Problem Set 4: Math 454 Spring 2017
Due Thursday February 9

January 6, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Prove that $\mathbf{F}_p^\times$ is a cyclic group.

Problem 2. Let R, R' be commutative rings with identity and let $\psi\colon R \to R'$ be an isomorphism of rings.

(a) If $\mathfrak{a} \lhd R$ is an ideal with $\mathfrak{a}' = \psi(\mathfrak{a})$, prove that there exists a ring isomorphism $\overline{\psi}\colon R/\mathfrak{a} \to R'/\mathfrak{a}'$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, "\psi_{\mathfrak{a}}"'] & R \arrow[d, "\psi_{\mathfrak{a}'}"] \\ R/\mathfrak{a} \arrow[r, "\overline{\psi}"'] & R'/\mathfrak{a}' \end{tikzcd}$$

(b) Prove that if $\mathfrak{p} \lhd R$ is a prime ideal, then $\psi(\mathfrak{p})$ is a prime ideal in R'.

(c) Prove that if $\mathfrak{m} \lhd R$ is a maximal ideal, then $\psi(\mathfrak{m})$ is a maximal ideal in R'.

Problem 3. Let F be a finite field. Prove that $|F| = p^\ell$ for some $\ell \in \mathbf{N}$ and prime p.

Problem 4. Prove that $F[t]$ is an integral domain. Prove that if E is a field and $F \leq E$ is a subfield, then $F[t] \leq E[t]$ is a subring.

Problem 5. Find an example of a ring R with subring R_0 and a maximal ideal $\mathfrak{m} \lhd R$ such that $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$ is not a maximal ideal of R_0. Prove that $\mathfrak{m}_0$ is a prime ideal.

Problem 6. Prove that if R is a ring with $\mathrm{char}(R) = p$ and p is a prime, then $(r+s)^p = r^p + s^p$ for any $r, s \in R$.

Problem 7. Let F be a field and $R \subseteq F$ a subring of F. Prove that R is an integral domain.