# Applications of Sylow's Theorems

<u>Example 1</u> Any group of order 66 contains a subgp isomorphic to $\mathbb{Z}_{33}$ $\boxed{66 = 2 \times 3 \times 11}$

$H_p$ = a Sylow p-subgp $n_p$ = # of Sylow p-subgps

Then $n_{11} \mid 6$ and $n_{11} \equiv 1 \mod 11$ (by Sylow's thm)

$\Rightarrow n_{11} = 1 \Rightarrow H_{11}$ is normal subgp

now $H_3 H_{11} = H_{11} H_3$ is a subgp (since $H_{11}$ is normal)

$H_3 \cap H_{11} = \{e\} \Rightarrow |H_3 H_{11}| = \dfrac{|H_3||H_{11}|}{|H_3 \cap H_{11}|} = 3 \times 11 = 33$

$\Rightarrow H_3 H_{11}$ is a subgp of order 33

N.B. Any group of order 33 is isomorphic to $\mathbb{Z}_{33}$

( $pq$ such that $p \leq q$ and $p \nmid q-1$ )

In fact, we can completely classify all groups of order 66 (Example 7 on page 420)

There are exactly 4 such group (up to $\cong$)

- $\mathbb{Z}_{66}$ $\qquad \langle 2 \rangle \leq \mathbb{Z}_{66}$ subgp of order 33
- $D_{33}$ $\qquad \{\text{Rotations}\} \leq D_{33}$ $\quad$ - - - - - -
- $D_{11} \oplus \mathbb{Z}_3$ $\qquad \mathbb{Z}_{11} \oplus \mathbb{Z}_3 \leq D_{11} \oplus \mathbb{Z}_3$ $\quad$ - - -
- $\mathbb{Z}_{11} \oplus D_3$ $\qquad \mathbb{Z}_{11} \oplus \mathbb{Z}_3 \leq \mathbb{Z}_{11} \oplus D_3$ $\quad$ - - -

<u>Example 2</u> let $G$ be a group of order $20 = 2^2 \times 5$ that is not abelian, then $G$ has 5 Sylow 2-gps.

By Sylow's Thm $\quad n_5 \mid 4$ and $n_5 \equiv 1 \mod 5 \Rightarrow n_5 = 1$

$n_2 \mid 5$ and $n_2 \equiv 1 \mod 2$

$\Rightarrow n_2 = 1$ or $n_2 = 5$

Suppose $n_2 = 1$ then $H_2 \triangleleft G$ and $H_5 \triangleleft G$.

$H_2 \cap H_5 = \{e\}$ $\quad |H_2 H_5| = \dfrac{|H_2| \cdot |H_5|}{|H_2 \cap H_5|} = 4 \times 5 = 20$

$$\left.\begin{array}{l} \Rightarrow G = H_2 \times H_5 \cong H_2 \oplus H_5 \\ \text{but } |H_2| = 4 \Rightarrow H_2 \cong \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \quad |H_5| = 5 \Rightarrow H_5 \cong \mathbb{Z}_5 \end{array}\right\} \Rightarrow \underbrace{G = \text{abelian}}_{\text{contradiction}}$$

Therefore $n_2 = 5$

<u>Example 3</u> Classify groups of order $255 = 3 \times 5 \times 17$

$n_{17} \mid 15$ and $n_{17} \equiv 1 \mod 17$ (Sylow's thm)

$\Rightarrow n_{17} = 1 \Rightarrow \mathbb{Z}_{17} \cong H_{17} \triangleleft G \Rightarrow N(H_{17}) = G$

By N/C Thm

$$N(H_{17})/C(H_{17}) \leq \mathrm{Aut}(H_{17})$$

$$|G/C(H_{17})| \;\Big|\; |\mathrm{Aut}(\mathbb{Z}_{17})| = |U(17)| = 16$$

$$|G/C(H_{17})| \;\Big|\; |G| = 255 = 3 \times 5 \times 17$$

$$\Rightarrow |G/C(H_{17})| \;\Big|\; \gcd(16, 255) = 1$$

$\Rightarrow C(H_{17}) = G$

(i.e., elements of G commutes with any elements in $H_{17}$)

$\Rightarrow \quad H_{17} \leq Z(G) \Rightarrow 17 \mid |Z(G)|$

Therefore $|Z(G)| = 17, 3\times17, 5\times17, 3\times5\times17$

$(|Z(G)| \mid 255$ and $17 \mid |Z(G)|)$

i.e. $|G/Z(G)| = 15, 5, 3$ or $1$

But any group of order $15, 5, 3$, or $1$ is cyclic

(pq such that $p \le q$, $p \nmid q-1$)

recall: If $G/Z(G)$ = cyclic, then $G$ = abelian

So $G$ is an abelian group

Now by Fundamental Thm, $G \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{17}$

$(\cong \mathbb{Z}_{255})$