3. 42, 52, 58

4. 2, 8, 10, 41, 62, 64, 72

42) $H \leq G$, $C(H) = \{x \in G \mid xh = hx \ \forall h \in H\}$

Prove $C(H) \leq G$

$x \in G \ \forall x \in C(H)$ by def $C(H) \Rightarrow C(H) \subseteq G$

WTS $a, b \in C(H) \Rightarrow ab \in C(H)$, $b^{-1} \in C(H)$

1) $abx = axb = xab \ \forall x \in C(H) \Leftarrow$ def $C(H)$

✓ $ab \in C(H)$

2) $b \in C(H) \Rightarrow bx = xb \ \forall x \in C(H)$

$(b^{-1}x)^{-1} = x^{-1}b = bx^{-1} = (xb^{-1})^{-1} \Rightarrow b^{-1}x = xb^{-1}$

$b^{-1} \in C(H)$

$\therefore C(H) \leq G$

52) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

$A, B \in SL(2, \mathbb{R})$

Find $|A|, |B|, |AB|$

$4 \quad 3 \quad \infty$

58) $U(15)$ Find all 6 cyclic subgroups

$U(15)$ = Numbers < and coprime to 15

$\{1 = e, 2, 4, 7, 8, 11, 13, 14\} \Rightarrow |U(15)| = 8$

$U(15)$

$\langle 1\rangle = \{1\}$

$\langle 2\rangle = \{1, 2, 4, 8\} = \langle 8\rangle$

$\langle 4\rangle = \{1, 4\}$

$\langle 7\rangle = \{1, 7, 4, 13\} = \langle 13\rangle$

$\langle 11\rangle = \{1, 11\}$

$\langle 14\rangle = \{1, 14\}$

4. 2, 8, 10, 41, 62, 64, 72

2) $|\langle a\rangle| = 6 \quad |\langle b\rangle| = 8 \quad |\langle c\rangle| = 20$ Find generators of each

$\langle a\rangle = \{a, a^2, a^3, a^4, a^5\}$

$\langle b\rangle = \{b, b^2, \ldots, b^7\}$

$\langle c\rangle = \{c, \ldots, c^{19}\}$

By cor, a^k generates $\langle a\rangle \Leftrightarrow |\langle a\rangle| \perp k$. coprime

$\therefore |\langle a\rangle| = 6 \Rightarrow a^5, a$

$|\langle b\rangle| = 8 \Rightarrow b^7, b^5, b^3, b$

$|\langle c\rangle| = 20 \Rightarrow c, c^3, c^7, c^9, c^{11}, c^{13}, c^{17}, c^{19}$

8) $a \in G$, $|a| = 15$ Find orders of

a) a^3, a^6, a^9, a^{12} b) a^5, a^{10} c) a^2, a^4, a^8, a^{14}

$\langle a\rangle = \{a, a^2, \ldots, a^{14}\}$

a) $|a^3| \overset{\text{by cor}}{=} |\langle a^3\rangle| = 5$

$|a^6| = |\langle a^6\rangle| = |\langle a^{\gcd(6,15)}\rangle| = |\langle a^3\rangle| = 5$

6 12 18 24 30 36
3 9 0 6

$|a^9| = |\langle a^9\rangle| = |\langle a^d\rangle| = |\langle a^3\rangle| = 5$

$d = \gcd(9, 15)$

$|a^{12}| = \cdots = 5$

b) $|a^5| = |\langle a^5\rangle| = 3$

$|a^{10}| = |\langle a^{\gcd(10,15)}\rangle| = |\langle a^5\rangle| = 3$

c) $|a^2| = |\langle a^2\rangle|$ $\gcd(2,15)=1 \Rightarrow |\langle a^2\rangle| = 15$

$|a^4| = 15$

$|a^8| = 15$

$|a^{14}| = 15$

10) $\mathbb{Z}_{24}$ list all generators for sgp of ord 8
Let $G = \langle a\rangle$, $|a| = 24$

By FTCG, the subgroup $\langle a^{n/k}\rangle$ is the unique sgp of $\langle a\rangle$ of order k, where k is pos divisor of $\underset{|\langle a\rangle|}{\overset{\shortparallel}{n}}$

$\therefore$ sgp of order $8 = \langle a^3\rangle$. $1 \perp 3$ and $2 \perp 3 \Rightarrow$ generated by a, a^2

41) $a, b \in G$ $ab = ba$ $|a| = m$ $|b| = n$
If $\langle a\rangle \cap \langle b\rangle = \{e\}$, prove G contains g st $|g| = \text{lcm}(m,n)$
Show this need not be true if $ab \neq ba$

Consider $ab \in G$. Since $ab = ba$, $(ab)^k = a^k b^k$, $k \in \mathbb{Z}$

let $l = \mathrm{lcm}(a,b)$ and $|ab| = r$

$(ab)^r = a^r b^r = e \Rightarrow a^r = b^{-r} \Rightarrow a^r = e$

$a^r \in \langle a, b \rangle = \{e\}$

$a^m = a^r = e \Rightarrow m | r$

Similarly $b^r = e \in \langle a, b \rangle \Rightarrow n | r$

$m|r, n|r$

$\Downarrow$

$l|r$

Since $l = \alpha m = \beta n$ for some $\alpha, \beta \in \mathbb{Z}$,

$(ab)^l = a^l b^l = (a^m)^\alpha (b^n)^\beta = e$

$\Rightarrow r | l \Rightarrow r = l$

$\langle S \rangle \leq G$

62 $S \subseteq \langle S \rangle$ and if $\exists H \leq G$ st $S \leq H$,

64 then $\langle S \rangle \leq H$

72