Midterm 1 Review
Math 453 Spring 2019

February 14, 2019

1 Exam focus

1.1 Overview

The exam will focus on groups. The main topics of the exam are subgroups and homomorphisms. Below, the exam format is described. There are practice problems and a practice exam. The practice exam is identical in format to your actual exam. I tried to pick similar problems to the ones that appear on the exam. The problems on your exam are fairly different however and it is good to work on the practice problems as well.

You can use any of the results from problem sets. You can use any result from the group theory fact sheet *except* if the problem on the exam is to prove one of the results from the fact sheet. If you are unsure, please ask me during the exam.

1.2 Important Concepts

Below are some important concepts that you should know for the exam. I would review material beyond just what is below though.

Definition 1 (Subgroup)**.** Given a group G, we say that a subset $S \subset G$ is a **subgroup** if the following holds for all $s, s' \in S$:

(1) $S \neq \emptyset$.

(2) $ss' \in S$.

(3) $s^{-1} \in S$.

We write $S \leq G$ when S is a subgroup of G.

Definition 2 (Homomorphism)**.** Given groups G, H, we say that a function $\varphi\colon G \to H$ is a **homomorphism** if

$$\varphi(gg') = \varphi(g)\varphi(g')$$

for all $g, g' \in G$.

Product sets appear in two of the problems. Recall that if $S, T \subset G$ where G is a group, we can define a **product set**

$$ST = \{st \ : \ s \in S,\ t \in T\}.$$

Cosets gH and Hg of a subgroup are examples of product sets.

A subgroup N of G is **normal** if any of the following equivalent conditions holds:

(1) $gNg^{-1} = N$ for all $g \in G$.

(2) $gN = Ng$ for all $g \in G$.

(3) For each $g \in G$ and $n \in N$, we have $gng^{-1} \in N$.

(4) For each $g \in G$ and $n \in N$, there exists $n' \in N$ such that $gn = n'g$.

We write $N \lhd G$ when N is a normal subgroup.

The concept of a finite group appears in one problem. A finite group is just a finite set with a group operation. One thing that is nice about finite groups is that you can use counting methods and elementary combinatorics to solve some problems. The problem on your practice exam is an example of such problems. The problem involving finite groups on your exam is basic. It does require that the group be finite, so be mindful of that.

The kernel of a homomorphism appears on the exam. Recall that if $\varphi \colon G \to H$ is a homomorphism between two groups G, H, the **kernel** is defined to be

$$\ker(\varphi) = \{g \in G \;:\; \varphi(g) = 1\}.$$

You have proven on your problem sets that $\ker(\varphi)$ is a normal subgroup of G.

2 Exam format

The first midterm exam will have 6 problems. I will describe each problem in the order it appears on the exam.

Problem 1. (10 points) You will be asked to state two definitions from the material covered. It is highly probable that the two definitions will be the definition of a subgroup and the definition of a group homomorphism.

Problem 2. (10 points) You will be asked 4 true/false questions. You do not need to justify your answers. Two of the questions have a true answer and two of the questions have a false answer.

Problem 3. (10 points) This problem will have two parts; you will prove two things are equivalent. The concepts that appear in this problem are group homomorphisms, one-to-one/onto, and normal subgroups.

Problem 4. (10 points) This problem has one part. This problem is on subgroups of finite groups. This might be the easiest problem on the exam.

Problem 5. (10 points) This problem has two parts. The concepts that appear in this problem are subgroups, normal subgroups, and products sets.

Problem 6. (10 points) This problem has two parts; you will prove two things are equivalent. The concepts that appear in this problem are subgroups and products sets.

3 Practice Problems

Below are some practice problems. Working these problems should be helpful for preparing for the exam. It is likely unnecessary to work all of the problems. The practice exam problems are worth working on. One might consider saving those problems until after working some of the other practice problems. You might even consider taking it as an exam first and seeing how far you get. The practice exam and real exam are both a bit too long. I expect a decent number of students will not finish the exam. Hopefully you all will manage to get through a good chunk of it.

Problem 1. Let $H, K \leq G$ be subgroups.

(a) Prove that $H \cap K$ is a subgroup of G.

(b) Prove that if $H \lhd G$, then $H \cap K \lhd K$.

Problem 2. Let $\varphi, \psi \colon G \to H$ be homomorphisms and define $\phi \colon G \to H \times H$ by $\phi(g) = (\varphi(g), \psi(g))$.

(a) Prove that ϕ is a homomorphism.

(b) Prove that $\ker(\phi) = \ker(\varphi) \cap \ker(\psi)$.

Problem 3. Let $\varphi \colon G \to H$ and $\psi \colon H \to K$ be homomorphisms.

(a) Prove that $\psi \circ \varphi \colon G \to K$ is a homomorphism.

(b) Prove that $\ker(\psi \circ \varphi) = \varphi^{-1}(\ker(\psi))$.

(c) If $G_0, G_1, \dots, G_n$ are groups and $\varphi_i \colon G_{i-1} \to G_i$ are homomorphisms for $i = 1, \dots, n-1$, prove that

$$\ker(\varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_2 \circ \varphi_1) = \varphi_1^{-1}(\varphi_2^{-1}(\dots \varphi_{n-2}^{-1}(\ker(\varphi_{n-1}))\dots)).$$

Problem 4. Let G be a commutative group. For each $g \in G$, the **order of** g is defined to be the smallest positive integer n such that $g^n = 1$ and we denote the order of g by $\mathrm{ord}(g)$. If $g^n \neq 1$ for all $n \in \mathbf{N}$, then we define $\mathrm{ord}(g) = \infty$.

(a) Prove that the subset

$$\Omega_n(G) = \{g \in G \ : \ \mathrm{ord}(g) = n\}$$

is a subgroup of G.

(b) Let $\varphi \colon G \to H$ be a surjective homomorphism. Prove that H is commutative.

(c) Prove that

$$\varphi(\Omega_n(G)) \leq \Omega_{n_\varphi}(H)$$

where

$$n_\varphi = \mathrm{lcm}(\{\mathrm{ord}(\varphi(g)) \ : \ g \in \Omega_n(G)\}).$$

Problem 5. Let G be a cyclic group. Prove G is commutative.

Problem 6. Let $m, n \in \mathbf{N}$ with $m \mid n$. Prove that $\varphi \colon \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z}$ given by $\varphi(\bar{i}) = \bar{i}$ is an onto homomorphism.

Problem 7. Let X be a set and let G be a group. The set $\mathrm{Fun}(X, G)$ is the set of functions $f \colon X \to G$. For $f, g \in \mathrm{Fun}(X, G)$, define $f \cdot g \in \mathrm{Fun}(X, G)$ by (the right hand side is the product of $f(x), g(x) \in G$).

$$(f \cdot g)(x) = f(x)g(x).$$

Let $\mathbf{1}_X$ denote the function $\mathbf{1}_X \colon X \to G$ defined by $\mathbf{1}_X(x) = 1$ where $1 \in G$ is the identity in G.

(a) Prove that $\mathrm{Fun}(X,G)$ is a group with group operation $\cdot$ and identity $\mathbf{1}_X$.

(b) Let $f\colon X \to Y$ be a function. Define $f^\star\colon \mathrm{Fun}(Y,G) \to \mathrm{Fun}(X,G)$ by $f^\star(g) = g \circ f$ where $g \in \mathrm{Fun}(Y,G)$. Prove that $f^\star$ is a homomorphism.

(c) Prove that if f is injective, then $f^\star$ is surjective.

(d) Prove that if f is surjective, then $f^\star$ is injective.

(e) Prove that f is a bijection if and only if $f^\star$ is an isomorphism.

Problem 8. Let G be a group, $H \lhd G$, and $K \leq H$ a characteristic subgroup of H. Prove that $K \lhd G$.

Problem 9. Let G be a group and let $\mathrm{Aut}(G)$ be the group of automorphisms of G. Define $\mathrm{Ad}\colon G \to \mathrm{Aut}(G)$ by

$$\mathrm{Ad}(g)(h) = ghg^{-1}.$$

We previously showed that Ad is a homomorphism.

(a) Prove that if $\psi, \mathrm{Ad}(g) \in \mathrm{Aut}(G)$ for some $g \in G$, then

$$\psi \circ \mathrm{Ad}(g) \circ \psi^{-1} = \mathrm{Ad}(\psi(g)).$$

(b) Set $\mathrm{Inn}(G) = \mathrm{Ad}(G)$. Prove that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

Problem 10. Let G be a group such that $\mathrm{Aut}(G) = \{1\}$. Prove that G is commutative.

Problem 11. Let G be a group such that $g^2 = 1$ for all $g \in G$. Prove that G is commutative.

Problem 12. Let G be a group such that $Z(G) = \{1\}$. Prove that $|\mathrm{Aut}(G)| \geq |G|$.

4 Practice Exam

Midterm 3: Math 341, Fall 2016

Work the problems below. Write clearly and in complete sentences. Justify your work.

Problem 1. (10 Points) State the following definitions:

(a) Given a group, for a subset $S \subset G$ to be a subgroup.

(b) Given groups G, H, for a function $\varphi\colon G \to H$ to be a homomorphism.

Problem 2. (10 points) State without justification whether the following statements are "false" or "true".

(a) If $H \leq G$ and $K \leq H$, then $K \leq G$.

(b) If $\varphi\colon G \to H$ is a homomorphism, then $\varphi(g^{-1}) = (\varphi(g))^{-1}$.

(c) If $H \leq G$, then $H \leq C_G(H)$.

(d) If $\varphi\colon G \to H$ and $\psi\colon H \to K$ are homomorphisms such that $\psi \circ \varphi$ and ψ are onto, then φ is onto.

Problem 3. (10 points) Let $\varphi\colon G \to H$ and $\psi\colon G \to K$ be onto homomorphisms. Prove that the following are equivalent:

(a) There exists a homomorphism $\phi\colon H \to K$ such that $\psi = \phi \circ \varphi$.

(b) $\ker(\varphi) \subset \ker(\psi)$.

Problem 4. (10 points) Let G be a finite group and let $H \leq G$ be a proper subgroup. Prove that

$$\bigcup_{g \in G} gHg^{-1} \neq G.$$

Problem 5. (10 points) Let G be a group with $H, K \leq G$. Prove the following are equivalent:

(a) $H \cup K \leq G$.

(b) $HK = H$ or $HK = K$.

Problem 6. (10 points) Let G_1, G_2 be groups witht $H_1 \leq G_1$, $H_2 \leq G_2$.

(a) Prove that $H_1 \times H_2 \leq G_1 \times G_2$ where

$$H_1 \times H_2 = \{(h_1, h_2) \in G_1 \times G_2 \ : \ h_1 \in H_1,\ h_2 \in H_2\}.$$

(b) Prove that if $H_1 \lhd G_1$ and $H_2 \lhd G_2$, then $H_1 \times H_2 \lhd G_1 \times G_2$.