Problem Set 5: Math 453 Spring 2019
Due Friday March 1

February 26, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Friday March 1 in class.

Problem 1. Let G be a group with subgroup $H,K,L \leq G$ and $H \leq K$. Prove that if

$$H \cap L = K \cap L, \quad HL = KL,$$

then $H = K$.

Problem 2. Let G be a group with subgroup $H,K,L \leq G$ and $H \leq K$. Prove that

$$HL \cap K = H(L \cap K).$$

Problem 3. Let G be a group and let $[G,G]$ denote the commutator subgroup. Let $\varphi \colon G \to G/[G,G]$ denote the quotient homomorphism.

(a) Prove that $G/[G,G]$ is a commutative group.

(b) Prove that if $\psi \colon G \to C$ is a surjective homomorphism with C commutative, then there exists a surjective homomorphism $\phi \colon G/[G,G] \to C$ such that $\psi = \phi \circ \varphi$.

Problem 4. Let $\mathbf{C}$ denote the complex numbers. Given $z = a + bi \in \mathbf{C}$, we define

$$\overline{z} = a - bi, \quad |z|^2 = z\overline{z}.$$

Let

$$S = \left\{ z \in \mathbf{C} \ : \ |z|^2 = 1 \right\}.$$

$\mathbf{C}^\times = \mathbf{C} - \{0\}$ is a group under multiplication.

(a) Prove that S is a subgroup of $\mathbf{C}^\times$.

(b) Prove that $\mathbf{R}/\mathbf{Z}$ and S are isomorphic groups.

Problem 5. Prove that $\mathrm{GL}(m,\mathbf{R})/\,\mathrm{SL}(m,\mathbf{R})$ is isomorphic to $\mathbf{R}^\times$.