v_i' \mid v_i \in S\} \cup S$. ■

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