Problem Set 1: Math 453 Spring 2019
Due Wednesday January 23

January 20, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write you name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Wednesday January 23 in class.

In what follows below, $\mathbf{N}$ is the set of natural numbers, $\mathbf{Z}$ is the set of integers, $\mathbf{R}$ is the set of real numbers, and $\mathbf{C}$ is the set of complex numbers. Throughout, if G is a group, we will denote the identity element by 1_G. The group operation on G will be given in multiplicative notation. That is, if $g,h \in G$, the product of g and h in the group G will be denoted by gh. For each $g \in G$, we will denote the inverse of g by g^{-1}.

Problem 1. Let G be a group. Prove that if $g \in G$ satisfies $gh = h$ for all $h \in G$, then $g = 1_G$.

Problem 2. Let G be a group and $g \in G$. Prove that if $h \in G$ satisfies $gh = 1_G$, then $h = g^{-1}$.

Problem 3. Let X be a set of size n and let $\mathrm{Bi}(X)$ denote the group of bijective functions $f \colon X \to X$ with the group operation given by composition $\circ$ and the identity element being Id_X. Prove that $|\mathrm{Bi}(X)| = n!$.

Problem 4. Let G,H be groups and define a binary operation $\star$ on the product set $G \times H$ by

$$(g_1,h_1) \star (g_2,h_2) = (g_1g_2, h_1h_2).$$

Prove that $G \times H$ is a group with group operation $\star$ and identity element $1_{G\times H} = (1_G, 1_H)$.

Problem 5. Let G be a group. For $g,h \in G$, the **commutator** of g and h is denoted by $[g,h]$ and is defined to be

$$[g,h] = g^{-1}h^{-1}gh.$$

Prove that $gh = hg$ if and only if $[g,h] = 1_G$.

Problem 6. Let $\mathrm{GL}(2,\mathbf{R})$ denote the group of two by two real matrices with non-zero determinant. This is a group under matrix multiplication and with identity element

$$\mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $A,B,C \in \mathrm{GL}(2,\mathbf{R})$ and assume that

$$A,B,C \neq \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

for all $\alpha \in \mathbf{R}$. Prove that if $A,B,C \in \mathrm{GL}(2,\mathbf{R})$ satisfy $[A,B] = [B,C] = \mathrm{I}_2$, then $[A,C] = \mathrm{I}_2$

Problem 7. Let X be a finite set with $|X| \geq 5$. Construct elements $\sigma, \eta, \theta \in \mathrm{Bi}(X)$ such that

$$[\sigma,\eta] = [\eta,\theta] = \mathrm{Id}_X, \quad [\sigma,\theta] \neq \mathrm{Id}_X.$$