MA 34100 Homework 7

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Exercise 5.2.1

(a) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1} \qquad (x \in \mathbb{R})$

Notice f is a rational function.

Ex 5.2.3(a) gives us that any rational function is continuous for every $x \in \mathbb{R}$ for which it is defined.

The domain of f is $\mathbb{R}$, so it follows that $f(x)$ is continuous on $\mathbb{R}$.

(b) $g(x) = \sqrt{x + \sqrt{x}} \qquad (x \geq 0)$

Let $f(x) = x$ for all $x \in \mathbb{R}$ and $h(x) = \sqrt(x)$ for $x \geq 0$.

It is trivial that $f(x)$ is continuous on $\mathbb{R}$, so by Thm 5.2.5(b) $h(x)$ is continuous on $\mathbb{R}^+ \cup \{0\}$.

We know that $(f + h)(x) = f(x) + h(x) = x + \sqrt{x}$, so from Thm 5.2.2(a) it follows that $f + h$ is continuous on $\mathbb{R}^+ \cup \{0\}$.

Then Thm 5.2.7 gives that $g(x) = h \circ (f + h)$ is continuous on $\mathbb{R}^+ \cup \{0\}$.

(c) $h(x) = \dfrac{\sqrt{1 + |\sin x|}}{x} \qquad (x \neq 0)$

Define $\phi(x) = x$ for $x \in \mathbb{R}$, $\gamma(x) = \sin(x)$ for $x \in \mathbb{R}$, $\mu(x) = 1 + x$ for $x \in \mathbb{R}$, $\chi(x) = |x|$ for $x \in \mathbb{R}$, and $\eta(x) = \sqrt{x}$.

Then, we can write $h = \dfrac{\eta \circ (\mu \circ (\chi \circ (\gamma \circ \phi)))}{\phi}$.

We know ϕ, γ, μ, and χ are continuous over $\mathbb{R}$, so by Thm 5.2.7 $\mu \circ (\chi \circ \gamma)$ is continuous over $\mathbb{R}$.

We also know that the domain of η is $\mathbb{R}^+ \cup \{0\}$ and the range of $\mu \circ (\chi \circ \gamma)$ is a subset of $\mathbb{R}^+ \cup \{0\}$.

So, the range of $\eta \circ \mu \circ (\chi \circ (\gamma \circ \phi))$ is $\mathbb{R}$.

So, $h(x)$ is continuous on $\mathbb{R} \setminus \{0\}$ by Thm 5.2.2(b).

(d) $k(x) = \cos \sqrt{1 + x^2} \qquad (x \in \mathbb{R})$

Let $\alpha(x) = x^2$ for $x \in \mathbb{R}$, $\beta(x) = 1 + x$ for $x \in \mathbb{R}$, $\gamma(x) = \sqrt{x}$ for $x \geq 0$, and $\epsilon(x) = \cos(x)$ for $x \in \mathbb{R}$.

We know α is continuous on $\mathbb{R}$, so from Thm 5.2.7 it follows that $\beta \circ \alpha$ is continuous on $\mathbb{R}$.
Notice that the domain of γ is $\mathbb{R}^+ \cup \{0\}$, of which the range of $\beta \circ \alpha$ is a subset.
So, $\gamma \circ (\beta \circ \alpha)$ is continuous on $\mathbb{R}$.
Then from Thm 5.2.7 it follows that $\eta \circ (\gamma \circ (\beta \circ \alpha))$ is continuous on $\mathbb{R}$.

Exercise 5.2.3

Consider the functions f and g where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{1}$$

Notice they are both discontinuous everywhere on $\mathbb{R}$. We can see that

$$(f+g)(x) = \begin{cases} 1+0 & \text{if } x \in \mathbb{Q} \\ 0+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \implies (f+g)(x) = 1 \quad \forall x \in \mathbb{R} \tag{2}$$

and

$$fg(x) = \begin{cases} 0 \cdot 1 & \text{if } x \in \mathbb{Q} \\ 1 \cdot 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \implies fg(x) = 0 \quad \forall x \in \mathbb{R}, \tag{3}$$

so $(f+g)$ and fg are both continuous for any $c \in \mathbb{R}$.

Exercise 5.2.7

Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}, \tag{4}$$

where f is known to be everywhere discontinuous on $\mathbb{R}$ (and subsequently on [0,1]). Then,

$$|f(x)| = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}. \tag{5}$$

Notice that $|f(x)| = 1$ for all $x \in \mathbb{R}$, so it is trivially continuous on [0,1].

Exercise 5.2.8

True. Let $c \in \mathbb{R} \setminus \mathbb{Q}$. By the density theorem, there exists a sequence of rational numbers (x_n) such that $(x_n) \to c$. Then, $r_n \in \mathbb{R} \setminus \mathbb{Q}$. We are given that f and g are both continuous on $\mathbb{R}$, so by the Sequential Criterion for continuity:

$$(f(x_n)) \to f(c) \qquad\qquad (g(x_n)) \to g(c). \tag{6}$$

Then, for all $n \in \mathbb{N}$, we have that $f(x_n) = g(x_n)$. This implies that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) \quad \implies \quad f(c) = g(c). \tag{7}$$

Since c is an arbitrary element of $\mathbb{R} \setminus \mathbb{Q}$, we can deduce that

$$f(c) = g(c) \quad \forall c \in \mathbb{R} \setminus \mathbb{Q} \implies f(n) = g(n) \quad \forall n \in \mathbb{R} \tag{8}$$

Exercise 5.3.3

f is continuous on $I \implies |f|$ is continuous on I.
Let $\alpha = \inf |f|(I)$.
By the Maximum-Minimum Thm, there exists some $c \in I$ such that $|f|(c) = |f(c)| = \alpha$.
Now, we conjecture that $\alpha = 0$. For contradiction, we assume $\alpha \neq 0$.

Proof. We know that for any $x \in I$, there exists some $y \in I$ with the property that $|f(y)| \leq \frac{1}{2}|f(c)| = \frac{1}{2}m$.
Then, $\alpha \neq 0 \implies \alpha > 0 \implies |f(y)| < m = \inf |f|(I)$.
However this is a contradiction by definition of infimum.
Thus our assumption that $\alpha \neq 0$ must be false. $\square$

So, there necessarily exists some $c \in I$ with $f(c) = 0$.

Exercise 5.3.17

Yes, $f : [0,1] \to \mathbb{R}$ is a constant function.

Proof. Suppose for contradiction that f was not a constant function.
Then, there must exist numbers $\alpha, \beta \in [0,1]$ such that $f(\alpha) \neq f(\beta)$.
Without loss of generality, let $f(\alpha) < f(\beta)$.
By the Density Theorem, there exists some $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ such that $f(\alpha) < \gamma < f(\beta)$.
By the Intermediate Value Theorem, there exists some $\delta \in [0,1]$ with $f(\delta) = \gamma$.

We know f only produces rational numbers so $\gamma \in \mathbb{Q}$.
However, this contradicts with $\gamma \in \mathbb{R} \setminus \mathbb{Q}$.
Thus our assumption that f is not a constnat function must be false. Thus f is necessarily a constant function. $\square$

Exercise 5.4.2

Let $\varepsilon > 0$ be given and let $\delta = \frac{\varepsilon}{2}$.
Then for all $x, y \in A$ if $|x - y| < \delta$, then

$$
\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{x^2} - \frac{1}{u^2} \right| && (9) \\
&= \left(\frac{y+x}{x^2 y^2} \right) |y - x| && (10) \\
&= \left(\frac{1}{x^2 y} + \frac{1}{x y^2} \right) |y - x| && (11) \\
&\leq 2\,|x - u| && (12) \\
&< 2\frac{\varepsilon}{2} = \varepsilon && (13)
\end{aligned}
$$

So f is uniformly continuous over A.
For B, suppose we have sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$.
It follows then that $|x_n - y_n| \to 0$, but

$$
\begin{aligned}
|f(x_n) - f(y_n)| &= \left| n^2 - n^2 + 2n + 1 \right| && (14) \\
&= |2n + 1| \geq 1 \text{ for all } n && (15)
\end{aligned}
$$

Thus f is not uniformly convergent on B.

Exercise 5.4.5

Let $|x - y| < \min\left\{ \delta_f(\frac{\varepsilon}{2}), \delta_g(\frac{\varepsilon}{2}) \right\}$. Then,

$$
\begin{aligned}
|(f+g)(x) - (f+g)(y)| &= |(f(x) + g(x)) - (f(y) + g(y))| && (16) \\
&\leq |f(x) + f(y)| + |g(x) + g(y)| && (17) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && (18)
\end{aligned}
$$

Exercise 5.4.6

Let $|x - y| < \min\left\{\delta_f(\frac{\varepsilon}{2M}), \delta_g(\frac{\varepsilon}{2M})\right\}$ and let M be an upper bound for f and g. Then,

$$\begin{aligned}
|(fg)(x) - (fg)(y)| &= |f(x)g(x) - f(x)g(y) + f(y)g(x) - f(y)g(y)| && (19)\\
&= |f(x)g(x) - f(x)g(y)| + |f(y)g(x) - f(y)g(y)| && (20)\\
&= |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| && (21)\\
&\leq M\,|g(x) - g(y)| + M\,|f(x) - f(y)| && (22)\\
&\leq M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} && (23)\\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon && (24)
\end{aligned}$$