SOLUTION KEY
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Problems:
Chap 24: 1, 10, 20, 21, 22, 28, 32, 52, 60, 61

Problem 24.1. Show that conjugacy is an equivalence relation on a group.

Solution:
Let G be a group. Say that $a \sim b$ if there exists $x \in G$ such that $a = xbx^{-1}$.
Reflexive property:
$a \sim a$ since $a = eae^{-1}$.
Symmetry property:
If $a \sim b$ then there exists $x \in G$ such that $a = xbx^{-1}$. Hence $b = x^{-1}ax$ so $b \sim a$.
Transitive property:
Suppose that $a \sim b$ and $b \sim c$ with $a = xbx^{-1}$ and $b = ycy^{-1}$. Then $a = xycy^{-1}x^{-1} = xyc(xy)^{-1}$. Hence $a \sim c$. ■

Problem 24.10. Exhibit a Sylow 2-subgroup of S_4. Describe an isomorphism from this group to D_4.

Solution:
A Sylow 2-subgroup of S_4 is $H = \{e, (1234), (12)(34), (13)(24), (24), (13), (1432), (14)(32)\}$. Note that H is generated by (1234) and (12)(34).

Define $\phi : H \to D_4$ by $\phi((1234)) = R_{90}$ and $\phi((12)(34)) = V$. Then, ϕ maps generators to generators, $(1234)(12)(34)(1234) = (12)(34)$ matches the relation $R_{90}VR_{270} = V$, and H and D_4 have the same order, these groups are isomorphic. ■

Problem 24.20. Let G be a noncyclic group of order 21. How many Sylow 3-subgroups does G have?

Solution:
By Theorem 24.5, the number of Sylow 3-subgroups of G, n_3 is equal to 1 mod 3 and divides 7. Hence n_3 is either 1 or 7. Similarly, there can only be one Sylow 7-subgroup, since $n_7 \equiv 1 \bmod 7$ and $n_7|3$. By the Corollary on pg 416, the unique Sylow 7-subgroup of order 7, call it H, is normal in G.

Suppose that $n_3 = 1$. Then the unique Sylow 3-subgroup of order 3, call it K, is normal in G. Now since the order of H and K are relatively prime, $H \cap K = \{e\}$, and hence $|HK| = \frac{|H||K|}{|H \cap K|} = 21$. Thus $G = H \times K$. But then $G \approx H \oplus K \approx \mathbb{Z}_7 \oplus \mathbb{Z}_3$ is cyclic and we assumed G was not cyclic.

Hence $n_3 = 7$. ■

Problem 24.21. Prove that a noncyclic group of order 21 must have 14 elements of order 3.

Solution:
By the previous problem, we know that such a group must have 7 Sylow 3-subgroups which are conjugate to each other. Any element of order 3 must be contained in one of these subgroups. Since 3 is prime, each of these subgroups is cyclic. Hence there are $7 \times (3-1) = 14$ elements of order 3 contained in these subgroups. ■

Problem 24.22. How many Sylow 5-subgroups of S_5 are there? Exhibit two.

Solution:
Let n_5 be the number of Sylow 5-subgroups of S_5. Then $n_5 \mid 24$ and $n_5 \equiv 1 \bmod 5$. Hence $n_5 \in \{1, 6\}$.

Since $\langle(12345)\rangle$ and $\langle(21345)\rangle$ are subgroups of order 5 with $\langle(12345)\rangle \neq \langle(21345)\rangle$, we must have that $n_5 = 6$. ■

Problem 24.28. Determine the number of Sylow 2-subgroups of D_{2m}, where m is an odd integer at least 3.

Solution:
Let $D_{2m} = \langle R, V\rangle$ where $|R| = 2m$, $|V| = 2$, and $VRV = R^{-1}$. There are only two subgroup types for D_{2m}: $\langle R^d\rangle$ or $\langle R^d, R^k V\rangle$ where $d \mid 2m$ and $0 \leq k < d$. By **Problem 3.27**, D_{2m} has at least one subgroup of order 4, given by $H = \langle R^m, V\rangle$. Any other Sylow 2-subgroup is conjugate to this group. We need only check the conjugates of the generators of H.

For any k, $VR^kV = (VRV)^k = R^{-k}$, hence $VR^mV = R^m$. Now $R^kVR^{-k} = R^{2k}V$. Hence $R^kHR^{-k} = \langle R^m, R^{2k}V\rangle$ are the other subgroups of order 4. Hence there are m Sylow 2-subgroups. ■

Problem 24.32. Prove that a group of order 375 has a subgroup of order 15.

Solution:
Since $375 = 5^3 \times 3$, the number of Sylow 5-subgroups, n_5, must satisfy $n_5 \mid 3$ and $n_5 \equiv 1 \bmod 5$. Hence $n_5 = 1$. The number of Sylow 3-subgroups, n_3, must satisfy $n_3 \mid 125$ and $n_3 \equiv 1 \bmod 3$, hence $n_3 = 1$ or 25.

If $n_3 = 1$, let H be the unique (normal) cylic subgroup of order 3. There is also a unique (normal) cyclic subgroup of order 5, K. Since 3 and 5 are prime and relatively prime, $H \cap K = \{e\}$. Hence HK has order 15.

Now suppose that $n_3 = 25$. Let L be a Sylow 3-subgroup. Since the number of Sylow 3-subgroups is 25, $|G : N(L)| = 25$. Thus $|N(L)| = |G|/25 = 15$, and $N(L)$ is a subgroup of order 15. ■

Problem 24.52. Let G be a group of order p^2q^2, where p and q are distinct primes, $q \nmid p^2-1$, and $p \nmid q^2-1$. Prove that G is Abelian. List three pairs of primes that satisfy these conditions.

Solution:
By Sylow's Third Theorem, $n_p \mid q^2$ and $n_p = 1 + kp$, for some k. Suppose that $k > 0$. Then $1 + kp \mid q^2$ and hence $kp \mid q^2 - 1$. But p is prime and $p \nmid q^2 - 1$, so $k = 0$. Hence $n_p = 1$. Similarly, by switching the roles of p and q, $n_q = 1$. Let H be the unique (normal) Sylow p-subgroup and let K be the unique (normal) Sylow q-subgroup. By the Corollary on pg. 411, both H and K are Abelian. Furthermore, $H \cap K = \{e\}$ and $|H||K| = p^2q^2 = |G|$, so that $G = H \times K$. Thus, G is Abelian.

Three examples of pairs of primes which satisfy this property are: (5,13), (11,13), (13,17) ■

Problem 24.60. Determine the groups of order 45.

Solution:
A group G of order $45 = 5 \times 3^2$ must have a unique Sylow 5-subgroup since $n_5 \mid 9$ and $n_5 \equiv 1 \bmod 5$ has only the solution $n_5 = 1$. Let H be the unique Sylow 5-subgroup.

Similarly, it has a unique Sylow 3-subgroup since $n_3 \mid 5$ and $n_3 \equiv 1 \bmod 3$ has only the solutions $n_3 = 1$. Let K be the unique Sylow 3-subgroup.

Since $\gcd(|H|, |K|) = 1$, $H \cap K = \{e\}$. Further, $|H||K| = 45 = |G|$. Hence $G = H \times K$. The order of H is prime, hence H is cylic and Abelian. The order of K is 9, a prime squared, hence it is Abelian.

Hence G must be an Abelian group of order 45 and isomorphic to either $\mathbb{Z}_9 \oplus \mathbb{Z}_5$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

■

Problem 24.61. Show that there are at most three nonisomorphic groups of order 21.

Solution:
In fact, there are exactly two nonisomorphic groups of order 21 (see **Figure 24.2**). Two groups cannot be isomorphic if they do not have the same number of Sylow p-subgroups, for a given prime p.

For a group of order 21, the number of Sylow 7-subgroups, n_7, must satisfy $n_7 \mid 3$ and $n_7 \equiv 1 \bmod 7$. Hence $n_7 = 1$. The number of Sylow 3-subgroups, n_3, must satisfy $n_3 \mid 7$ and $n_3 \equiv 1 \bmod 3$. Hence $n_3 \in \{1, 7\}$.

If $n_3 = 1$, then $G \approx \mathbb{Z}_3 \oplus \mathbb{Z}_7$, since 3 and 7 are distinct primes. Hence we need only show that there are at most two nonisomorphic groups of order 21 with $n_3 = 7$. Any such group must be non-Abelian.

Let G have order 21 with 7 Sylow 3-subgroups. Let H be a Sylow 3-subgroup and K the Sylow 7-subgroup. Then $H \cap K = \{e\}$ and $HK = G$, since $|H||K| = 21$ and K is normal in G.

Let h be a generator of H and k a generator of K. Since K is normal, $hkh^{-1} = k^i$, for some i. The group G will be entirely determined by the value of i in this relation, since we have already specified the generators. Since $h^3 = 1$:

$$\begin{aligned}
k &= h^3 k h^{-3} \\
&= h^2 (hkh^{-1}) h^{-2} \\
&= h^2 k^i h^{-2} \\
&= h(hk^i h^{-1}) h^{-1} \\
&= hk^{i^2} h^{-1} \\
&= k^{i^3}
\end{aligned}$$

Hence $i^3 - 1 \mid 7$. This has only two solutions $i = 0$ and $i = 1$. Hence there can be at most two such group, up to isomorphism. ■