16 Discrete Groups

16.1 Review

Last time, we looked at discrete[57] subgroups $G \leq M_2$. Then, we looked at a projection π:

$$\begin{aligned} \ker(\pi) = (\mathbb{R}^2, +) \subset M_2 &\xrightarrow{\pi} O_2, \\ t_b \circ A &\mapsto A; \end{aligned}$$

essentially, it gets rid of the translation part of an isometry.

We can restrict π to G to get a mapping

$$G \xrightarrow{\pi|_G} O_2,$$

and we call the kernel,

$$L \coloneqq \ker(\pi|_G),$$

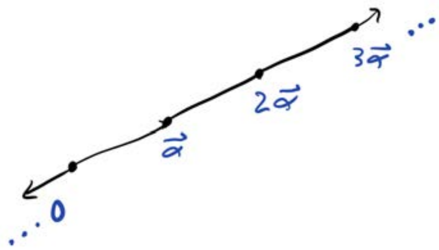
and it consists of all the translations in G.

The image of G in O_2, denoted $\overline{G} \coloneqq \pi(G)$, is called the *point group* of G. For some element $g \in G$, its image $\overline{g} \coloneqq \pi(g) \in \overline{G}$ only "remembers" the angle of rotation or the slope of the reflection line.

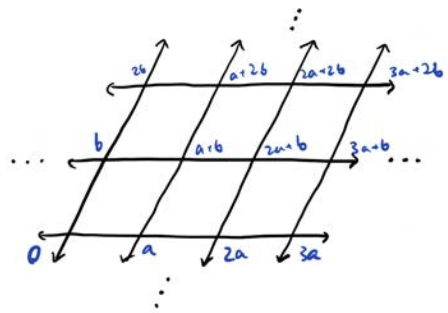
If $\overline{G}$ is discrete, it is either C_n or D_n, which we proved earlier.

If $L \subseteq \mathbb{R}^2$ is discrete, then we obtained three possible cases.

(i) $L = \{0\}$;



(ii) $L = \mathbb{Z}\alpha$ where $\alpha \neq 0$;



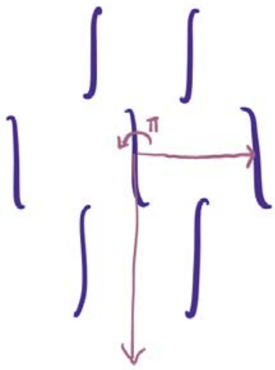
(iii) $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$, where α, β are linearly independent.[58]

16.2 Examples for L and $\overline{G}$

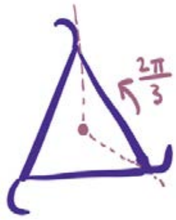
For a given plane figure, it is actually not difficult to see what L and $\overline{G}$ are! For the translation subgroup L, since it must either be the identity translation, $\mathbb{Z}\alpha$, or a lattice, it is possible to simply eyeball which translations preserve the figure. Let's consider the following plane figures. Later in this lecture, we will discuss the possibilities for $\overline{G}$; it consists of the (untranslated) rotations and reflections preserving a figure.

[57] The translations and rotations that cannot be arbitrarily small

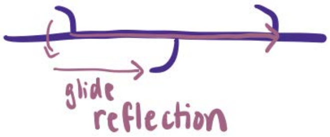
[58] When you look at two vectors and everything you generate from them, it's called a lattice.



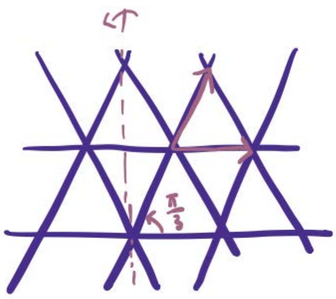
Example 16.1 (A)
For this first figure, say figure A, the translation subgroup L is a rectangular lattice generated by two translation vectors, to the right and upward. Also, $\overline{G}$ is D_2, since it contains a reflection as well as rotation by π.



Example 16.2 (B)
For figure B, the translation subgroup is trivial, consisting of 0. Also, $\overline{G}$ is C_3, since there cannot be any reflections but rotation by $2\pi/3$ or $4\pi/3$ around the center both preserve the figure.



Example 16.3 (C)
For figure C, the translation subgroup is generated by one vector, so $L = \mathbb{Z}\alpha$ where $\alpha = (1, 0)$. Also, $\overline{G}$ is D_1, since there is a reflection (corresponding to a glide reflection in G) and no rotations possible.



Example 16.4 (D)
For figure D, the translation subgroup is a triangular lattice generated by two vectors at an angle of $\pi/3$ to each other.[a] The point group is $\overline{G} = D_6$, since rotation at a lattice point by any multiple of $\pi/3$ preserves the figure, as well as reflection.

[a] Or two vectors at an angle of $2\pi/3$.

16.3 Crystallographic Restriction

Now that we have decomposed studying G into studying groups we understand better, L, a subgroup of translations, and $\overline{G} \subseteq$, the point group, we can actually constrain G further!

Recall that

- The translation subgroup $L \subseteq (\mathbb{R}^2, +)$[59] must be one of three possibilities, which we get from studying discrete subgroups of $\mathbb{R}^2$;
- $\overline{G}$ must be C_n or D_n, which we get from studying discrete subgroups of O_2.

Now that we understand the components L and $\overline{G}$ separately, we want to use this knowledge to understand G better.

Guiding Question
How do L and $\overline{G}$ interact with each other?

Example 16.5
Consider our earlier example 16.4. In this case, any element of the point group D_6 preserved the triangular lattice.

In fact, $\overline{G}$ acts on L for any discrete group $G \subseteq M_2$; this is a very strong constraint on how $\overline{G}$ and L interact.

Theorem 16.6
For the point group $\overline{G} \leq O_2$ of some discrete subgroup G of M_2, and the translation subgroup $L \subset \mathbb{R}^2$, the group $\overline{G}$ must map L to itself.

For any element $A \in \overline{G}$ and $b \in L$, the image of b under the action of A is

$$b \mapsto Ab \in L.$$

We already know that O_2 and thus $\overline{G}$ acts on the plane $\mathbb{R}^2$ and therefore L. The surprising part is that under the action of any element of $\overline{G}$, an element of L is actually mapped to another element in L!

Proof. Since $A \in \overline{G}$, it is the image of an element of G, say $t_{\vec{c}} \circ A \in G$ for some $\vec{c} \in \mathbb{R}^2$. Then, $\vec{b} \in L$, so $t_{\vec{b}} \in G$. The key observation in this proof is that $L = \ker(\pi|_G)$ is the kernel of a homomorphism! Thus, the subgroup $L \trianglelefteq G$ is actually normal, so conjugating an element of L by anything in G stays in L.

Then for $t_{\vec{b}} \in L$,

$$(t_{\vec{c}} \circ A) \cdot t_{\vec{b}} \cdot (t_{\vec{c}} \circ A)^{-1} \in L$$

also. As isometries in M_2, we know how to manipulate these products, and so expanding out this expression gives us

$$\begin{aligned} t_{\vec{c}} \cdot A \cdot t_{\vec{b}} \cdot A^{-1} \cdot t_{\vec{c}}^{-1} &= t_{\vec{c}} t_{A\vec{b}} \cdot A \cdot A^{-1} \cdot t_{-\vec{c}} \\ &= t_{\vec{c}} t_{A\vec{b}} t_{-\vec{c}} \\ &= t_{A\vec{b}} \in L. \end{aligned}$$

[59] The translation subgroup L is sometimes written ambiguously in one of two equivalent ways; an element of L can either be the translation $t_{\vec{b}} \in L$ considered as an element in G, or simply the vector $\vec{b} \in L$ considered as an element in $\mathbb{R}^2$. So L could be considered either as a subgroup of G or of $\mathbb{R}^2$.

Thus, conjugating $t_{\vec{b}} \in L$ by $t_{\vec{c}} \circ A$ gives $t_{A\vec{b}} \in L$. Using the identification of L with $\mathbb{R}^2$, $A\vec{b} \in L \subset \mathbb{R}^2$, and so every $A \in \overline{G}$ takes vectors $\vec{b}$ in L to other vectors in L, preserving the translation subgroup. $\square$

Student Question. *We're studying discrete groups, which are groups with the requirement that the translations or rotations can't be arbitrarily small. Are we also requiring that they have to be groups preserving a given diagram, or can they be any discrete groups of isometries?*

Answer. *Earlier on in this lecture, we saw some* ***examples*** *of discrete groups G that came from the symmetry groups of certain diagrams, but what we are actually doing is simply looking at groups G with the condition that the rotations and translations must be arbitarily small*[60]*, and classifying them; mathematically, there is no requirement that they come from pictures.*

However, the way that these discrete groups actually show up and the way that we find them is by drawing these kinds of pictures; this is one of the main reasons why we care about them! In fact, for every discrete subgroup $G \subseteq M_2$, there will be *some picture that produces the group G as its symmetry group. The pictures in this lecture are mainly so that there are concrete examples to look at and think about.*

In Section 16.2, each of the examples has a symmetry group G consisting of the isometries of the plane sending the picture to itself.[61] *For example, in Example 16.2, rotation by 120 degrees preserves the "triangle," while 5 degrees does not, so $\rho_{2\pi/3} \in G$, whereas $\rho_{\pi/36} \notin G$.*

Theorem 16.6 states that the point group $\overline{G}$, which is a different group from G, actually preserves $L \subseteq \mathbb{R}^2$, the translation group.

In Example 16.4, L is generated by

$$\mathbb{Z}(1,0)^t + \mathbb{Z}(1/2, 3/2)^t,$$

the two sides of an equilateral triangle, and the point group is D_6. Any element of D_6 will send an element of L to a different element in L.

In fact, when L is a lattice, preservation by some point group $\overline{G}$ is a strong constraint on the possible angles that show up in the lattice; only certain angles are allowed. Given $\overline{G}$, most lattices are not preserved by every element. Thus, the theorem constrains $\overline{G}$ and L together — not on each of them separately, but on how they interact.

The groups that show up this way are often called crystallographic groups. [62] They are well-studied; in fact, there are only finitely many.

Theorem 16.7 (Crystallographic Restriction)
[a] Let $L \neq \{0\}$. Then $\overline{G} = C_n$ or D_n, where $n = 1, 2, 3, 4,$ or 6.

[a]The theorem name comes from the fact that it restricts the possible crystallographic groups.

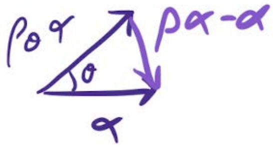
Although we could imagine that there are lots of possibilities for $\overline{G}$ and L, the fact that $\overline{G}$ preserves L constrains the possible point groups to finitely many, and there are also only certain choices of L allowed for a given n.

Proof. The group $\overline{G}$ is a discrete subgroup of O_2, and so it is C_n or D_n for some integer n.

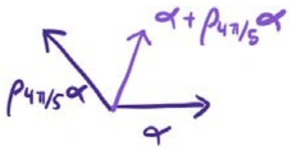
Since L is discrete, there is a (non-unique) shortest nonzero vector $\alpha \neq 0$. Consider a rotation $\rho = \rho_\theta \in G$. The result of rotating α by θ is another vector in L, and since rotations are length-preserving, $\rho_\theta \alpha$ is also a vector of shortest length. Since both vectors are in L, $\rho\alpha - \alpha$ is also in L.[63] If θ is too small, $\rho\alpha - \alpha$ will have a shorter length, and there will be a contradiction.

In particular, if $\theta < 2\pi/6$, $\rho\alpha - \alpha$ is shorter than α, so $\theta \geq 2\pi/6$. Since C_n and D_n contain $\rho_{2\pi/n}$, it must be the case that $n \leq 6$.

[60]These are called discrete groups
[61]Not each point individually is sent to itself; the picture as a whole is sent to an identical copy of itself.
[62]Especially when L is a lattice, and there are two different directions to translate.
[63]Since L is a subgroup, it is closed under addition/subtraction.



A similar argument holds to rule out $n = 5$. The vector $\alpha + \rho_{4\pi/5}\alpha$ will be shorter than any α, which is also a contradiction.[64]



So $n = 1, 2, 3, 4$, or 6. □

Actually, for C_n or D_n where $n = 1, 2, 3, 4$, or 6, it is possible to constrain the translation subgroups L that can simultaneously show up.

For instance, when L is a lattice,[65] there are only 17 possible symmetry groups G that can occur. When L is 0, $\overline{G}$ can be C_n or D_n for any arbitrary n, but allowing nontrivial translations constrains $\overline{G}$ significantly.

Student Question. *How much does constraining $\overline{G}$ and L constrain the actual symmetry group G itself?*

Answer. *Finding G from $\overline{G}$ and L is precisely the same as figuring out the 17 plane symmetry groups,*[66] *and is precisely the last step! We will do one example now.*

Let's consider a specific group $\overline{G}$ and try to figure out what the actual symmetry group G can be!

[64]This question is equivalent to the feasibility of tiling the plane with a regular pentagon, and in fact that is not possible!

[65]When L is a lattice, it is two-dimensional, and it is $\mathbb{Z}\vec{a} + \mathbb{Z}\vec{b}$ for generating vectors $\vec{a}$ and $\vec{b}$. It is also possible for L to be $\mathbb{Z}\vec{a}$, which is one-dimensional.

[66]These are called wallpaper groups, since wallpapers are 2-dimensional patterns that usually have nontrivial symmetry groups.

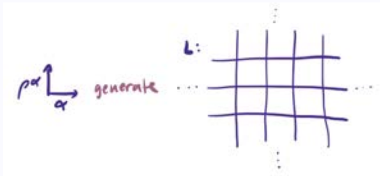
Example 16.8 (C_4)
Suppose $\overline{G} = C_4$.[a] Then $L \subset G \xrightarrow{\pi|_G} C_4$, and the index $[G : L] = 4$.[b]

Also, $\overline{\rho} = \rho_{\pi/2} \in \overline{G}$ is a generator of $\overline{G}$. Where α is some shortest-length vector in L, it's possible to show[c] that $\rho\alpha$ and α do generate L. Thus,

$$L = \mathbb{Z}\alpha + \mathbb{Z}(\rho\alpha),$$

a square lattice.

Also, there exists some rotation $\rho \in G$ giving $\pi(\rho) = \overline{\rho}$. Then ρ is in fact a rotation by $\pi/2$ around some other point, which we will call the origin.[d] The group G contains L, of index 4, as well as some rotation by $\pi/2$, ρ.[e]



Thus, G is "generated" by L and ρ, and must consist of everything of the form

$$G = \{t_v \circ \rho^i : \vec{v} \in L, i = 0, 1, 2, 3\}.$$

Also, $\rho t_v = t_{\rho v} \circ \rho$, so the group multiplication can be written down, and G is completely determined by knowing that $\overline{G}$ was C_4; this is 1 out of the 17 wallpaper groups!

[a] Rotations by 90 degrees, but no reflections.
[b] The index $[G : \ker(\pi|_G)] = [G : L]$ is equal to the size of the image under $\pi|_G$, which is $\overline{G} = C_4$.
[c] There is a more involved argument there, but it is not super relevant here.
[d] In the discussion of the four kinds of isometries in M_2, the elements which were mapped to rotations were in fact rotations around some point.
[e] The rotation ρ is $\overline{\rho}$, lifted to be in G, and it is an element of G not in L which generates the quotient, C_4.

Student Question. *Can you explain where ρ came from? Why is it a rotation?*

Answer. *By definition, $\overline{G}$ is the image of G under $\pi : M_2 \to O_2$ taking $t_b \circ A \mapsto A$. Then there are four possibilities for elements in M_2: translation, rotations, reflections, and glide reflections. The first two are orientation-preserving, and the last two are orientation-reversing. Reflections and glide reflections map to reflections in O_2[67] under π, translations will map to the identity, and rotations will map to rotations (around the origin). So ρ has an image of $\overline{\rho}$, which is a rotation, and thus ρ is a rotation around some point.*

If $\overline{\rho}$, the element in $\overline{G}$, were a reflection instead of a rotation, the preimage in G could have been either a reflection or a glide reflection, so when the point group $\overline{G} = D_n$, one of the dihedral groups, instead of C_n, the analysis is more subtle. In fact, there might not be any reflections in G at all. (In Example 16.3, there were no reflections, only glide reflections.)

Example 16.9
If $\overline{r} = \pi(r)$ where r, then $\overline{r} = \pi(t_b \circ r_\ell)$, where b is some zero[a] or nonzero[b] vector parallel to the line ℓ. Does this mean there are uncountably many possibilities for b and therefore r? In fact, b is constrained a bit more: $t_b r_\ell t_b r_\ell = t_{2b}$, so $2b \in L$. Thus, there are two possible situations:

- The vector is in the lattice: $b \in L$;
- The vector b is halfway between two lattice points, as in Example 16.1.

[a] reflection
[b] glide reflection

From these two examples, we see that given some $\overline{G}$, of which there are finitely many, and working through the information that is present, there aren't too many possibilities for G, and in fact there are finitely many — 17

[67] Reflections across lines through the origin

in total.