1 Separability

Definition 1 (Separable)**.** Let K be a field.

(i) An irreducible polynomial $f \in K[t]$ is *separable over* K if it has no multiple roots, meaning that $f = \lambda(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_d)$, where $\alpha_1, \ldots, \alpha_d \in \overline{K}$ are distinct.

(ii) A non-zero polynomial $f \in K[t]$ is *separable over* K if its irreducible factors in $K[t]$ are separable over K.

(iii) When $L : K$ is a field extension, we say that $\alpha \in L$ is *separable over* K when α is algebraic over K and μ_α^K is separable.

(iv) An algebraic extension $L : K$ is *a separable extension* if every $\alpha \in L$ is separable over K.

Lemma 1.1. Suppose that $L : M : K$ is a tower of algebraic field extensions. Assume that $K \subseteq M \subseteq L \subseteq \overline{K}$, and suppose that $f \in K[t] \setminus K$ satisfies the property that f is separable over K. If $g \in M[t] \setminus M$ has the property that $g \mid f$, then g is separable over M. Thus, if $\alpha \in L$ is separable over K then α is separable over M, and if $L : K$ is separable then so is $L : M$.

Lemma 1.2.
1. If $L : M$ is an algebraic field extension, $\alpha \in L$ and $\sigma : M \to \overline{M}$ is a homomorphism, then $\sigma(\mu_\alpha^M)$ is separable over $\sigma(M) \iff \mu_\alpha^M$ is separable over M.
2. If $L : K$ is a splitting field extension for $f \in K[t]$ and f is separable over K, then $L : K$ is separable.

Theorem 1.3. Let $L : K$ be a finite extension with $K \subseteq L \subseteq \overline{K}$, whence $L = K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n \in L$. Put $K_0 = K$, and for $1 \leq i \leq n$, set $K_i = K_{i-1}(\alpha_i)$. Finally, let $\sigma_0 : K \to \overline{K}$ be the inclusion map.

(i) If α_i is separable over K_{i-1} for $1 \leq i \leq n$, then there are $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \to \overline{K}$.

(ii) If α_i is not separable over K_{i-1} for some i with $1 \leq i \leq n$, then there are fewer than $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \to \overline{K}$.

Theorem 1.4. Let $L : K$ be a finite extension with $L = K(\alpha_1, \ldots, \alpha_n)$. Set $K_0 = K$, and for $1 \leq i \leq n$, inductively define K_i by putting $K_i = K_{i-1}(\alpha_i)$. Then the following are equivalent:

(i) the element α_i is separable over K_{i-1} for $1 \leq i \leq n$;

(ii) the element α_i is separable over K for $1 \leq i \leq n$;

(iii) the extension $L : K$ is separable.

Corollary 1. Suppose that $L : K$ is a finite extension. If $L : K$ is a separable extension, then the number of K-homomorphism $\sigma : L \to \overline{K}$ is $[L : K]$, and otherwise the number is smaller than $[L : K]$.

Corollary 2. Suppose that $f \in K[t] \setminus K$ and that $L : K$ is a splitting field extension for f. Then $L : K$ is a separable extension $\iff$ f is separable over K. More generally, suppose that $L : K$ is a splitting field extension for $S \subseteq K[t] \setminus K$. Then $L : K$ is a separable extension $\iff$ each $f \in S$ is separable over K.