1 SOLUBLE GROUPS II

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1 Soluble Groups II

Theorem 1.1 (Theorem - Definition). Let G be a group. Then the following are equivalent:

- 0. G is a (finite) soluble group;
- 1. There exists some $n \in \mathbb{Z}^+$ such that $G^{(n)} = \{e\}$;
- 2. There exists a normal series

$$\{Id.\} = G_n \triangleleft G_{n-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G$$

such that all quotients G_{j-1}/G_j are abelian;

3. There exists a subnormal series such that quotients G_{j-1}/G_j are abelian.

Definition 1 (Derived group). Let G be a group. Then the *derivative of* G is $G' = \langle [x,y] : x,y \in G \rangle = [G,G]$ where $[x,y] = xyx^{-1}y^{-1}$ is the *commutator* of x and y, and (G')' = G''.

Definition 2 (Derived series). The *derived series* of G is $G^{(n)} = (G^{(n-1)})'$ and $\{\text{Id.}\} = G^{(n)} \triangleleft G^{(n-1)} \triangleleft \cdots \triangleleft G' \triangleleft G$ (not to be confused with $G_{n+1} = [G_n, G]$, the *lower central series*).

Lemma 1.2. Let $\varphi: G \mapsto H$ be an epimorphism. Then $\varphi(G') = H'$.

Definition 3 (Composition series). Let G be a group. Then a composition series of G is a subnormal series of finite length

$$\{Id.\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{\ell-1} \triangleleft G_{\ell} = G$$

such that G_j/G_{j-1} is a simple group for all j.

Theorem 1.3 (Jordan-Hölder). Any 2 composition series of some group G are equivalent up to permutation and isomorphism.

Theorem 1.4. Let K be a field with char $K \neq 2$ and let $f \in K[t]$ be a separable polynomial with splitting field L. Then f = 0 is solvable by quadratic radicals $\iff [L:K] = 2^t$.