MA 34100 Homework 7

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Exercise A: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove there exists a point $x_{min} \in [a, b]$ such that $f(x_{min}) \leq f(x)$ for every $x \in [a, b]$.

f is given to be continuous on, so it is closed and bounded on $[a, b]$. Suppose we have that the greatest lower bound for f is m. Also assume that there is no value $c \in [a, b]$ such that $f(c) = m$. Then, $f(x) > m$ for all $x \in [a, b]$. If we define a second function $g(x) = \frac{1}{f(x)-m}$, we can see that $g(x) > 0$ for any $x \in [a, b]$. Thus, $g(x)$ must also be bounded on the interval $[a, b]$. Then, there must exist some $\alpha > 0$ such that $g(x) \leq \alpha$ for all $x \in [a, b]$. This gives us that $\frac{1}{f(x)-m} \leq \alpha \implies f(x) \geq \frac{1}{\alpha} - m$. However, this contradicts that m is our greatest lower bound. Thus, our assumption that there does not exist some value $c \in [a, b]$ with $f(c) = m$ must be false. That is, there exists some $c = x_{min} \in [a, b]$ with $f(x_min) \leq f(x)$ for all $x \in [a, b]$.

Exercise 4.1.1

a) If $|x - 1| \leq 1$, then

$$|x + 1| \leq 3 \tag{1}$$

$$|x + 1| |x - 1| \leq 3 |x - 1| \tag{2}$$

$$\left|x^2 - 1\right| \leq 3 |x - 1| \tag{3}$$

so $|x - 1| < \frac{1}{6}$ satisfies the inequality.

b) $|x - 1| < 1$

c) $|x - 1| < \frac{1}{3n}$

d) $|x - 1| < \frac{1}{7n}$

Exercise 4.1.9

b) Let $\varepsilon > 0$. Let $\delta = \min\{1, \varepsilon\}$. Suppose $0 < |x-1| < \delta$. Then

$$\begin{aligned}
\left|\frac{x}{1+x} - \frac{1}{2}\right| &= \left|\frac{2x-(1+x)}{2+2x}\right| && (4)\\
&= \frac{|x-1|}{2\,|x+1|} && (5)\\
&< \frac{\delta}{2} && (6)\\
&< \delta < \varepsilon && (7)
\end{aligned}$$

d) Let $\varepsilon > 0$. Let $\delta = \min\{1, \frac{2\varepsilon}{3}\}$. Suppose $0 < |x-1| < \delta$. Then

$$\begin{aligned}
\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| &= \left|\frac{(2x^2-2+2)-(x+1)}{2x+2}\right| && (8)\\
&= \left|\frac{2x^2-3x+1}{2x+2}\right| && (9)\\
&= \frac{|2x-1|}{2\,|x+1|}\,|x-1| && (10)\\
&< \frac{|2(2)-1|}{2\,|0+1|}\,|x-1| && (11)\\
&< \frac{3}{2}\delta && (12)\\
&\leq \frac{3}{2}\frac{2\varepsilon}{3} = \varepsilon && (13)
\end{aligned}$$

Exercise 4.1.10

a) Let $\varepsilon > 0$. Let $\delta = \min\{1, \frac{\varepsilon}{9}\}$. Suppose $0 < |x-2| < \delta$. Then

$$\begin{aligned}
\left|x^2+4x-12\right| &= |x+6|\,|x-2| && (14)\\
&= |x+6|\,\delta && (15)\\
&< 9\delta && (16)\\
&\leq 9\frac{\varepsilon}{9} = \varepsilon && (17)
\end{aligned}$$

Exercise 4.1.12

b) Consider the sequence $(a_n) = x^{-2}$. Then

$$f(a_n) = \frac{1}{\sqrt{\frac{1}{x^2}}} = x,$$

which certainly converges at $x = 0$.

Exercise 4.3.5

b) The limit does not exist, as the left hand limit and the right hand limit diverge to $-\infty$ and ∞ respectively. We begin by proving $\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$.
Let $M < 0$. Set $\delta = -\frac{1}{M}$. Suppose $0 < 1 - x < \delta$. Then

$$\begin{aligned} \frac{x}{x-1} &= x\frac{1}{x-1} && (18)\\ &< -x\frac{1}{\delta} && (19)\\ &< -\frac{1}{\delta} && (20)\\ &= -\frac{1}{-\frac{1}{M}} = M && (21) \end{aligned}$$

Next, we wish to prove $\lim_{x\to 1^+} \frac{x}{x-1} = \infty$.
Let $N > 0$. Set $\delta = \frac{1}{M}$. Suppose $0 < x - 1 < \delta$. Then

$$\begin{aligned} \frac{x}{x-1} &= x\frac{1}{x-1} && (22)\\ &> x\frac{1}{\delta} && (23)\\ &> \frac{1}{\frac{1}{M}} = M && (24) \end{aligned}$$

We conclude that the limit of $\frac{x}{x-1}$ can not be a real number, nor can it be

d) The function $\frac{x+2}{\sqrt{x}}$ is bounded below by $\sqrt{x}$, and as $x \to \infty$, $\sqrt{x} \to \infty$. So, $\lim_{x\to\infty} \frac{x+2}{\sqrt{x}} = \infty$.

Exercise 5.1.3

The function f is given to be continuous at b, so given some $\varepsilon > 0$, there exists some $\alpha > 0$ such that $b - \alpha < x < b \implies |f(x) - f(b)| < \varepsilon$. Likewise there exists some $\beta > 0$ such that $b < x < b + \beta \implies |g(x) - g(b)| < \varepsilon$. Setting $\delta = \min\{\alpha, \beta\}$ allows $|h(x) - h(b)| < \varepsilon$ when $|x - b| < \delta$, and thus $h(x)$ is continuous at b.

Exercise 5.1.8

The function f is given to be continuous for all $x \in \mathbb{R}$, so $f(x) = \lim(f(x_n)) = 0 \implies x \in S$.

Exercise 5.1.10

$||x| - |y|| \leq |x - y|$, so for any $\varepsilon > 0$, there exists some $\delta > 0$ with $|x - t| < \delta \implies ||f(x)| - |f(t)|| \leq |f(x) - f(t)| < \varepsilon$.