# Math 453 Spring 2019, Midterm 1

Work the problems below. Write clearly and in complete sentences. Justify your work. Believe that you are amazing and capable of anything, and remember, it is not a lie if you believe it!

**Problem 1. (10 Points)** State the following definitions:

(a) Given a group $G$, for a subset $S \subset G$ to be a subgroup.

(b) Given groups $G, H$, for a function $\varphi\colon G \to H$ to be a homomorphism.

**Problem 2. (10 points)** State whether the following statement is "false" or "true". You do not need to justify your answer.

(a) If $K \lhd H$ and $H \lhd G$, then $K \lhd G$.

(b) If $\varphi\colon G \to H$ is a homomorphism, then $\varphi(1_G) = 1_H$.

(c) If $\varphi\colon G \to H$ and $\psi\colon H \to K$ are homomorphisms, then $\psi \circ \varphi\colon G \to K$ is a homomorphism.

(d) If $H, K \leq G$ are subgroups, then $H \cup K$ is a subgroup.

**Problem 3. (10 points)** Let $G, H$ be group and let $\varphi\colon G \to H$ be a homomorphism. Prove that the following are equivalent:

(a) $\varphi$ is one-to-one (or injective).

(b) $\ker(\varphi) = \{1\}$.

**Problem 4. (10 points)** Let $G$ be a finite group and let $S \subset G$ be a non-empty subset. We say that $S$ is **closed under multiplication** if for every $s, t \in S$, we have $st \in S$. Prove that if $S$ is closed under multiplication, then $S$ is a subgroup of $G$.

**Problem 5. (10 points)** Let $G$ be a group, $H \lhd G$ a normal subgroup, and $K \leq G$ a subgroup.

(a) Prove that
$$HK = \{hk \,:\, h \in H,\ k \in K\}$$
is a subgroup of $G$.

(b) Prove that if $K$ is also normal in $G$, then $HK$ is normal in $G$.

**Problem 6. (10 points)** Let $G$ be a group and let $H, K \leq G$ be subgroups. Define

$$HK = \{hk \,:\, h \in H,\ k \in K\}, \quad KH = \{kh \,:\, k \in K,\ h \in H\}.$$

Prove the following are equivalent:

(a) $HK$ is a subgroup of $G$.

(b) $HK = KH$.