MA 341 lecture notes

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Lecture 8

Def: Let $A \subseteq \mathbb{R}$ be nonemppty. $s \in \mathbb{R}$ is the supremum of A, written $s = \sup(A)$ if:

(i) s is an upper bound for A

(ii) If s' is an upper bound for A, then $s \leq s'$

Def: l is the infimum of A if:

(i) l is a lower bound for A

(ii) If l' is a lower bound for A, then $l' \leq l$

Suppose that s, r satisfy i) and ii) in def of sup. Then $s \leq r$ and $r \leq s$. So $s = r$.
i.e. the supremum, when it exists, is unique (same for infimum)

$$s = \sup(A),\ A \subseteq \mathbb{R},\ A \neq \emptyset$$

The following statements are equivalent to ii) below.

ii)' If $z > s$, then z is <u>not</u> an upper bound for A.

ii)' If $z < s$, then $\exists x \in A$ such that $x > z$.

Lecture 11

Section 2.? — Nested Intervals

Thm: If $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n \forall n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$
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Thm: $\mathbb{R}$ is uncountable.

Proof. It is enough to show taht $[0, 1]$ is uncountable.
Suppose not, i.e. $[0, 1] = \{x_k \mid k \in \mathbb{N}\}$. x_1 lies somewhere between 0 and 1, so choose a nested interval $I_1 = [a_1, b_1]$ such that $x_1 \notin I_1$. Next, choose $I_2 \subseteq I$, such that $x_2 \notin I_2$. Continuing in this fashion, find $I_n \subseteq I_{n-1}$ such that $x_1, x_2, \ldots, x_n \notin I_n$. By Thm, $\eta \in \bigcap_{n=1}^{\infty} I_n$, so $\eta \neq x_n$ for any $n, \eta \in [0, 1]$. It follows that [0,1] is not countable. $\square$

Section 3.1 — Sequences

Def: A sequence is a function $a : \mathbb{N} \to \mathbb{R}$ (usually written $f(n) = x_n$)

Ex: $x_n = b, b \in \mathbb{R}$

Ex: $x_n = \frac{1}{2^n}$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

Ex: $x_1 = 2, x_{n+1} = x_n + 2$

$$2, 4, 6, 8, \ldots$$

Ex: $x_1 = 1, x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$

Def(WILL BE ON MIDTERM): Let (x_n) be a sequence. We say that the limit of x_n is equal to some number $L \in \mathbb{R}$, written

$$\lim_{n \to \infty} x_n = L$$

if $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ whenever $n \geq N(\epsilon)$.

Ex: Let $x_n = b \quad \forall n, b \in \mathbb{R}$.

1. Let $\epsilon > 0$ be given
2. Analyze $|x_n - L|$

   $(L = b, x_n = b)$

   Want $|b - b| < \epsilon \iff 0 < \epsilon$, so choose $N(\epsilon) = 1$
3. Given $\epsilon > 0$ choose $N(\epsilon) = 1$. Then, if $n \geq N(\epsilon)$, then $|x_n - L| = 0 < \epsilon$

Ex: Show $\lim_{n\to\infty} \frac{1}{n} = 0$.

1. Let $\epsilon > 0$ be given
2. Analyze $|x_n - L|$

$$|x_n - L| = \left|\frac{1}{n} - 0\right| \tag{1}$$
$$= \frac{1}{n} \tag{2}$$
$$\tag{3}$$

   Choose $N(\epsilon) = \frac{1}{n} + 1$
3. Given $\epsilon > 0$, set $N(\epsilon) = \frac{1}{\epsilon} + 1$. Then, if $n \geq N(\epsilon)$, we have

   $$|x_n - 0| = \frac{1}{n} < \epsilon$$

Ex. $x_n = \frac{1}{n^2+1}$ $\forall n$, show $\lim_{n\to\infty} = 0$.

1. Let $\epsilon > 0$ be given
2. Analyze $|x_n - L|$

$$|x_n - L| = \left|\frac{1}{n^2+1} - 0\right| \tag{4}$$
$$= \frac{1}{n^2+1} \tag{5}$$

   Want $\frac{1}{n^2+1} < \epsilon$

$$\frac{1}{n^2+1} < \epsilon \tag{6}$$
$$\frac{1}{\epsilon} < n^2 + 1 \tag{7}$$
$$\sqrt{\frac{1}{\epsilon} - 1} < n \tag{8}$$

   So, choose $N(\epsilon) = \sqrt{\frac{1}{\epsilon} - 1} + 1$
3. Given $\epsilon > 0$, choose $N(\epsilon) = \sqrt{\frac{1}{\epsilon} - 1} + 1$. Then, if $n \geq N(\epsilon)$, we have

   $|x_n - L| < \epsilon$ if $n \geq N(\epsilon)$

   So, $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

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Section 3.3 — Monotone Sequences

Definition. A sequence (x_n) is

1. increaseing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$
2. decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$

(x_n) is monotone if 1 or 2 holds.

Theorem. A monotone sequence converges iff it is bounded. If (x_n) is increasing and bounded, then $(x_n) \to \sup\{x_n | n \in \mathbb{N}\}$. If (x_n) is decreasing and bounded, then $(x_n) \to \inf\{x_n | n \in \mathbb{N}\}$.

Proof. If (x_n) converges then (x_n) is bdd. Only need to show bounded implies convergent.

(i) Suppose (x_n) is increasing and bounded. Then $\{x_n | n \in \mathbb{N}\}$ is bounded above so $s = \sup\{x_n | n \in \mathbb{N}\}$. Let ϵ be given. Then $\exists N \in \mathbb{N}$ such that

$$s - \epsilon < x_N \leq s \quad \implies \quad |x_n - s| < \epsilon$$

If $n \geq N$, then

$$s - \epsilon < x_N \leq x_n \leq s.$$

So, $(x_n) \to s$.

(ii) Suppose (x_n) bounded, decreasing., Then $r = \inf\{x_n | n \in \mathbb{N}\}$ exists. Let ϵ be given. Then, $\exists N \in \mathbb{N}$ such that

$$r \leq x_N < r + \epsilon \quad \implies \quad |x_N - r| < \epsilon \quad \implies \quad |x_n - r| < \epsilon$$

for all $n \geq \mathbb{N}$

$\square$

Example. $x_1 = 1, \quad x_{n+1} = \frac{1}{4}(2x_n + 3)$.
Show (x_n) is increasing

$$1 = x_1 \leq x_2 = \frac{5}{4}$$

Assume $x_n \geq x_n$. Show $x_{n+2} \geq x_{n+1}$. We have

$$x_{n+2} \geq x_{n+1} \iff \frac{1}{4}(2x_{n+1} + 3) \geq \frac{1}{4}(2x_n + 3) \iff x_{n+1} \geq x_n$$

So (x_n) is increasing. Let's show $x_n < 2$.

$$x_1 = 1 < 2$$

Assume $x_n < 2$. Show $x_{n+1} < 2$. We have

$$x_{n+1} = \frac{1}{4}(2x_n + 3) < \frac{1}{4}(2 \cdot 2 + 3) = \frac{1}{4}(4+3) = \frac{7}{4} < 2$$

Thus $x_n \to L \in \mathbb{R}$.
$L = \sup\{x_n | n \in \mathbb{N}\}$.
L "should" satisfy $L = \frac{1}{4}(2L+3)$.
Need to show

$$\left| L - \frac{1}{4}(2L+3) \right| < \epsilon, \; \forall \epsilon > 0$$

$$\begin{aligned}
\left| L - \frac{1}{4}(2L+3) \right| &= \left| L - x_{n+1} + x_{n+1} - \frac{1}{4}(2L+3) \right| \\
&= |x_{n+1} - L| + \left| \frac{1}{4}(2x_n + 3) - \frac{1}{4}(2L+3) \right| \\
&= |x_{n+1} - L| + \frac{1}{2}|x_n - L| \\
&= |x_{n+1} - L| + |x_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

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Example. $x_1 = 1, \quad x_{n+1} = \sqrt{2x_n}$

$$x_1 = 1, \; x_2 = \sqrt{2}, \; x_3 = \sqrt{2\sqrt{2}}$$

Show (x_n) is increasing.

$$x_1 = 1 \leq x_2 = \sqrt{2}$$

Assume $x_n \leq x_{n+1}$. Show $x_{n+1} \leq x_{n+2}$. We have

$$x_{n+1} \leq x_{n+2} \iff \sqrt{2x_n} \leq \sqrt{2x_{n+1}} \iff 2x_n \leq 2x_{n+1} \iff x_n \leq x_{n+1}$$

Thus (x_n) is increasing. Let's show $x_n < 2$.

$$x_1 = 1 < 2$$

Assume $x_n < 2$. Show $x_{n+1} < 2$. We have

$$x_{n+1} = \sqrt{2x_n} < \sqrt{2 \cdot 2} = \sqrt{4} = 2$$

(x_n) is bounded, so $x_n \to L$. L must satisfy $L = \sqrt{2L}$.

$$\begin{aligned}
\left| L - \sqrt{2L} \right| &\implies L^2 = 2L \\
&\implies L(L-2) = 0 \\
&\implies L = 2
\end{aligned}$$

Section 3.4 — Subsequences

Definition. Let (x_n) be a sequence and let $n_1 < n_2 < n_3 < \ldots$ be a strictly increasing sequence in $\mathbb{N}$. Then, the sequence $(x_{n_k})_{k=1}^{\infty}$ is called a subsequence of (x_n).

Example. $x_n = n, \quad n_k = 2^k$

$(x_{n_k}) = (2, 4, 8, 16, 32, \ldots)$

- (x_n) diverges if either of the following holds:
  1. (x_n) has two convergent subsequences (x_{n_k}), $(x_{n_{k'}})$ with different limits
  2. (x_n) is unbounded
- Every sequence has a monotone subsequence.

Definition. Given (x_n), say that x_m is a *peak* if $x_m \geq x_n$ for all $n \geq m$.

Proof. Let (x_n) be given.

Case 1: (x_n) has infinitely many peaks

$$x_{m_1}, x_{m_2}, x_{m_3}, \ldots \qquad m_1 < m_2 < m_3 < \cdots \qquad x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \ldots$$

So (x_{m_k}) is decreasing.

Case 2: (x_n) has finitely many peaks (maybe zero)

$$x_{m_1}, x_{m_2}, \ldots, x_{m_r}$$

Set $s_1 = m_r + 1$. Then x_s is NOT a peak so $\exists s_2 > s_1$ such that $x_{s_1} < x_{s_2}$, $s_2 > m_r$ so x_{s_2} is NOT a peak and $\exists s_3 > s_2$ such that $x_{s_2} < x_{s_3}$, etc.

$\square$

(Bolzano-Weierstrass Theorem). Every bounded sequence has a convergent subsequence.

Proof #1. (x_n) has a monotone subseq, (x_{n_k}), and (x_{n_k}) is bounded (and thus converges). If $I = [a, b]$, set $U_I = [\frac{a+b}{2}, b]$, $L_I = [a, \frac{a+b}{2}]$. $\square$

Proof #2. (x_n) bounded so $(x_n) \in [-M, M] = I_0$

Step 1) Either U_{I_0} or L_{I_0} contains infinitely many terms of (x_n). Call this interval I_1 and choose $(x_{n_1}) \in I_1$.

Step 2) I_1 contains infinitely many (x_n)'s so one of U_{I_1} or L_{I_1} contains infinitely many (x_n)'s. Call this I_2 and choose $(x_{n_2}) \in I_2$ AND $n_2 > n_1$.

Assume we have found I_n which is either $U_{I_{n-1}}$ or $L_{I_{n-1}}$ and contains infinitely many (x_n)'s, and contains (x_{n_k}) with $n_k > n_{k-1}$ (where $x_{n_{k-1}} \in I_{k-1}$). One of U_{I_k} or L_{I_k} has inifintely many elements. Call this I_{k+1} and choose $x_{n_{k+1}} \in I_{k+1}$ with $n_{k+1} > n_k$.

Now $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ are nested closed bounded intervals. Moreover, the length of I_k is

$$\frac{2M}{2^k} = \frac{M}{2^{k-1}} \xrightarrow{k \to \infty} 0$$

So by prev Thm, $\bigcap_{k=1}^{\infty} = \{\eta\}$

Let $\epsilon > 0$ be given. Find $K \in \mathbb{N}$ such that

$$\frac{M}{2^{k-1}} < \epsilon \quad \forall k \geq K.$$

Then

$$|(x_{n_k}) - \eta| \leq \frac{M}{2^{k-1}} < \epsilon \quad \forall k \geq K$$

since

$$(x_{n_k}), \eta \in I_k \quad \forall k \geq K$$

Thus, $\lim_{k\to\infty}(x_{n_k}) = \eta$. $\square$

Limsup and Liminf

Let (x_n) be a bounded sequence. Consider $\mathbb{L} = \{l \in \mathbb{R} \mid \exists (x_{n_k}) \text{ s.t. } x_{n_k} \to l\}$

Definition. $\mathbb{L}$ is the set of *subsequential limits*.

By B-W, $\mathbb{L} \neq \emptyset$, $\mathbb{L}$ is bounded.

$$\limsup x_n = \sup \mathbb{L}$$

$$\liminf x_n = \inf \mathbb{L}$$

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Section 3.5 — Cauchy Criterion

Definition. (x_n) is Cauchy if $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ such that $|x_n - x_m| \, \forall n, m \geq N$

Theorem. If $x_n \to L$ then (x_n) is Cauchy.

Proof. Let $\varepsilon > 0$ be given. We must find some $N \in \mathbb{N}$ such that

$$|x_n - L| < \frac{\varepsilon}{2},\ \forall n \in \mathbb{N}$$

Now

$$\begin{aligned} &|x_n - x_m| = |x_n - L + L - x_m| \\ &\leq |x_n - L| + |x_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ if } n, m \geq N \end{aligned}$$

$\square$

Theorem. If (x_n) is Cauchy, then (x_n) is bounded.

Proof. Set $\varepsilon = 1$, find N such that $|x_n - x_m| < 1$ for all $n, m > \mathbb{N}$. In particular

$$|x_N - x_m| < 1 \ \forall m \geq N$$

By the Triangle Inequality, $|x_m| < |x_N| + 1 \ \forall m \geq N$.
Set $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$. Then $|x_n| \leq M,\ \forall n \in \mathbb{N}$. $\square$

Theorem. (x_n) is convergent $\iff$ (x_n) is Cauchy.

Proof. Convergent $\implies$ Cauchy, so assume (x_n) is Cauchy. Then (x_n) is bounded so $\exists$ a subsequence (x_{n_k}) such that $(x_{n_k}) \to L$. Let $\varepsilon > 0$ be given. Find $N \in \mathbb{N}$ such that

$$|x - n - x_m| < \frac{\varepsilon}{2},\ \forall n, m \geq N$$

Find $k \geq N$ such that $|x_{n_k} - L| < \frac{\varepsilon}{2}$. Now

$$\begin{aligned} |x_n - L| &= |x_n - x_{n_k} + x_{n_k} - L| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} + L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

$\square$

Example. $x_n = \frac{1}{n}$ in $(0,1)$

$$\begin{aligned}|x_n - x_m| &= |\frac{1}{n} - \frac{1}{m}| \\ &= |\frac{m-n}{mn} \to 0 \text{ as } m, n \to \infty\end{aligned}$$

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A function $f : A \to \mathbb{R}$ is boudned if $\exists M > 0$ such taht $|f(x)| \leq M \forall x \in A$

Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded.

Proof. Assume f is <u>not</u> bounded. That is, for each $n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $|f(x_n)| \geq n$.
By B-W theorem, $\exists (x_{n_k})$ such that $x_{n_k} \to x, x \in [a, b]$.
If f is continuous, then we would have that $f(x_{n_k})$ converges.
This means $|f(x_{n_k})|$ is bounded, but $|f(x_{n_k})| \geq n_k \geq k$, a contradiction.
Taking the contrapositive gives the theorem. □

Theorem. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\exists x_{min}, x_{max} \in [a.b]$ such that $f(x_{max}) \geq f(x)$ and $f(x_{min}) \leq f(x)$, $\forall x \in [a, b]$

Proof. $S = \{f(x) : x \in [a, b]\} = f([a, b])$ is a bounded set, nonempty.
Thus $s = \sup(S)$, $l = \inf(S)$ exist.
For $n \in \mathbb{N}, s - \frac{1}{n} < s$, so $\exists f(x_n) \in S$ such that $s - \frac{1}{n} < f(x_n) < s$.
(x_n) is bounded, so $\exists (x_{n_k})$ such that $x_{n_k} \to x_{max} \in [a, b]$
Now, $f(x_{max}) = \lim_{k\to\infty} f(x_{n_k}) \leq s$
Also $\lim_{k\to\infty} f(x_{n_k}) \leq s$
□

Theorem (Location of Roots). suppose $f : [a, b] \to \mathbb{R}$ continuous and either $f(a) < 0 < f(b)$ <u>or</u> $f(a) > 0 > f(b)$. Then $\exists c \in (a, b)$ such that $f(c) = 0$.

Proof. Set $I_1 = [a_1, b_1] = [a, b]$. Assume $f(a) < 0 < f(b)$
Let $p_1 = \frac{a_1+b_1}{2}$.
If $f(p_1) = 0$, done. Otherwise, $f(p_1) > 0$ or $f(p_1) < 0$
If $f(p_1) > 0$, set $I_2 = [a_2, b_2] = [a_1, p_1]$ so $f(a_1) < 0 < f(b_2)$
If $f(p_1) < 0$, set $I_2 = [a_2, b_2] = [p_1, b_1]$ so $f(a_2) < 0 < f(b_2)$
Now, set $p_2 = \frac{a_2+b_2}{2}$
If $f(p_2) = 0$, done. Otherwise, $f(p_2) > 0$ or $f(p_1) < 0$.
etc etc.
continuing inductively in this manner has two possible outcomes:
1. $f(p_n) = 0$ for some n, done.
2. $f(p_n) \neq 0, \forall n$

By nested intervals thm, $\bigcap_{n=1}^{\infty} I_n = \{c\}$, since the lengths of the I_n's goes to zero.
In fact, $a_n \to c$ and $b_n \to c$ as $n \to \infty$.
Since f is continuous, $f(c) = \lim_{n\to\infty} f(a_n) \leq 0$ since $f(a_n) < 0$
Also, $f(c) = \lim_{n\to\infty} f(b_n) \geq 0$ since $f(b_n) > 0$.
$\implies f(c) = 0$ □

Theorem (Intermediate Value). suppose $f : [a,b] \to \mathbb{R}$. If $f(a) < k < f(b)$ and f continuous then $\exists c \in (a,b)$ such that $f(c) = k$.

Proof. $f(a) - k < 0 < f(b) - k$
$g(x) = f(x) - k$ continuous.

$\square$

5.4 — Uniform Continuity

Definition. Let $f : A \to \mathbb{R}$. f is <u>uniformly continuous</u> on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$, $\forall x, y \in A$.

Theorem. Let $f : A \to \mathbb{R}$. The following are equivalent:

(i) f is not uniformly continuous on A

(ii) $\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, $\exists x, y \in A$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.

(iii) $\exists \varepsilon_0 > 0$ and sequences $(x_n), (u_n)$ in A such that $|x_n - u_n| \to 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0|$

Example. $f(x) = \frac{1}{x}$. Choose $\varepsilon_0 = 1$.

$$x_n = \frac{1}{n} \quad u_n = \frac{1}{n+1}$$

$$|f(x_n) - f(u_n)| = |n - (n+1)| = 1 \geq \varepsilon_0$$

$|x_n - u_n| = |\frac{1}{n} - \frac{1}{n+1}| \leq \frac{1}{n} + \frac{1}{n+1} \to 0$

Theorem. If $f : [a,b] \to \mathbb{R}$ is continuous then it is uniformly continuous on $[a,b]$.

Proof. Suppose f <u>not</u> uniformly conitinuous, so that $\exists (x_n), (u_n)$ in $[a,b]$ and $\varepsilon_0 > 0$ such that $|x_n - u_n| \to 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$. By the Bolzano-Weierstrauss Theorem, $\exists (x_{n_k})$ such that $x_{n_k} \to z \in [a,b]$.
Also, $|u_{n_k} - z| \leq |u_{n_k} - x_{n_k}| + |x_{n_k} - z| \to 0$.
Thus, u_{n_k} converges to 0. If f were cotinuous, we woul dhave that

$$f(x_{n_k}) \to f(z) \qquad\qquad f(u_{n_k}) \to f(z)$$

Given $\frac{\varepsilon_0}{2}$, can find $K \in \mathbb{N}$ such that

$$|f(x_{n_k}) - f(u_{n_k})| \leq |f(x_{n_k}) - f(z)| + |f(z) - f(u_{n_k})| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0, \ \forall k \in K.$$

$\square$

Lipschitz Functions

Definition. A function $f : A \to \mathbb{R}$ is Lipschitz with constant $K > 0$ if

$$|f(x) - f(y)| < k|x - y|, \; \forall x, y \in A$$

$$\left(\frac{|f(x) - f(y)|}{|x - y|} \right)$$

Example. Consider $f(x) = x^2$ on $A = (0, 2)$.

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| && (9) \\ &= |x + y||x - y| && (10) \\ &\leq 8|x - y| && (11) \end{aligned}$$

Theorem. If f is Lipschitz with constant K on A, then f is uniformly continuous on A.

Proof. Let $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{K}$. Then, the distance from $|f(x) - f(y)| \leq K|x - y| < K\frac{\varepsilon}{K} = \varepsilon$ if $|x - y| < \delta$. ☐

Definition. If $f : A \to \mathbb{R}$ and $B \supseteq A$, then $g : B \to \mathbb{R}$ is an <u>extension</u> of f if $g(x) = f(x) \forall x \in A$

Definition. If f and g are continuous on A adn B, respectively, we say g is a <u>continuous extension</u> of f.