

Exercise 11.5 Let $x_1, \dots, x_n \in \mathbb{N}$. For each of the following problems, either (a) design and analyze a polynomial time algorithm (the faster the better), or (b) prove that a polynomial time algorithm would imply a polynomial time algorithm for SAT.²

²You can use the solution of one subproblem to solve another, as long as there's no circular dependencies overall.

Exercise 11.5.1. The *partition problem* asks if one can partition x_1, \dots, x_n into two parts such that the sums of each part are equal.

Solution. We claim that a polynomial time solution for the partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from subset sum, a problem known to be hard, to the partition problem.

Notation. Given some set $S = \{s_1, \dots, s_n\}$, we denote the sum $\sum_{s \in S} s$ with ΣS .

Consider an arbitrary instance of subset sum. That is, suppose we have a set of positive integers

$$A := \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{N}$$

and a positive integer target value $T \in \mathbb{N}$.

Now, let $x := 2T - \Sigma A$ and define a new set $\bar{A} := A \cup \{x\}$.

Note that if $T < \Sigma A/2$, the value of x becomes less than 0. However, if there exists some set $B = \{\beta_1, \dots, \beta_k\} \subseteq A$ such that $\Sigma B = T$, then $A \setminus B$ sums to $\Sigma A - T \leq \Sigma A/2$. Thus, WLOG we can simply rephrase the problem to use $\Sigma A - T$ as the target value instead.

Then, notice that

$$\begin{aligned} \Sigma \bar{A} &= \Sigma A + 2T - \Sigma A \\ &= 2T. \end{aligned}$$

Correctness. (SS \implies PP) Suppose there exists some $B \subseteq A$ such that $\Sigma B = T$. Consider the partition of \bar{A} defined as $\bar{B} = B \cup \{x\}$. Then,

$$\Sigma \bar{B} = T + 2T - \Sigma A.$$

The remaining partition is then $C := \bar{A} \setminus \bar{B}$, and

$$\begin{aligned} \Sigma C &= 2T - T + 2T - \Sigma A \\ &= T + 2T - \Sigma A, \end{aligned}$$

and we can see that these two sums are equal. Hence, the partition problem is solved.

(SS \Leftarrow PP) Suppose the set \bar{A} has a valid partition such that each of the two subsets sum to T . Recall that \bar{A} is defined as the union of A and the singleton set $\{x\}$. By the pigeonhole principle, we know that one of these subsets of \bar{A} is a subset of A , whence the subset sum problem is solved. ■

Since each step in the reduction process takes only $O(1)$ or $O(n)$ time, the entire reduction can be done in polynomial time relative to the size of A . Thus, a polynomial time solution for the partition problem implies a polynomial time solution for SAT. □

Exercise 11.5.2. The *3-partition problem* asks if one can partition x_1, \dots, x_n into 3 parts such that the sums of each part are all equal.

Solution. We claim that a polynomial time solution for the 3-partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, which we showed to be hard in 11.5.1.

Notation. Given sets A and B , we denote the *disjoint union* of A and B by $A \sqcup B$.

Consider an arbitrary instance of the partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{N}.$$

We wish to find two disjoint subsets $B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C$.

Note that if ΣA is odd or the cardinality of A is less than 2, then the problem becomes impossible. Thus, WLOG we may assume that $\Sigma A = 2n$ for some $n \in \mathbb{N}$ and that A contains at least 2 elements.

Now, let $x := n$ and define a new set $\bar{A} := A \cup \{x\} \implies \Sigma \bar{A} = 3n$.

Correctness. (PP \implies 3P) Assume that $\exists B \subseteq A$ such that $\Sigma B = n$. Let $C := A \setminus B$ and notice that $\Sigma C = 2n - n = n$. By construction,

$$\bar{A} = C \sqcup B \sqcup \{x\} \text{ and } \Sigma C = \Sigma B = \Sigma \{x\}.$$

Hence, the 3-partition problem is solved.

(PP \Leftarrow 3P) Assume that there exists a valid 3-partition of \bar{A} . That is, assume that there exist $A_1, A_2, A_3 \subseteq \bar{A}$ such that

$$\Sigma A_1 = \Sigma A_2 = \Sigma A_3 = n \text{ and } A_1 \sqcup A_2 \sqcup A_3 = \bar{A}.$$

We already know $\{x\} \subseteq \bar{A}$ and $x = n$, so WLOG we can set $A_1 := \{x\}$. Then, we have that $A_2 \sqcup A_3 = \bar{A} \setminus A_1 = A$, and we know $\Sigma A_2 = \Sigma A_3 = n$, whence the partition problem is solved. ■

This reduction can obviously be done in polynomial time relative to the size of A . Thus a polynomial time solution for the 3-partition problem would imply a polynomial time solution for the partition problem, which we have already shown would imply a polynomial time solution for SAT. □

Exercise 11.5.3. The *any- k -partition problem* asks if one can partition x_1, \dots, x_n into k parts, for any integer $k \geq 2$, such that the sums of each part are all equal.

Solution. We claim that a polynomial time solution for the k -partition problem would imply a polynomial time solution for SAT. To see this, we present an inductive proof of a polynomial time reduction from the 3-partition problem to the k -partition problem.

As stated, our base case will be the 3-partition problem, which we showed to be hard in 11.5.2. Assume that we have shown that the ℓ -partition problem is hard for all $3 \leq \ell < k$.

Consider an arbitrary instance of the $(k-1)$ -partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{N}.$$

We wish to find $k-1$ pairwise disjoint subsets $A_1, A_2, \dots, A_{k-1} \subseteq A$ such that

$$\bigsqcup_{1 \leq i \leq k-1} A_i = A \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \dots = \Sigma A_{k-1}.$$

Note that if ΣA is not divisible by $k-1$ or if the cardinality of A is less than $k-1$, then the problem is rendered impossible. Thus, WLOG we may assume that $\Sigma A = (k-1)n$ for some $n \in \mathbb{N}$ and that A contains at least $k-1$ elements.

By our inductive hypothesis, we have that the existence of a polynomial time solution for the $(k-1)$ -partition problem implies the existence of a polynomial time solution for SAT.

Now, let $x := n$ and define a new set $\bar{A} := A \cup \{x\} \implies \Sigma \bar{A} = kn$.

Correctness. $((k-1)P \implies kP)$ Assume that there exists a valid $(k-1)$ -partition for \bar{A} . That is, assume that there exist $k-1$ pairwise disjoint subsets $A_1, \dots, A_{k-1} \subseteq \bar{A}$ such that

$$\bigsqcup_{1 \leq i \leq k-1} A_i = \bar{A} \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \dots = \Sigma A_{k-1} = n.$$

By construction, we have that

$$\bar{A} = \left[\bigsqcup_{1 \leq i \leq k-1} A_i \right] \sqcup \{x\} \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \dots = \Sigma A_{k-1} = \Sigma \{x\} = n.$$

Hence, the k -partition problem is solved.

$((k-1)P \Leftarrow kP)$ Assume that there exists a valid k -partition of \bar{A} . That is, assume there exist k pairwise disjoint subsets $A_1, A_2, \dots, A_k \subseteq \bar{A}$ such that

$$\bar{A} = \bigsqcup_{1 \leq i \leq k} A_i \quad \text{and} \quad \Sigma A_1 = \Sigma A_2 = \dots = \Sigma A_k = n.$$

We already know $\{x\} \subseteq \bar{A}$ and $x = n$, so WLOG we can set $A_1 := \{x\}$. Then, we have that

$$\bigsqcup_{2 \leq i \leq k} A_i = \bar{A} \setminus A_1 = A \quad \text{and} \quad \Sigma A_2 = \Sigma A_3 = \dots = \Sigma A_k = n,$$

whence the $(k-1)$ -partition problem is solved. ■

This reduction can obviously be done in polynomial time relative to the size of A . Thus, a polynomial time solution for the k -partition problem implies a polynomial time solution for the $(k-1)$ -partition problem, and by induction does so for the 3-partition problem (and equivalently for SAT). □

Exercise 11.5.4. The *almost-partition problem* asks if one can partition x_1, \dots, x_n into two parts such that the two sums of each part differ by at most 1.

Solution. We claim that a polynomial time solution for the almost-partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, which we proved in 11.5.1 to be hard.

Suppose we want to solve the partition problem on a set of positive integers $A := \{a_1, \dots, a_n\} \subseteq \mathbb{N}$, given a solution to the almost-partition problem as a blackbox. We transform A into the set

$$A' := \{2a_i : a_i \in A\} = \{2a_1, \dots, 2a_n\}$$

and then apply the almost-partition solution to A' . Since all elements in A' are even, it is impossible for partitions to differ by exactly 1. Hence, we claim A has a partition if and only if A' has an almost-partition.

Correctness. To prove correctness, let us first assume that A' has an almost-partition; that is, there exists some $B' \subseteq A'$ for which

$$\sum B' = \sum (A' \setminus B') \quad \text{or} \quad \sum B' = \sum (A' \setminus B') \pm 1$$

B' and $A' \setminus B'$ are both subsets of A' , so we have $2 \mid \sum B'$ and $2 \mid \sum (A' \setminus B')$.

Since $\sum B' = \sum (A' \setminus B') \pm 1$ cannot be true, we must have $\sum B' = \sum (A' \setminus B')$, which can be rewritten $2 \sum B' = \sum A'$.

Let $B := \{a_i : 2a_i \in B'\} \subseteq A$. Then $2 \sum B = \sum B' = \frac{1}{2} \sum A' = \sum A$; hence, A has an exact partition.

Conversely, we now assume that A has an exact partition given by $\sum B = \sum (A \setminus B)$ for some $B \subseteq A$. If we define $B' := \{2a_i : a_i \in B\} \subseteq A'$, then A' also has an exact partition given by $\sum B' = \sum (A' \setminus B')$ which is, by definition, an almost-partition of A' . ■

This reduction can clearly be performed in polynomial time relative to the input size of A and expression size of the integers in A . Since we proved above that almost-partition can be used to solve the exact partition problem, which is known to be hard, we can conclude that a polynomial time solution for the almost-partition problem would also imply a polynomial-time solution for SAT. □

Exercise 11.5.5. ³Let n be even. The *perfect partition problem* asks if one can partition x_1, \dots, x_n into two parts such that

- (a) Each part has the same sum.
- (b) Each part contains the same number of x_i 's.

³IMO, this one is the trickiest.

Solution. We claim that a polynomial time solution for the partition problem would imply a polynomial time solution for SAT. To see this, we present a polynomial time reduction from the partition problem, a problem known we showed to be hard in 11.5.1, to the perfect partition problem.

Consider an arbitrary instance of the partition problem. That is, consider a set of positive integers

$$A := \{\alpha_1, \dots, \alpha_m\} \subseteq \mathbb{N}.$$

We wish to find two disjoint subsets $B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C = \Sigma A/2$.

Note that if ΣA is odd or the cardinality of A is less than 2, then the problem becomes impossible. Thus, WLOG we may assume that $\Sigma A = 2n$ for some $n \in \mathbb{N}$ and that A contains at least 2 elements.

Correctness. (PP \implies PPP) Assume the partition problem is solved. That is, assume that $\exists B, C \subseteq A$ such that $B \sqcup C = A$ and $\Sigma B = \Sigma C = n$. Then, we can say

$$B := \{\beta_1, \dots, \beta_i\} \quad \text{and} \quad C := \{\gamma_1, \dots, \gamma_j\},$$

such that $\{\beta_1, \dots, \beta_i, \gamma_1, \dots, \gamma_j\} = A$.

WLOG we can say $i \geq j$, that is, $|B| \geq |C|$, so define $\varphi = i - j$. Now, we have 2 cases depending on the value of φ .

Case 1 ($\varphi = 0$). This case is trivial; if $i - j = 0$ then B and C have equal cardinalities and by our hypothesis we have that $B \sqcup C = A$ and $\Sigma B = \Sigma C$. Thus the perfect partition problem is solved. ■

Case 2 ($\varphi \geq 1$). Let $S := \{s_1, \dots, s_{\varphi+1}\}$ such that $s = 1$ for each $s \in S$. Next, let $\psi := \varphi + 1$ and $\Psi = \{\psi\}$. Note that $\Sigma S = \Sigma \Psi = \varphi + 1$. Then by hypothesis

$$B \sqcup C = A \quad \text{and} \quad \Sigma(B \cup \Psi) = \Sigma(C \cup S).$$

Now, notice that

$$\begin{aligned} |C \cup S| &= |C| + \varphi + 1 \\ &= |C| + |B| - |C| + 1 \\ &= |B| + 1 \\ &= |B \cup \Psi| \end{aligned}$$

Hence the perfect partition problem is solved. ■

(PP \Leftarrow PPP) Assume the perfect partition problem is solved. That is, assume that there exist disjoint $B, C \subseteq A$ such that

$$B \sqcup C = A, \quad \Sigma B = \Sigma C = n, \quad \text{and} \quad |B| = |C|.$$

Obviously, the partition problem is solved. ■

Each step of the reduction process can be done in polynomial time relative to the size of A , so it follows that the a polynomial time solution to the perfect partition problem implies a polynomial time solution to the partition problem, whence a polynomial time solution to SAT. □