15 Finite and Discrete Subgroups

15.1 Review

Last time, we began studying certain subgroups of M_2. The group of *isometries* of $\mathbb{R}^2$ is precisely

$$M_2 = \{t_{\vec{b}} \circ A : \vec{b} \in \mathbb{R}^2, A \in O_2\},$$

where O_2 is the group of orthogonal matrices.

Guiding Question

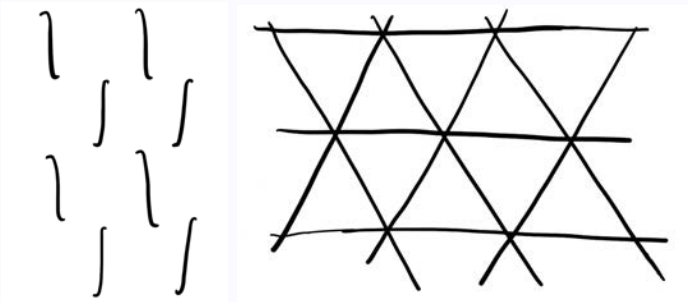
What are the finite subgroups of O_2?[a]

[a] The discrete subgroups of O_2 turn out to be the same as the finite subgroups, either C_n or D_n (we omit the proof, as it is in the homework.)

One way in which subgroups of M_2 naturally arise is with symmetries of plane figures.

Example 15.1

For the following two plane figures, they both have discrete symmetries including translations, rotations, and glide reflections.



Last time, we looked at finite subgroups of the orthogonal matrices $G \subseteq O_2$. We found the following theorem which greatly restricts the possibilities for such subgroups:

Theorem 15.2

Any *finite* subgroup $G \subseteq O_2$ is either

- $G \cong C_n = \langle \rho_{2\pi/n} \rangle$, the cyclic group generated by a *rotation* by $2\pi/n$; or
- $G \cong D_n = \langle \rho_{2\pi/n}, r \rangle$ which is the group C_n with an extra reflection r.

The elements of the form $\rho_{2\pi/n}$, which are rotations by $2\pi/n$, are orientation-preserving, while elements of the form $\rho_{2\pi/n}r$, which are reflections over certain lines through the origin, are orientation-reversing.

15.2 Finite Subgroups of M_2

Now that we have found the finite and discrete subgroups of O_2, we bring our attention to finite subgroups $G \subseteq M_2$.

Guiding Question

What are the finite subgroups of M_2? Do we get more subgroups now that we have more elements?

In fact, there are *no* new finite subgroups obtained from allowing G to be in M_2 instead of O_2.

Theorem 15.3

Any finite subgroup $G \subseteq M_2$ is also isomorphic to C_n or D_n.

Proof. In order to show that G is isomorphic to C_n or D_n, it is enough to find $s_0 \in \mathbb{R}^2$ such that $g(s_0) = s_0$ for all $g \in G$. Then, by changing coordinates such that s_0 is the new origin[53], G fixes the origin (formerly s_0) and so $G \subseteq O_2$. As a result, by applying Theorem 15.2, G must in fact be isomorphic to C_n or D_n.

- **Step 1.** First, we find some finite set S fixed by every element g: we require that $gS = S$ for all $g \in G$. For any $p \in \mathbb{R}^2$, let
$$S = \{g(p) \in \mathbb{R}^2 : g \in G\}^{54}.$$
Then, for any element $s \in S$, it is equal to $s = g'(p)$ for some $g' \in G$, by the definition of S. In addition, for any $g \in G$, the action of g on s is
$$g(s) = g(g'(p')) = (gg')(p) \in S,$$
again by how S is defined. So
$$gS = S.$$

- **Step 2.** Intuitively, to find s_0, we would take the average, or the center of mass, of all the points. For example, for the set of rotations $\langle 2\pi/3 \rangle$, S would be 3 equidistant points, and the center of the equilateral triangle would be fixed by such rotations. From this intuition, we can apply the following averaging trick. This is where G being finite is required, as we wouldn't be able to take the average otherwise.

  Where $S = \{s_1, \cdots, s_n\}$, let
$$s_0 = \frac{1}{n}(s_1 + \cdots + s_n)$$
be the average of all the elements in S. For any isometry $f = t_b \circ A$,
$$\begin{aligned} f(s_0) &= t_b\left(\frac{1}{n}(As_1 + \cdots + As_n)\right) \\ &= \frac{1}{n}((As_1 + b) + \cdots + (As_n + b)) \\ &= \frac{1}{n}(f(s_1) + \cdots + f(s_n)), \end{aligned}$$
since A is a linear operator.

  As a result, for any $g \in G$,
$$\begin{aligned} g(s_0) &= \frac{1}{n}(g(s_1) + \cdots + g(s_n)) \\ &= \frac{1}{n}(s_1 + \cdots + s_n) \\ &= s_0, \end{aligned}$$
since g permutes the elements in S.

  So we see that G does fix s_0, and by changing coordinates so that s_0 is the origin, G must in fact be isomorphic to C_n or D_n.

□

15.3 Discrete Subgroups of M_2

No new finite subgroups are obtained by taking elements in M_2 instead of O_2; what if we take discrete subgroups[55] instead of finite subgroups?

> **Guiding Question**
> What about discrete subgroups of M_2?

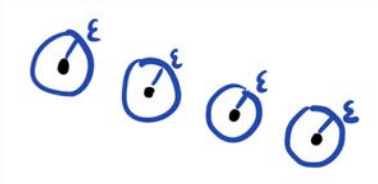
The definition of discreteness in M_2 combines the two definitions for the rotations and translations.

[53] We take $t_{-s_0} G t_{s_0}$

[54] This is called the *orbit* of p, since it is all the points that p can reach by some transformation in G, or all the points that p orbits to.

[55] We will formalize the notion of discreteness in M_2 now!

Definition 15.4
A group $G \subseteq M_2$ is discrete if there exists some $\varepsilon > 0$ such that any translation in G has distance $\geq \varepsilon$ and any rotation in G has angle $\geq \varepsilon$.[a]



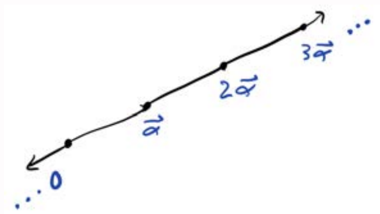
[a] In fact, for discreteness, it would make more sense to require two different ε_1 and ε_2 for translations and rotations, just to ensure that there are not continuously many translations and rotations. In this case, we can simply acquire the ε for this definition by taking the minimum of the two; then any translation in G has distance $\geq \varepsilon_1 \geq \varepsilon$ and any rotation has angle $\geq \varepsilon_2 \geq \varepsilon$.

15.3.1 Discrete Subgroups of $\mathbb{R}^2$

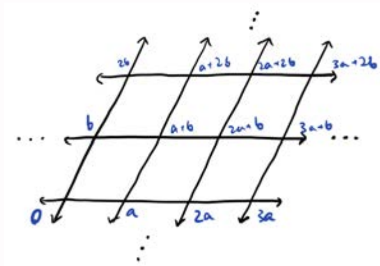
As a warmup, let's consider the copy of the plane inside M_2, $(\mathbb{R}^2, +) \subseteq M_2$, consisting of the translations t_b. What are the discrete subgroups of $(\mathbb{R}^2, +)$? The result and argument is similar to the discrete subgroups of $(\mathbb{R}, +)$ that we covered last week.

Theorem 15.5
If $G \subseteq \mathbb{R}^2$ is discrete, then

1. G $= \{0\}$; or
2. there exists some $\vec{\alpha} \in \mathbb{R}^2$ such that $G = \mathbb{Z}\vec{\alpha}$; or



3. there exist linearly independent vectors $\vec{a}, \vec{b} \in \mathbb{R}^2$ such that $G = \mathbb{Z}\vec{a} + \mathbb{Z}\vec{b}$. This is called a *lattice* inside $\mathbb{R}^2$.



Proof. First pick any $\hat{\alpha} \neq 0 \in G$. The intersection $G \cap \mathbb{R}\hat{\alpha}$ must be discrete, so there is some smallest length vector in $G \cap \mathbb{R}\hat{\alpha}$; call it α. Then if $G \cap \mathbb{R}\hat{\alpha} = G$, then $G \cap \mathbb{R}\hat{\alpha} = \mathbb{Z}\alpha$, and we are done.

Otherwise, pick $\beta \in G$ such that $\beta \notin \mathbb{R}\alpha$, minimizing the distance from β to $\mathbb{R}\alpha$. There exists such a β because in any bounded region of $\mathbb{R}^2$, there can only be finitely many points of G; then we can simply pick the point in G closest to $\mathbb{R}\alpha$.

Claim: $G = \mathbb{Z}\alpha + \mathbb{Z}\beta$. If this were not true, then there would exist a point $\gamma \in G$ that is not on the lattice formed by α and β. Thus, by shifting by α and β, the parallelogram with sides α and β would contain a point closer to $\mathbb{R}\alpha$, violating the minimality of β. ☐

15.3.2 Back to Discrete Subgroups of M_2!

Now that we have considered the translations in M_2, which are isomorphic to the plane $\mathbb{R}^2$, we can move on to the entire M_2.

Guiding Question
How can we study discrete groups $G \subseteq M_2$?

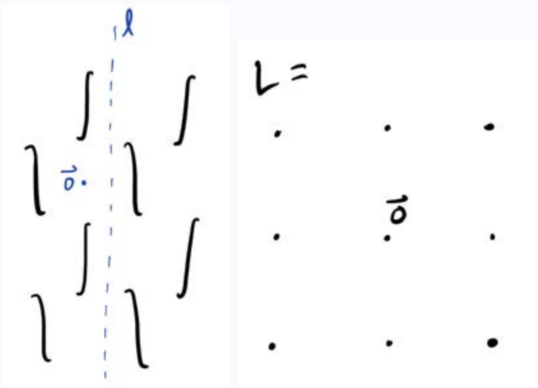
Recall that there exists a projection π from M_2 to O_2, where $\mathbb{R}^2$, the group of translations, is the kernel. The projection takes

$$\begin{aligned} \ker(\pi) = \mathbb{R}^2 \hookrightarrow {}^{56} M_2 &\xrightarrow{\pi} O_2 \\ t_{\vec{b}} \circ A &\mapsto A. \end{aligned}$$

The restriction of π to G takes $\pi|_G : G \longrightarrow O_2$. The kernel $L = \ker(\pi|_G)$ consists of the translations in G. Under this map, the image of G is a subgroup $\overline{G} \coloneqq \pi(G) \subseteq O_2$, known as the **point group** of G. The projection takes

$$\ker(\pi|_G) = L \subseteq G \xrightarrow{\pi|_G} \overline{G}.$$

Example 15.6
For this infinite plane figure, the group of translations L in the symmetry group G is a rectangular lattice. The point group $\overline{G}$ contains rotation by π around $\vec{0}$ and reflection across ℓ; as a result, $\overline{G}$ is isomorphic to D_2.



As we can see in the example, by using the projection π, each G can be decomposed into a discrete point group $\overline{G}$ isomorphic to C_n or D_n, and a discrete group $L \subseteq \mathbb{R}^2$, classified in Theorem 15.5. In fact, we can constrain the possibilities even more! The following proposition is a start.

Proposition 15.7
Every $A \in \overline{G}$ maps L to L.

Proof. Next time! $\square$