14 Symmetry Groups

So far, in this class, we've covered *groups* and *linear algebra*. Now, we are looking at groups of symmetries that preserve extra forms of structure.

14.1 Review

Last week, we looked at the *orthogonal matrices*.

Definition 14.1
The **orthogonal matrices** O_n are matrices that preserve *distance*. It is the set

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n : |Tv| = |v| \text{ for all } v \in \mathbb{R}^n.$$

Definition 14.2
The set M_n of **isometries** from $\mathbb{R}^n$ to itself is

$$\{f : \mathbb{R}^n \longrightarrow \mathbb{R}^n : |f(u) - f(v)| = |u - v|\}.$$

The orthogonal matrices are the subset of isometries that are *linear transformations*. In class, we showed that every isometry f is of the form $f(x) = Ax + b$ where $A \in O_n$ and $b \in \mathbb{R}^n$.

Then, we looked at O_2, the orthogonal matrices in two dimensions. There are two possibilities for a transformation in O_2.

- Rotations around 0: these have determinant 1 and are called SO_2. [47]
- Reflections across a line through $\vec{0}$: these have determinant -1

Then the *isometries* of two-dimensional space, M_2, also fit into several categories.[48]

- Translations
- Rotations around p
- Reflections across a line
- A glide reflection[49]

14.2 Examples of Symmetry Groups

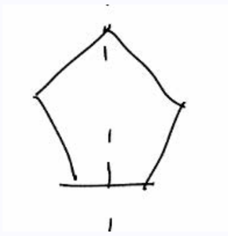
Now, we want to add some additional structure to preserve.

Guiding Question
What isometries of $\mathbb{R}^2$ fix some shape inside $\mathbb{R}^2$?

We call the group of such isometries *symmetry groups* for that shape. Let's start with a couple examples of shapes and their symmetry groups.

[47] The special orthogonal group
[48] This is quite surprising, since a priori, an isometry could take many different forms.
[49] A reflection in addition to a parallel translation

Example 14.3
For a regular pentagon, the group of symmetries are rotations by multiples of $\frac{2\pi}{5}$, and reflections across lines. This group of symmetries is what we would call *discrete.*[a]



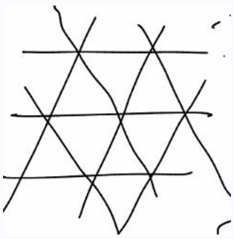
[a]This will be formalized later on.

Next, we look at a group that is not discrete.

Example 14.4
For a circle centered at the origin, every rotation or reflection will fix it, and so its symmetry group is all of O_2. This group of symmetries is *not discrete.*

We can also look at infinitely large shapes.

Example 14.5
For a triangular lattice, certain translations, reflections over lines, rotations, and glide reflections all preserve it. It is a *discrete* symmetry group.



14.3 Discrete Subgroups of $\mathbb{R}$

From our examples, we see that some symmetry groups are "discrete" and some are not.

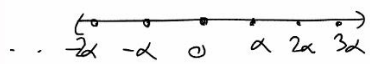
Guiding Question
How can the notion of a *discrete group* be formalized?

We can start with an easier notion, which is a discrete group inside $(\mathbb{R}, +)$.

Definition 14.6
A group $G \leq (\mathbb{R}, +)$ is discrete if there exists $\varepsilon > 0$ such that any $g \in G$ such that $g \neq 0$ satisfies $|g| > \varepsilon$. Equivalently, for $a, b \in G$ and $a \neq b$, then it must be true that $|a - b| > \varepsilon$ for a discrete group.

The discreteness tells us some important information about G.

Theorem 14.7
If $G \leq (\mathbb{R}, +)$ is discrete, then $G = \{0\}$ or $G = \mathbb{Z}\alpha$ for some real number $\alpha > 0$.



This theorem is very similar to the theorem we had about subgroups of $\mathbb{Z}$, where we showed they were either trivial or of the form $k\mathbb{Z}$.

Proof. Assume that $G \neq \{0\}$. Then there is some smallest positive element $\alpha \in G$. To see why it is possible to find a smallest element, we start by taking any $g > 0$ in G. By discreteness, in the interval from $[0, g]$, we have at most g/ε elements of G inside of the interval. We can then pick the smallest one because the set is finite.

We now claim that $G = \mathbb{Z}\alpha$. Why is this true? If $2\alpha < x < 3\alpha$ for some $x \in G$, then $0 < x - 2\alpha < \alpha$, where $x - 2\alpha \in G$, which is a contradiction. [50] □

14.4 Finite subgroups of O_2

So what are all the finite subgroups of O_2? Let's first try to create some examples to get some intuition about them.

Example 14.8
Let x be a rotation by $\frac{2\pi}{n}$. Then $C_n = \langle x \rangle$[a], the cyclic group of order n, is generated by x, and is a finite subgroup of O_2.

[a] $\{1, x, \cdots, x^{n-1}\}$

Another possible finite subgroup can be created by expanding C_n a little bit.

Example 14.9
Let y be a reflection across a line ℓ through $\vec{0}$. Notice that the relations $yx = x^{-1}y, y^2 = e$, and $x^n = e$ hold, and so any product $y^{a_1}x^{a_2}y^{a_3}\cdots$ can be written as $x^i y^j$, where $0 \leq i < n$ and $0 \leq j < 2$. Then the group generated by x and y is

$$D_n := \langle x, y \rangle = \{e, x, x^2, \cdots, x^{n-1}, y, xy, x^2y, \cdots, x^{n-1}y\},$$

which is called the dihedral group. It has order $2n$.

For $n \geq 3$, D_n is the group of symmetries of a regular $n-$gon.[51] The dihedral group for $n = 1$ is $D_1 \cong C_2$ and for $n = 2$, $D_2 \cong C_2 \times C_2$. For $n = 3$, $D_3 \cong S_3$, and larger dihedral groups can also be studied.

Now, we have two families of finite subgroups of O_n, the cyclic groups of rotations, and the dihedral groups. It turns out that these are actually all the finite subgroups of O_2. This provides yet another classification theorem.

Let's start with a simpler version.

Theorem 14.10
If a subgroup $H \leq SO_2$ is finite, then H is isomorphic to C_n for some n.

Proof. Let ρ_θ be $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then let

$$S = \{\theta \in \mathbb{R} \text{ such that } \rho_\theta \in H\}.$$

Under the homomorphism $\pi : \theta \mapsto \rho_\theta$, $S = \pi^{-1}(H)$. Since S is a preimage, we know that S is a subgroup of $(\mathbb{R}, +)$.

If H is finite, then S must be discrete, and so by Theorem 14.7, S is $\mathbb{Z}\alpha$ for some α. Also, $2\pi \in S$ because a rotation by 2π is the identity in H, and so $\alpha = \frac{2\pi}{n}$. So $\boxed{H = C_n.}$ □

[50] The discreteness guarantees that we can find a smallest positive element! This is definitely *not* the case for $\mathbb{R}$ in general (it is a fundamental property of $\mathbb{R}$ that there is *no* smallest positive element.)

[51] In general, if x is a rotation by an angle that is not a rational multiple of 2π, then we do not get a rational group. We would get a non-discrete subgroup of SO_2.

Theorem 14.11
Any finite subgroup of O_2 is isomorphic to C_n or D_n.

Now, we can prove Theorem 15.2.

Proof. There are two cases:

- **Case I.** If $G \subseteq SO_2$, by the above theorem, $G \cong C_n$ for some n.
- **Case II.** If G is not a subset of SO_2,, then take the restriction of the determinant function on O_2 to G. It takes
$$G \xrightarrow{\det} \{\pm 1\}.$$
By the assumption that G isn't a subset of SO_2, this is surjective. Let
$$H = \ker(G \xrightarrow{\det} \{\pm 1\}).$$
Then, $H \trianglelefteq G$ is a normal subgroup of index 2. So $\det^{-1}(\{-1\})$ is a nontrivial coset of H, and so it is Hr for some $r \in G$ such that $\det(r) = -1$. Then r must be a reflection across some line ℓ.[52] Then, it is clear by definition that $H \leq SO_2$, and so $H = C_n$ for some n, and it is generated by some $x = \frac{2\pi\rho}{n}$, and then we have
$$G = \left\langle \frac{2\pi\rho}{n}, r \right\rangle \cong D_n.$$

□

14.5 More Discrete Subgroups

Next, what are the finite or discrete subgroups of M_2? Let's start with a couple of definitions.

Definition 14.12
A subgroup $G \leq O_2$ is **discrete** if there exists some $\varepsilon > 0$ such that all nontrivial rotations in G have angle θ such that $|\theta| > \varepsilon$.[a]

[a]Here, discrete implies finite, which implies that it is C_n or D_n.

Definition 14.13
A subgroup $G \leq M_2$ is **discrete** if there exists some $\varepsilon > 0$ such that all translations in G are by vectors b with $|b| > \varepsilon$, and all rotations in G have angle θ such that $|\theta| > \varepsilon$.

This ends up being quite a strong constraint on what the discrete subgroups look like, even though there could be lots of different possibilities. We'll talk about this more next time.

[52]Note that we have many options for ℓ because any $r \in Hr$ generates Hr. In particular, these are all the rotations of ℓ.