MA 450 Homework 2

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Exercise 0.58

Suppose $a \in S$.
Trivially $a - a = 0$, an integer.
Thus $a \sim a$ by def $\sim$.
Thus $\sim$ is reflexive.

Suppose $a, b \in S$ such that $a \sim b$.
Notice that $a - b \in \mathbb{Z} \implies -(a-b) = b - a \in \mathbb{Z}$.
Then $b \sim a$ by def $\sim$.
Thus $\sim$ is symmetric.

Suppose we have $a, b, c \in S$ such that $a \sim b$ and $b \sim c$.
That is, $a - b, b - c \in \mathbb{Z}$.
By closure of the integers, $(a - b) + (b - c) = a - c \in \mathbb{Z}$.
So $a \sim c$ by def $\sim$.
Thus $\sim$ is transitive.

Thus $\sim$ is an equivalence relation by def equivalence relation.

The equivalence classes represent the real numbers between 0 and 1.
As an example, suppose we let $a = 25.3245$ and $b = 20.3245$.
Then, $a \sim b$ since $a - b = 5 \in \mathbb{Z}$.
So, $a, b \in [0.3245]$.

Exercise 0.59

No. Notice that $1 * 0 \geq 0$ and $0 * -1 \geq 0$, but $1 * -1 \not\geq 0$. Thus R fails to be transitive and can not be an equivalence relation.

Exercise 2.6

a) In $\mathbb{C}^*$, $7 + 5i)(-3 + 2i) = -21 + 14i - 15i - 10 = -31 - i$

b) In $GL(2, \mathbb{Z}_{13})$, $\det \begin{bmatrix} 7 & 4 \\ 1 & 5 \end{bmatrix} = 9 - 4 = 8$

c) In $GL(2, \mathbb{R})$, ${}^{-1}\begin{bmatrix} 6 & 3 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{2}{-12} & \frac{-3}{-12} \\ \frac{-8}{-12} & \frac{6}{-12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{4} \\ \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$

d) In $SL(2, \mathbb{Z}_{13})$, ${}^{-1}\begin{bmatrix} 6 & 3 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -8 & 6 \end{bmatrix}$

Exercise 2.16

Let G be the set $\{5, 15, 25, 35\}$ with multiplication modulo 40. We wish to show that G is a group.
First, notice that

$$25 * 5 = 125 \equiv 5 \pmod{40} \tag{1}$$
$$25 * 15 = 375 \equiv 15 \pmod{40} \tag{2}$$
$$25 * 25 = 625 \equiv 25 \pmod{40} \tag{3}$$
$$25 * 35 = 875 \equiv 35 \pmod{40} \tag{4}$$

This tells us that 25 must be the identity element.
Then, it is easy to see that ${}^{-1}5$ is 5.
We also know that ${}^{-1}15 = 15$, as $15^2 = 225 \equiv 25 \pmod{40}$.
The inverse of 35 is also easy to test, as $35^2 = 1225 \equiv 25 \pmod{40}$.
Then G has (1) an associative operation, (2) an identity element, and (3) is closed under inverses.
The group axioms are satisfied, so G is a group.

If we divide all the values by 5, it becomes $\{1, 3, 5, 7\}$ under multiplication modulo 8.
This is exactly $U(8)$, the group of positive nonzero integers less than 8 and coprime to 8.

Exercise 2.18

We are given that $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.

First we identify elements of H.
Trivially, the square of an identity is the identity, so $R_0 \in H$.
Notice that the square of any reflective element of a dihedral group will equal the identity, so no reflective elements of D_4 generate new elements of H.
Squaring rotations gives $R_0^2 = R_0,\ R_{90}^2 = R_{180},\ R_{180}^2 = R_0,\ R_{270}^2 = R_{180}$.
Thus $H = \{R_0, R_{180}\}$.

Next, we identify elements of K.
Again, the identity is trivial and $R_0 \in K$.
As stated above, the square of every reflective element is the identity.
Thus $D, D', F, F' \in K$, where D and D' are diagonal reflections, F is a horizontal flip and F' is a vertical flip.
We also found above that the only rotations whose squares are equal to the identity are $R_0, R_{180} \in K$.
Thus $K = \{R_0, R_{180}, D, D', F, F'\}$.

Exercise 2.31

Let $*$ represent the group operation.
Assume we have some group table with a row (or column) containing an element, say a, twice.
This would mean that there are two distinct elements, say r_1, and r_2, that combine with a third element, say s, to create a.

$$\begin{array}{c|ccc} & r_1 & r_2 & \cdots \\ \hline \vdots & \vdots & \vdots & \cdots \\ s & a & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

This would imply that $r_1 * s = a$ and $r_2 * s = a$.
It follows that $r_1 * s = r_2 * s$.
By Theorem 2.2 on page 50, we can cancel the s on both sides to find $r_1 = r_2$.
However, this contradicts the assertion that r_1 and r_2 are distinct.
Thus each element in a row (or column) of a Cayley table must be unique.

Exercise 2.32

We wish to construct a Cayley table for $U(12)$.

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

The identity row and column are trivial.
Moving down on the main diagonal, $5^2 = 24 + 1 \equiv 1$, $7^2 = 48 + 1 \equiv 1$, $11^2 = 120 + 1 \equiv 1 \pmod{12}$.
Then, $7 * 5 = 24 + 11 \equiv 11$, $11 * 5 = 48 + 7 \equiv 7$, and $11 * 7 = 72 + 5 \equiv 5 \pmod{12}$.
Since $U(12)$ is abelian, the entries of the table are reflected over the main diagonal.

Exercise 2.33

We wish to fill in the following Cayley table.

	e	a	b	c	d
e	e	__	__	__	__
a	__	b	__	__	e
b	__	c	d	e	__
c	__	d	__	a	b
d	__	__	__	__	__

The identity row and column are trivial.
By uniqueness of inverses, $da = ad = e$ and $cb = bc = e$. So

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	__	__	e
b	b	c	d	e	__
c	c	d	e	a	b
d	d	e	__	__	__

By problem 2.31, each element of the group appears exactly once in each row and each column. Thus $db = a$ and $dd = c$.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	__	__	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	__	__	c

Note that if we let $ca = x$, then $cad = xd \implies c = xd \implies x = d$.
Invoking 2.31 again, we find that $cd = b$, $ab = c$, and $bd = a$

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Exercise 2.42

Suppose $F_1F_2 = F_2F_1$ in D_n such that $F_1 \neq F_2$.
Since both are reflections, F_1F_2 must represent some rotation on D_n.
Notice that $(F_1F_2)^2 = F_1F_2F_2F_1 = F_1F_1 = e$.
The only rotation with order 2 is R_{180}, so $F_1F_2 = F_2F_1 = R_{180}$.

Exercise 2.45

(a) First, notice that we can rewrite R^5 as R. Then the expression becomes $FR^{-2}FR$.

Lemma 0.1. $FR^mF = R^{-m}$ for any $m \in \mathbb{Z}$.

Proof. We know FR^m is a reflection for arbitrary m, so $(FR^m)(FR^m) = R^0$.
Multiplying both sides by R^{-m} gives $FR^mF = R^{-m}$. □

By this lemma, $(FR^{-2}F)R = R^2R = R^3$.

(b) By the lemma above, $R^{-3}(FR^4F)R^{-2} = R^{-3}R^{-4}R^{-2} = R^2RR^3 = R$

(c) Note that $R^5 = R^{-1}$. By the lemma above, $(FR^{-1}F)R^{-2}F = RR^{-2}F = R^{-1}F = R^5F$.