17 Group Actions

Today, we will discuss group operations or actions[68] on a set.

Guiding Question
How can a group be seen as a group of transformations?

17.1 Review

Last time, we finished talking about (discrete) subgroups of isometries of the plane. Finite subgroups of M_2 are isomorphic to C_n or D_n, and there are only finitely many isomorphism classes of infinite discrete subgroups of M_2.[69]

It is also possible to go up a dimension and classify discrete subgroups of isometries of $\mathbb{R}^3$, although it is more complicated; there are 200 or so.

17.2 Motivating Examples

The idea of a group action will generalize and make more abstract an idea that has been present throughout the class so far. Let's start with the following motivating example.

Example 17.1 (GL_n)
Given $g \in GL_n(\mathbb{R})$ and a column vector $v \in \mathbb{R}^n$, the matrix g can be seen as a transformation on $\mathbb{R}^n$, taking $v \mapsto g(v) \in \mathbb{R}^n$.

The data of $GL_n(\mathbb{R})$ acting on $\mathbb{R}^n$ can be packaged together by a map

$$\begin{aligned} GL_n(\mathbb{R}) \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (g, \vec{v}) &\longmapsto g(\vec{v}). \end{aligned}$$

The same principle applies to S_n, the group of permutations on $\{1, \cdots, n\}$.

Example 17.2 (S_n)
The symmetric group S_n can also be viewed as acting on a set. More or less by definition, given a number between 1 and n, and a permutation, it's possible to spit out the result of permutation acting on that number. So S_n permutes the set $[n] = \{1, \cdots, n\}$. This gives us another mapping encoding this information:

$$\begin{aligned} S_n \times \{1, \cdots, n\} &\longrightarrow \{1, \cdots, n\} \\ (\sigma, i) &\longmapsto \sigma(i). \end{aligned}$$

Our last example is one we have been considering for the past few lectures.

Example 17.3
The set M_2, isometries of 2-space, acts on $\mathbb{R}^2$: given some vector in the plane and some isometry, the isometry will return some other vector in the plane. This information is again encoded by a mapping

$$\begin{aligned} M_2(\mathbb{R}) \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (f, \vec{x}) &\longmapsto f(\vec{x}). \end{aligned}$$

[68] They are different terms for the same idea. Artin uses group operations, while Professor Maulik prefers to call them group actions.

[69] In fact, with *any* metric space, which is a set with some distance on it (as discussed in 18.100, for example), it's possible to consider isometries, distance-preserving transformations, in the same way as we considered the plane $\mathbb{R}^2$. Depending on the metric space, the groups can look very different! One example of this is the hyperbolic plane, which is the upper half-plane of $\mathbb{R}^2$ with a non-Euclidean metric, or distance, on it, and the discrete subgroups of isometries on it. There are infinitely many discrete subgroups of isometries on it, even though it is 2-dimensional, just like $\mathbb{R}^2$. The question of why it is so different from the $\mathbb{R}^2$ case is really a geometry question, rather than an algebra question.

17.3 What is a group action?

These are all examples of group operations on a set, and they motivate the following definition of a group operation in general.

Definition 17.4
Given a group G and a set S, a **group action**[a] on a set S is a mapping

$$\begin{aligned} G \times S &\longrightarrow S \\ (g, s) &\longmapsto gs. \end{aligned}$$

It must satisfy the following axioms:

- The identity maps every element of the set back to itself: $es = s$ for all $s \in S$.[b]
- The mapping must respect the group multiplication: $(gh)s = g(hs)$ for $s \in S$ and $g, h \in G$.[c]

[a] or group operation
[b] This corresponds to the identity multiplication rule.
[c] This corresponds to the associativity rule.

Essentially, given an element $g \in G$ and $s \in S$, the mapping returns another element of S depending on g, and the mapping must respect the group structure on G. All of the groups that we have seen so far show up as symmetries of some set, maybe preserving some extra structure, so all the groups that we usually think about already come with an action on some set S. Furthermore, a group G can act on many different sets at the same time in different ways, which gives insight into the group itself.

Let's look at a couple of examples.

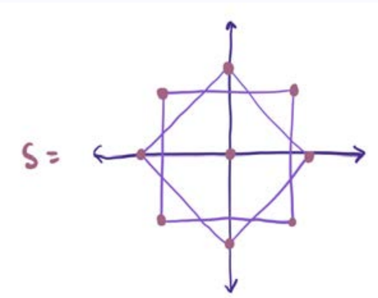
Example 17.5 (S_4)
The symmetric group S_4, permutations on 4 elements, acts on $S = \{1, 2, 3, 4\}$. It can *also* act on a different set, $T = \{\text{unordered pairs in } S\} = \{(12), (13), (14), (23), (24), (34)\}$. The set T has 6 elements, and S_4 acts on T as well as acting on S. Given a permutation $\sigma \in S_4$, and an unordered pair $\{i, j\}$, it acts by taking

$$\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$$

for a permutation $\sigma \in S_4$.

So the group action on S leads to a different group action on a different set, T. The existence of a group action on a given set actually yields a lot of information about the group G, as will be explored in the next few lectures. Let's see a different example.

Example 17.6 (D_2)
Let $G = D_2$, which contains rotation by π as well as reflection across the x-axis (and then reflection across the y-axis.) As a subgroup of O_2,[a] D_2 will act on all of $\mathbb{R}^2$. It also acts on the set S consisting of the vertices of a square and a diamond, as well as the center.



[a] 2×2 orthogonal matrices

A group G can also act on itself viewed as a set.

Example 17.7
Given a group G, there is a mapping

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, g') &\longmapsto gg', \end{aligned}$$

and this is a valid group action.

When G acts on itself, the first G in $G \times G$ is seen as a group, while the second G is seen as a set, since the axioms of a group action don't care about the group operation on the second instance of G.

Let's see one last example.

Example 17.8
Taking a vector space V over a field F, the group $F^\times$, the nonzero elements of the field, which is a group with respect to multiplication, acts on V by scaling:

$$\begin{aligned} F^\times \times V &\longrightarrow V \\ (a, \vec{v}) &\longmapsto a\vec{v}. \end{aligned}$$

Scaling by nonzero scalars defines a group operation! It satisfies each of the axioms.

Student Question. *What type of element is $g(s)$?*

Answer. *It depends on what S is! It is the type of element that is in S. Two group actions of G on S and S' might not have anything to do with each other, other than the fact that they both involve G; G can act on wildly different types of sets, and show up in different contexts.*

Say we fix an element $g \in G$, we can define the group action of g on S, a mapping $\tau_g : S \longrightarrow S$ sending $s \longmapsto g(s)$.[70] We can show that τ_g is a bijection from S to itself because it has an inverse map, $\tau_{g^{-1}}$, coming from the fact that g is invertible. Because τ_g is a bijection, it actually *permutes* the elements of S, and so it is a permutation of S. Thus, each element of G can be mapped to a permutation by a map

$$\tau : G \longrightarrow \mathrm{Perm}(S),$$

which takes $g \longmapsto \tau_g \in \mathrm{Perm}(S)$. From the group action axioms, τ is a group homomorphism. In Example 17.6, D_2 is acting on a set with $|S| = 9$, so there exists a homomorphism from $D_2 \longrightarrow \mathrm{Perm}(S) = S_9$.

Note that τ does not have to be injective; there may be some action $g \in G$ such that $g \neq e$ but G fixes each $s \in S$, which would make $\tau(g)$ the identity permutation.

17.4 The Counting Formula

Definition 17.9
Given $s \in S$, the **orbit** of s is

$$O_s = Gs := \{gs : g \in G\} \subseteq S.$$

For instance, in Example 17.6, there are several orbits of different sizes. The top and bottom vertices of the diamond are in the same orbit (size 2), the left and right vertices of the diamond are in the same orbit (size 2), all the vertices of the square are in the same orbit (size 4), and the origin is in an orbit by itself (size 1), just by applying each of the group elements to an element of the set.

Definition 17.10
The group G acts **transitively** on S if $S = O_s$ for some $s \in S$.

For example, S_n acts transitively on $\{1, \cdots, n\}$, since given an element $i \in \{1, \cdots, n\}$, there is some permutation mapping it to any other element i'.

[70] This notation is not standard and may not correspond with the textbook.

Student Question. *Does this have to be true for all $s \in S$, or just one?*

Answer. *If it is true for one $s \in S$, it is true for all $s \in S$. Try checking it!*

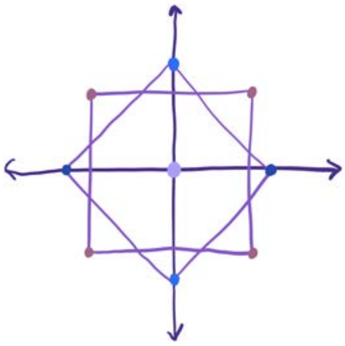
So a transitive group action is one where there is only one orbit consisting of the entire set S; in particular, any element of s can be carried to any other element when acted on by some $g \in G$.

Definition 17.11
The **stabilizer** of s is

$$G_s = \mathrm{Stab}_G(s) := \{g \in G : gs = s\},$$

and it is a subgroup of G.

For Example 17.6, the top and bottom vertices of the diamond are stabilized by the reflection across the y-axis, whereas the stabilizer group of a vertex of the square is just the identity element.



Proposition 17.12
The orbits of G form a *partition* of S.[a] In particular, S is the disjoint union of the orbits: $S = \amalg O_i$ where $O_i \cap O_j = \varnothing$.

[a]The set can be cut into non-overlapping pieces by the orbits.

Proof. The orbits clearly cover S, since every element $s \in S$ is also an element of O_s, its own orbit. Also, they are disjoint. If $O_s \cap O_{s'} \neq \varnothing$, then there is some element in their intersection $t = gs = g's'$. Then $s = (g^{-1}g')s'$, which is in $O_{s'}$. So every element of O_s is in $O_{s'}$, and by the same logic $O_{s'} \subseteq O_s$. Then $O_s = O_{s'}$. So if two orbits have nonempty intersection, they are in fact the same orbit. □

For a finite set, the size of S can be obtained from the sizes of the orbits.

Corollary 17.13
If S is a finite set, and $O_1, \cdots, O_k$ are the orbits, then

$$|S| = \sum_{i=1}^{k} |O_i|,$$

since each of the orbits cover S exactly.

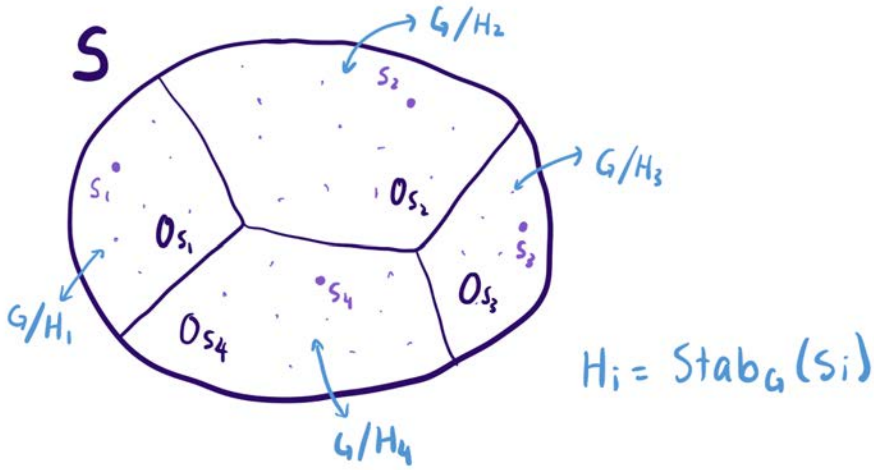
In Example 17.6, this gives $9 = 4 + 2 + 2 + 1$.

Guiding Question
What does each orbit look like?

For this, we use the notion of a *stabilizer* of an element.

Proposition 17.14
Fix some $s \in S$ and let $H \coloneqq \text{Stab}(s)$. Then there exists a bijection ε from the quotient group G/H to the orbit of s, O_s. It takes

$$\begin{aligned} G/H &\xrightarrow{\varepsilon} O_s \\ gH &\mapsto gS. \end{aligned}$$



Proof. Consider g and γ in G. Then their cosets map to the same element if $gs = \gamma s$, which is equivalent to saying that $g^{-1}\gamma s = s$. Since H is the stabilizer of S, this means that $g^{-1}\gamma \in H$; equivalently, $\gamma \in gH$. Since each of these conditions were equivalent conditions, $gs = \gamma s$ if and only if $\gamma \in gH$, and thus ε must be bijective: two elements in G/H map to the same element in O_s if and only if they are the same element. $\square$

Corollary 17.15 (Counting Formula for Orbits)
As a result, the number of cosets of H, which is the order $|G/H|$, is equal to the size of the orbit of s, since there is a bijective correspondence between them. So

$$|O_s| = [G : \text{Stab}(s)].$$

In particular, the size of the orbit of any element $|O_s|$ divides $|G|$ when G is a finite group. We have

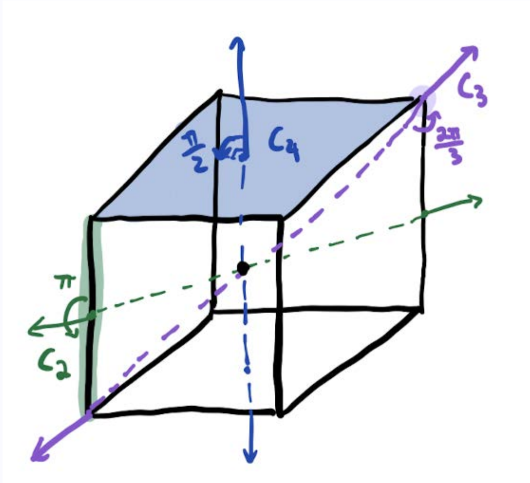
$$|O_s| \cdot |\text{Stab}(s)| = |G|.$$

These theorems are similar to the Counting Formula and Lagrange's Theorem from Chapter 2. In particular, let $\mathcal{C}$ be the set of left cosets of a given subgroup H. Then G acts on $\mathcal{C}$; an element $g \in G$ takes $C \longmapsto gC$. Every coset can be mapped to any other coset by some element of G. For example, g_1H is mapped to g_2H by $g_2g_1^{-1} \in G$. So there is only one orbit, the entire set $\mathcal{C}$. The stabilizer of the identity coset, which is $eH = H$, is $\text{Stab}(eH) = H$, because some element $g \in G$ carries $h \in H$ to $h' \in H$ if and only if $gh = h'$, which implies that $g = h'h^{-1} \in H$. Thus, the Orbit-Stabilizer Theorem states that

$$|G| = |H|[G : H],$$

since $|H|$ is the stabilizer of the identity in G/H and $[G : H]$ is the size of the identity orbit.

Example 17.16
Consider the subgroup $G \leq SO_3$ consisting of rotational symmetries of a cube centered at the origin.



- Let S be the set of faces of the cube; it has order 6 since there are 6 faces. For every face of the cube, there is some element in G mapping it to any other face in the cube (G acts transitively on the faces), so the orbit of a given face is the set of all the other faces, which is S. The stabilizer $\mathrm{Stab}_G(\text{face}) = C_4$, since a given face, which is a square, is preserved by rotation by $\pi/2$ around the axis through the center of the face. Then
$$|G| = |S| \cdot |\mathrm{Stab}_G(\text{face})| = 6 \cdot 4 = 24.$$
- Similarly, any vertex can be mapped to any other vertex by some element of G. The stabilizer $\mathrm{Stab}_G(\text{vertex}) = C_3$, since a vertex is preserved under rotation by $2\pi/3$ around the axis from the vertex to the opposite vertex. Again,
$$|G| = |\{\text{vertices}\}| \cdot |\mathrm{Stab}_G(\text{vertex})| = 8 \cdot 3 = 24.$$
- Again, G acts transitively on the set of edges. The stabilizer of an edge is $\mathrm{Stab}_G(\text{edge}) = C_2$. Then
$$|G| = |\{\text{edges}\}| \cdot |\mathrm{Stab}_G(\text{edge})| = 12 \cdot 2 = 24.$$