# 31 One-Parameter Subgroups

## 31.1 Review

Last time, we talked about one-parameter subgroups.

**Definition 31.1**
A **one-parameter group** in $GL_n(\mathbb{C})$ is a differentiable homomorphism $\varphi : \mathbb{R} \longrightarrow GL_n(\mathbb{C})$.

For a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, the matrix exponential is

$$e^A := 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots,$$

which converges to a matrix in $GL_n(\mathbb{C})$.[97] For example, $\varphi_A(t) = e^{tA}$ is a one-parameter group.[98]

**Example 31.2**
If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $n \geq 1$. Then

$$e^A = \sum_{n\geq 0} \frac{1}{n!}A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n\geq 1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}.$$

**Example 31.3**
Similarly, for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A^3 = \cdots$. Then

$$e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

## 31.2 Properties of the Matrix Exponential

The matrix exponential fulfills several nice properties.

- The product is the exponential of the sum: $e^{sA}e^{tA} = e^{(s+t)A}$. In fact, if $AB = BA$, then $e^Ae^B = e^{A+B}$, but they must commute.[99]
- If $A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$, then $e^A = \begin{pmatrix} e_1^\lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n^\lambda \end{pmatrix}$.
- If $B = PAP^{-1}$, then $e^B = Pe^AP^{-1}$. This allows us to easily take the matrix exponential of any diagonalizable matrix.

  **Example 31.4**
  If $A = \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}$, it has eigenvalues $2\pi i$ and $-2\pi i$, so diagonalizing gives $PAP^{-1} = \begin{pmatrix} 2\pi i & 0 \\ 0 & 2\pi i \end{pmatrix}$. Then $Pe^AP^{-1} = e^{PAP^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since $e^{2\pi i} = 1$. Since $e^A$ is conjugate to the identity matrix, $e^A$ itself must be the identity matrix.

  In particular, $e^{\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}} = e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$, and so the matrix exponential is not injective, unlike the normal exponential.

[97] With the metric $||M|| = \max_{i,j}|m_{ij}|$, every entry converges.
[98] It is called a one-parameter "subgroup," but it does not have to be injective; it can wrap around.
[99] The key fact here is that $\frac{1}{n!}(A+B)^n = \sum_{k+\ell=n} \frac{A^k}{k!}\frac{B^\ell}{\ell!}$ when $AB = BA$; matrix multiplication is not commutative so it is not always true.

- Defining the derivative of a matrix to be $\frac{d}{dt}\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} = \begin{pmatrix} a'(t) & b'(t) \\ c'(t) & d'(t) \end{pmatrix}$, the derivative is

$$\begin{aligned} \frac{d}{dt}(e^{tA}) &= \frac{d}{dt}\left(I + tA + \frac{t^2}{2}A^2 + \cdots\right) \\ &= {}^{100} 0 + A + tA^2 + \frac{t^2}{2}A^3 + \cdots \\ &= Ae^{tA}, \end{aligned}$$

  similarly to the normal exponential.

## 31.3 One-Parameter Subgroups

The matrix exponential is related to one-parameter subgroups in the following manner.

**Proposition 31.5**
Every one-parameter group in $GL_n(\mathbb{C})$ is of the form $\varphi(t) = e^{tA}$ for a unique matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

*Proof.* We prove uniqueness and existence.

- **Uniqueness.** If $\varphi(t) = e^{tA}$, then $\varphi'(t) = Ae^{tA}$, so $\varphi'(0) = A$. So the coefficient $A$ in the one-parameter subgroup is given by taking the derivative and evaluating at 0.[101]
- **Existence.** Given $\varphi(t)$, set $A := \varphi'(0) \in \mathrm{Mat}_{n\times n}$. Since $\varphi$ is a homomorphism, $\varphi(s+t) = \varphi(s)\varphi(t)$ for all $s$ and $t$. Taking the derivative $\frac{\partial}{\partial s}$,
$$\varphi'(s+t) = \varphi'(s)\varphi(t).$$
  Plugging in $s = 0$, we get
$$\varphi'(t) = A\varphi(t),$$
  and we also have $\varphi(0) = I_n$. Since this is a linear first-order ordinary differential equation with an initial condition, there is a unique solution, which is $\varphi(t) = e^{tA}$.

$\square$

**Definition 31.6**
For $G \leq GL_n(\mathbb{C})$, a **one-parameter group in** $G$ is a one-parameter group $\varphi(t)$ in $GL_n(\mathbb{C})$ such that $\varphi(t) \in G$ for all $t \in \mathbb{R}$.

For a one-parameter group in $G$, $\varphi(t) = e^{tA}$ for some $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ as well.

**Guiding Question**
Given a group $G$, what are the one-parameter groups in $G$? What is the corresponding set of matrices $A$ for which $e^{tA} \in G$ for all $t$?

Let's see an example.

**Example 31.7** (Diagonal Matrices)
Let
$$G = \left\{\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}\right\} \leq GL_n(\mathbb{C})$$
where $\lambda_i \neq 0$. The one-parameter groups in $G$ are determined by the matrices $A$ such that $e^{tA} \in G$ for all $t \in \mathbb{R}$. Here, $e^{tA} \in G$ for all $t \in \mathbb{R}$ if and only if $A$ is diagonal.

[101] Thinking of $\varphi$ as a trajectory, $A$ is essentially the velocity of the particle when it is passing through the identity.

*Proof.* If

$$\varphi(t) = e^{tA} = \begin{pmatrix} \lambda_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n(t) \end{pmatrix},$$

then $\varphi'(t) = \begin{pmatrix} \lambda_1'(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n'(t) \end{pmatrix}$. Then

$$A = \varphi'(0) = \begin{pmatrix} \lambda_1'(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n'(0) \end{pmatrix}$$

must be diagonal.

If $A = \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix}$ is diagonal, then $tA$ is diagonal, and so $e^{tA} = \begin{pmatrix} e^{ta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{ta_n} \end{pmatrix} \in G$. So every diagonal matrix $A$ does correspond to a one-parameter subgroup in $G$. □

We can also do the same with upper triangular invertible matrices.

**Example 31.8** (Upper Triangular Matrices)
Let $G = \left\{ \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn} \end{pmatrix} \right\} \leq GL_n(\mathbb{C})$, where $c_{ii} \neq 0$ for all $i$. Then $e^{tA} \in G$ for all $t \in \mathbb{R}$ if and only if $A = \begin{pmatrix} a_{11} & \cdots & \star \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}$.

*Proof.* If $\varphi(t)$ is upper triangular, then $A = \varphi'(0) = \begin{pmatrix} c_{11}'(0) & \cdots & c_{1n}'(0) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{nn}'(0) \end{pmatrix}$ must also be upper triangular.

Also, if $A$ is upper triangular, so is $A^n$ for all $n$, and thus so is $e^{tA}$. So the image of $\varphi$ is in $G$. □

**Problem 31.9**
For

$$G = \begin{pmatrix} 1 & \cdots & \star \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \leq GL_n(\mathbb{C}),$$

what are the corresponding matrices $A$?[a]

[a] The answer is that $A$ is of the form $\begin{pmatrix} 0 & \cdots & \star \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$.

We can also look at the one-parameter groups for unitary matrices.

**Example 31.10** (Unitary Matrices)
For $U_n = \{M^* = M^{-1}\} \leq GL_n(\mathbb{C})$, $e^{tA} \in U_n$ if and only if $A^* = -A$ is skew-Hermitian for some matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

*Proof.* We have
$$(e^A)^* = \left(I + A + \frac{A^2}{2!} + \cdots\right)^* = I^* + A^* + \frac{(A^*)^2}{2!} + \cdots = e^{(A^*)}.$$
If $e^{tA}$ is unitary, then $(e^{tA})^* = (e^{tA})^{-1}$, so $e^{tA^*} = e^{-tA}$. Differentiating gives $A^*e^{tA^*} = -Ae^{-tA}$, and taking $t = 0$ gives $A^* = -A$.

Conversely, if $A^* = -A$, then $(e^{tA})^* = e^{tA^*} = e^{-tA} = (e^{tA})^{-1}$, and so $e^{tA} \in U_n$ for all $t$. $\square$