SOLUTION KEY
Produced by: Kyle Dahlin

Problems:
Chap 9: 14, 30, **34**, 42, **48**, 54
Chap 10: 6, 8, 14, 18, 20, **24**

Problem 9.14. What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?

Solution:
$|14 + \langle 8 \rangle| = 4$ since $\langle 14 + \langle 8 \rangle \rangle = \{14 + \langle 8 \rangle, 4 + \langle 8 \rangle, 18 + \langle 8 \rangle, \langle 8 \rangle\}$, where we have used the fact that $28 \equiv 4 \bmod 24$ and $32 \equiv 8 \bmod 24$. ∎

Problem 9.30. Express $U(165)$ as an internal direct product of proper subgroups in four different ways.

Solution:
Since $165 = 3 \cdot 5 \cdot 11$, there is an isomorphism, ϕ, from $U(165)$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{10}$. Let $\psi = \phi^{-1}$. Then $G_1 = \psi(\mathbb{Z}_2)$, $G_2 = \psi(\mathbb{Z}_4)$, and $G_3 = \psi(\mathbb{Z}_{10})$ are disjoint normal proper subgroups of $U(165)$. Let $H_1 = G_1 \times G_2$, $H_2 = G_2 \times G_3$, and $H_3 = G_1 \times G_3$. Then $U(165)$ can be written in the following four different ways:

$$\begin{aligned} U(165) &= G_1 \times G_2 \times G_3 \\ &= H_1 \times G_3 \\ &= H_2 \times G_1 \\ &= H_3 \times G_2 \end{aligned}$$

∎

Problem 9.34*.** In $\mathbb{Z}$, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $\mathbb{Z} = HK$. Does $\mathbb{Z} = H \times K$?

Solution:
First note that $HK = \{5s + 7t | s, t \in \mathbb{Z}\}$. Since $\gcd(5, 7) = 1$, there exists $s, t \in \mathbb{Z}$ such that $5s + 7t = 1$. Hence $\mathbb{Z} = \langle 1 \rangle \subseteq HK$ and thus $HK = \mathbb{Z}$. However, $\mathbb{Z} \neq H \times K$ since $H \cap K \neq \emptyset$. For example, $35 \in H \cap K$. ∎

Problem 9.42. An element is called a *square* if it can be expressed in the form b^2 for some b. Suppose that G is an Abelian group and H is a subgroup of G. If every element of H is a square and every element of G/H is a square, prove that every element of G is a square. Does your proof remain valid when "square" is replaced by "nth power" where n is any integer?

Solution:
Let $g \in G$. Then $gH = b^2 H$ for some $b \in G$. Hence there exist h_1 and h_2 in H such that $gh_1 = b^2 h_2$. Further, there are c_1 and c_2 in G such that $gc_1^2 = b^2 c_2^2$. Thus, since G is Abelian, $g = (bc_2 c_1^{-1})^2$ is a square.

The proof remains valid for any value of n as long as G is Abelian by replacing the power of "2" with "n" in the proof above. ∎

Problem 9.48*.** If G is a group and $|G : Z(G)| = 4$, prove that $G/Z(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Solution:
By Example 3 (pg. 163) a group of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Suppose that $G/Z(G)$ is cyclic. Then by Theorem 9.3, G is Abelian and $G = Z(G)$, which is impossible. Since $G/Z(G)$ is not cylic, it must be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. ■

Problem 9.54. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1$, $-i = (-1)i$, $1^2 = (-1)^2 = 1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$.

a. Construct the Cayley table for G.

b. Show that $H = \{1, -1\} \triangleleft G$.

c. Construct the Cayley table for G/H. Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

Solution:

a. The Cayley table for G is:

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	i	$-i$	1	$-k$	$-j$
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

b. By observation of the Cayley table for G, we see that $H = Z(G)$. Hence $H \triangleleft G$.

c. The Cayley table for G/H is:

	$\langle -1\rangle$	$i\langle -1\rangle$	$j\langle -1\rangle$	$k\langle -1\rangle$
$\langle -1\rangle$	$\langle -1\rangle$	$i\langle -1\rangle$	$j\langle -1\rangle$	$k\langle -1\rangle$
$i\langle -1\rangle$	$i\langle -1\rangle$	$\langle -1\rangle$	$k\langle -1\rangle$	$j\langle -1\rangle$
$j\langle -1\rangle$	$j\langle -1\rangle$	$k\langle -1\rangle$	$\langle -1\rangle$	$i\langle -1\rangle$
$k\langle -1\rangle$	$k\langle -1\rangle$	$j\langle -1\rangle$	$i\langle -1\rangle$	$\langle -1\rangle$

Since $|G/H| = 8/2 = 4 = |G : Z(G)|$, $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by Problem 9.48.
■

Problem 10.6. Let G be the group of all polynomials with real coefficients under addition. For each f in G, let $\int f$ denote the antiderivative of f that passes through the point $(0, 0)$. Show that the mapping $f \mapsto \int f$ from G to G is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of f that passes through $(0, 1)$?

Solution:
Let $\phi : G \to G$ be the map defined in the problem statement. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ be arbitrary elements of G. Then $(f+g)(x) = \sum_{i=0}^{\max\{m,n\}} (\hat{a}_i + \hat{b}_i)x^i$ where $\hat{a}_i = a_i$ if $i \leq n$ and $\hat{a}_i = 0$ otherwise. Define $\hat{b}_i$ similarly.

Now $\phi(f)(x) = \sum_{i=0}^{n} \left(\frac{1}{i+1}\right) a_i x^{i+1}$ and $\phi(g)(x) = \sum_{i=0}^{m} \left(\frac{1}{i+1}\right) b_i x^{i+1}$ and

$$\begin{aligned}\phi(f+g)(x) &= \sum_{i=0}^{\max\{m,n\}} \left(\frac{1}{i+1}\right)(\hat{a}_i + \hat{b}_i)x^i \\ &= \phi(f)(x) + \phi(g)(x)\end{aligned}$$

Hence ϕ is a homomorphism. If $f \in \ker\phi$, then $\phi(f) = \sum_{i=0}^{n} \left(\frac{1}{i+1}\right) a_i x^{i+1} = 0$ implies $f = 0$. Hence $\ker\phi = \{0\}$.

The map $\psi : G \to G$ where $\phi(f)$ denotes the antiderivative of f that passes through $(0,1)$ is <u>not</u> a homomorphism since $\psi(f+g)(0) = 1 \neq 2 = \psi(f)(0) + \psi(g)(0)$ for any $f, g \in G$. ■

Problem 10.8. Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this homomorphism allow you to conclude that A_n is a normal subgroup of S_n of index 2? Why does this prove Exercise 23 of Chapter 5?

Solution:
First note the following facts about the parity of a permutation:

	even	odd
even	even	odd
odd	odd	even

which is exactly the multiplication table for -1 and 1. Hence it is clear that $\operatorname{sgn}(\sigma\gamma) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\gamma)$ and sgn is a homomorphism.

$\ker \operatorname{sgn} = \{\sigma \in G | \operatorname{sgn}(\sigma) = 1\} = \{\text{even permutations in } G\}$.

If $G = S_n$, then $\ker(\operatorname{sgn}) = A_n$ and $|S_n : A_n| = |\{1, -1\}| = 2$.

If $H \subseteq S_n$ and sgn is defined on H, then either $H \in \ker(\operatorname{sgn})$ or $\ker(\operatorname{sgn}) \lhd H$. In the first case, every element of H is even. In the second, since $H/\ker(\operatorname{sgn}) \cong \mathbb{Z}_2$, exactly half the elements of H are even. Thus this proves Exercise 5.23. ■

Problem 10.14. Explain why the correspondence $x \mapsto 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.

Solution:
Let ϕ be the map defined above. Then $\phi(4 \cdot 3) = \phi(0) = 0$ but $\phi(4) \cdot \phi(3) = 12 \cdot 9 = 8$. ■

Problem 10.18. Can there be a homomorphism from $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ onto $\mathbb{Z}_8$? Can there be a homomorphism from $\mathbb{Z}_{16}$ onto $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? Explain your answers.

Solution:
Suppose ϕ is an onto homormophism from $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ to $\mathbb{Z}_8$. Then there exists $(a, b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ such that $\phi((a, b)) = 1$. Every element of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has order 1, 2, or 4 and $|\phi((a, b))| = 8$ must divide $|(a, b)|$, which is impossible. Hence no onto homomorphism can exist.

Suppose ψ is an onto homomorphism from $\mathbb{Z}_{16}$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then since $\psi(2) = \psi(1+1) = \psi(1) + \psi(1) = 0$, then $\langle 2 \rangle \in \ker \psi$ and $|\ker \psi| \geq 8$. However,

$$4 = |\mathbb{Z}_2 \oplus \mathbb{Z}_2| = |\mathbb{Z}_{16} : \ker \psi| = \frac{16}{|\ker \psi|},$$

and $\ker \psi$ must have order 4, a contradiction. ■

Problem 10.20. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$? How many are there to $\mathbb{Z}_8$?

Solution:
Suppose ϕ is an onto homomorphism from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$. Then there is an $a \in \mathbb{Z}_{20}$ such that $\phi(a) = 1$ and hence $|\phi(a)| = 8$. However, by property 3 of Theorem 10.1, $|\phi(a)|$ divides $|a| \in \{1, 2, 4, 5, 10, 20\}$, which is impossible.

By property 2 of Theorem 10.1, any homomorphism from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$ is completely specified by where it sends 1. Define $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_8$ and $\phi(1) = a$. Lagrange's Theorem and property 3 of Theorem 10.1 require that $|a|$ divide both 20 and 8, so $|a| \in \{1, 2, 4\}$. Hence $a \in \{0, 2, 4, 6\}$. Each value of a specifies a distinct homomorphism. Hence there are four homormorphisms from $\mathbb{Z}_{20}$ into $\mathbb{Z}_8$. ■

Problem 10.24*.** Suppose that $\phi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine $\phi(x)$.

b. Determine the image of ϕ.

c. Determine the kernel of ϕ.

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

Solution:

a. Let $a = \phi(1)$. Then $\phi(7) = \phi(7 \cdot 1) = 7\phi(1) = 7a = 6$. Hence $a = 3$ and hence $\phi(x) = 3x$.

b. $\phi(\mathbb{Z}_{50}) = \langle 3 \rangle \subset \mathbb{Z}_{15}$.

c. $\ker \phi = \langle 5 \rangle \subset \mathbb{Z}_{50}$.

d. By property 6 of Theorem 10.1, $\phi^{-1}(3) = 1 + \ker \phi = 1 + \langle 5 \rangle \subset \mathbb{Z}_{50}$.

■