1 Splitting Fields, Abel-Ruffini

Definition 1 (Splitting field). *Let $L : K$ with embedding $\varphi : K \to L$ and $f \in K[t] \setminus K$. We say f splits over L if $\varphi(f) = c \prod_{j=1}^{n} (x - \alpha_j)$ for $\alpha_j \in L$ and $c \in \varphi(K)$. If f splits over L and $\varphi(K) \subseteq M \subseteq L$, then we say that $M : K$ is a splitting field extension for f if M is the smallest subfield of L containing $\varphi(K)$ over which f splits.*

Lemma 1.1. *Let $L : K$ be a splitting field extension for $f \in K[t]$ relative to the embedding $\varphi : K \to L$, and let $\alpha_j \in L$ be roots of $\varphi(f)$. Then $L = \varphi(K)(\alpha_1, \ldots, \alpha_n)$.*

Lemma 1.2. *Let $L : K$ be a splitting field extension for $f \in K[t] \setminus K$. Then $[L : K] \leq (\deg f)!$.*

Lemma 1.3. *Let $L : K$ and $M : K$ be splitting field extensions for $f \in K[t] \setminus K$. Then $L \cong M$ (in particular, $[L : K] = [M : K]$).*

Definition 2 (Radical, radical extension, solvability by radicals). *Let $L : K$ and $\beta \in L$. We say that β is radical over K when $\beta^n \in K$ for some $n \in \mathbb{N}$ (so $\beta = \alpha^{1/n}$ for some $\alpha \in K$ and some $n \in \mathbb{N}$). We say that $L : K$ is an extension by radicals when there is a tower of field extensions $L = L_r : L_{r-1} : \cdots : L_0 = K$ such that $L_i = L_{i-1}(\beta_i)$ with β_i radical over L_{i-1} (for $1 \leq i \leq r$). We say $f \in K[t]$ is solvable by radicals if there is a radical extension of K over which f splits.*

Theorem 1.4 (Abel-Ruffini). *Let $K = \mathbb{C}(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are formal variables. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be the generic polynomial of degree $n \geq 5$ over K. Then $f(x)$ is not solvable by radicals.*