MATH 45000 - Final Exam

December 11, 2024

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NAME: Solution

PUID: ____________________

(1) No textbook or notes.

(2) No calculators or portable electronic devices.

(3) Show your work to all problems except Question 1.

(4) There are **eleven** questions.

(5) The total score of this exam is 120.

1. (30 points) True or False. Please circle your answer. Circle more than one answer will receive no credits.

   (a) The group $U(200)$ has order 80.

   True False

   (b) The subgroup $G = \langle (123)(456), (78) \rangle \leq S_8$ is isomorphic to $\mathbb{Z}_6$.

   True False

   (c) A group of order 99 must be abelian.

   True False

   (d) The set $\{0, 3, 6, 9, 12\}$ under addition and multiplication modulo 15 is a commutative ring with a multiplicative identity.

   True False

   (e) The ring $\mathbb{Z} \oplus \mathbb{Z}_2$ is an integral domain.

   True False

   (f) The principal ideal $\langle 1 + i \rangle$ is a maximal ideal in the ring $\mathbb{Z}[i]$.

   True False

2. (10 points) Find a subgroup of order 6 in $\mathbb{Z}_{24}$, and list all its generators.

$\langle 4 \rangle = \{0, 4, 8, 12, 16, 20\}$

generators: 4, 20

3. (10 points) Consider the element

$$\alpha = (24)(12345) \in S_5.$$

(a) Write α as a product of disjoint cycles.

(b) Write α^{100} as a product of 2-cycles (use least number of 2-cycles in your expression).

(a) $\alpha = (1\,4\,5)(2\,3)$

(b) $\alpha^{100} = (145)^{100} \cdot (23)^{100}$

(145) has order 3. $(145)^{100} = (145)$

(23) has order 2 $(23)^{100} = id.$

$\Rightarrow \alpha^{100} = (1\,4\,5) = (15)(14)$

4. (10 points) Suppose N is a normal subgroup of G and $|G : N| = n$. Prove that $g^n \in N$ for all $g \in G$.

Consider the factor group G/N

$|G/N| = |G : N| = n$

By Lagrange, $|gN|_{G/N}$ divides n.

$\Rightarrow g^n N = N$ in G/N

$\Rightarrow g^n \in N$.

5. (10 points) How many Sylow 2-subgroups of D_6 are there? Prove your result.

$|D_6| = 12 = 2^2 \times 3$

$n_2 | 3$ and $n_2 \equiv 1 \mod 2$

$\Rightarrow n_2 = 1$ or $n_2 = 3$.

if $n_2 = 1$ then at most 3 order 2 elements while D_6 has more than 3 elements of order 2

$\Rightarrow n_2 = 3$

6. (10 points) Classify all abelian groups of order 75. How many subgroups of order 15 does each of them have? Prove your result.

$75 = 3 \times 5^2$

$\mathbb{Z}_{25} \oplus \mathbb{Z}_3$ or $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$.

- $\mathbb{Z}_{25} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_{75}$ cyclic $\Rightarrow$ unique order 15 subgp.

- $\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3$: count order 15 elements.

(a, b, c) :

$\lvert a\rvert = 5$	$\lvert b\rvert = 1$	$\lvert c\rvert = 3$	8
$\lvert a\rvert = 1$	$\lvert b\rvert = 5$	$\lvert c\rvert = 3$	8
$\lvert a\rvert = 5$	$\lvert b\rvert = 5$	$\lvert c\rvert = 3$	32.

there are totally $32 + 8 + 8 = \underline{48}$ elements of order 15, since any subgp of order 15 is cyclic, it gives $\phi(15) = 8$ elements of order 15 (and none of them are equal).

$\Rightarrow$ there are $\frac{48}{8} = 6$ subgps of order 15.

7. (10 points)

(a) Find all *group isomorphisms* from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$.

(b) Find all *ring homomorphisms* from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$.

(a) $x \longrightarrow ax \mod 10$ isomorphism

where $a \in \{0, 1, \cdots, 9\}$ $\Longleftrightarrow a = \underbrace{1, 3, 7, 9}_{\text{these are generators of } \mathbb{Z}_{10}}$

(b) $x \longrightarrow bx \mod 10$

so that $b^2 \equiv b \mod 10$ $b \in \{0, 1, \cdots 9\}$

now $0^2 \equiv 0$ $1^2 \equiv 1$ $5^2 \equiv 5$ $6^2 \equiv 6$

are the solutions

so four ring homomorphisms

$x \to 0$

$x \to x$

$x \to 5x$

$x \to 6x$.

8. (5 points) Let R be a commutative ring with a multiplicative identity. Suppose $a \in R$ is a unit and $b \in R$ satisfies $b^3 = 0$. Prove that $a - b$ is a unit of R.

$$(a-b)(a^2+ab+b^2) = a^3-b^3 = a^3$$

$$\Rightarrow (a-b)\cdot\left[a^{-3}(a^2+ab+b^2)\right] = 1$$

$\Rightarrow$ $a-b$ is a unit

9. (10 points) Prove that $\mathbb{Z}_5[x]/\langle x^3 + 2x + 1\rangle$ is a field with 125 elements.

enough to show $x^3 + 2x + 1$ is irreducible over $\mathbb{Z}_5$

enough to show $x^3 + 2x + 1$ has no zero in $\mathbb{Z}_5$

case-by-case:

$$0^3 + 2\cdot 0 + 1 = 1$$
$$1^3 + 2\cdot 1 + 1 = 4$$
$$2^3 + 2\cdot 2 + 1 = 3$$
$$3^3 + 2\cdot 3 + 1 = 4$$
$$4^3 + 2\cdot 4 + 1 = 3$$

in $\mathbb{Z}_5$.

10. (10 points) Determine whether the following polynomials are irreducible over $\mathbb{Q}$. Prove your claim.

(a) $x^4 + 5x + 3$.

(b) $\frac{5}{6}x^5 + 3x^4 + 10x^3 + \frac{5}{2}$.

(a) mod 2 test.

enough to show $x^4 + x + 1$ is irreducible over $\mathbb{Z}_2$.

it has no zeros ($0^2 + 0 + 1 = 1^2 + 1 + 1 = 1$ in $\mathbb{Z}_2$).

The only irreducible polynomial of degree 2 is $x^2 + x + 1$: check $(x^2+x+1) \nmid x^4+x+1$.

~~but~~ $(x^4+x+1) = (x^2+x+1)(x^2+x) + 1$

$\Rightarrow$ $x^4 + x + 1$ is irreducible over $\mathbb{Z}_2$.

(b) enough to show

$5x^5 + 18x^4 + 60x^3 + 15$ is irreducible over $\mathbb{Q}$

Eisenstein test with $p = 3$

$p \nmid 5$ $\quad p \mid 18$ $\quad p \mid 60$ $\quad p \mid 15$ $\quad p^2 \nmid 15$.

11. (5 points) Prove that there is no integral domain with exactly 6 elements.

Suppose there is such an integral domain.

as abelian gps it is $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

(by fundamental thm).

now the element $(1,1)$ has order 6

it follows that characteristic is 6.

but for an integral domain, the characteristic

is either 0 or a prime. contradiction.