

1 Soluble Groups II

Theorem 1.1 (Theorem - Definition). Let G be a group. Then the following are equivalent:

0. G is a (finite) soluble group;
1. There exists some $n \in \mathbb{Z}^+$ such that $G^{(n)} = \{e\}$;
2. There exists a normal series

$$\{Id.\} = G_n \leq G_{n-1} \leq \cdots \leq G_1 \leq G_0 = G$$

such that $G_j \triangleleft G$ and all quotients G_{j-1}/G_j are abelian;

3. There exists a subnormal series such that quotients G_{j-1}/G_j are abelian.

Definition 1 (Derived group). Let G be a group. Then the *derivative of G* is $G' = \langle [x, y] : x, y \in G \rangle = [G, G]$ where $[x, y] = xyx^{-1}y^{-1}$ is the *commutator* of x and y , and $(G')' = G''$.

Definition 2. The *derived series* of G is $G^{(n)} = (G^{(n-1)})'$ and $\{Id.\} = G^{(n)} \triangleleft G^{(n-1)} \triangleleft \cdots \triangleleft G' \triangleleft G$ (not to be confused with $G_{n+1} = [G_n, G]$, the *lower central series*).

Lemma 1.2. Let $\varphi : G \mapsto H$ be an epimorphism. Then $\varphi(G') = H'$.

Definition 3 (Composition series). Let G be a group. Then a *composition series* of G is a subnormal series of finite length

$$\{Id.\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{\ell-1} \triangleleft G_\ell = G$$

such that G_j/G_{j-1} is a simple group for all j .

Theorem 1.3 (Jordan-Hölder). Any 2 composition series of some group G are equivalent up to permutation and isomorphism.

Theorem 1.4. Let K be a field with $\text{char } K \neq 2$ and let $f \in K[t]$ be a separable polynomial with splitting field L . Then $f = 0$ is solvable by *quadratic radicals* $\iff [L : K] = 2^t$.