Problem Set 3: Math 453 Spring 2019
Due Wednesday February 6

February 7, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Wednesday February 6 in class.

Problem 1. Let G be a group and $g \in G$.

(a) Prove that the centralizer $C_G(g)$ of g in G is a subgroup of G. Here

$$C_G(g) = \{h \in G \, : \, gh = hg\}.$$

(b) Prove that $g^n \in C_G(g)$ for all $n \in \mathbf{Z}$.

Problem 2. Let G be a group and $H \leq G$ a subgroup.

(a) Prove that the centralizer $C_G(H)$ of H in G is a subgroup of G. Here

$$C_G(H) = \{g \in G \, : \, gh = hg \text{ for all } h \in H\}.$$

(b) Prove that the normalizer $N_G(H)$ of H in G is a subgroup of G. Here

$$N_G(H) = \left\{g \in G \, : \, gHg^{-1} = H\right\}$$

where

$$gHg^{-1} = \left\{ghg^{-1} \, : \, h \in H\right\}.$$

(c) Prove that $C_G(H) \leq N_G(H)$.

(d) Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$.

(e) Find a necessary and sufficient condition on H to ensure that $H \leq C_G(H)$.

Problem 3. Let G be a group and $H \leq G$ a subgroup.

(a) Prove that $H \leq N_G(H)$ and that H is a normal subgroup of $N_G(H)$.

(b) Prove that if N is a subgroup of G and $H \lhd N$, then $N \leq N_G(H)$.

Problem 4. Let G be a group.

(a) Prove that the center $Z(G)$ of G is a normal subgroup of G. Here
$$Z(G) = \{g \in G \,:\, gh = hg \text{ for all } h \in H\}\,.$$

(b) Prove that the commutator subgroup $[G,G]$ is a normal subgroup of G. Here
$$[G,G] = \langle\{[g,h] \,:\, g,h \in G\}\rangle\,.$$

(c) Prove that the center $Z(G)$ of G is a characteristic subgroup of G.

(d) Prove that the commutator subgroup $[G,G]$ is a characteristic subgroup of G.

Problem 5. Let $\psi\colon G \to H$ be a homomorphism.

(a) Prove that $(\psi(g))^{-1} = \psi(g^{-1})$.

(b) Prove that $\psi(1) = 1$.

(c) Prove that the kernel $\ker(\psi)$ of ψ is a normal subgroup of G.

Problem 6. Let $\psi\colon G \to H$ be a surjective group homomorphism.

(a) Prove that if $N \lhd G$, then $\psi(N) \lhd H$.

(b) Prove that if $N \lhd H$, then $\psi^{-1}(N) \lhd G$.

(c) Prove that if $N \leq H$, then $\ker(\psi) \leq \psi^{-1}(N)$.