30 The Special Unitary Group SU_2

30.1 Review

Last time, we started looking at subgroups of the group of invertible matrices. We saw that one thing these groups have in common, that finite or discrete groups don't, is that they have some sort of shape or geometry. In particular, we looked at the group

$$SU_2 := \{A \in GL_2(\mathbb{C}) \mid A^* = A^{-1}, \det A = 1\},$$

the special unitary group. By playing around with the definitions, we found that SU_2 sits inside the quaternions

$$\mathbb{H} = \{x_0 I + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\},$$

where we defined

$$\mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

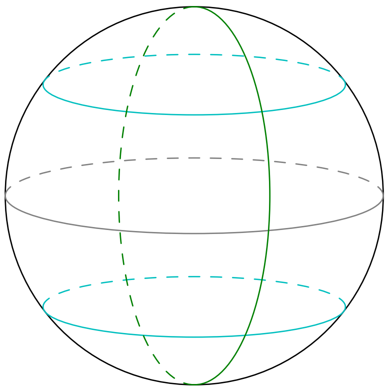
In particular, SU_2 is the subset with $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, corresponding to the 3-sphere S^3 in $\mathbb{R}^4$.

Note 30.1
We've seen that the 1-dimensional sphere and 3-dimensional spheres both have a group structure, so we can ask whether the same is true for other dimensions. It turns out the answer is no – there are no other n-dimensional spheres which can be made into groups. (This is a deeper fact.)

Last class, we started taking geometric properties of the 3-sphere and seeing how they correspond to the group structure. In particular, we looked at the latitudes – the horizontal slices $\text{Lat}_c = \{x_0 = c\} \cap S^3$ for $-1 \le c \le 1$. We proved last class that these latitudes are precisely the conjugacy classes of SU_2. We call Lat_0 the **equator**, denoted $\mathbb{E}$.

30.2 Longitudes

Another thing we can think about are the **longitudes** – the circles which go through the north and south pole.



We can define these more precisely: for each $x \in \mathbb{E}$, the longitude containing x is $\text{Long}_x := \text{Span}(I, x) \cap S^3$. Here $\text{Span}(I, x)$ is a 2-dimensional plane, so we're taking the unit circle of a 2-dimensional plane.

Theorem 30.2
For each $x \in \mathbb{E}$, Long_x is a *subgroup* of SU_2. In fact, given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the map $\theta \mapsto \cos\theta I + \sin\theta x$ is an isomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and Long_x.

What this means is that the longitudes aren't just circles as *shapes*, they're also circles as *groups* (since we've seen that the unit circle as a group is isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$.

Proof. To see this is true, we'll first consider the special case $x = \mathbf{i}$. Then we can check that given two points in $\text{Long}_{\mathbf{i}}$, we have

$$(c + s\mathbf{i})(c' + s'\mathbf{i}) \in \text{Span}(I, \mathbf{i})$$

as well, since $\mathbf{i}^2 = -I$. Meanwhile, since both elements are in SU_2, their product must be as well; so their product is in $SU_2 \cap \text{Span}(I, \mathbf{i}) = \text{Long}_{\mathbf{i}}$. So this longitude is closed under multiplication, and is therefore a subgroup.

We won't check the isomorphism to $\mathbb{R}/2\pi\mathbb{Z}$ here, but it's possible to check this directly by multiplying out.

We can then use this to solve the general case – for any $x \in \mathbb{E}$, we know x is conjugate to $\mathbf{i}$ (since we saw that the equator is a conjugacy class). So then we can write $x = P^{-1}\mathbf{i}P$, and then $\text{Long}_x = P^{-1}\,\text{Long}_{\mathbf{i}}\,P$ is conjugate to $\text{Long}_{\mathbf{i}}$. But when we conjugate a subgroup, we get another subgroup.

So not only are the longitudes all circle subgroups, but they're also all conjugate to each other. □

30.3 More Group Theoretic Properties

When we studied conjugacy classes, we also studied centralizers:

Guiding Question
What is the centralizer of $\mathbf{i}$?

Recall that this means the set of elements for which if we conjugate $\mathbf{i}$ by them, we get back $\mathbf{i}$.

We know that $\text{Long}_{\mathbf{i}}$ is a subgroup of SU_2, and it's *abelian* (since it's isomorphic to the circle). Since $\mathbf{i}$ is in this longitude, this means it commutes with everything in this longitude. So $Z(\mathbf{i}) \supset \text{Long}_{\mathbf{i}}$. In fact, this turns out to be an equality – we have $Z(\mathbf{i}) = \text{Long}_{\mathbf{i}}$. This is true for any other point on the equator as well – its longitude is exactly its centralizer.

Another thing we saw when studying conjugacy classes was that there's a bijection between the conjugacy class $C(g)$ and the cosets of the centralizer $G/Z(g)$. In our case, this is still true, but now both sides are geometric objects. If we fix a point g on the equator, then $C(g) = \mathbb{E}$ is a 2-dimensional sphere. Meanwhile, $G/Z(g)$ corresponds to taking cosets of a longitude. We're taking a 3-sphere and covering it in cosets – so we have a map $S^3 \to S^2$, where the fibers are circles (the cosets of the longitude).

This is really hard to picture, but the idea is that we start with the 3-sphere and a given longitude, and we're taking its cosets (which correspond to circles not necessarily through the north and south pole) and covering the entire 3-sphere in these circles. When we collapse all these circles to a point, we get a copy of the 2-sphere.

Note 30.3
This is really difficult to think about, but it's a construction in topology relating spheres of dimensions 1, 2, and 3, and it can also be thought of in this group theory setting.

What we would like to illustrate is that group theoretic facts about this group also become interesting geometric facts; it doesn't really matter if you don't understand all of them.

30.4 Conjugation and the Orthogonal Group

There's another thing we can look at: we know the equator is a conjugacy class, so SU_2 acts on $\mathbb{E}$ transitively (with the action given by conjugation). In fact, SU_2 acts on the space $\{x_0 = 0\} \subset \mathbb{H}$ (which is the 3-dimensional vector space containing the equator), and it preserves the equator $\mathbb{E}$ inside this space.

Conjugation by an element of SU_2 is a linear map, so it defines a group homomorphism $\rho : SU_2 \to GL_3(\mathbb{R})$, where $\rho(g)$ is the matrix such that $\rho(g)\vec{v} = gvg^{-1}$. But $\rho(g)$ preserves $\mathbb{E}$, so since it preserves vectors of length 1, this means it must preserve length in general. So $\rho(g)$ is actually an isometry – which means this map is actually $\rho : SU_2 \to O_3$.

In fact, we can say even more. We've seen that orthogonal matrices in 3 dimensions are either reflections or rotations – and you can tell which by looking at the determinant (which is always ± 1). But SU_2 is connected (we can get from any point to any other point by following some path), so $\det(\rho(g))$ can't jump between ± 1 (since ρ is continuous). So then $\det(\rho(g))$ is constant as g varies. We know that $\det(\rho(I)) = 1$, so then $\det(\rho(g))$ is always 1. So in fact, this is a homomorphism $\rho : SU_2 \to SO_3$.

Note 30.4
We could write down this homomorphism in terms of the matrix entries – we start with a 2×2 complex matrix and create a 3×3 real one, and we could explicitly write down the homomorphism. But it's more interesting to think about it geometrically, by considering the action of SU_2 on one of its conjugacy classes.

Note 30.5
You can go further with this – given a point on the 3-sphere, we can ask how to figure out what angle and axis of rotation it corresponds to. This is written up in the notes, but we won't discuss it here. But you can go really far by playing around with the group-theoretic constructions we've seen earlier and trying to picture what they mean.

Student Question. *Did we show that ρ was continuous?*

Answer. *No, we did not. In order to check that it's continuous, you can write down the map in terms of the entries. But this shouldn't be surprising – we have $\rho(g)v = gvg^{-1}$, and we can write down a explicit formula for g^{-1} in terms of g. So the matrix $\rho(g)$ is something we can write down explicitly in terms of coordinates. (In fact, you can also use this explicit formula to show it's in SO_3, but it's nicer to just show that it's continuous and then deduce it's in SO_3 by thinking about it geometrically.)*

Student Question. *Was the action SU_2 defined on $\mathbb{E}$ just left multiplication?*

Answer. *No, it's conjugation. The idea is that $\mathbb{E}$ is one of the conjugacy classes of the group, and all the conjugacy classes are orbits with respect to the conjugation action. (This is why the action is transitive as well.)*

30.5 One-Parameter Groups

Now we'll return to looking at linear groups more generally – subgroups G of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ which satisfy some condition (for example, preserving volume or a bilinear form).

Definition 30.6
A **one-parameter group** (in $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$) is a differentiable homomorphism from $\mathbb{R} \to GL_n(\mathbb{R})$ or $\mathbb{R} \to GL_n(\mathbb{C})$.

In otherwords, it's a function $\varphi : \mathbb{R} \to GL_n(\mathbb{C})$ with $t \mapsto \varphi(t)$. It should be a group homomorphism, so $\varphi(s+t) = \varphi(s) + \varphi(t)$, and it should be differentiable (where we think of $GL_n(\mathbb{C})$ as sitting inside $\mathbb{R}^{2n^2}$ – then each entry of the function should be a differentiable function on $\mathbb{R}$).

One way to think of this definition is as an analog of when we looked at maps $\mathbb{Z} \to G$. The integers are in some sense the simplest group we can write down – it has just one generator and no relations – and we can look at maps $\mathbb{Z} \to G$, to help us study G.

The idea here is that $(\mathbb{R}, +)$ is basically the simplest one-dimensional group. (We haven't defined dimension, but you can think of dimension as how many parameters we have. There are other one-dimensional groups, like a circle, but the real numbers are simpler because we don't have relations like $2\pi = 0$ here.)

We've already seen a few examples of one-parameter groups:

Example 30.7
In SU_2, the map $\theta \mapsto \cos\theta I + \sin\theta x$ (for any $x \in \mathbb{E}$) is a one-parameter group.

These one-parameter groups are the longitudes. We've seen that every point lies in some longitude, and therefore some one-parameter group; in general, that isn't always true.

Let's see another example, when $n = 1$.

Example 30.8
When $n = 1$, the map $\varphi : \mathbb{R} \to \mathbb{C}^\times$ with $\varphi(t) = e^{\alpha t}$ (for any $\alpha \in \mathbb{C}$) is a one-parameter group.

Here one-dimensional matrices are just numbers. This construction works because we have

$$\varphi(s+t) = e^{\alpha s + \alpha t} = e^{\alpha s} e^{\alpha t} = \varphi(s)\varphi(t).$$

Note 30.9

This isn't an analysis class, so we won't check the differentiability of these maps. In this example, it can be done by writing everything down in terms of sines and cosines.

Guiding Question

Is there a version of this construction for $n > 1$?

The answer is yes – if we have $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, we can try to define e^A. Taking a number to the power of a matrix doesn't make any sense, but the exponential function also has a description using power series: we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

This is a very nice power series – it converges to e^x everywhere. In fact, you can even take this as the definition of e^x, and you can take the derivative term-by-term.

So we can use this to define e^A as well:

Definition 30.10

The exponential of a matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \in \mathrm{Mat}_{n\times n}(\mathbb{C}).$$

This also converges uniformly as A varies in a bounded region, meaning that for every entry of the matrix, if we take the corresponding entries of each term, then we get a convergent series. As we vary A in a neighborhood, this convergence of that series is uniform. So this gives us a well-defined $n \times n$ matrix e^A. (To be more precise, we can actually put a metric on the space of matrices, and use this to be careful about the notion of convergence.)

This exponential has several nice properties, similarly to the normal exponential.

- The exponential interacts well with conjugation – we have

  $$\begin{aligned} P^{-1}e^AP &= P^{-1}IP + P^{-1}AP + P^{-1}A^2/2P + \cdots \\ &= I + P^{-1}AP + \frac{(P^{-1}AP)^2}{2} + \cdots \\ &= e^{P^{-1}AP}. \end{aligned}$$

  (To be more careful, the LHS and RHS are both defined by limits – where we take the power series and truncate it. The equality is true on the level of these truncations, so the limits are equal as well.)

- If v is an eigenvector of A with eigenvalue λ, then v is also an eigenvector of e^A with eigenvalue e^λ.
- We have $e^{sA}e^{tA} = e^{(s+t)A}$. To prove this, we can expand the RHS out using the Binomial Theorem as

  $$\sum_{n\geq 0} \frac{(s+t)^n A^n}{n!} = \sum_{k,\ell\geq 0} \frac{s^k t^\ell}{(k+\ell)!} \cdot \frac{(k+\ell)!}{k!\ell!} A^k A^\ell.$$

  Then using uniform convergence, we can factor out the infinite sum as

  $$\left(\sum_{k\geq 0} \frac{s^k A^k}{k!}\right)\left(\sum_{\ell\geq 0} \frac{t^\ell A^\ell}{\ell!}\right) = e^{sA}e^{tA}$$

  (the fact that we can rearrange in this way is the result of the strong convergence properties).

  In particular, $e^A e^{-A} = I$, so we actually have $e^A \in GL_n(\mathbb{C})$. (We're working in the complex case, but this works equally well in the real case.)

The last result means that for any $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, $\varphi(t) = e^{tA}$ is a one-parameter group in $GL_n(\mathbb{C})$.

We can ask two questions about these one-parameter groups:

Guiding Question
Is every one-parameter group of this form?

The answer will be yes, and we will see why in future lectures!

Guiding Question
Given a subgroup $G \leq GL_n$, what are the one-parameter subgroups living inside of G?

We will discuss this question in future lectures as well.