1 Fixed Fields

Definition 1 (Fixed field). Let $L : K$ be a field extension and $G \leqslant \mathrm{Aut}(L)$. Then the fixed field of G is

$$\mathrm{Fix}_L(G) = L^G = \{\alpha \in L : g\alpha = \alpha \ \forall g \in G\}$$

Theorem 1.1. *Let $K, M \subseteq L$ be fields and $G, H \leqslant \mathrm{Aut}(L)$. Then*

1) if $K \subseteq M$, then $\mathrm{Gal}(L : K) \geqslant \mathrm{Gal}(L : M)$;

2) if $G \leqslant H$, then $\mathrm{Fix}_L(G) \supseteq \mathrm{Fix}_L(H)$;

3) $K \subseteq \mathrm{Fix}_L(\mathrm{Gal}(L : K))$;

4) $G \leqslant \mathrm{Gal}(L : \mathrm{Fix}_L(G))$;

5) $\mathrm{Gal}(L : K) = \mathrm{Gal}(L : \mathrm{Fix}_L(\mathrm{Gal}(L : K)))$;

6) $\mathrm{Fix}_L(G) = \mathrm{Fix}_L(\mathrm{Gal}(L : \mathrm{Fix}_L(G)))$.

Definition 2 (Galois Extension). Let $L : K$ be a field extension. Then $L : K$ is a Galois extension if it is normal and separable.

Theorem 1.2. *Let $L : K$ be algebraic. Then $L : K$ is Galois $\iff K = \mathrm{Fix}_L(\mathrm{Gal}_K(L))$*

Theorem 1.3. *Suppose that L is a field, $G \leqslant \mathrm{Aut}(L)$ such that $|G| < \infty$, and put $K = \mathrm{Fix}_L(G)$. Then $L : K$ is a finite Galois extension with $[L : K] = |\mathrm{Gal}(L : K)|$, and furthermore $G = \mathrm{Gal}_K(L)$.*

Theorem 1.4. *Let $L : K$ be finite.*

1. *If $L : K$ is a Galois extension, then $|\mathrm{Gal}(L : K)| = [L : K]$ and $K = \mathrm{Fix}_L(\mathrm{Gal}(L : K))$.*
2. *If $L : K$ is not Galois, then $|\mathrm{Gal}(L : K)| < [L : K]$ and K is a proper subfield of $\mathrm{Fix}_L(\mathrm{Gal}(L : K))$.*

Corollary 1. *Let $L : M : K$ be a tower such that $L : K$ is Galois. Then $L : M$ is Galois.*