MA 450 Homework 12

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Exercise 12.2

We wish to find the unity in the ring $R = \{0,\ 2,\ 4,\ 6,\ 8\}$ under addition and multiplication modulo 10.

That is, we wish to find an element $u \in R$ such that $a \cdot u \equiv a \pmod{10}$. Notice that

$$\begin{aligned}
0 \cdot 6 = 0 &\equiv 0 \pmod{10} \\
2 \cdot 6 = 12 &\equiv 2 \pmod{10} \\
4 \cdot 6 = 24 &\equiv 4 \pmod{10} \\
6 \cdot 6 = 36 &\equiv 6 \pmod{10} \\
8 \cdot 6 = 48 &\equiv 8 \pmod{10}
\end{aligned}$$

By theorem, the unity of a ring is unique. Thus the unity of R is 6.

Exercise 12.6

Consider the ring $\mathbb{Z}_6$.

The first property does not hold because $3^2 = 9 \equiv 3 \pmod 6$.

The second property does not hold because $2 \cdot 3 = 12 \equiv 0 \pmod 6$.

The third property does not hold because $2 \cdot 3 \equiv 4 \cdot 3 \equiv 0 \pmod 6$ and $2 \neq 4$ but $3 = 3$.

Thus $n = 6$. No, $n = 6$ is not prime.

Exercise 12.18

Let $a, b \in S$.

Then, $(a - b)x = ax - bx = 0 - 0 = 0$ and $(ab)x = a(bx) = a \cdot 0 = 0$ by def ring.

Thus, S is a subring of R by the subring test.

Exercise 12.22

By the one step test, we must show $ab^{-1} \in U(R)$ whenever $a, b \in U(R)$.

That is, we must show ab^{-1} is a unit in R whenever a, b are.

Let $a, b \in U(R)$. Then, $a^{-1}, b^{-1} \in U(R)$ by def unit.

To show ab^{-1} is a unit in R, we must show there exists some $c \in R$ such that $ab^{-1}c = 1$.

Let $c = ba^{-1}$. Then, $(ab^{-1})(ba^{-1}) = ab^{-1}ba^{-1} = aa^{-1} = 1$.

Thus $U(R)$ is a group under the multiplication of R.

Exercise 12.23

Trivially we can see that ± 1 and $\pm i$ are units in $\mathbb{Z}[i]$.

Let $x \in \mathbb{Z}[i]$. By def $\mathbb{Z}[i]$, $x = a + bi$ for some $a, b \in \mathbb{Z}$.

To find any other units, we must find $x \in \mathbb{Z}[i]$ such that $x^{-1} \in \mathbb{Z}[i]$ by def unit.

We know that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$, and we know how to find multiplicative inverse in $\mathbb{C}$.

$$\begin{aligned} x^{-1} = \frac{1}{a+bi} &= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \\ &= \frac{a-bi}{a^2+b^2} \\ &= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i \end{aligned}$$

However, notice that $\frac{a}{a^2+b^2} \in \mathbb{Z} \iff a^2+b^2=1$, but this only holds for ± 1 and $\pm i$.

Thus $U(\mathbb{Z}[i]) = \{\pm 1, \pm i\}$.

Exercise 12.31

Consider the ring $R = M_2(\mathbb{Z})$. Let $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then, $ab = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $ba = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$.

Exercise 12.44

Since n is even, we know $n = 2k$ for some $k \in \mathbb{Z}$.

By properties of rings, $(-a)^2 = (-a)(-a) = aa = a^2$.

Then, $a = (a)^n = a^{2k} = (a^2)k$ and $-a = (-a)^n = (-a)^{2k} = ((-a)^2)^k$.

Exercise 12.50

Let $a, b \in R$. Then, $a + b = (a+b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$ by def R.

This implies $ab + ba = 0 \iff ab = -ba$.

By properties of rings and def R, $-ba = (-ba)^2 = (ba)^2 = ba$.

Thus $ab = ba$ and R is commutative.