MATH 45000 - Exam I September 20, 2024

Instructor: Linquan Ma

NAME: Solution

PUID: ______________________

(1) No textbook or notes.

(2) No calculators or portable electronic devices.

(3) You must show your work to all problems.

(4) There are **six** questions.

(5) The total score of this exam is 100.

1. (20 points)

(a) Prove that for any integer n, $n^3 \equiv n \mod 6$.

(b) Prove that for any integer n, $2n+3$ and $3n+5$ are relatively prime.

(a) We work mod 6, it is enough to show

$n^3 \equiv n \mod 6$ when $n = 0, 1, 2, 3, 4, 5$

$0^3 = 0 \quad 1^3 = 1 \quad 2^3 = 8 \equiv 2 \mod 6 \quad 3^3 = 27 \equiv 3 \mod 6$

$4^3 = 64 \equiv 4 \mod 6 \quad 5^3 = 125 \equiv 5 \mod 6$

[alternatively, $n^3 - n = (n-1)n(n+1)$ is a product of 3 consecutive integers, at least one is even and at least one is a multiple of 3. so $n^3 - n$ is divisible by 2 and 3]

so $n^3 - n$ is divisible by 6

(b) $2(3n+5) - 3(2n+3) = 6n + 10 - (6n+9) = 1$

$\Rightarrow \gcd(2n+3, 3n+5) = 1$

2. (20 points)

(a) Find all subgroups of $\mathbb{Z}_{18}$.

(b) Show that $U(14)$ is a cyclic group, and find one generator.

(a) All divisors of 18 are: 1. 2. 3. 6. 9. 18

$\langle 0 \rangle = \{0\}$

$\langle 9 \rangle = \{0. 9\}$

$\langle 6 \rangle = \{0. 6, 12\}$

$\langle 3 \rangle = \{0. 3. 6. 9. 12, 15\}$

$\langle 2 \rangle = \{0. 2, \cdots 16\}$

$\langle 1 \rangle = \mathbb{Z}_{18}$

(b) $U(14) = \{1, 3. 5. 9. 11. 13\}$

3 is a generator: $3^2 = 9$ $3^3 = 27 \equiv 13 \mod 14$

$3^4 \equiv 13 \times 3 \equiv 39 \equiv 11 \mod 14$

$3^5 \equiv 11 \times 3 \equiv 33 \equiv 5 \mod 14$

$3^6 \equiv 5 \times 3 \equiv 15 \equiv 1 \mod 14$

(the other generator is 5)

3. (20 points)

(a) Prove that D_6 has no element of order 4.

(b) Find a *subgroup* of D_6 of order 4.

(a) all reflections have order 2

the rotations form a cyclic subgp of order 6

$\Rightarrow$ no rotation could have order 4 since $4 \nmid 6$

(you can also just compute the order of each rotation)

(b) $\{R_0, R_\pi, L_0, L_{\frac{\pi}{2}}\}$ is a subgp of order 4

To check this is a subgp.

- can write down explicit Caylay table
- use the fact that R_π, L_0 both have order 2 and $R_\pi L_0 = L_0 R_\pi = L_{\frac{\pi}{2}}$

(note: you can use any reflection L, not necessarily L_0.
then $\{R_0, R_\pi, L, R_\pi \circ L\}$ is a subgp.
by the second method)

4. (20 points) Define an operation on $\mathbb{Z}$ by

$$x * y = x + y + 4$$

for any $x, y \in \mathbb{Z}$. Prove that $(\mathbb{Z}, *)$ is a group. What is the identity element and what is the inverse of x?

Assoziativity:
$$\begin{aligned}(x*y)*z &= (x+y+4)*z\\ &= x+y+4+z+4\\ &= x+y+z+8\end{aligned}$$

$$\begin{aligned}x*(y*z) &= x*(y+z+4)\\ &= x+y+z+4+4\\ &= x+y+z+8\end{aligned}$$

Identity is -4: $x*(-4) = (-4)*x = x-4+4 = x$

Inverse of x is $-8-x$:

$$\begin{aligned}x*(-8-x) = (-8-x)*x &= x-8-x+4\\ &= -4 \leftarrow \text{identity}.\end{aligned}$$

5. (10 points) In S_7, write the element $(12)(423157)(64)$ as a product of disjoint cycles and compute its order.

$(12)(423157)(64) = (15746)(23)$

order $= \operatorname{lcm}(5,2) = 10$

6. (10 points) Let G be a group with identity element e. Suppose $a, b \in G$ such that $aba^{-1} = b^{-1}$.

(1) Prove that $ab^{-1}a^{-1} = b$.

(2) Suppose in addition $a^3 = e$. Prove that $b^2 = e$.

(1) $aba^{-1} = b^{-1}$

$(aba^{-1})^{-1} = (b^{-1})^{-1} = b$

$(aba^{-1})^{-1} = (a^{-1})^{-1} b^{-1} a^{-1} = ab^{-1}a^{-1}$

$\Rightarrow ab^{-1}a^{-1} = b$

(2) $aba^{-1} = b^{-1}$

$\rightsquigarrow a^2 b a^{-2} = ab^{-1}a^{-1} \overset{(1)}{=} b$

$\rightsquigarrow a^3 b a^{-3} = aba^{-1} = b^{-1}$

since $a^3 = e = a^{-3}$

$\rightsquigarrow ebe = b^{-1} \Rightarrow b = b^{-1} \Rightarrow b^2 = e$