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MA 450 HW9
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9.14) Want smallest $k \in \mathbb{Z}_{>0}$ s.t. $k(14+\langle 8\rangle) = \langle 8\rangle$ by def order

$|\langle 8\rangle| = 3 \Rightarrow |\mathbb{Z}_{24}/\langle 8\rangle| = 8$

We want $14k \equiv 0 \pmod 8$

Then $14 = 2\cdot 7 \Rightarrow 4\cdot 14 = 8\cdot 7 \equiv 0 \pmod 8$

So the order of $14+\langle 8\rangle = 4$

9.30) By thm 8.3 + cor and thm 9.6,

$$m = \prod_{i=1}^{k} n_i \quad \Rightarrow \quad U(m) = U_{m/n_1}(m) \times \cdots \times U_{m/n_k}(M)$$
$$\cong U(n_1) \oplus \cdots \oplus U(n_k)$$

where $U_\alpha(\beta) = \{x \in U(\beta) \mid x \bmod \beta = 1\}$

For simplicity of notation let $U = U(165)$

Since $165 = 3\cdot 5\cdot 11$,

$$U \cong U_3 \times U_5 \times U_{11} \cong U_{15} \times U_{11} \cong U_3 \times U_{55} \cong U_5 \times U_{33}$$

9.34) Bezout's Identity $\Rightarrow \exists s,t \in \mathbb{Z}$ s.t. $5s+7t=1$

Then $\forall k \in \mathbb{Z}$, $k = (5s+7t)k = 5ks + 7kt$

That is, $\exists \alpha \in \langle 5\rangle, \beta \in \langle 7\rangle$ s.t. $k = \alpha + \beta \ \forall k \in \mathbb{Z}$

Thus $\langle 5\rangle\langle 7\rangle = HK = \mathbb{Z}$

No, $5\cdot 7 = 35 \in H \cap K \Rightarrow H \cap K$ nontrivial

So there is no internal direct product by def IDP

9.42) Supp. $g \in G$. Then $\exists \alpha \in G$ such that $gH = (\alpha H)^2 = \alpha^2 H$

$\Rightarrow g \in \alpha^2 H$ by prop. of cosets

$\Rightarrow g = \alpha^2 h$ for some $h \in H$

By def H, $\exists \beta \in H$ such that $h = \beta^2$ — G abelian

So $g = \alpha^2 \beta^2 = \alpha\alpha\beta\beta = \alpha\beta\alpha\beta = (\alpha\beta)^2$

Now, I claim $\alpha^n \beta^n = (\alpha\beta)^n$ holds $\forall n \in \mathbb{Z}$

Base case when $n=1$ is trivial. $\underbrace{\text{Assume this holds for } n.}_{(*)}$

Notice $\alpha^{n+1}\beta^{n+1} = \alpha^n \alpha \beta^n \beta = \alpha^n \beta^n \alpha\beta$ $\Leftarrow$ G abelian

$\overset{(*)}{=} (\alpha\beta)^n \alpha\beta = (\alpha\beta)^{n+1}$

Similarly, $\alpha^{n-1}\beta^{n-1} = (\alpha\beta)^{n-1}$

Thus this holds $\forall n \in \mathbb{Z}$

9.48) By cor of Lagrange's Thm, $|G:Z(G)| = \frac{|G|}{|Z(G)|} = 4$

We know from lecture that $|G/Z(G)| = \frac{|G|}{|Z(G)|} = 4$

By example 8.3, $|H| = 4 \Rightarrow H \cong \mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

9.54) a.

	1	i	-1	-i	-k	-j	k	j
1	1	i	-1	-i	-k	-j	k	j
i	i	-1	-i	1	j	-k	-j	k
-1	-1	-i	1	i	k	j	-k	-j
-i	-i	1	i	-1	-j	k	j	-k
-k	-k	-j	k	j	-1	-i	1	i
-j	-j	k	j	-k	i	-1	-i	1
k	k	j	-k	-j	1	i	-1	-i
j	j	-k	-j	k	-i	1	i	-1

b. Notice $Z(G) = \{-1, 1\} = H$

From lecture, $Z(G) \triangleleft G$

So $H \triangleleft G$

c.

	1	i	j	k
1	1	i	j	k
i	i	1	k	j
j	j	k	1	i
k	k	j	i	1

Yes.

$|G/H| = \frac{|G|}{|H|} = 4$

exercise 9.48

$\Downarrow$

$G/H \cong \mathbb{Z}_4$

$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

10.6) Let $\varphi: G \to G$ where $f \mapsto \int f$

Notice $\varphi(f+g) = \int(f+g) = \int f + \int g = \varphi(f) + \varphi(g)$

Also, $(\varphi(f)+\varphi(g))(0) = \varphi(f)(0) + \varphi(g)(0) = 0+0=0$

$Ker(\varphi) = \{x \in G \mid \varphi(x) = e\} \implies$ want x s.t $\int x = 0$

The only constant function that passes through $(0,0)$ is $f=0$

Thus $Ker(\varphi) = \{0\}$

No, $f+f = 2f \in G$ but $\int(2f)(0) = 2\left(\int f\right)(0) = 2\cdot 1 = 2$ so φ is not operation preserving

10.14) Notice that $\varphi(7+7) = \varphi(2) = 6$ but $\varphi(7)+\varphi(7) = 1+1 = 2 \neq 6$

Thus φ is not operation preserving

10.20) Supp. $\varphi: \mathbb{Z}_{20} \to \mathbb{Z}_8$ where $\varphi(1) = 1$

Supp. $g \in \mathbb{Z}_{20}$ s.t. $\varphi(1) = k$ for some $k \in \mathbb{Z}_8$

Then by Thm 10.1.2, $\varphi(n) = nk \quad \forall n \in \mathbb{Z}_{20}$

So φ must be such that $\varphi(20) = 20k \equiv 0 \pmod 8$

$\iff 5k \equiv 0 \pmod 2$

Since $k \in \mathbb{Z}_8$, possible values of k are even numbers in $\mathbb{Z}_8$, namely $0, 2, 4, 6$

$\implies$ 4 homomorphisms

To be onto, φ must "hit" every elt in $\mathbb{Z}_8$.

That is, since 1 generates $\mathbb{Z}_{20}$, $|\varphi(1)|$ should be 8.

But $|0| = 1$, $|2| = 4$, $|4| = 2$, and $|6| = 4$, so no such $\varphi(1)$ exists.

$\implies \nexists$ surjective homomorphism from $\mathbb{Z}_{20} \to \mathbb{Z}_8$

10.24) a. Supp $\varphi(1) = m$. Then $\varphi(7) = 7m \equiv 6 \pmod{15}$.

Notice $7\cdot 3 = 21 \equiv 6 \pmod{15} \implies m = 3 \implies \varphi(x) = 3x \pmod{15}$

b. Since $\varphi: \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ where $\varphi(x) = 3x \pmod{15} \implies Img(\varphi) = \langle 3 \rangle$

c. Want all $\beta \in \mathbb{Z}_{50}$ s.t. $3\beta \equiv 0 \pmod{15} \implies n$ multiple of $\frac{15}{\gcd(3,15)} = 5$ by Thm 4.2

So $0 \leq 5k < 50$ for $k \in \mathbb{Z} \implies Ker(\varphi) = \langle 5 \rangle$