Problem Set 2: Math 453 Spring 2019
Due Wednesday January 30

January 23, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Wednesday January 30 in class.

In what follows below, **N** is the set of natural numbers, **Z** is the set of integers, **R** is the set of real numbers, and **C** is the set of complex numbers. Throughout, if G is a group, we will denote the identity element by $1 = 1_G$. The group operation on G will be given in multiplicative notation. That is, if $g, h \in G$, the product of g and h in the group G will be denoted by gh. For each $g \in G$, we will denote the inverse of g by g^{-1}.

The following problem deals with what we will later call the **left action** of a group G on itself.

Problem 1 (Left twix)**.** Let G be a group. For each $g \in G$, we define $L_g\colon G \to G$ by $L_g(h) = gh$.

(a) Prove that L_g is a bijective function.

(b) Define $\Psi_G\colon G \to \mathrm{Bi}(G)$ by $\Psi_G(g) = L_g$. Prove that Ψ_G is a group homomorphism. The group operation on $\mathrm{Bi}(X)$ is, as always, composition of functions.

(c) Prove that Ψ_G is injective.

(d) Decide under what conditions on G imply Ψ_G is surjective. [This is challenging]

The following problem deals with what we will later call the **right action** of a group G on itself.

Problem 2 (Right twix)**.** Let G be a group. For each $g \in G$, we define $R_g\colon G \to G$ by $R_g(h) = hg$.

(a) Prove that R_g is a bijective function.

(b) Define $\Psi_G\colon G \to \mathrm{Bi}(G)$ by $\Psi_G(g) = R_g$. Prove that Ψ_G is a group homomorphism. The group operation on $\mathrm{Bi}(X)$ is, as always, composition of functions.

(c) Prove that Ψ_G is injective.

(d) Decide under what conditions on G imply Ψ_G is surjective. [How does this relate to (d) from Problem 1?]

The following problem deals with what we will later call the **conjugate or adjoint action** of a group G on itself.

Problem 3. Let G be a group and $g \in G$. For each $g \in G$, we define $\mathrm{Ad}_g\colon G \to G$ defined by $\mathrm{Ad}_g(h) = ghg^{-1}$ or $L_g \circ R_{g^{-1}}$.

(a) Prove that Ad_g is a bijective function.

(b) Prove that Ad_g is a group homomorphism.

(c) Define $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$ by $\mathrm{Ad}(g) = \mathrm{Ad}_g$. Prove that Ad is a group homomorphism. The group $\mathrm{Aut}(G)$ is the set of all bijective group homomorphism $\varphi\colon G \to G$ with group operation given by composition of functions. The group $\mathrm{Aut}(G)$ is called the **automorphism group of** G.

(d) Prove that $g \in \ker(\mathrm{Ad})$ if and only if $g \in Z(G)$, the center of G.

Problem 4. Let $A \in \mathrm{M}(m,\mathbf{R})$. Recall that

$$\mathrm{O}(m) = \{A \in \mathrm{M}(m,\mathbf{R}) \ : \ Ax \cdot Ay = x \cdot y \text{ for all } x,y \in \mathbf{R}^m\}.$$

Prove the following are equivalent:

(a) $A \in \mathrm{O}(m)$.

(b) $AA^T = \mathrm{I}_m$. The matrix A^T is the transpose of A.

(c) The column vectors $a_1, a_2, \dots, a_m \in \mathbf{R}^m$ of A are an orthonormal set (i.e. $a_i \cdot a_j = \delta_{i,j}$).

(d) The row vectors $a^1, a^2, \dots, a^m \in \mathbf{R}^m$ of A are an orthonormal set (i.e. $a^i \cdot a^j = \delta_{i,j}$).

Problem 5. Let $A, B \in \mathrm{GL}(m,\mathbf{R})$.

(a) Prove that $Ax \cdot y = x \cdot A^T y$ for all $x,y \in \mathbf{R}^m$.

(b) Prove that $\det(A) = \det(A^T)$.

(c) Prove that $(AB)^T = B^T A^T$.

(d) Prove that $(A^{-1})^T = (A^T)^{-1}$.

(e) Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Problem 6. Given $f \in \mathrm{Bi}(X)$, we define the **support of** f to be the subset of X defined by

$$\mathrm{supp}(f) = \{x \in X \ : \ f(x) \neq x\}.$$

(a) Prove that if $f,g \in \mathrm{Bi}(X)$ and $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, then $[f,g] = 1$.

(b) Prove that there exist $f,g \in \mathrm{Bi}(X)$ with $\mathrm{supp}(f) = \mathrm{supp}(g) \neq \emptyset$ and with $[f,g] = 1$ provided $|X| > 2$.

(c) Prove that if $f,g \in \mathrm{Bi}(X)$, then
$$\mathrm{supp}(fg) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g).$$

(d) Prove that if $f \in \mathrm{Bi}(X)$, then
$$\mathrm{supp}(f) = \mathrm{supp}(f^{-1}).$$