Exercise 7.1. Let $K = \mathbb{Q}$, $M = \mathbb{Q}(2^{1/3})$ and $L = \mathbb{Q}(2^{1/3}, \sqrt{3}, i)$. Prove that $L : K$ and $L : M$ are normal but $M : K$ is not normal.

Solution. We know that a field extension $F_1 : F_2$ is normal iff it is a splitting field extension for some $f \in F_2[t]$.

$\square$

Exercise 7.2.1. Let $K - L$ be algebraic, $\alpha \in L$ and $\sigma : K \to \overline{K}$ be a homomorphism. Prove that μ_α^K is separable over K iff $\sigma(\mu_\alpha^K)$ is separable over $\sigma(K)$.

Solution. Since we have a homomorphism from $K \to \overline{K}$, we know that the extension $\overline{K} : K$ exists. Moreover, it is obviously algebraic by definition of $\overline{K}$. Thus there exists some homomorphism $\overline{\sigma} : \overline{K} \to \overline{K}$ extending σ, and we note that $\overline{\sigma}|_K = \sigma$. Since $K - L$ is algebraic we know that μ_α^K exists. Further, since all coefficients of μ_α^K are in K and $K \subseteq \overline{K}$, we can say $\mu_\alpha^K(t) \in \overline{K}[t]$. By definition of algebraic closure, observe that we can split μ_α^K over $\overline{K}[t]$ in the following form:

$$\mu_\alpha^K(t) = \prod_{i=1}^{d} (t - \alpha_i)^{r_i}, \quad r \in \mathbb{N}$$

□

Exercise 7.2.2. Let $L : K$ be a splitting field for $f \in K[t]$. Prove that if f is separable, then $L : K$ is separable.

Solution. □

Exercise 7.3. Let $L : K$ be a splitting field extension for a polynomial $f \in K[t]$. Then $L : K$ is separable iff f is separable over K.

Solution. □

Exercise 7.4. Let $K - M - L$ be an algebraic extension. Prove that $K - L$ is separable iff $K - M$ and $M - L$ are separable.

Solution. $\square$