Problem Set 6: Math 453 Spring 2019
Due Friday March 8

March 4, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Friday March 8 in class.

Problem 1. Let R be a commutative ring. Prove the following are equivalent:

(a) R is an integral domain.

(b) For each $r \in R$ with $r \neq 0$, the function $f_r \colon R \to R$ given by $f_r(a) = ra$ is injective.

(c) R has the left/right cancellation property.

Problem 2. For $m \in \mathbf{N}$, we define

$$m\mathbf{Z} = \{jm \ : \ j \in \mathbf{Z}\}.$$

(a) Prove that $m\mathbf{Z}$ is an ideal in $\mathbf{Z}$.

(b) Prove that $m\mathbf{Z}$ is a prime ideal if and only if m is prime.

(c) Prove that $m\mathbf{Z}$ is a maximal ideal if and only if m is prime.

Problem 3. Let R be a commutative ring and let $\mathfrak{a}$ be an ideal in R. Prove that if $r \in \mathfrak{a}$ is a unit, then $\mathfrak{a} = R$

Problem 4. Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a commutative ring R. We define

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{j=1}^{n} a_j b_j \ : \ n \in \mathbf{N}, \ a_1 \dots, a_n \in \mathfrak{a}, \ b_1, \dots b_n \in \mathfrak{b} \right\}.$$

(a) Prove that $\mathfrak{a}\mathfrak{b}$ is an ideal in R.

(b) Prove that $\mathfrak{a} \cap \mathfrak{b}$ is an ideal in R.

(c) Prove that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$.

Problem 5. Let R be a commutative ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in R. Define

$$\mathfrak{a} + \mathfrak{b} = \{a + b \ : \ a \in \mathfrak{a}, \ b \in \mathfrak{b}\}.$$

(a) Prove that $\mathfrak{a} + \mathfrak{b}$ is an ideal in R.

(b) Prove that if $\mathfrak{m}$ is an ideal in R such that $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{m}$, then $\mathfrak{a} + \mathfrak{b} \subset \mathfrak{m}$.