Exercise 13.4. Let $G = (V, E)$ be a directed graph. We say that a set of vertices is *almost independent* if each $v \in S$ has at most one neighbor in S.[5] Consider the problem of computing the maximum carcdinality of any almost independent set of vertices. For this problem, either (a) design and analyze a polynomial time algorithm (the faster the better), or (b) prove that a polynomial time algorithm would imply a polynomial time algorithm for SAT.

[5] Two vertices u and v are neighbors if they are connected by an edge.

Solution. We claim that the maximum cardinality almost independent set (MCAIS) problem is NP-complete. To see this, we present a polynomial time reduction from the maximum cardinality independent set (MCIS) problem, a problem we saw to be NP-complete in class, to MCAIS.

Note. Since that the condition for vertices $v, w \in V$ to be neighbors only requires that there be a connection between the two, we need not be concerned with the directional property of the edges in E. Hence, we will interpret G as an undirected graph.

Consider an arbitrary instance of independent set. That is, suppose we have a graph $G = (V, E)$. The MCAIS problem seeks the largest $S \subset V$ such that $(v, w) \notin E$ for all $v, w \in S$.

Create an auxiliary graph $G' = (V', E')$, where $V' = \bigcup_{v_i \in V} \{v_i, v_i'\}$ and $E' = \{(v_i, v_i') \mid v_i \in V\} \cup E$. Thus, V' contains duplicates of each vertex in V and E' contains each edge in E, as well as an additional edge connecting each pair of vertices v_j and v_j'. We can think of G' as a bilayered graph, where G is the top layer and every vertex has a copy of itself 'hanging' below. Importantly, a 'hanging' vertex is only connected to the original vertex.

Correctness. (MCIS $\implies$ MCAIS) Assume we have that a maximum cardinality independent set of G has cardinality n. Notice that a subset S' of V' containing v_i' for every i is independent over G' with cardinality $|V|$, since no two hanging vertices are connected by an edge. We know there exists some independent set S of cardinality n over V. Then $S'' = S \cup S'$ must be almost independent over G' with cardinality $|V| + n$, since each vertex in S' gains at most one neighbor in the top layer of G'. Now assume ad absurdum that we have some $H \subseteq V'$ such that $|H| > |S''|$. If there exists some hanging vertex $v_k' \notin H$, then it must be the case that $v_k \in H$ with some neighboring vertex $v_{k+1} \in H$. However, this implies that $v_{k+1}' \notin H$, so we can always replace v_k, v_{k+1} with their hanging counterparts. Thus WLOG every hanging $v_i' \in H$. So $|H| = |V| + m$ and by hypothesis $|V| + m > |V| + n$. Then we can remove the hanging vertices from H to get an independent set over G with cardinality $m > n$. However, this contradicts that the maximum cardinality of an independent set over G is n. Thus S'' must be a maximum cardinality almost independent set of G' with cardinality $|V| + n$.

(MCIS $\impliedby$ MCAIS) Assume we have that a maximum cardinality almost independent set S of G' has cardinality n. As above, we can say WLOG S contains every hanging vertex. Assume ad absurdum that S contains adjacent vertices $v_i, v_{i+1} \in V$. But $v_i' \in S$ whence v_i has two neighbors, a contradiction. Thus no two vertices in S are adjacent in the top layer of G', which we know to be isomorphic to G. Hence by removing every hanging vertex from S, we can get an independent set H over G of cardinality $|H| = n - |V|$. Once again, assume ad absurdum that there exists some independent set K over G with cardinality $m > n - |V|$. Then by adding every hanging vertex to the set, we obtain an almost independent set with cardinality $m + |V| > |H| + |V| = n$. However, this contradicts that n is the maximum cardinality for an almost indepedent set over G' whence H must be a maximum cardinality independent set over G. ■

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