MATH 45000 - Exam II November 11, 2024

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NAME: Solution

PUID: ____________________

(1) No textbook or notes.

(2) No calculators or portable electronic devices.

(3) You must show your work to all problems.

(4) There are **six** questions.

(5) The total score of this exam is 100.

1. (20 points)

(a) How many elements of order 6 does $\mathbb{Z}_3 \oplus \mathbb{Z}_{12}$ have?

(b) How many Sylow 3-subgroups does S_4 have?

(a) $\langle a, b \rangle$ such that $\text{lcm}(|a|, |b|) = 6$

① $|a| = 1 \quad |b| = 6 \qquad 1 \times \phi(6) = 1 \times 2 = 2$ elts

② $|a| = 3 \quad |b| = 6 \qquad \phi(3) \times \phi(6) = 2 \times 2 = 4$ elts

③ $|a| = 3 \quad |b| = 2 \qquad \phi(3) \times \phi(2) = 2 \times 1 = 2$ elts

totally $2 + 4 + 2 = 8$ elts of order 6

(b) $|S_4| = 24 = 2^3 \times 3$

$n_3 \equiv 1 \bmod 3$ and $n_3 \mid 8 \Rightarrow n_3 = 1$ or $n_3 = 4$

but $n_3 \neq 1$ since there are more than 3 elements in S_4 that has order 3 (e.g. $(123), (132), (124), (142)$ etc.)

so $n_3 = 4$

2. (20 points)

(a) Prove that every subgroup of D_n of odd order is cyclic.

(b) Prove that $U(15)$ is isomorphic to $U(20)$.

(a) Suppose $H \leq D_n$ and $|H| = \text{odd}$

then H cannot contain any reflection

(otherwise as any refletion has order 2

we have $2 \mid |H|$ contradiction)

So $H \leq \{R_0, R_{\frac{2\pi}{n}}, \cdots, R_{\frac{(n-1)}{n}\cdot 2\pi}\}$

$\Rightarrow$ H is a subgp of a cyclic gp

$\Rightarrow$ H is cyclic (by Fundamental thm of cyclic gp)

(b) $U(15) \cong U(3) \oplus U(5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$

$U(20) \cong U(4) \oplus U(5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$

So $U(15) \cong U(20)$

3. (20 points)

(a) Find all abelian groups of order 200 (up to isomorphism).

(b) In each case above, find a subgroup of order 20.

Use Fundamental Thm of abelian gp.

$200 = 2^3 \times 5^2$

$20 = 2^2 \times 5 = 2 \times 2 \times 5$

$\mathbb{Z}_8 \oplus \mathbb{Z}_{25} \geq \langle 2 \rangle \oplus \langle 5 \rangle$

$\mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \geq \langle 2 \rangle \oplus \langle 1 \rangle \oplus \langle 0 \rangle$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \geq \langle 1 \rangle \oplus \langle 0 \rangle \oplus \langle 5 \rangle$

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \geq \langle 1 \rangle \oplus \langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle 0 \rangle$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{25} \geq \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 0 \rangle \oplus \langle 5 \rangle$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5 \geq \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle 0 \rangle$

4. (15 points) Recall that for $K \leq G$, we have the normalizer $N(K) = \{g \in G \mid gK = Kg\}$ and the centralizer $C(K) = \{g \in G \mid gk = kg, \forall k \in K\}$. Now let $H \leq D_6$ be the subgroup consisting of all rotations.

(a) Find $N(H)$ and $C(H)$.

(b) Prove that $N(H)/C(H)$ is isomorphic to $\mathrm{Aut}(H)$.

(a) Since $|D_6 : H| = 2$

we have $H \lhd D_6 \Rightarrow N(H) = D_6$

Since rotations commute with each other,

and reflections do not commute with rotations

$\Rightarrow C(H) = H$

(b) $N(H)/C(H) \cong D_6/H \cong \mathbb{Z}_2$

$\mathrm{Aut}(H) \cong \mathrm{Aut}(\mathbb{Z}_6) \cong U(6) \cong U(2) \oplus U(3) \cong \mathbb{Z}_2$

$\Rightarrow N(H)/C(H) \cong \mathrm{Aut}(H)$

5. (10 points)

(a) Prove that the map $\mathbb{Z}_6 \to \mathbb{Z}_{12}$ sending x to $3x$ is *not* a group homomorphism.

(b) Find a group homomorphism $\mathbb{Z}_6 \to \mathbb{Z}_{12}$ such that the kernel contains exactly two elements.

(a) The map sends 1 to 3

but $|1|_{\mathbb{Z}_6} = 6$ while $|3|_{\mathbb{Z}_{12}} = 4$

since $4 \nmid 6$, this map is not a group homomorphism.

(b) Consider the map $\mathbb{Z}_6 \to \mathbb{Z}_{12}$

that sends $x \longrightarrow 4x$

(this is a gp homomorphism: $|4|_{\mathbb{Z}_{12}} = 3$ divides $|1|_{\mathbb{Z}_6} = 6$)

The kernel is $\{0.3\}$

6. (15 points) Prove that a group of order 105 contains a subgroup of order 35.

$|G| = 105 = 3 \times 5 \times 7$

$n_7 \equiv 1 \mod 7$ and $n_7 | 15 \Rightarrow n_7 = 1$ or $n_7 = 15$

$n_5 \equiv 1 \mod 5$ and $n_5 | 21 \Rightarrow n_5 = 1$ or $n_5 = 21$

Suppose $n_7 = 15$ <u>and</u> $n_5 = 21$

then G contains $6 \times 15 = 90$ elts of order 7

(each $H_7 \cong \mathbb{Z}_7$ has 6 elts of order 7.

no two H_7's could have elts in common except $\{e\}$)

similarly, G contains $4 \times 21 = 84$ elts of order 5

$\Rightarrow |G| \geq 90 + 84 = 174$ ~~elt~~ contradicting $|G| = 105$

$\Rightarrow$ either $n_5 = 1$ or $n_7 = 1$

$\Rightarrow$ either $H_5 \triangleleft G$ or $H_7 \triangleleft G$

$\Rightarrow H_5 H_7 \leq G$ and $|H_5 H_7| = \dfrac{|H_5| \cdot |H_7|}{|H_5 \cap H_7|} = 5 \times 7 = 35.$