Exercise 9.1.1. Let L be the splitting field of the polynomial $t^{13} - 1$. Find all subgroups of $\mathrm{Gal}_{\mathbb{Q}}(L)$.

Solution. $\square$

Exercise 9.1.2. How many intermediate subfields are there in the extension $L : \mathbb{Q}$?

Solution. □

Exercise 9.2. Draw the lattice of subfields and corresponding lattice of subgroups of $\mathrm{Gal}_{\mathbb{F}_3}(\mathbb{F}_{3^8})$. Find orders of all subgroups of $\mathrm{Gal}_{\mathbb{F}_3}(\mathbb{F}_{3^8})$.

Solution. □

Exercise 9.3. Prove Artin's theorem: let $[L : K] < \infty$, $G := \mathrm{Gal}_K(L)$. Then $[L : L^G]$ is a Galois extension.

Solution. □

Exercise 9.4. Let $L : K$ be a finite Galois extension, $G := \mathrm{Gal}_K(L)$. For any $\alpha \in L$ define

$$\mathrm{Tr}(\alpha) = \sum_{g \in G} g(\alpha) \quad \text{and} \quad \mathrm{Norm}(\alpha) = \prod_{g \in G} g(\alpha).$$

Prove that for an arbitrary $\alpha \in L$ one has $\mathrm{Tr}(\alpha), \mathrm{Norm}(\alpha) \in K$.

Solution. $\square$

Exercise 9.4.1. Find all of the subfields of $\mathbb{Q}(2^{1/3}, \exp(2\pi i/3))$.

Solution. $\square$

Exercise 9.4.2. Draw the lattice of subfields and corresponding lattice of subgroups of $\mathrm{Gal}_{\mathbb{Q}}(\mathbb{Q}(2^{1/3}, \exp(2\pi i/3)))$.

Solution. $\square$