SOLUTION KEY
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Problems:
Chap 5: 2, 6, 28, 32, 69, 71
Chap 6: 2, 10, 28, 34

Problem 5.2. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$. Write α, β, and $\alpha\beta$ as

a. products of disjoint cycles;

b. products of 2-cycles.

Solution:

a. $\alpha = (12345)(678)$,
$\beta = (23847)(56)$, and
$\alpha\beta = (12485736)$

b. $\alpha = (15)(14)(13)(12)(68)(67)$,
$\beta = (27)(24)(28)(23)(56)$, and
$\alpha\beta = (16)(13)(17)(15)(18)(14)(12)$

■

Problem 5.6. What is the order of each of the following permutations?

a. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

Solution:
By Theorem 5.3, the order of these permutations is the least common multiple of the lengths of the cycles when they are written in disjoint cycle form. Re-writing the permutations in disjoint cycle form, we obtain:

a. $(12)(356)(4)$

b. $(1753)(264)$

Hence the permutation in a. has order 6 and the permutation in b. has order 12. ■

Problem 5.28. How many elements of order 5 are in S_7?

Solution:
Since 5 is prime and by Theorem 5.3, we need only count the number of permutations of the form $(a_1a_2a_3a_4a_5)$. There are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$ such quintuples, but this double-counts each permutation 5 times since

$$(a_1a_2a_3a_4a_5) = (a_5a_1a_2a_3a_4) = (a_4a_5a_1a_2a_3) = (a_3a_4a_5a_1a_2) = (a_2a_3a_4a_5a_1).$$

Thus the number of permutations in S_7 of order 5 is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3/5 = 504$. ■

Problem 5.32. Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form.

Solution:
We first write β in disjoint cycle form in order to determine its order: $\beta = (14523)$.
Since β has order 5, $\beta^{99} = \beta^{19 \cdot 5+4} = \beta^4$. Hence,

$$\beta^{99} = (14523)(14523)(14523)(14523) = (13254)$$

■

Problem 5.69. Prove that every element of S_n $(n > 1)$ can be written as a product of elements of the form $(1k)$.

Solution:
By Theorem 5.4, every permutation can be written as a product of 2-cycles. We thus need only to show that any two cycle can be written as a product of elements of the form $(1k)$. Let $a, b \in \{2, \dots, n\}$. Then $(ab) = (1b)(1a)(1b)$ since this permutation switches a and b and fixes 1. ■

Problem 5.71. Show that a permutation with odd order must be an even permutation.

Solution:
Let α be a permutation in S_n with odd order. Write α in disjoint cycle form, say $\alpha = C_1 C_2 \cdots C_m$. Since $|\alpha|$ is odd, and the order is the least common multiple of the lengths of the cycles $C_1, C_2, \dots C_m$, we must have that the lengths of each of the C_i's is odd. Hence, if we can show that every **cycle** of odd length is an even permutation, we are done.

Let $C = (a_1 a_2 \cdots a_k)$, where k is odd, be a cycle. Then since C can be written as a product of $(k-1)$ 2-cycles

$$C = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_2),$$

it is an even permutation. ■

Problem 6.2. Find $\text{Aut}(\mathbb{Z})$.

Solution:
Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be an automorphism. Since the only generators of $\mathbb{Z}$ are 1 and -1, by Theorem 6.2 property 4, $\phi(1) \in \{1, -1\}$. If $\phi(1) = 1$, then for any $n \in \mathbb{Z}$, $\phi(n) = \phi(n \cdot 1) = n\phi(1) = n$, so ϕ is the identity homomorphism. Now if $\phi(1) = -1$, then $\phi(n) = -n$.

Hence $\text{Aut}(\mathbb{Z}) = \{n \mapsto n, n \mapsto (-n)\}$. ■

Problem 6.10. Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all g in G is an automorphism if and only if G is Abelian.

Solution:
First, $\alpha(g)$ is always bijective since groups are closed under inverses and inverses are unique. Hence the only property we need concern ourselves with is whether or not α is a group homomorphism.

Let $a, b \in G$ be group elements. Then α is a group homomorphism if and only if

$$\alpha(ab) = (ab)^{-1} = b^{-1}a^{-1} = \alpha(b)\alpha(a) = \alpha(ba)$$

if and only if a and b commute, since α is necessarily one-to-one. Hence α is a group homomorphism when, and only when, every pair of elements of G must commute, that is, when G is Abelian. ■

Problem 6.28. The group $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| \, a \in \mathbb{Z} \right\}$ is isomorphic to what familiar group? What if $\mathbb{Z}$ is replaced by $\mathbb{R}$?

Solution:
Call the group defined in the problem G. Define $\phi : G \to \mathbb{Z}$ by $\phi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right) = a$. Then since

$$\begin{aligned} \phi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} 1 & b+a \\ 0 & 1 \end{bmatrix}\right) \\ &= b + a \\ &= \phi\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right), \end{aligned}$$

ϕ is a homomorphism. ϕ is surjective since for any $z \in \mathbb{Z}$, $\phi\left(\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}\right) = z$. Further, $\phi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)$, if and only if $x = y$. Hence G and $\mathbb{Z}$ are isomorphic.

If $\mathbb{Z}$ is replaced by $(\mathbb{R}, +)$, then G is isomorphic to $(\mathbb{R}, +)$. ■

Problem 6.34. Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

Solution:
We will show that $U(24)$ has only elements of order 1 or 2 while $U(20)$ has at least one element of order 4. Then by Theorem 6.2 Property 5, there can be no isomorphism between these two groups.

The elements of each group are: $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$ and $U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\}$. It can be checked that each element of $U(24)$ has order 1 or 2. However, in $U(20)$ the order of 3 is 4. Hence these groups are not isomorphic. ■