MA 34100 Homework 3

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Section 2.2

2.2.2) If $a, b \in \mathbb{R}$, show that $|a+b| = |a| + |b|$ ***iff*** $ab \geq 0$

To show this, we must prove both directions of the biconditional statement.

1) Assume $ab \geq 0$ is true.

This means that either a and b are both non-negative, or both non-positive. Then, we can split the proof into two cases.

(i) If a and b are both non-negative, $a+b$ is necessarily non-negative, and $|a+b| = a+b$. Because they are non-negative, we know $|a| = a$ and $|b| = b$, so it follows that $|a|+|b| = |a+b|$ is true.

(ii) If a and b are both non-positive, notice that $|a+b| = -(a+b)$. Also, it follows that $|a| = -a$ and $|b| = -b$, so $|a| + |b| = |a+b|$ is true.

2) Assume $|a| + |b| = |a+b|$ is true.

From this, we know that $a+b$ does not change signs, so a and b must be both non-positive or non-negative. If a and b have different signs, we know $|a+b|$ must be less than the sum of $|a|$ and $|b|$, because the negative and positive would partially cancel each other. This contradicts our assumption that $|a| + |b| = |a+b|$, so a and b must have the same sign. We have now shown that both directions of the biconditional statment are dependant on each other $\square$

2.2.9) Find all values that satify the following inequalities.

a) $|x-2| \leq x+1$

For the first case, assume $x \leq 2$. Then, $|x-2| = 2-x$, and the inequality becomes

$$2 - x \leq x + 1 \quad \Rightarrow \quad \frac{1}{2} \leq x. \tag{1}$$

Thus, case one gives us $\frac{1}{2} \leq x \leq 2$. For the second case, assume $x \geq 2$. Then, $|x-2| = x-2$ and the inequality becomes

$$x - 2 \leq x + 1. \tag{2}$$

This simplifies to be trivially true, so the solution set is $x \in [\frac{1}{2}, \infty)$.

b) $3|x| \leq 2 - x$

For the first case, assume $x \geq 0$. Then, $3|x| = 3x$, and the inequality becomes

$$3x \leq x - 2 \quad \Rightarrow \quad x \leq -1. \tag{3}$$

This contradicts our assumption, so $x \geq 0$ must not be possible. Then, for the second case, assume $x < 0$. Then, $3|x| = -3x$ and the inequality becomes

$$-3x \leq x - 2 \quad \Rightarrow \quad x \geq \frac{1}{2}. \tag{4}$$

This gives us that $0 \geq x \geq \frac{1}{2}$, so the expression is valid for all $x \in [0, \frac{1}{2}]$.

2.2.10) Find all $x \in \mathbb{R}$ that satisfy the following inequalities.

a) $|x - 1| > |x + 1|$

For the first case, consider $x > 1$. Then, $|x - 1| = x - 1$ and $|x + 1| = x + 1$, so the inequality becomes

$$x - 1 > x + 1. \tag{5}$$

This is trivially false, so $x > 1$ is not a valid solution. For the second case, consider $x < 1$. Then, $|x - 1| = 1 - x$ and $|x + 1| = 1 + x$, so the inequality becomes

$$1 - x > 1 + x. \tag{6}$$

This simplifies to $-x > x$, which is true for all $x < 0$. Thus, the solution set is $x \in (-\infty, 1)$.

b) $|x| + |x + 1| < 2$

For the first case, consider $x \geq 0$. Then, $|x| = x$ and $|x + 1| = x + 1$, so the inequality becomes

$$x + x + 1 < 2 \quad \Rightarrow \quad x < \frac{1}{2}. \tag{7}$$

This gives us that $x \in [0, \frac{1}{2})$. For the second case, consider $x < 0$. Then, $|x| = -x$ and $|x + 1| = -(x + 1)$, so the inequality becomes

$$-x - x - 1 < 2 \quad \Rightarrow \quad x > -\frac{3}{2}. \tag{8}$$

This gives us that $x \in (-\infty, 0)$. Thus, the solution set is $x \in (-\infty, \frac{1}{2}]$.

2.2.12) Find all $x \in \mathbb{R}$ that satisfy the inequality $4 < |x + 2| + |x - 1| < 5$.

For the first case, consider $x < -2$, so $|x + 2| = -x - 2$ and $|x - 1| = -x + 1$. Then, the inequality becomes

$$4 < -x - 2 - x + 1 = -2x - 1 < 5 \quad \Rightarrow \quad -3 < x < -\frac{5}{2}. \tag{9}$$

This gives that $x \in (-3, -\frac{5}{2})$. For the second case, consider $-2 \leq x < 1$, so $|x+2| = x+2$ and $|x-1| = -x+1$. Then, the inequality becomes

$$4 < x + 2 - x + 1 = 3 < 5. \tag{10}$$

This is trivially false, so $-2 \leq x < 1$ is not a valid solution. For the third case, consider $x \geq 1$, so $|x+2| = x+2$ and $|x-1| = x-1$. Then, the inequality becomes

$$4 < x + 2 + x - 1 = 2x + 1 < 5 \quad \Rightarrow \quad \frac{3}{2} < x < 2. \tag{11}$$

This gives that $x \in (\frac{3}{2}, 2)$. Thus, the solution set is $x \in (-3, -\frac{5}{2}) \cup (\frac{3}{2}, 2)$.

Section 2.3

2.3.5) Find the infimum and supremum, if they exist, of the following sets.

a) $A := \{x \in \mathbb{R} : \ 2x - 5 > 0\}$

$\inf A = -\frac{5}{2}$, and $\sup A$ does not exist.

b) $B := \{x \in \mathbb{R} : \ x + 2 \geq x^2\}$

$\inf B = -2$, and $\sup B = 2$.

c) $C := \{x \in \mathbb{R} : \ x < 1/x\}$

$\inf C = -1$, and $\sup C = 1$.

d) $D := \{x \in \mathbb{R} : \ x^2 - 2x - 5 < 0\}$

$\inf D = 1 - \sqrt{6}$, and $\sup D = 1 + \sqrt{6}$.

2.3.6) Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$

If S is a nonempty subset of $\mathbb{R}$ that is bounded below, then we know that $\inf S$ exists. Then, we can define the set $T := \{-s : s \in S\}$. Obviously, $\inf S \in S$, so $-\inf S$ is in T. By definition of infimum, there are no smaller elements in S, so $-\inf S$ is the largest element in T. Thus, $-\inf S = \sup T$. Then, by definition of T, we know that $\sup T = \sup\{-s : s \in S\}$, so $\inf S = -\sup\{-s : s \in S\}$ $\square$

2.3.10) Show that if A and B are bounded subsets of R, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

If A and B are bounded subsets of $\mathbb{R}$, then we know that $\sup A$ and $\sup B$ exist. Then, we can define the set $C := A \cup B$. By definition of supremum, we know that $\sup A$ and $\sup B$ are the least upper bounds of A and B, respectively. Then, by definition of least upper bound, we know that $\sup A$ and $\sup B$ are greater than or equal to all elements in A and B, respectively. Thus, $\sup A$ and $\sup B$ are greater than or equal to all elements in C. Then, by definition of supremum, we know that $\sup A$ and $\sup B$ are the least upper bounds of C. Thus, $\sup A \cup B = \sup\{\sup A, \sup B\}$ $\square$

2.3.11) Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$

If S is a bounded set in $\mathbb{R}$, then we know that $\inf S$ and $\sup S$ exist. Then, we can define the set $T := S_0$. By definition of infimum, we know that $\inf S$ is the greatest lower bound of S, so it is less than or equal to all elements in S_0. Then, by definition of infimum, we know that $\inf S_0$ is the greatest lower bound of S_0, so it is less than or equal to all elements in S_0. Then, by definition of supremum, we know that $\sup S_0$ is the least upper bound of S_0, so it is greater than or equal to all elements in S_0. Then, by definition of supremum, we know that $\sup S$ is the least upper bound of S, so it is greater than or equal to all elements in S_0. Thus, $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$. $\square$