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MA450
HW6

5.11 a) $(135)=(15)(13) \Rightarrow$ even

b) $(1356)=(16)(15)(13) \Rightarrow$ odd

c) $(13567)=(17)(16)(15)(13) \Rightarrow$ even

d) $(12)(134)(152)=(12)(14)(13)(12)(15) \Rightarrow$ odd

e) $(1243)(3521)=(13)(14)(12)(31)(32)(35) \Rightarrow$ even

5.24) $\boxed{\text{Supp } H \text{ is a sg of } S_n \text{ of odd order. Prove } H \leq A_n.}$

Claim] $H \leq S_n \Rightarrow$ every element of H is an even perm or half are even, half odd.

Pf of claim] Trivial if all even; assume $\exists$ at least 1 odd perm α.

Since odd+odd=even, $\alpha\beta$ will be even $\forall$ odd perms $\beta \in H$.

Also $\beta_1 \neq \beta_2 \Rightarrow \alpha\beta_1 \neq \alpha\beta_2 \quad \forall \beta_1, \beta_2 \in H$.

$\Rightarrow$ # even perms $\in H \geq$ # odd perms $\in H$. (1)

Since odd+even=odd, $\alpha\gamma$ will be odd $\forall$ odd $\gamma \in H$.

Also, $\gamma_1 \neq \gamma_2 \Rightarrow \alpha\gamma_1 \neq \alpha\gamma_2 \quad \forall \gamma_1, \gamma_2 \in H$

$\Rightarrow$ # even perms $\in H \leq$ # odd perms $\in H$. (2)

(1) and (2) $\Rightarrow$ # even perms $\in H =$ # odd perms $\in H$.

Then by claim, if H contained n odd permutations, $|H|=2n$ is even.

Thus H must only have even perms $\Rightarrow H \leq A_n$. $\square$

6.24) $\boxed{\text{Supp } \phi: \mathbb{Z}_{20} \to \mathbb{Z}_{20} \text{ is automorphism and } \phi(5)=5. \text{ What are possibilities of } \phi(x)?}$

By thm, $\phi: G \to \overline{G}$ iso $\Rightarrow G=\langle a \rangle \Leftrightarrow \overline{G}=\langle \phi(a) \rangle$.

Since $\mathbb{Z}_{20}=\langle 1 \rangle$, want $\phi(1)=y$

By def isomorphism, $\phi(5)=\phi(\sum_0^5 1)=\sum_0^5 \phi(1)=5\phi(1)=5y$

Then $5y \equiv 5 \pmod{20}$

$\Rightarrow 5(y-1) \equiv 0 \pmod{20}$

$\Rightarrow y \in \{1, 5, 9, 13, 17\}$ but $\phi(5)=5$ and ϕ one to one, so $\phi(1) \in \{1, 9, 13, 17\}$

$\Rightarrow \phi(x)=x, 9x, 13x, 17x$ $\square$

7.1) $H=\{(1),(12)(34),(13)(24),(14)(23)\}=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$

$\alpha_5 H=\{\alpha_5\cdot\alpha_1,\alpha_5\cdot\alpha_2,\alpha_5\cdot\alpha_3,\alpha_5\cdot\alpha_4\}=\{\alpha_5,\alpha_8,\alpha_6,\alpha_7\}$

$\alpha_9 H=\{\alpha_9\cdot\alpha_1,\alpha_9\cdot\alpha_2,\alpha_9\cdot\alpha_3\cdot\alpha_9\cdot\alpha_4\}=\{\alpha_9,\alpha_{11},\alpha_{12},\alpha_{10}\}$

7.2) Q7.1 $\Rightarrow H\leq A_4$, and $A_4\leq S_4$ by def A_4 $\Rightarrow H\leq S_4$

Lagrange's thm $\Rightarrow$ #distinct left cosets of H in $S_4=\frac{|S_4|}{|H|}=\frac{4!}{4}=3!=6$

7.6) Properties of cosets $\Rightarrow \alpha H=\beta H \Leftrightarrow a-b\in H$

$\Rightarrow \alpha-\beta=kn$ for some $k\in\mathbb{Z}$ $\Rightarrow$ $\alpha\equiv\beta \pmod n$

This equivalence relation gives n distinct equiv. classes $\Rightarrow \exists n$ left cosets $\square$

7.8) Cor $\Rightarrow |a|=15 \Leftrightarrow |\langle a\rangle|=15$

Thm $\Rightarrow |a^5|=\frac{|a|}{\gcd(5,15)}=\frac{15}{5}=3$

By def index, $|G:H|=\frac{|G|}{|H|}$ if $|G|\neq\infty$.

So $|\langle a\rangle:\langle a^5\rangle|=5$; namely H, aH, a^2H, a^3H, a^4H

7.18) Notice that $\phi(n)=|U(n)|$

By def $U(n)$, $a\in U(n)$. Cor 4 $\Rightarrow a^{\phi(n)}=a^{|U(n)|}=1$

By division algo, $a=nm+r$, $0\leq r<n$

$\Rightarrow a \bmod n=r \Rightarrow a^{\phi(n)} \bmod n=r^{\phi(n)}=1$ $\square$

7.30) Consider $g\in G$ s.t. $g\neq e$. By cor 2, $|g|\,\big|\,|G|$ so $|g|\in\{1,2,4,8\}$

$|g|\neq1 \Leftarrow g\neq e$ $\qquad$ $|g|=2 \Rightarrow |g|=2$

$|g|=4 \Rightarrow |g^4|=2$ $\qquad$ $|g|=8 \Rightarrow |g^4|=2$ $\square$

7.42) Thm $\Rightarrow n\perp k \Rightarrow \exists s,t\in\mathbb{Z}$ s.t. $ns+kt=1$

Consider $\alpha^k=\beta^k$. By cor 4, $\alpha=\alpha^{ns+kt}=(\alpha^n)^s(\alpha^k)^t=(\alpha^k)^t=(\beta^k)^t=(\beta^n)^s(\beta^k)^t=\beta^{ns+kt}=\beta$

Thus mapping is one to one.

Consider $h\in G$. WTS $\exists g\in G$ s.t. $g^k=h$.

Note $ns+kt=1 \Rightarrow kt=1 \pmod n \Rightarrow t\equiv k^{-1} \pmod n$

Let $g=h^t$. Then $g^k=(h^t)^k=h^{tk}=h$.

Thus mapping is onto by def onto

Thus mapping automorphism by def auto. $\square$