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HW5

MA450 - Algebra Honors

Fall 2024

5.2) $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$

$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$

$\alpha = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$
$= (1\ 5)(2\ 5)(3\ 5)(4\ 5)(6\ 8)(7\ 8)$

$\beta = (2\ 3\ 8\ 4\ 7)(5\ 6)$
$= (2\ 7)(3\ 7)(8\ 7)(4\ 7)(5\ 6)$

$\alpha\beta = (1\ 2\ 4\ 8\ 5\ 7\ 3\ 6)$
$= (1\ 6)(1\ 3)(1\ 7)(1\ 5)(1\ 8)(1\ 4)(1\ 2)$

5.6) What is the order of the foll.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix} = (1\ 2)(3\ 5\ 6)$

$\text{lcm}(2,3) = \underline{6}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} = (1\ 7\ 5\ 3)(2\ 6\ 4)$

$\text{lcm}(3,4) = \underline{12}$

5.28) How many elt of order 5 $\in S_7$?

$\text{lcm} = 5 \implies 5+1+1$

$(a_1 \cdots a_5) \implies \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5} = 7 \cdot 6 \cdot 4 \cdot 3$

5.32) Let $\beta=(1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} in DC form

$$=\begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\4&2&3&5&1\end{pmatrix}$$

$$=(1\ 4\ 5\ 2\ 3)$$

$$\beta^{99}=(14523)^{99}=\left[(14523)^{5}\right]^{19}(14523)^{4}$$

$$=(14523)^{4}$$

$$=(13254)$$

5.69) Prove every $g\in S_n$ $(n>1)$ can be written as a product of elements of form $(1\ k)$

Thm 5.4 $\Rightarrow$ All perms can be written as 2-cycles

$\Rightarrow$ ETS all 2 cycles cbw in form $(1\ k)$

$\Rightarrow$ given $a\neq 1\neq b$, $(a\ b)=(1\ a)(1\ b)(1\ a)$

$\Rightarrow$ all perms can be written in form $(1\ k)$

5.71) Show perm w/ odd order must be even

Odd order $\Rightarrow$ lcm(lengths) = odd $\Rightarrow$ lengths all odd

$$(a_1\ a_2 \cdots a_{2n+1})=\underbrace{(a_1\ a_{2n+1})(a_1\ a_{2n})\cdots(a_1 a_2)}_{\substack{2n+1-1=2n\text{ terms}\\ \Rightarrow \text{even perm}}}$$

6.2) Find $Aut(\mathbb{Z})$ $\qquad \mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$

$\overset{\text{Thm}}{\Longrightarrow} G = \langle a \rangle \Leftrightarrow \overline{G} = \langle \phi(a) \rangle$

$\Rightarrow \phi(1) = 1^{(*)}$ or $\phi = -1^{(**)}$

def iso $\Rightarrow \phi(k) = k\phi(1)$

$\phi(1) = 1 \Rightarrow \phi(x) = x \Leftrightarrow id$

$\phi(1) = -1 \Rightarrow \phi(x) = -x$

$Aut(\mathbb{Z}) = \{id, \phi\}$ where $\phi(x) = -x$

6.10) G group. Prove $\alpha(g) = g^{-1}$ $\forall g \in G$ is an aut. $\Leftrightarrow$ G abelian

($\Rightarrow$) Supp $\alpha \in Aut(G)$.

Then $\alpha(xy) = (xy)^{-1}$ and $\alpha(xy) = \alpha(x)\alpha(y)$
$= x^{-1}y^{-1}$

$\Rightarrow (xy)^{-1} = x^{-1}y^{-1}$

$\Rightarrow xy = yx \Rightarrow$ G abelian

($\Leftarrow$) Supp G abelian $\Rightarrow xy = yx$

$\Rightarrow (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \phi(x)\phi(y) = \phi(xy)$

6.28) group $\Psi = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in \mathbb{Z} \right\}$ iso to what familiar gp?

$\phi: \Psi \to \mathbb{Z}$

$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \to a$

- 1-1: $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \Rightarrow a = b$
- onto: $\forall z \in \mathbb{Z}$ clearly $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$ maps to z
- o.p.: $\begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

$$\phi(ab) = a + b = \phi(a)\phi(b)$$

6.34) Prove (or disprove) that $U(20) \cong U(24)$

Let $\alpha(G) = \text{card}\{g \mid g \in G, |g| = 2\}$

$U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\}$

$\lvert 1 \rvert = 1$	$\lvert 5 \rvert = 2$	$\lvert 7 \rvert = 2$	$\lvert 11 \rvert = 2$
$\lvert 13 \rvert = 2$	$\lvert 17 \rvert = 2$	$\lvert 19 \rvert = 2$	$\lvert 23 \rvert = 2$

$\Rightarrow \alpha(U(24)) = 7$

$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$

$|1| = 1 \quad |3| = 4 \quad \Rightarrow \quad \alpha(U(20)) \geq 8 - 2 = 6$

$\Rightarrow \alpha(U(20)) \neq \alpha(U(24))$

Thm 6.2.7 $\Rightarrow U(24) \not\cong U(24)$