SOLUTION KEY
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Problems:
Chap 14: 30, **38**, **53**, 56
Chap 15: 6, **8**, 10, 12

Problem 14.30. Show that $A = \{(3x, y) \mid x, y \in \mathbb{Z}\}$ is a maximal ideal in $\mathbb{Z} \oplus \mathbb{Z}$. Generalize. What happens if $3x$ is replaced by $4x$? Generalize.

Solution:
First note that the ring $\mathbb{Z} \oplus \mathbb{Z}$ is generated by the elements $(1, 0)$ and $(0, 1)$. Suppose that there exists an ideal B with $A \subset B \subseteq \mathbb{Z} \oplus \mathbb{Z}$. Then there is some element $(x, y) \in B \setminus A$. Clearly $(0, 1) \in A \subset B$ and $(x, 0) = (x, y) \cdot (1, 0) \in B$ since B is an ideal. We proceed to show that $(1, 0) \in B$ and hence $B = \mathbb{Z} \oplus \mathbb{Z}$.

Now x must be congruent to 1 or 2 modulo 3. If $x \equiv 1 \bmod 3$, then $x = 3q + 1$ for some $q \in \mathbb{Z}$. Since $(3q, 0) \in A \subset B$, $(1, 0) = (x, 0) - (3q, 0) \in B$ (since B is a subring) and therefore $B = \mathbb{Z} \oplus \mathbb{Z}$. Alternatively, if $x \equiv 2 \bmod 3$, then $x = 3q + 2$. By a similar argument, $(2, 0) \in B$. Hence $(1, 0) = (3, 0) - (2, 0) \in B$, and $B = \mathbb{Z} \oplus \mathbb{Z}$.

If $3x$ is replaced by $4x$, then A would be not be a maximal ideal since it would be contained in the ideal $B = \{(2x, y) \mid x, y \in \mathbb{Z}\}$.

In general, if p is prime then $\{(px, y) \mid x, y, \in \mathbb{Z}\}$ is maximal but if p is composite, it is not maximal. ∎

Problem 14.38. Prove that $I = \langle 2 + 2i \rangle$ is not a prime ideal of $\mathbb{Z}[i]$. How many elements are in $\mathbb{Z}[i]/I$? What is the characteristic of $\mathbb{Z}[i]/I$?

Solution:
I is not a prime ideal since $2(1 + i) = 2 + 2i$ but 2 and $1 + i$ are not in I since otherwise we arrive at a contradiction:

$$(a + bi)(2 + 2i) = 2 \implies 4a = 2$$

which has no integer solutions. Similarly,

$$(a + bi)(2 + 2i) = 1 + i \implies 4a = 2.$$

Since $\mathbb{Z}[i]/I$ has a unity, $1 + I$, we can check the characteristic using Theorem 13.3:

$$0 + I = (1 - i)(2 + 2i) + I = 4 + I = 4(1 + I),$$

and hence it has characteristic 4.

The elements of $\mathbb{Z}[i]/I$ are:

$$I, 1 + I, 2 + I, 3 + I, i + I, (1 + i) + I, (2 + i) + I, (3 + i) + I$$

since in this ring $2 = 2i$. Hence there are eight elements. ∎

Problem 14.53. Show that $\mathbb{Z}_3[x]/\langle x^2 + x + 1 \rangle$ is not a field.

Solution:
Note that $x^2 + x + 1 = (x + 2)^2$ in $\mathbb{Z}_3[x]$. Hence $\langle x^2 + x + 1 \rangle \subset \langle x + 2 \rangle \subset \mathbb{Z}_3$, with strict inclusions since $x + 2 \notin \langle x^2 + x + 1 \rangle$ and $1 \notin \langle x + 2 \rangle$. Since $\langle x^2 + x + 1 \rangle$ is thus not a maximal ideal, $\mathbb{Z}_3[x]/\langle x^2 + x + 1 \rangle$ cannot be a field. ∎

Problem 14.56. Show that $\mathbb{Z}[i]/\langle 1-i \rangle$ is a field. How many elements does this field have?

Solution:
Since in this ring, $-1 = (i + (1-i))^2 = 1^2 = 1$, we must have that $2 = 0$. Furthermore, since $0 + \langle 1-i \rangle = (1-i) + \langle 1-i \rangle$, we get that $1 + \langle 1-i \rangle = i + \langle 1-i \rangle$. Finally, $0 + \langle 1-i \rangle \neq 1 + \langle 1-i \rangle$ since, if this were true there would exist $a, b \in \mathbb{Z}$ such that:

$$\begin{aligned} a(1-i) &= 1 + b(1-i) \\ a - ai - 1 - b + bi &= 0 \\ (a-b-1) + (-a+b)i &= 0 \end{aligned}$$

which implies that $a = b$ and $a = b + 1$, which has no solutions in $\mathbb{Z}$.

Therefore $\mathbb{Z}[i]/\langle 1-i \rangle = \{0 + \langle 1-i \rangle, 1 + \langle 1-i \rangle\}$. Since every non-zero element (namely $1 + \langle 1-i \rangle$) has an inverse (in this case, itself) this is a field. ■

Problem 15.6. Show that the correspondence $x \mapsto 3x$ from $\mathbb{Z}_4$ to $\mathbb{Z}_{12}$ does not preserve multiplication.

Solution:
Let ϕ represent this correspondence. Then $\phi(1 \times 1) = 3 \neq 9 = \phi(1) \times \phi(1)$. ■

Problem 15.8. Prove that every ring homomorphism ϕ from $\mathbb{Z}_n$ to itself has the form $\phi(x) = ax$, where $a^2 = a$.

Solution:
Let $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ be a ring homomorphism. Let $a = \phi(1)$. Since any $x \in \mathbb{Z}_n$ is uniquely defined by repeated addition of 1 x times, i.e. $x = x \cdot 1$, we know that $\phi(x) = \phi(x \cdot 1) = x \cdot \phi(1) = ax$. Finally, $a = \phi(1) = \phi(1^2) = \phi(1)^2 = a^2$. ■

Problem 15.10.

a. Is the ring $2\mathbb{Z}$ isomorphic to the ring $3\mathbb{Z}$?

b. Is the ring $2\mathbb{Z}$ isomorphic to the ring $4\mathbb{Z}$?

Solution:

a. No. Suppose there exists an isomorphism ϕ between $2\mathbb{Z}$ and $3\mathbb{Z}$. Then $\phi(2) = 3z$ for some $0 \neq z \in \mathbb{Z}$. Then since ϕ is a homomorphism it must preserve both addition and multiplication, meaning that:

$$6z = \phi(2) + \phi(2) = \phi(2+2) = \phi(4) = \phi(2^2) = \phi(2)^2 = 9z^2$$

Since $z \neq 0$, z must be an integer solution to the equation $3z = 2$, which is impossible. Hence no such isomorphism can exist.

b. Proceed similarly as in (a.): If there were an isomorphism, it would have the property $\phi(2) = 4z$ for some z. Then

$$8z = \phi(2 + 2) = \phi(2^2) = 16z^2$$

which has no non-zero integer solutions. Hence there can be no isomorphism.

■

Problem 15.12. Let $\mathbb{Z}_3[i] = \{a+bi \mid a, b \in \mathbb{Z}_3\}$. Show that the field $\mathbb{Z}_3[i]$ is ring-isomorphic to the field $\mathbb{Z}_3[x]/\langle x^2 + 1\rangle$.

Solution:
Define $\phi : \mathbb{Z}_3[x] \to \mathbb{Z}_3[i]$ by $\phi(f(x)) = f(i)$. Then clearly ϕ is an onto correspondence, since $\phi(a + bx) = a + bi$ for any $a, b \in \mathbb{Z}_3$.

ϕ is clearly a ring homomorphism since

$$\phi(f(x) + g(x)) = f(i) + g(i)$$

and

$$\phi(f(x)g(x)) = f(i)g(i)$$

is just evaluation of a polynomial, similar to **Example 3**.

Claim: $\ker\phi = \langle x^2 + 1\rangle$.

Clearly $\langle x^2 + 1\rangle \subseteq \ker\phi$ since $\phi(x^2 + 1) = i^2 + 1 = 2 + 1 = 0$. Now suppose that $f(x) \in \ker\phi$ but $f(x) \notin \langle x^2 + 1\rangle$. By basic results about polynomials over real numbers, $f(i) = 0$ implies that $f(2i) = 0$ and hence $(x-i)(x-2i) = x^2+1$ divides $f(x)$, a contradiction. Hence $\ker\phi = \langle x^2 + 1\rangle$.

By the First Isomorphism Theorem for Rings, $\mathbb{Z}_3[x]/\ker\phi = \mathbb{Z}_3[x]/\langle x^2 + 1\rangle \approx \mathbb{Z}_3[i]$. ■