Exercise 13.4. Let G = (V, E) be a directed graph. We say that a set of vertices is almost independent if each $v \in S$ has at most one neighbor in S.⁵ Consider the problem of computing the maximum carcdinality of any almost independent set of vertices. For this problem, either (a) design and analyze a polynomial time algorithm (the faster the better), or (b) prove that a polynomial time algorithm would imply a polynomial time algorithm for SAT.

Solution. We claim that the maximum cardinality almost independent set (MCAIS) problem is NP-complete. To see this, we present a polynomial time reduction from the maximum cardinality independent set (MCIS) problem, a problem we saw to be NP-complete in class, to MCAIS.

Note. Since that the condition for vertices $v, w \in V$ to be neighbors only requires that there be a connection between the two, we need not be concerned with the directional property of the edges in E. Hence, we will interpret G as an undirected graph.

Consider an arbitrary instance of independent set. That is, suppose we have a graph G = (V, E). The MCAIS problem seeks the largest $S \subset V$ such that $(v, w) \notin E$ for all $v, w \in S$.

Create an auxiliary graph G' = (V', E'), where $V' = \bigcup_{v_i \in V} \{v_i, v_i'\}$ and $E' = \{(v_i, v_i') \mid v_i \in V\} \cup E$. Thus,

V' contains duplicates of each vertex in V and E' contains each edge in E, as well as an additional edge connecting each pair of vertices v_j and v'_j . We can think of G' as a bilayered graph, where G is the top layer and every vertex has a copy of itself 'hanging' below. Importantly, a 'hanging' vertex is only connected to the original vertex.

Correctness. (MCIS \Longrightarrow MCAIS) Assume we have that a maximum cardinality independent set of G has cardinality n. Notice that a subset S' of V' containing v'_i for every i is independent over G' with cardinality |V|, since no two hanging vertices are connected by an edge. We know there exists some independent set S of cardinality n over V. Then $S'' = S \cup S'$ must be almost independent over G' with cardinality |V| + n, since each vertex in S' gains at most one neighbor in the top layer of G'. Now assume ad absurdum that we have some $H \subseteq V'$ such that |H| > |S''|. If there exists some hanging vertex $v'_k \not\in H$, then it must be the case that $v_k \in H$ with some neighboring vertex $v_{k+1} \in H$. However, this implies that $v'_{k+1} \not\in H$, so we can always replace v_k, v_{k+1} with their hanging counterparts. Thus WLOG every hanging $v'_i \in H$. So |H| = |V| + m and by hypothesis |V| + m > |V| + n. Then we can remove the hanging vertices from H to get an independent set over G with cardinality m > n. However, this contradicts that the maximum cardinality of an independent set over G is n. Thus S'' must be a maximum cardinality almost independent set of G' with cardinality |V| + n.

(MCIS \Leftarrow MCAIS) Assume we have that a maximum cardinality almost independent set S of G' has cardinality n. As above, we can say WLOG S contains every hanging vertex. Assume ad absurdum that S contains adjacent vertices $v_i, v_{i+1} \in V$. But $v_i' \in S$ whence v_i has two neighbors, a contradiction. Thus no two vertices in S are adjacent in the top layer of G', which we know to be isomorphic to G. Hence by removing every hanging vertex from S, we can get an independent set G of cardinality G or eagain, assume ad absurdum that there exists some independent set G over G with cardinality G or G and G in the property definition of G is the maximum cardinality G or an almost independent set over G whence G is a maximum cardinality independent set over G.

 $^{^{5}}$ Two vertices u and v are neighbors if they are connected by an edge.