Problem Set 7: Math 453 Spring 2019
Due Friday April 5 (Optional)

March 19, 2019

Solve the problems below. Make your arguments as clear as you can; clear will matter on the exams. Make sure to write your name and which section you are enrolled in (that is, either 1030 or 1130 depending on when your class begins). I encourage you to collaborate with your peers on this problem set. Collaboration is an important part of learning, and I believe an important part for success in this class. I will ask that you write the names of your collaborators on the problem set. You can simply write their names near where you sign your name, though be clear that these are your collaborators. This problem set is due on Friday April 5 in class.

Problem 1. Let R be a commutative ring and let $S \subset R$ is closed under multiplication. Let $\mathfrak{a}$ to be an ideal in R such that $\mathfrak{a} \cap S = \emptyset$. We further assume that if $\mathfrak{b}$ is an ideal in R such that $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$, then $\mathfrak{b} \cap S \neq \emptyset$. Prove that $\mathfrak{a}$ is a prime ideal in R.

Problem 2. Let R be a commutative ring and take $S = \{1\}$. Prove that if $\mathfrak{a}$ is an ideal satisfying the condition in the Problem 1 for $S = \{1\}$. Prove that $\mathfrak{a}$ is a maximal ideal.

Problem 3. Let R be a commutative ring and let $\mathfrak{m}$ be a maximal ideal. Prove that $\mathfrak{m}$ satisfies the condition in Problem 1 for $S = \{1\}$.

Problem 4. Let R be a commutative ring. Prove that every maximal ideal in R is a prime ideal.

Problem 5. Let $\mathfrak{a}$ be an ideal in $\mathbf{Z}$. Define $m_{\mathfrak{a}}$ to be the smallest positive integer $m_{\mathfrak{a}} \in \mathfrak{a}$.

(a) Prove that if $\ell \in \mathfrak{a}$, then $m_{\mathfrak{a}}$ divides ℓ.

(b) Prove that $\mathfrak{a} = m_{\mathfrak{a}}\mathbf{Z}$ where
$$m_{\mathfrak{a}}\mathbf{Z} = \{jm_{\mathfrak{a}} \ : \ j \in \mathbf{Z}\}.$$

Problem 6. Let R be a finite, integral domain. Prove that R is a field.

Problem 7. Let $\overline{a} \in \mathbf{Z}/m\mathbf{Z}$ with $\overline{a} \neq \overline{0}$. Prove that $\overline{a}$ is invertible in $\mathbf{Z}/m\mathbf{Z}$ if and only if $\gcd(a,m) = 1$.

Problem 8. Prove that $\mathbf{Z}/m\mathbf{Z}$ is a field if and only if m is prime.

Problem 9. Let $p \in \mathbf{N}$ be a prime such that $p \equiv 1 \mod 4$. Prove that $p = a^2 + b^2$ for some $a, b \in \mathbf{Z}$.

Problem 10. Let R be a ring with the following property: For all $r, s, t \in R$ such that $rs = tr$, we have $s = t$. Prove that R is commutative.

Problem 11. Let R be a commutative ring with the property that every ideal $\mathfrak{a}$ of R with $\mathfrak{a} \neq R$ is a prime ideal. Prove that R is a field.

Problem 12. Let R be a commutative ring with the following property: for each $r \in R$, there exists $n_r \in \mathbf{N}$ such that $r^{n_r} = r$. Prove that if $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p}$ is a maximal ideal in R.

Problem 13. Let R be a commutative ring and let S be the subset of R comprised of all $r \in R$ that are not invertible (i.e. have no multiplicative inverse). Prove that following two statements are equivalent:

(a) R has a unique maximal ideal $\mathfrak{m}$.

(b) S is an ideal in R.

Problem 14. Let R, S be commutative rings and $\psi \colon R \to S$ a ring homomorphism.

(a) Let $\mathfrak{a} \lhd S$ an ideal. Prove that $\psi^{-1}(\mathfrak{a})$ is an ideal in R.

(b) Let $\mathfrak{p} \lhd S$ a prime ideal. Prove that $\psi^{-1}(\mathfrak{p})$ is a prime ideal in R.

Problem 15. Let R be a commutative ring.

(a) Prove that there exists a ring homomorphism $\psi \colon \mathbf{Z} \to R$.

(b) Prove that if $\psi_1, \psi_2 \colon \mathbf{Z} \to R$ are any pair of ring homomorphisms, then $\psi_1 = \psi_2$.

(c) Prove that $\ker \psi = m_R\mathbf{Z}$ for some $m_R \in R$. The positive integer m_R is called the **characteristic** of the ring. Note that if ψ is injective, the characteristic is defined to be 0.

(d) Prove if R is an integral domain, then m_R is either a prime or zero.

Problem 16. Let R be a commutative such that there exists a prime ideal $\mathfrak{p} \lhd R$ with no nonzero zero divisor. That is, if $a \in \mathfrak{p}$ and $a \neq 0$, then a is not a zero divisor. Prove that R is an integral domain.

Problem 17. Let R be a commutative ring. We say that an ideal $\mathfrak{a}$ of R is **irreducible** if whenever $\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2$ for ideals $\mathfrak{a}_1, \mathfrak{a}_2 \lhd R$, then $\mathfrak{a} = \mathfrak{a}_1$ or $\mathfrak{a} = \mathfrak{a}_2$. Prove that if $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p}$ is irreducible.

Problem 18. Let R be a commutative ring and let $r \in R$ be such that $r^n = 0$ for some positive integer n. Prove that $1 - ra$ is a unit in R for all $a \in R$.

Problem 19. Let R be a commutative ring with $r \in R$ such that $r^2 = r$ and $r \neq 0, 1$. Prove that if $\mathfrak{p}$ is a prime ideal in R, then there exists $s \in \mathfrak{p}$ such that $s^2 = 2$ and $s \neq 0, 1$.

Problem 20. Let R be a commutative ring and let $\mathfrak{a}$ be an ideal in R such that $1 + a$ is a unit for all $a \in \mathfrak{a}$. Prove that if $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{a} \subset \mathfrak{m}$.

Problem 21. Let R be a commutative ring and let $r \in R$ be such that $r^n = 0$ for some $n > 0$. Prove that if $\mathfrak{p}$ is a prime ideal in R, then $r \in \mathfrak{p}$.

Problem 22. Let R be a commutative ring and let $x \in R$ be such that $x^n \neq 0$ for all $n \in \mathbf{N}$. Prove that there exists a prime ideal $\mathfrak{p}$ in R with $x \notin \mathfrak{p}$.

Problem 23. Let R be a finite, commutative ring. Prove that if $\mathfrak{p}$ is a prime ideal in R. then $\mathfrak{p}$ is a maximal ideal.

Problem 24. Let R be a commutative ring with $x, y \in R$ such that x is a unit and $y^n = 0$ for some $n > 0$. Prove that $x + y$ is a unit.

Problem 25. Let R be a commutative ring with exactly three ideals $\{0\}, R$, and $\mathfrak{a}$.

(a) Prove that if $x \notin \mathfrak{a}$, then x is a unit.

(b) Prove that if $x \in \mathfrak{a}$, then there exists $r \in R$ with $rx = 0$ and $r \neq 0$.