1 Introduction I

Definition 1 (Symmetric function)**.** A function $\varphi(x_1, \ldots, x_n)$ is called *symmetric* if

$$\varphi(x_1, \ldots, x_n) = \varphi(x_{\omega(1)}, \ldots, x_{\omega(n)})$$

for all $\omega \in S_n$.

Definition 2 (Elementary symmetric polynomial)**.**

$$\begin{aligned}
\sigma_1 &= \sigma_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n \\
\sigma_2 &= \sigma_2(x_1, \ldots, x_n) = x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\
&\vdots \\
\sigma_k &= \sigma_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \\
&\vdots \\
\sigma_n &= \sigma_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i
\end{aligned}$$

Theorem 1.1. For any symmetric function $\psi(x_1, \ldots, x_n)$, there exists a unique polynomial $P(t_1, \ldots, t_n)$ such that $\psi(x_1, \ldots, x_n) = P(\sigma_1, \ldots, \sigma_n)$.

Vieta formulae:

$$\begin{aligned}
r_1 + r_2 + \cdots + r_n &= -\frac{a_{n-1}}{a_n} \\
\sum_{1 \le i < j \le n} r_i r_j &= \frac{a_{n-2}}{a_n} \\
\sum_{1 \le i < j < k \le n} r_i r_j r_k &= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
r_1 r_2 \cdots r_n &= (-1)^n \frac{a_0}{a_n}
\end{aligned}$$

Corollary 1. The discriminant D of $f \in R[x]$, where R is a ring and $f = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, is a polynomial in $a_1, \ldots, a_n$ and coefficients from R (i.e. $D \in R[a_1, \ldots, a_n]$).

Note: Any cubic equation can be converted to a depressed cubic by

$$x^3 + Ax^2 + Bx + c = \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q.$$

Vieta's method: Using the trigonometric formula $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$, we can solve certain cubic equations. For example, consider $4x^3 - 3x = -\frac{1}{2}$. Let $x = \cos\varphi$. Then

$$\begin{aligned}
\cos 3\varphi = -\frac{1}{2} &\iff 3\varphi = \pm\frac{2\pi}{3} + 2\pi k \quad \text{for } k \in \mathbb{Z} \\
&\iff \varphi = \pm\frac{2\pi}{9} + 2\pi k \\
&\iff x \in \left\{\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}\right\}.
\end{aligned}$$

In general, we can use this method to solve $4x^3 - 3x = a \implies x = \cos\varphi,\ \cos 3\varphi$ and $\cos : \mathbb{C} \to \mathbb{C}$ is now a complex function. For $x^3 + px + q = 0$, set $x = ky$ such that $\frac{k^3}{pk} = \frac{-4}{3} \implies k = \pm\frac{\sqrt{-4p}}{3}$.

Definition 3 (Ferrari resolvent)**.** Let $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$ be a quartic polynomial over a field K of characteristic not 2. We define the *Ferrari resolvent* of f to be the associated cubic resolvent polynomial $R(z) \in K[z]$ given by

$$R(z) = z^3 - 2bz^2 + (b^2 - 4d + ac)z - c^2 - a^2d + 4bd.$$

Solving the resolvent allows one to reduce solving f to solving a system of quadratics.

Lagrange's method: Suppose $f(x) = x^3 + px + q$ is a depressed cubic with roots x_1, x_2, x_3. *Lagrange's method* finds expressions involving the roots that take only a few values under permutation, then uses symmetry to connect them to the coefficients.

For instance, define

$$y_1 = x_1 + \zeta x_2 + \zeta^2 x_3,$$

where $\zeta = e^{2\pi i/3}$ is a primitive cube root of unity. Then define

$$y_2 = x_1 + \zeta^2 x_2 + \zeta x_3.$$

These expressions are not symmetric, but they only take a few values when the x_i's are permuted. In particular, y_1^3 and y_2^3 are symmetric functions of the roots and thus can be written as polynomials in p and q.

Since the roots x_i are related to y_1 and y_2, we can use symmetric combinations such as

$$x = \frac{1}{3}(y_1 + y_2)$$

to recover the original roots of $f(x)$.