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MA450 Ma
HW7 F24

7.22) Trivially $H \cap K \leq H$ and $H \cap K \leq K$, so by Lagrange's thm $|H \cap K|$ divides $|H|$ and $|K| \Rightarrow \gcd(12, 35) = 1$.

$|H \cap K| = \gcd(|H|, |K|)$ $\square$

7.26) Supp $|G| = \infty$ and $g \in G$ such that $|g| = \infty$.

Then $\langle g^2 \rangle < G$ is a contradiction.

Then $|g|$ must be finite $\forall g \in G$.

But then $\langle g \rangle < G$ is a contradiction $\forall e \neq g \in G$.

So $|G| \neq \infty$. Consider $g \in G$ such that $g \neq e$.

Then $\langle g \rangle = G$, lest it be nontrivial/proper.

Then $|\langle g \rangle| = |G| = n$, and by FTCG $\langle g \rangle$ has exactly one subgroup per divisor of n.

Since G has no proper subgroups, only divisors of n must be 1 and $n \Rightarrow n$ prime $\square$

7.40) Let $|G| = 63$ and $g \in G$ s.t. $g \neq e$.

Cor of LT $\Rightarrow |g| \in \{3, 7, 9, 21, 63\}$

$|g| = 3$ trivial $\qquad |g| = 9 \Rightarrow |g^3| = 3$

$|g| = 21 \Rightarrow |g^7| = 3 \qquad |g| = 63 \Rightarrow |g^{21}| = 3$

Assume $|g|=7\ \forall g \overset{\neq e}{\in} G$. There are 62 such g.
But Thm 4.4 cor $\Rightarrow$ # elt $\in G$ w/order $7 = \phi(7) = 6$.
Since $6 \nmid 62$, our assumption must be false □

7.44) Recall all reflections have order 2.
If a sgp of D_n contains a reflection, by closure half of the sgp must be reflections $\Rightarrow$ |sgp| is even.
Thus any sgp of D_n of odd order must only contain rotations $\Rightarrow$ it is cyclic. □

8.2) $h=(a,b,c)\ \forall h \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where $a,b,c \in \{0,1,2\}$.
There are $2^3 = 8$ elements incl. $e=(0,0,0)$ in $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, forming 7 subgroups.
Each of these has order 2 by Thm 8.1. □

8.8) No. If $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \cong \mathbb{Z}_{27}$, $\exists g \in \mathbb{Z}_3 \oplus \mathbb{Z}_9$ such that $|g|=27$.
Then $g=(a,b)$ where $\text{lcm}(a,b)=27$ by Thm 8.1.
But there is no such $(a,b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_9$, as $\text{lcm}(3,9)=9$. □

8.10) # elt $\in \mathbb{Z}_3 \oplus \mathbb{Z}_9$ of ord 9

case 1: $|a|=1$ $\quad |b|=9 \Rightarrow a=0 \quad b=1,2,4,5,7,8$

case 2: $|a|=3$ $\quad |b|=9 \Rightarrow a=1,2 \quad b=1,2,4,5,7,8$

$6+(2\cdot 6)=18$ elements $\square$

8.54) By property of isomorphisms, generator $\mapsto$ gentr.

Thus $\phi(1) \mapsto (1,1)$.

By preservation of operation, $\phi(n)=\phi(1^n)=\phi(1)^n=(1,1)^n$

$=(n \bmod 4, n \bmod 3)$ $\square$