MA 34100 Homework 5

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Exercise 3.1.2

a) $$X = (3 + 2n : n \in \mathbb{N})$$

b) $$X = \left(\frac{1}{2^n}(-1)^n : n \in \mathbb{N}\right)$$

c) $$X = \left(\frac{n}{n+1} : n \in \mathbb{N}\right)$$

d) $$X = \left(n^2 : n \in \mathbb{N}\right)$$

Exercise 3.1.5

a) Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$|x_n - x| = \left|\frac{n}{n^2+1} - 0\right| < \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

That is, $|x_n - x| < \varepsilon$ for all $n \geq N$. Thus, $\lim x_n = 0$.

d) Fix any $\varepsilon > 0$. Set $N = \sqrt{\frac{5-6\varepsilon}{4\varepsilon}}$. Then for all $n \geq N$,

$$\begin{aligned} |x_n - x| &= \frac{5}{4n^2+6} < \frac{5}{4N^2+6} = \frac{5}{4\left(\frac{5-6\varepsilon}{4\varepsilon}\right)+6} \\ &= \frac{5}{\left(\frac{5-6\varepsilon}{\varepsilon}\right)+6} = \frac{5}{\left(\frac{5}{\varepsilon}-6\right)+6} = \frac{5}{\frac{5}{\varepsilon}} = \varepsilon \end{aligned}$$

That is, $|x_n - x| < \varepsilon$ for all $n \geq N$. Thus, $\lim x_n = \frac{1}{2}$.

Exercise 3.1.6

a) Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon^2} - 7$. Then for all $n \geq N$,

$$|x_n - x| = \left|\frac{1}{\sqrt{n+7}} - 0\right| < \frac{1}{\sqrt{N+7}} = \frac{1}{\sqrt{\frac{1}{\varepsilon^2} - 7 + 7}} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

That is, $|x_n - x| < \varepsilon$ for all $n \geq N$. Thus, $\lim x_n = 0$.

c) Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$m\,|x_n - x| = \left|\frac{\sqrt{n}}{n+1} - 0\right| < \frac{\sqrt{N}}{N+1} = \frac{\sqrt{\frac{1}{\varepsilon}}}{\frac{1}{\varepsilon}+1} = \frac{\frac{1}{\sqrt{\varepsilon}}}{\frac{1+\varepsilon}{\varepsilon}} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$$

That is, $|x_n - x| < \varepsilon$ for all $n \geq N$. Thus, $\lim x_n = 0$.

Exercise 3.1.11

Fix any $\varepsilon > 0$. Set $N = \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$\begin{aligned} |x_n - x| &= \left|\frac{1}{n} - \frac{1}{n+1} - 0\right| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{n+1-n}{n(n+1)}\right| \\ &= \left|\frac{1}{n(n+1)}\right| < \frac{1}{N(N+1)} = \frac{1}{\frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)} \\ &= \frac{\varepsilon}{\frac{1+\varepsilon}{\varepsilon}} = \frac{\varepsilon^2}{1+\varepsilon} < \varepsilon \end{aligned}$$

That is, $|x_n - x| < \varepsilon$ for all $n \geq N$. Thus, $\lim_{n\to\infty} x_n = 0$.

Exercise 3.2.3

We know that X converges to x and $X + Y$ converges to $x + y$ by the squeeze theorem, so we can see that Y converges to $(x + y) - x = y$.

Exercise 3.2.4

We know that X converges to $x \neq 0$ and XY converges to xy by the squeeze theorem, so we can see that Y converges to $\frac{xy}{x} = y$.

Exercise 3.2.6

a) $$\lim_{n\to\infty}\left(\left(2 + \frac{1}{n}\right)^2\right) = 4$$

b) $$\lim_{n\to\infty}\left(\frac{(-1)^n}{n+2}\right) = 0$$

c) $$\lim_{n\to\infty}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right) = 1$$

d) $$\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt{n}}\right) = 0$$