1 Solvability by radicals and Galois theory I

Theorem 1.1. Let K be a field with $\operatorname{char} K = 0$. Then $f \in K[t]$ is solvable by radicals $\iff \operatorname{Gal}_K(f)$ is soluble.

Lemma 1.2. Let $\operatorname{char} K = 0$ and $R : K$ be a radical extension. Then there exists a tower $K - R - N$ such that $N : K$ is normal and radical.

Definition 1 (Cyclic extension)**.** Let L be the splitting field of some polynomial f over K. If $\operatorname{Gal}(L : K)$ is a cyclic group, then $L : K$ is a *cyclic* extension.

Lemma 1.3. Let $\operatorname{char} K = 0$ and let n be a positive integer such that $t^n - 1$ splits over K, and let $L : K$ be the splitting field extension for $t^n - a$ for some $a \in K$. Then $\operatorname{Gal}(L : K)$ is abelian.

Theorem 1.4. Let $\operatorname{char} K = 0$ and $L : K$ be Galois. Suppose there exists some extension $M : L$ such that $M : K$ is normal. Then $\operatorname{Gal}(L : K)$ is soluble.

Corollary 1. Let $\operatorname{char} K = 0$. Then $f \in K[t]$ is SBR $\implies \operatorname{Gal}_K(f)$ is soluble.