1 Soluble Groups II

Theorem 1.1. *Let G be a group. Then the following are equivalent:*

1. *G is a soluble group;*
2. *$\exists n \in \mathbb{Z}^+$ such that $G^{(n)} = \{e\}$;*
3. *$\exists$dnormal series*

$$\{Id.\} = G_n \leqslant G_{n-1} \leqslant \cdots \leqslant G_1 \leqslant G_0 = G$$

   such that $G_j \lhd G$ and all quotients G_{j-1}/G_j are abelian;

4. *$\exists$subnormal series such that quotients G_{j-1}/G_j are abelian.*

Remark 1. Recall that $G' = \langle [x, y] : x, y \in G \rangle$ where $[x, y] = xyx^{-1}y^{-1}$. Equivalently,

Exercise 1. G' is a minimal normal subgroup of G such that G/G' is abelian.

Finally, $G^{(n)} = \left(G^{(n-1)}\right)'$ and $\{\text{Id.}\} = G^{(n)} \lhd G^{(n-1)} \lhd \cdots \lhd G' \lhd G$ is the derived series (not to be confused with $G_{n+1} = [G_n, G]$, the lower central series).

Remark 2. ?? don't understand this one

Lemma 1.2. *Let $\varphi : G \mapsto H$ be an epimorphism. Then $\varphi(G') = H'$.*

Definition 1 (Composition series)**.** Let G be a group. Then a composition series of G is a subnormal series of finite length

$$\{Id.\} = G_0 \lhd G_1 \lhd \cdots \lhd G_{\ell-1} \lhd G_\ell = G$$

such that G_j/G_{j-1} is a simple group for all j.

Theorem 1.3 (Jordan-Hölder)**.** *Any 2 composition series of some group G are equivalent up to permutation and isomorphism.*

Theorem 1.4. *Let K be a field with $\operatorname{char} K \neq 2$ and let $f \in K[t]$ be a separable polynomial with splitting field L. Then $f = 0$ is solvable by* quadratic *radicals* $\iff [L : K] = 2^t$.