Problem 1. Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with "T", and those which may be false with "F".

(a) Let $f \in \mathbb{Z}[t]$ be a polynomial, every root of which has multiplicity 2024. Then f is not separable over $\mathbb{Q}$.

(b) If $L : K$ is an algebraic extension of fields with $K \subseteq L$, then the algebraic closure $\overline{L}$ of L is isomorphic to the algebraic closure $\overline{K}$ of K.

(c) Every algebraic extension of $\mathbb{Q}$ is separable.

(d) Suppose that K and L are fields with $K \subseteq L$, and L is algebraically closed. Then the field extension $L : K$ is normal.

(e) Suppose that $L : M$ and $M : K$ are field extensions with $L : K$ normal. Then $L : M$ is a normal field extension.

(f) Let $f \in \mathbb{Z}[x]$ be a polynomial having prime degree p, and let θ be any root of f in a splitting field extension for f over $\mathbb{Q}$. Then $[\mathbb{Q}(\theta) : \mathbb{Q}] = p$.

Problem 2.

(a) Define what it means for a field extension $L : K$ to be a splitting field extension.

(b) Define what it means for a field extension $L : K$ to be normal.

(c) Let $L : K$ be a field extension. Define what it means for an element $\alpha \in L$ to be separable over K.

(d) Define what it means for a field extension $L : K$ to be separable.

Problem 3. This question concerns the polynomial $f(t) = t^4 - (t+1)^2 \in \mathbb{Q}[t]$.

(a) Find a splitting field extension $L : \mathbb{Q}$ for f, justifying your answer.

(b) Determine the degree of your splitting field extension $L : \mathbb{Q}$, justifying your answer.

(c) Determine the subgroup of S_4 to which $\mathrm{Gal}(L : \mathbb{Q})$ is isomorphic.

Problem 4. Suppose that $L : K$ is a splitting field extension for the polynomial $f \in K[t] \setminus K$. Prove that $[L : K]$ divides $(\deg f)!$.

Problem 5.

(a) Suppose that M is an algebraically closed field. Show that all polynomials in $M[t]$ are separable.

(b) Suppose that p is a prime number and t is an indeterminate, and let $L = \overline{\mathbb{F}}_p(t)$, where $\overline{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$. Are all polynomials in $L[X]$ separable? Justify your answer.

Problem 6. Throughout, let f denote the polynomial $t^5 - 9t - 3 \in \mathbb{Q}[t]$, let L be a splitting field for f over $\mathbb{Q}$, and let M be a field with $\mathbb{Q} \subsetneq M \subsetneq L$ (that is, a field strictly intermediate between $\mathbb{Q}$ and L).

(a) Show that, for any $\sigma \in \mathrm{Gal}(L : \mathbb{Q})$, and for any $\alpha \in M$, the polynomial $\sigma(m_\alpha(\mathbb{Q}))$ is monic and irreducible over $\mathbb{Q}$. Here $m_\alpha(\mathbb{Q})$ denotes the minimal polynomial of α over $\mathbb{Q}$.

(b) Suppose that $M : \mathbb{Q}$ is normal and that f factors as a product of monic irreducibles $f_1, \ldots, f_r$ (of positive degree) over $M[t]$. Show that $\deg(f_i) = \deg(f_1)$ for each i.

(c) Show that if $M : \mathbb{Q}$ is normal, then f remains irreducible over M.