SOLUTION KEY
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Problems:
Chap 8: 12,14,22,58,70
Chap 9: 5,6,8

Problem 8.12. Give examples of four groups of order 12, no two of which are isomorphic. Give reasons why no two are isomorphic

Solution:
Four groups of order 12 are: Z_{12}, D_6, $Z_4 \oplus Z_3$, and $Z_2 \oplus Z_2 \oplus Z_3$.

These groups are differentiated from each other as follows:

1. Z_{12} is cyclic.
2. D_6 is non-Abelian.
3. $Z_4 \oplus Z_3$ has 2 elements of order 4: $(1, 0)$ and $(3, 0)$.
4. $Z_2 \oplus Z_2 \oplus Z_3$ is not cyclic, is Abelian, and has no elements of order 4.

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Problem 8.14. The dihedral group D_n of order $2n$ $(n \geq 3)$ has a subgroup of n rotations and a subgroup of order 2. Explain why D_n cannot be isomorphic to the external direct product of two such groups.

Solution:
The subgroup of n rotations is cyclic and hence is isomorphic to Z_n. The subgroup of order 2 is isomorphic to Z_2. Hence if D_n were isomorphic to the external direct product of these two groups, then it must be isomorphic to $Z_n \oplus Z_2$. However, $Z_n \oplus Z_2$ is Abelian and D_n is not. ■

Problem 8.22. Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $Z_{30} \oplus Z_{20}$.

Solution:
An element (a, b) of this group has order 15 when $\text{lcm}(|a|, |b|) = 15$. We run through all the possibilities in the following cases:

1. $|a| = 15, |b| = 1$: $a \in \{2, 4, 8, 14, 16, 22, 26, 28\}$ and $b = 0$
2. $|a| = 15, |b| = 5$: $a \in \{2, 4, 8, 14, 16, 22, 26, 28\}$ and $b \in \{4, 8, 12, 16\}$
3. $|a| = 3, \quad |b| = 5$: $a \in \{10, 20\}$ and $b \in \{4, 8, 12, 16\}$

There are 8 options in Case 1, $8 \times 4 = 32$ in Case 2, and $2 \times 4 = 8$ in Case 3 making for a total of $8 + 32 + 8 = 48$ elements of order 15.

While there are 48 total elements of order 15, the cyclic groups generated by these elements overlap. Any cyclic subgroup of order 15, H, has $\phi(15) = 8$ elements of order 15, i.e. generators. Hence each cyclic subgroup of order 15 contains 8 "repeats" that appear as a subgroup generated by the above 48 elements. Therefore there are $48/8 = 6$ total cyclic subgroups of order 15 in $Z_{30} \oplus Z_{20}$. ■

Problem 8.58. Prove that $Z_5 \oplus Z_5$ has exactly six subgroups of order 5.

Solution:
Every element of $Z_5 \oplus Z_5$ (except the identity) has order 5 since 5 is prime. Hence there are 24 elements of order 5. Any subgroup of order 5 must be cyclic. There are $\phi(5) = 4$ elements of order 5 in each (cyclic) subgroup of order 5. Hence there are 24/4=6 total subgroups of order 5. ■

Problem 8.70. Without doing any calculations in $U(27)$, decide how many subgroups $U(27)$ has.

Solution:
Since $U(27) \cong Z_{18} \cong Z_9 \oplus Z_2$ and Z_9 has 3 subgroups and Z_2 has 2, $U(27)$ has 6 subgroups. ■

Problem 9.5. Show that if G is the internal direct product of H_1, H_2,..., H_n and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$.

Solution:
We will use the Second Principle of Mathematical Induction with the base case of $n = 2$. If G is the internal direct product of H_1 and H_2 then by definition, $H_1 \cap H_2 = \{e\}$.

Now suppose that for any $k < n$, if K is the direct product of H_1, H_2,..., H_k, then $H_i \cap H_j = \{e\}$ when $i \neq j$. Define $K = H_1 \times H_2 \times \cdots \times H_{n-1}$. Then $H_i \cap H_j = \{e\}$ for $i \neq j$ when $1 \leq i, j \leq n+1$. Now $G = K \times H_n$. By definition $K \cap H_n = \{e\}$. Let $g \in H_i \cap H_n$ for some $1 \leq i \leq n-1$. Then $g \in K \cap H_n = \{e\}$. Hence $H_i \cap H_n = \{e\}$ for $i \neq n$.

By induction, the statement is true for all n. ■

Problem 9.6. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$?

Solution:
Let's rewrite the elements of H in the form $M = \begin{bmatrix} f & g \\ 0 & h \end{bmatrix}$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of $GL(2, \mathbb{R})$. We will show H is not a normal subgroup using Theorem 9.1, the Normal Subgroup Test.

$$\begin{aligned} AMA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f & g \\ 0 & h \end{bmatrix} \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \left(\frac{1}{ad - bc} \right) \begin{bmatrix} af & ag + bh \\ cf & cg + dh \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \left(\frac{1}{ad - bc} \right) \begin{bmatrix} adf - acg - bch & -abf + a^2g + abh \\ cdf - c^2g - cdh & -bcf + acg + adh \end{bmatrix}, \end{aligned}$$

which is only in H if $cdf - c^2g - cdh = 0$, which is not true in general. Hence H is not a normal subgroup. ■

Comment: This problem can also be done quickly by taking $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and showing that $AMA^{-1} = \begin{bmatrix} h & 0 \\ g & f \end{bmatrix} \notin H$ if $g \neq 0$.

Problem 9.8. Viewing $\langle 3\rangle$ and $\langle 12\rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3\rangle / \langle 12\rangle$ is isomorphic to Z_4. Similarly, prove that $\langle 8\rangle / \langle 48\rangle$ is isomorphic to Z_6. Generalize to arbitrary integers k and n.

Solution:
Elements of $\langle 3\rangle / \langle 12\rangle$ have the form $a + \langle 12\rangle$ for $a \in \langle 3\rangle$ so

$$\langle 3\rangle / \langle 12\rangle = \{\langle 12\rangle, 3 + \langle 12\rangle, 6 + \langle 12\rangle, 9 + \langle 12\rangle\}$$

since $12 + \langle 12\rangle = \langle 12\rangle$. This is a cyclic group of order 4 (generated by $3 + \langle 12\rangle$), hence it is isomorphic to Z_4 through the homomorphism that does $3 + \langle 12\rangle \mapsto 1$.

Similarly,

$$\langle 8\rangle / \langle 48\rangle = \{\langle 48\rangle, 8 + \langle 48\rangle, 16 + \langle 48\rangle, 24 + \langle 48\rangle\ 32 + \langle 48\rangle\}$$

since $48 + \langle 48\rangle = \langle 48\rangle$. This is a cyclic group of order 6 (generated by $8 + \langle 48\rangle$), hence it is isomorphic to Z_6 through the homomorphism that does $8 + \langle 48\rangle \mapsto 1$.

In general, for arbitrary integers k and n, we have the following possibilites:

1. If k does not divide n, then $\langle n\rangle$ is not a subgroup of $\langle k\rangle$.

2. If k divides n, then $\langle k\rangle / \langle n\rangle \cong Z_{n/k}$.

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Comment: Remember that $\mathbb{Z}$ is only a group with the operation of *addition*, so we use additive notation to mark the operation in $\mathbb{Z}$.