Problem Set 4: Math 453 Spring 2019
Due Wednesday February 13

February 7, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Wednesday February 13 in class.

Problem 1. Let $\mathbf{R}^+$ be the group of positive real numbers under multiplication and let $\mathbf{R}$ be the group of real numbers under addition. We have

$$\exp\colon \mathbf{R} \to \mathbf{R}^+, \quad \ln\colon \mathbf{R}^+ \to \mathbf{R}.$$

(a) Prove that $\exp, \ln$ are group homomorphisms.

(b) Prove that $\exp, \ln$ are group isomorphisms.

Problem 2. Let $\mathbf{R}^\times$ be the group of non-zero real numbers under multiplication and let $\det\colon \mathrm{GL}(n,\mathbf{R}) \to \mathbf{R}^\times$ be the determinant map.

(a) Prove that det is a homomorphism and determine $\ker(\det)$.

(b) Prove that $\mathrm{SL}(n,\mathbf{R})$ is a normal subgroup of $\mathrm{GL}(n,\mathbf{R})$.

Problem 3. Let G be a group and let $\mathbf{Z}$ be the group of integers under addition.

(a) Let $g \in G$ and define $\psi_g\colon \mathbf{Z} \to G$ by $\psi_g(n) = g^n$. Prove that ψ_g is a homomorphism.

(b) Let $\psi, \psi'\colon \mathbf{Z} \to G$ be group homomorphisms. Prove that $\psi = \psi'$ if and only if $\psi(1) = \psi'(1)$.

(c) Let $\psi\colon \mathbf{Z} \to G$ be a group homomorphism. Prove that $\psi = \psi_g$ for some $g \in G$.

Problem 4. Let $n \in \mathbf{N}$. Given $a, b \in \mathbf{Z}$, we will write $a \equiv b \mod n$ when n divides $a - b$.

(a) Prove that if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $a + c \equiv b + d \mod n$.

(b) Prove that if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $ac \equiv bd \mod n$.

(c) Prove that if n is prime, then $(a+b)^n \equiv a^n + b^n \mod n$.

(d) Prove that if n is a prime and $a \not\equiv 0 \mod n$, then $a^{n-1} \equiv 1 \mod n$.

Problem 5. Let $n \in \mathbf{N}$ and let $\mathbf{Z}/n\mathbf{Z} = \left\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\right\}$ where

$$\overline{i} = \{k \in \mathbf{Z} \,:\, i \equiv k \mod n\}.$$

The set $\mathbf{Z}/n\mathbf{Z}$ is a group with group operation defined by

$$\overline{i} + \overline{j} = \overline{i+j}.$$

(a) Prove that for each $d \in \mathbf{N}$ with $d \mid n$, there exists a subgroup $H \leq \mathbf{Z}/n\mathbf{Z}$ with $|H| = d$.

(b) Prove that $\mathbf{Z}/n\mathbf{Z}$ is a simple group if and only if n is a prime.

(c) Let $m, n \in \mathbf{N}$ with $\gcd(m,n) = 1$. Prove that $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ and $\mathbf{Z}/mn\mathbf{Z}$ are isomorphic groups.

Problem 6. Let $n, m \in \mathbf{N}$ and define

$$A = n\mathbf{Z} = \{kn \,:\, k \in \mathbf{Z}\}, \quad B = m\mathbf{Z} = \{km \,:\, k \in \mathbf{Z}\}.$$

and

$$AB = \{ab \,:\, a \in A,\ b \in B\}.$$

(a) Prove that $AB = \gcd(m,n)\mathbf{Z}$.

(b) Prove that $A \cap B = \operatorname{lcm}(m,n)\mathbf{Z}$.