4. Cyclic groups

Lemma 4.1. *Let G be a group and let H_i, $i \in I$ be a collection of subgroups of G.*

Then the intersection

$$H = \bigcap_{i \in I} H_i,$$

is a subgroup of G

Proof. First note that H is non-empty, as the identity belongs to every H_i. We have to check that H is closed under products and inverses.

Suppose that g and h are in H. Then g and h are in H_i, for all i. But then $hg \in H_i$ for all i, as H_i is closed under products. Thus $gh \in H$.

Similarly as H_i is closed under taking inverses, $g^{-1} \in H_i$ for all $i \in I$. But then $g^{-1} \in H$.

Thus H is indeed a subgroup. □

Definition-Lemma 4.2. *Let G be a group and let S be a subset of G.*

The **subgroup** $H = \langle S \rangle$ **generated by** S *is equal to the smallest subgroup of G that contains S.*

Proof. The only thing to check is that the word smallest makes sense.

Suppose that H_i, $i \in I$ is the collection of subgroups that contain S. By (4.1), the intersection H of the H_i is a subgroup of G.

On the other hand H obviously contains S and it is contained in each H_i.

Thus H is the smallest subgroup that contains S. □

Lemma 4.3. *Let S be a non-empty subset of G.*

Then the subgroup H generated by S is equal to the smallest subset of G, containing S, that is closed under taking products and inverses.

Proof. Let K be the smallest subset of G, closed under taking products and inverses.

As H is closed under taking products and inverses, it is clear that H must contain K. On the other hand, as K is a subgroup of G, K must contain H.

But then $H = K$. □

Definition 4.4. *Let G be a group. We say that a subset S of G* **generates** *G, if the smallest subgroup of G that contains S is G itself.*

Definition 4.5. *Let G be a group. We say that G is* **cyclic** *if it is generated by one element.*

Let $G = \langle a \rangle$ be a cyclic group. By (4.3)

$$G = \{\, a^i \mid i \in \mathbb{Z} \,\}.$$

Definition 4.6. *Let G be a group and let $g \in G$ be an element of G.*

The **order** *of g is equal to the cardinality of the subgroup generated by g.*

Lemma 4.7. *Let G be a finite group and let $g \in G$.*

Then the order of g divides the order of G.

Proof. Immediate from Lagrange's Theorem. $\square$

Lemma 4.8. *Let G be a group of prime order.*

Then G is cyclic.

Proof. If the order of G is one, there is nothing to prove. Otherwise pick an element g of G not equal to the identity. As g is not equal to the identity, its order is not one. As the order of g divides the order of G and this is prime, it follows that the order of g is equal to the order of G.

But then $G = \langle g \rangle$ and G is cyclic. $\square$

It is interesting to go back to the problem of classifying groups of finite order and see how these results change our picture of what is going on.

Now we know that every group of order 1, 2, 3 and 5 must be cyclic. Suppose that G has order 4. There are two cases. If G has an element a of order 4, then G is cyclic.

We get the following group table.

$*$	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

Replacing a^2 by b, a^3 by c we get

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Now suppose that G does not contain any elements of order 4. Since the order of every element divides 4, the order of every element must be 1, 2 or 4. On the other hand, the only element of order 1 is the identity element. Thus if G does not have an element of order 4, then every element, other than the identity, must have order 2.

In other words, every element is its own inverse.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	$?$	
b	b		e	
c	c			e

Now ? must in fact be c, simply by a process of elimination. In fact we must put c somewhere in the row that contains a and we cannot put it in the last column, as this already contains c. Continuing in this way, it turns out there is only one way to fill in the whole table

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

So now we have a complete classification of all finite groups up to order five (it easy to see that there is a cyclic group of any order; just take the rotations of a regular n-gon). If the order is not four, then the only possibility is a cyclic group of that order. Otherwise the order is four and there are two possibilities.

Either G is cyclic. In this case there are two elements of order 4 (a and a^3) and one element of order two (a^2). Otherwise G has three elements of order two. Note however that G is abelian.

So the first non-abelian group has order six (equal to D_3).

One reason that cyclic groups are so important, is that any group G contains lots of cyclic groups, the subgroups generated by the elements of G. On the other hand, cyclic groups are reasonably easy to understand. First an easy lemma about the order of an element.

Lemma 4.9. *Let G be a group and let $g \in G$ be an element of G.*

Then the order of g is the smallest positive number k, such that $a^k = e$.

Proof. Replacing G by the subgroup $\langle g \rangle$ generated by g, we might as well assume that G is cyclic, generated by g.

Suppose that $g^l = e$. I claim that in this case

$$G = \{ e, g, g^2, g^3, g^4, \dots, g^{l-1} \}.$$

Indeed it suffices to show that the set is closed under multiplication and taking inverses.

Suppose that g^i and g^j are in the set. Then $g^i g^j = g^{i+j}$. If $i + j < l$ there is nothing to prove. If $i + j \geq l$, then use the fact that $g^l = e$ to rewrite g^{i+j} as g^{i+j-l}. In this case $i+j-l>0$ and less than l. So the set is closed under products.

Given g^i, what is its inverse? Well $g^{l-i}g^i = g^l = e$. So g^{l-i} is the inverse of g^i. Alternatively we could simply use the fact that H is finite, to conclude that it must be closed under taking inverses.

Thus $|G| \leq l$ and in particular $|G| \leq k$. In particular if G is infinite, there is no integer k such that $g^k = e$ and the order of g is infinite and the smallest k such that $g^k = e$ is infinity. Thus we may assume that the order of g is finite.

Suppose that $|G| < k$. Then there must be some repetitions in the set

$$\{ e, g, g^2, g^3, g^4, \dots, g^{k-1} \}.$$

Thus $g^a = g^b$ for some $a \neq b$ between 0 and $k-1$. Suppose that $a < b$. Then $g^{b-a} = e$. But this contradicts the fact that k is the smallest integer such that $g^k = e$. $\square$

Lemma 4.10. *Let G be a finite group of order n and let g be an element of G.*

Then $g^n = e$.

Proof. We know that $g^k = e$ where k is the order of g. But k divides n. So $n = km$. But then

$$g^n = g^{km} = (g^k)^m = e^m = e. \qquad \square$$

Lemma 4.11. *Let G be a cyclic group, generated by a.*

Then

(1) G is abelian.

(2) If G is infinite, the elements of G are precisely

$$\dots a^{-3}, a^{-2}, a^{-1}, e, a, a^2, a^3, \dots$$

(3) If G is finite, of order n, then the elements of G are precisely

$$e, a, a^2, \dots, a^{n-2}, a^{n-1},$$

and $a^n = e$.

Proof. We first prove (1). Suppose that g and h are two elements of G.

As G is generated by a, there are integers m and n such that $g = a^m$ and $h = a^n$. Then

$$\begin{aligned} gh &= a^m a^n \\ &= a^{m+n} \\ &= a^{n+m} \\ &= hg. \end{aligned}$$

Thus G is abelian. Hence (1).

(2) and (3) follow from (4.9). □

Note that we can easily write down a cyclic group of order n. The group of rotations of an n-gon forms a cyclic group of order n. Indeed any rotation may be expressed as a power of a rotation R through $2\pi/n$. On the other hand, $R^n = 1$.

However there is another way to write down a cyclic group of order n. Suppose that one takes the integers $\mathbb{Z}$. Look at the subgroup $n\mathbb{Z}$. Then we get equivalence classes modulo n, the left cosets.

$$[0], [1], [2], [3], \ldots, [n-1].$$

I claim that this is a group, with a natural method of addition. In fact I define

$$[a] + [b] = [a + b].$$

in the obvious way. However we need to check that this is well-defined. The problem is that the notation

$$[a]$$

is somewhat ambiguous, in the sense that there are infinitely many numbers a' such that

$$[a'] = [a].$$

In other words, if the difference $a' - a$ is a multiple of n then a and a' represent the same equivalence class. For example, suppose that $n = 3$. Then $[1] = [4]$ and $[5] = [-1]$. So there are two ways to calculate

$$[1] + [5].$$

One way is to add 1 and 5 and take the equivalence class. $[1] + [5] = [6]$. On the other hand we could compute $[1] + [5] = [4] + [-1] = [3]$. Of course $[6] = [3] = [0]$ so we are okay.

So now suppose that a' is equal to a modulo n and b' is equal to b modulo n. This means

$$a' = a + pn$$

and

$$b' = b + qn,$$

where p and q are integers.

Then

$$a' + b' = (a + pn) + (b + qn) = (a + b) + (p + q)n.$$

So we are okay

$$[a + b] = [a' + b'],$$

and addition is well-defined. The set of left cosets with this law of addition is denote $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n. Is this a group? Well associativity comes for free. As ordinary addition is associative, so is addition in the integers modulo n.

[0] obviously plays the role of the identity. That is

$$[a] + [0] = [a + 0] = [a].$$

Finally inverses obviously exist. Given $[a]$, consider $[-a]$. Then

$$[a] + [-a] = [a - a] = [0].$$

Note that this group is abelian. In fact it is clear that it is generated by [1], as 1 generates the integers $\mathbb{Z}$.

How about the integers modulo n under multiplication? There is an obvious choice of multiplication.

$$[a] \cdot [b] = [a \cdot b].$$

Once again we need to check that this is well-defined. Exercise left for the reader.

Do we get a group? Again associativity is easy, and [1] plays the role of the identity. Unfortunately, inverses don't exist. For example [0] does not have an inverse. The obvious thing to do is throw away zero. But even then there is a problem. For example, take the integers modulo 4. Then

$$[2] \cdot [2] = [4] = [0].$$

So if you throw away [0] then you have to throw away [2]. In fact given n, you should throw away all those integers that are not coprime to n, at the very least. In fact this is enough.

Definition-Lemma 4.12. *Let n be a positive integer.*

The **group of units,** U_n, *for the integers modulo n is the subset of $\mathbb{Z}/n\mathbb{Z}$ of integers coprime to n, under multiplication.*

Proof. We check that U_n is a group.

First we need to check that U_n is closed under multiplication. Suppose that $[a] \in U_n$ and $[b] \in U_n$. Then a and b are coprime to n. This means that if a prime p divides n, then it does not divide a or b. But then p does not divide ab. As this is true for all primes that divide n, it follows that ab is coprime to n. But then $[ab] \in U_n$. Hence multiplication is well-defined.

This rule of multiplication is clearly associative. Indeed suppose that $[a]$, $[b]$ and $[c] \in U_n$. Then

$$\begin{aligned}([a] \cdot [b]) \cdot [c] &= [ab] \cdot c \\ &= [(ab)c] \\ &= [a(bc)] \\ &= [a] \cdot [bc] \\ &= [a] \cdot ([b] \cdot [c]).\end{aligned}$$

So multiplication is associative.

Now 1 is coprime to n. But then $[1] \in U_n$ and this clearly plays the role of the identity.

Now suppose that $[a] \in U_n$. We need to find an inverse of $[a]$. We want an integer b such that

$$[ab] = 1.$$

This means that

$$ab + mn = 1,$$

for some integer m. But a and n are coprime. So by Euclid's algorithm, such integers exist. □

Definition 4.13. *The* **Euler ϕ function** *is the function $\varphi(n)$ which assigns the order of U_n to n.*

Lemma 4.14. *Let a be any integer, which is coprime to the positive integer n.*

Then $a^{\phi(n)} = 1 \mod n$.

Proof. Let $g = [a] \in U_n$. By (4.10) $g^{\phi(n)} = e$. But then

$$[a^{\phi}(n)] = [1].$$

Thus

$$a^{\phi(n)} = 1 \mod n.$$ □

Given this, it would be really nice to have a quick way to compute $\varphi(n)$.

Lemma 4.15. *The Euler φ function is multiplicative.*

That is, if m and n are coprime positive integers,

$$\varphi(mn) = \varphi(m)\varphi(n).$$

Proof. We will prove this later in the course. □

Given (4.15), and the fact that any number can be factored, it suffices to compute $\varphi(p^k)$, where p is prime and k is a positive integer.

Consider first $\varphi(p)$. Well every number between 1 and $p-1$ is automatically coprime to p. So $\varphi(p) = p - 1$.

Theorem 4.16. *(Fermat's Little Theorem) Let a be any integer. Then $a^p = a \mod p$. In particular $a^{p-1} = 1 \mod p$ if a is coprime to p.*

Proof. Follows from (4.14). □

How about $\varphi(p^k)$? Let us do an easy example.

Suppose we take $p = 3$, $k = 2$. Then of the eight numbers between 1 and 8, two are multiples of 3, 3 and $6 = 2 \cdot 3$. More generally, if a number between 1 and $p^k - 1$ is not coprime to p, then it is a multiple of p. But there are $p^{k-1} - 1$ such multiples,

$$p = 1 \cdot p, 2p, 3p, \dots (p^{k-1} - 1)p.$$

Thus $(p^k - 1) - (p^{k-1} - 1) - p^k - p^{k-1}$ numbers between 1 and p^k are coprime to p. We have proved

Lemma 4.17. *Let p be a prime number. Then*

$$\varphi(p^k) = p^k - p^{k-1}.$$

Example 4.18. *What is the order of U_{5000}?*

$$5000 = 5 \cdot 1000 = 5 \cdot (10)^3 = 5^4 \cdot 2^3.$$

Now

$$\varphi(2^3) = 2^3 - 2^2 = 4,$$

and

$$\varphi(5^4) = 5^4 - 5^3 = 5^3(4) = 125 \cdot 4.$$

As the Euler-phi function is multiplicative, we get

$$\varphi(5000) = 4 \cdot 4 \cdot 125 = 2^4 \cdot 5^3 = 2000.$$

It is also interesting to see what sort of groups one gets. For example, what is U_6?

$\varphi(6) = \varphi(2)\varphi(3) = 1 \cdot 2 = 2$. Thus we get a cyclic group of order 2. In fact 1 and 5 are the only numbers coprime to 6.

$$5^2 = 25 = 1 \mod 6.$$

How about U_8? Well

$$\varphi(8) = 4.$$

So either U_8 is either cyclic of order 4, or every element has order 2. 1, 3, 5 and 7 are the numbers coprime to 2. Now

$$3^2 = 9 = 1 \mod 8,$$

$$5^2 = 25 = 1 \mod 8,$$

and

$$7^2 = 49 = 1 \mod 8.$$

So

$$[3]^2 = [5]^2 = [7]^2 = [1]$$

and every element of U_8, other than the identity, has order two. But then U_8 cannot be cyclic.