MA 34100
Homework 2

Josh Park

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Section 1.2

2) Our base case is when $n = 1$, because $1^3 = [\frac{1}{2}1(1+1)]^2 = 1^2 = 1$.
Assume that the statement holds for n. We wish to prove it holds for $n+1$.

$$\begin{align}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \left[\frac{1}{2}n(n+1)\right]^2 + (n+1)^3 \tag{1}\\
&= \left[\frac{1}{2}(n^2+n)\right]^2 + (n+1)^3 \tag{2}\\
&= \frac{n^4+2n^3+n^2}{4} + (n+1)^3 \tag{3}\\
&= \frac{n^4+6n^3+13n^2+12n+4}{4} \tag{4}\\
&= \frac{(n^2+3n+2)(n^2+3n+2)}{4} \tag{5}\\
&= \left[\frac{n^2+3n+2}{2}\right]^2 \tag{6}\\
&= \left[\frac{1}{2}(n^2+3n+2)\right]^2 \tag{7}\\
&= \left[\frac{1}{2}(n+1)(n+2)\right]^2 \tag{8}
\end{align}$$

The final expression is equivalent to the original statement with $n+1$ substituted for n, so the statement holds for $n+1$ and thus for all $n \in \mathbb{N}$.

7) Our base case is when $n = 1$, because $5^{2(1)} - 1 = 25 - 1 = 24 = 8 \cdot 3$.
Assume that the statement holds for n. We wish to prove it holds for $n+1$.

$$\begin{align}
5^{2(n+1)} - 1 &= 5^{2n+2} - 1 \tag{9}\\
&= 5^{2n}5^2 - 1 \tag{10}\\
&= ((5^{2n}-1)+1)5^2 - 1 \tag{11}\\
&= (5^{2n}-1)5^2 + 5^2 - 1 \tag{12}
\end{align}$$

If the quantity $5^{2n}-1$ is divisible by 8 for n, then any multiple of $5^{2n}-1$ is also divisible by 8. The final expression can be written $5^2(5^{2n}-1)+(5^2-1)$, and we can see that $5^2(5^{2n}-1)$ is divisible by 8. We also know that $5^2-1=24=8\cdot 3$, so the final expression is divisible by 8. Therefore, the statement holds for $n+1$ and thus for all $n\in\mathbb{N}$.

14) Our base case is when $n=4$, because $2^4=16<24=4!$.
Assume that the statement holds for n. We wish to prove it holds for $n+1$.

$$2^{n+1}=2\cdot 2^n<2\cdot n!<(n+1)\cdot n!=(n+1)! \tag{13}$$

The final expression is equivalent to the original statement with $n+1$ substituted for n, so the statement holds for $n+1$ and thus for all $n\in\mathbb{N}$.

Section 1.3

2) b) The set A has m elements, so there exists some bijection $f: A\mapsto\mathbb{N}_m$. Let $C=\{c\}$ where $c\in A$, and let $n=f^{-1}(c)$. Also define $g: A\mapsto\mathbb{N}_m$ such that

$$g(i)=\begin{cases} f(i), & i=1,2,\dots,n-1\\ f(i+1), & i=n,n+1,\dots,m-1\end{cases}$$

The only two cases are when $i\in\mathbb{N}_{n-1}$ and when $i\in\mathbb{N}_{m-1}\setminus\mathbb{N}_{n-1}$.
Case 1: Suppose $x,y\in\mathbb{N}_{n-1}$ where $x\neq y$ and $g(x),g(y)\in f(\mathbb{N}_{n-1})$. We know that f is injective, so $g(x)\neq g(y)$.
Case 2: Suppose $x,y\in\mathbb{N}_{m-1}\setminus\mathbb{N}_{n-1}$ where $x\neq y$ and $g(x),g(y)\in f(\mathbb{N}_{m-1}\setminus\mathbb{N}_{n-1})$. We know that f is injective, so $g(x)\neq g(y)$.
This proves injectivity, but not surjectivity.
Consider some element $q\in A\backslash C$. Because f is a bijection, there must exist some $p\in\mathbb{N}_m$ such that $f(p)=q$. If $p<n$, $p\in\mathbb{N}_{n-1}$ and $g(p)=f(p)=q$. If $n<p\leq m$, $p-1\in\mathbb{N}_{m-1}$ such that $g(p-1)=f(p)=q$. So, g is surjective.
Thus g is a bijection from $A\setminus C\mapsto\mathbb{N}_{m-1}$.

c) Suppose $C\setminus B$ is a finite set. If $C\setminus B$ is finite, then $|C\setminus B|\in\mathbb{N}$. Thus, we can write $C\setminus B=\{x_1,x_2,\dots,x_n\}$ where $n=|C\setminus B|$. If C is an infinite set, there must exist some elements of C that are not in B. Consider the element $\alpha\in C$ such that $\alpha\notin B$. However, $\alpha\notin\{x_1,x_2,\dots,x_n\}$. This is a contradiction, so $C\setminus B$ must be an infinite set.

4) Let O be the set of all odd numbers greater than 13. Consider the function $f: O\mapsto\mathbb{N}$ such that $f(x)=\frac{x-13}{2}$. This function is injective because if $f(x)=f(y)$, then $\frac{x-13}{2}=\frac{y-13}{2}$, so $x=y$. This function is also surjective because for any $n\in\mathbb{N}$, $f(2n+13)=n$. Thus, f is a bijection from $O\mapsto\mathbb{N}$.

6) Suppose we have some $m\in\mathbb{N}$. Consider the function $f:\mathbb{N}_m\mapsto\mathbb{N}$ such that $f(x)=x-m$. This function is injective because if $f(x)=f(y)$, then $x-m=y-m$, so $x=y$. This function is also surjective because for any $n\in\mathbb{N}$, $f(n+m)=n$. Thus, f is a bijection from $\mathbb{N}_m\mapsto\mathbb{N}$.

12) Our base case is when $n=0$, because the power set of the empty set only has $2^n=2^0=1$ element. Suppose this holds for n. If we add a new element q to S, then $|S|=n+1$ and $|\mathcal{P}(s)|=2|\mathcal{P}(s)\setminus\{q\}|=2*2^n=2^{n+1}$. Thus, the induction hypothesis must be true and $\mathcal{P}(S)$ must have 2^n elements for all $n\in\mathbb{N}$.