Exercise 10.1. Let $K, E, F \subseteq L$ be fields, $E : K, F : K$ be finite extensions. Prove

(a) if $E : K$ is separable, then $EF : F$ is separable;

(b) if $E : K$ and $F : K$ are both separable, then $EF : K$ and $E \cap F : K$ are both separable;

(c) if $E : K$ is Galois, then $EF : F$ is Galois;

(d) if $E : K$ and $F : K$ are both Galois, then $EF : K$ and $E \cap F : K$ are both Galois.

(a) *Solution.* Suppose $E : F$ is separable. □

(b) *Solution.* Suppose $E : K$ and $F : K$ are both separable. □

(c) *Solution.* Suppose $E : K$ is Galois. □

(d) *Solution.* Suppose $E : K$ and $F : K$ are both Galois. □

Exercise 10.2. (a) Find the splitting field L of the polynomial $f(t) = t^4 - 4t^2 + 5$.

(b) Prove that $[L : \mathbb{Q}]$ is either 4 or 8.

(c) Find 10 intermediate fields of the extension $L : \mathbb{Q}$ and their degrees.

(d) (for enthusiasts) Draw the lattice of subfields and corresponding lattice of subgroups of $\mathrm{Gal}_{\mathbb{Q}}(f)$.

(a) *Solution.* Notice if we set $t^4 - 4t^2 + 5 = 0$, then we can subtract 1 to see $t^4 - 4t^2 + 4 = (t^2 - 2)^2 = -1$. Hence $t^2 - 2 = \pm i$ and $t \in \left\{\pm\sqrt{2 \pm i}\right\}$. We note that if $w = \sqrt{a + bi}$ then $w^2 = a + bi$ and $\overline{w^2} = \overline{w}^2 = a - bi$, whence $\overline{w} = \sqrt{a - bi}$. That is, the square roots of complex conjugates are themselves complex conjugates. So it is enough to construct L by adjoining $\sqrt{2 + i}$ to $\mathbb{Q}$ and thus $L = \mathbb{Q}(\sqrt{2 + i})$. □

(b) *Solution.* Set $x = \sqrt{2 + i}$. Then

$$
\begin{aligned}
x^2 &= 2 + i \\
x^2 - 2 &= i \\
x^4 - 4x + 4 &= -1 \\
x^4 - 4x + 5 &= 0
\end{aligned}
$$

Hence the minimum polynomial for $\sqrt{2 + i}$ is $\mu^{\mathbb{Q}}_{\sqrt{2+i}}(x) = x^4 - 4x + 5 = f(x)$. The degree of a field extension is the degree of the minimum polynomial of the generator, so $[L : \mathbb{Q}] = 4$. □

(c) *Solution.* □

(d) *Solution.* □

Exercise 10.3. Draw the lattice of subfields and corresponding lattice of subgroups of $Gal_{\mathbb{Q}}(t^6 + 3)$.
Hint: Use the calculations (and the notation, if you like) from Lecture 18.

Solution. □