Exercise 8.1. Let $K \subseteq L$ be a splitting field extension for some $f \in K[t] \setminus K$. Then the following are equivalent:

(i) f has a repeated root over L;

(ii) $\exists \alpha \in L$ s.t. $0 = f(\alpha) = (\mathcal{D}f)(\alpha)$;

(iii) $\exists g \in K[t]$, $\deg g \geq 1$ s.t. g divides both f and $\mathcal{D}f$.

Solution. □

Exercise 8.2. Let K be a field, $\mathrm{char}(K) = p > 0$ and $f \in K[t^p]$ is an irreducible polynomial over K. Prove that f is inseparable.

Solution. □

Exercise 8.3. Let K be a field, $\mathrm{char}(K) = p > 0$ and $f \in K[t^p]$ is an irreducible polynomial over K. Prove that there is $g \in K[t]$ and a non-negative n such that $f(t) = g\left(t^{p^n}\right)$ and g is an irreducible and separable polynomial.

Solution. □

Exercise 8.4. Prove that $\prod_{\alpha \in \mathbb{F}_q^*} \alpha = -1$

Solution. □

Exercise 8.5.1. Let $\alpha \in \mathbb{F}_q$ and $\alpha = \beta - \beta^p$ for some $\beta \in \mathbb{F}_q$. Prove that $\mathrm{Tr}(\alpha) = 0$.

Solution. □

Exercise 8.5.2. Let $\alpha \in \mathbb{F}_q$ and $\alpha = \gamma^{1-p}$ for some nonzero $\gamma \in \mathbb{F}_q$. Prove that $\mathrm{Norm}(\alpha) = 1$.

Solution. □

Exercise 8.5.3. Let $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$. Prove that $\mathrm{Tr}(\alpha) = n\alpha$.

Solution. □

Exercise 8.5.4. Let $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$. Prove that $\mathrm{Norm}(\alpha) = \alpha^n$.

Solution. $\square$