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Homework 1 (Jan 16 – Jan 24).

1 (10+10) 1) Using Vieta's trigonometric method, solve $x^3 - 3x + 1 = 0$.
2) Applying the cube of sum formula, solve $x^3 - 3 \cdot 2^{1/3} x - 3 = 0$.

2 (10) Let x_1, x_2, x_3 be the roots of the cubic $x^3 + ax^2 + bx + c = 0$. Compute $x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1}$.

3 (10) Prove that the stabilizer of the polynomial $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$ is D_5, that is the subgroup of permutations $g \in S_5$ of the form $g : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ and $gx = \pm x + b$, where $b \in \mathbb{Z}/5\mathbb{Z}$.

4 (5+5) Let $H \le S_n$ be a subgroup and K be a field. Take any $f \in K[x_1, \dots, x_n]$ and form

$$F = F(f) = \sum_{h \in H} f(x_{h(1)}, \dots, x_{h(n)}) := \sum_{h \in H} h \cdot f \,,$$

where $h \cdot f$ and the natural action of S_n on $K[x_1, \dots, x_n]$ (i.e. $(h \cdot f)(x_1, \dots, x_n) := f(x_{h(1)}, \dots, x_{h(n)})$).
1) Prove that for any $h \in H$ one has $h \cdot F = F$.
2) Take $f = x_1 x_2^2 \dots x_n^n$ and prove that $h \cdot F = F$ iff $h \in H$.
3) *(for enthusiasts, does not affect the rating)* Is the second part true for any f?

5 (5+5+15) A complex polynomial $f(x_1, \dots, x_n)$ is called *skew–symmetric* if $h \cdot f = -f$ for any transposition h.
1) Prove that the ratio of any skew–symmetric polynomials is a symmetric rational function.
2) Let $D = D(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)^2$ be the discriminant and $\Delta = \Delta(x_1, \dots, x_n) = \prod_{i<j}(x_i - x_j)$, $\Delta^2 = D$. Prove that Δ is a skew–symmetric polynomial.
3) Prove that any symmetric polynomial f is a product of Δ and another symmetric polynomial g.

Solutions

General remark. If there is a typo in any task, then the maximum score will be awarded for that task.

1 1) Using Vieta's trigonometric method, solve $x^3 - 3x + 1 = 0$.

2) Applying the cube of sum formula, solve $x^3 - 3 \cdot 2^{1/3}x - 3 = 0$.

Solution: 1) Let $x = ky$. Then

$$k^3y^3 - 3ky = -1$$

and to apply Vieta's trigonometric method we need to take k such that

$$\frac{k^3}{-3k} = \frac{4}{-3}\,.$$

Thus, $k = 2$ and we arrive to

$$4y^3 - 3y = -\frac{1}{2}\,.$$

Let $y = \cos\alpha$. Then thanks to $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$, we have $\cos 3\alpha = -1/2$ and therefore $3\alpha = \pm 2\pi/3 + 2\pi n$, where $n \in \mathbb{Z}$. It follows that $x \in \{\cos 2\pi/9, \cos 4\pi/9, \cos 8\pi/9\}$.

2) One has

$$x^3 - 3 \cdot 2^{1/3}x - 3 = x^3 - 3abx - a^3 - b^3 = 0\,,$$

where $a = 1$, $b = 2^{1/3}$. Thus $x_1 = a + b = 1 + 2^{1/3}$ and $x_2 = \varepsilon + 2^{1/3}\varepsilon^2$, $x_3 = \varepsilon^2 + 2^{1/3}\varepsilon$, where $\varepsilon^3 = 1$, $\varepsilon \neq 1$ is a cube root of unity.

2 Let x_1, x_2, x_3 be the roots of the cubic $x^3 + ax^2 + bx + c = 0$. Compute $x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1}$.

Solution: We have

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3)\,,$$

and

$$x_1^{-1} + x_2^{-1} + x_3^{-1} = (x_1x_2 + x_2x_3 + x_1x_3)/(x_1x_2x_3)\,.$$

By Vieta's formulae, we have $x_1 + x_2 + x_3 = -a$, $x_1x_2 + x_2x_3 + x_1x_3 = b$ and $x_1x_2x_3 = -c$. Hence

$$x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1} = a^2 - 2b - b/c\,.$$

3 Prove that the stabilizer of the polynomial $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$ is D_5, that is the subgroup of permutations $g \in S_5$ of the form $g : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ and $gx = \pm x + b$, where $b \in \mathbb{Z}/5\mathbb{Z}$.

Solution: Fix $\eta \in \{-1, 1\}$, $b \in \mathbb{Z}/5\mathbb{Z}$ and the permutation $gx = \eta x + b$. We have

$$f := x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = \sum_{i=1}^{5} x_ix_{i+1}\,,$$

where $x_6 = x_1$. Then

$$g \cdot f = \sum_{i=1}^{5} x_{g(i)}x_{g(i+1)} = \sum_{i=1}^{5} x_{\eta i+b}x_{\eta i+b+\eta}\,.$$

Making the substitution $\eta i + b := j$, we get

$$g \cdot f = \sum_{j=1}^{5} x_jx_{j+\eta}\,.$$

Since $\mathbb{Z}/5\mathbb{Z}$ is a cyclic group, it is easy to see the last sum coincides with f.

Thus, we have proved that the stabilizer f contains D_5. The maximum score will be given even for this part. The other direction will be considered later in the lectures.

4 Let $H \le S_n$ be a subgroup and K be a field. Take any $f \in K[x_1, \dots, x_n]$ and form

$$F = F(f) = \sum_{h \in H} f(x_{h(1)}, \dots, x_{h(n)}) := \sum_{h \in H} h \cdot f \,,$$

where $h \cdot f$ and the natural action of S_n on $K[x_1, \dots, x_n]$ (i.e. $(h \cdot f)(x_1, \dots, x_n) := f(x_{h(1)}, \dots, x_{h(n)})$).
1) Prove that for any $h \in H$ one has $h \cdot F = F$.
2) Take $f = x_1 x_2^2 \dots x_n^n$ and prove that $h \cdot F = F$ iff $h \in H$.
3) *(for enthusiasts)* Is the second part true for any f?

Solution: 1) The first part follows from $hH = H$ for any $h \in H$. The maximum score will be awarded even in the case of a monomial f.
2) Take $g \in S_n \setminus H$. Then $g \cdot f \ne h \cdot f$ for any $h \in H$. Indeed, if $x_{h(1)} x_{h(2)}^2 \dots x_{h(n)}^n = x_{g(1)} x_{g(2)}^2 \dots x_{g(n)}^n$, then $h(j) = g(j)$ for all $j = 1, 2, \dots, n$ and hence $g = h$.
3) No. Take $n = 4$ and $f = x_1 x_2$ and $H = \{e, (12)(34), (13)(24), (14)(23)\}$, say. Then $F = 2(x_1 x_2 + x_3 x_4)$ and F is obviously invariant under $(12) \notin H$.

5 A complex polynomial $f(x_1, \dots, x_n)$ is called *skew–symmetric* if $h \cdot f = -f$ for any transposition h.
1) Prove that the ratio of any skew–symmetric polynomials is a symmetric rational function.
2) Let $D = D(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)^2$ be the discriminant and $\Delta = \Delta(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)$, $\Delta^2 = D$. Prove that Δ is a skew–symmetric polynomial.
3) Prove that any **skew**–symmetric polynomial f is a product of Δ and another symmetric polynomial g.

Solution: 1) and 2) are obvious. To obtain 3) it is enough to show that Δ divides f. Thus, it is enough to prove that $(x_i - x_j)$ divides f. Let $i = 1$ and $j = 2$ for concreteness. Let $x_1 = v + u$, $x_2 = v - u$, so $x_i = x_j$ iff $u = 0$ (we assume that the characteristic $\ne 2$). So

$$f(x_1, x_2, x_3, \dots, x_n) = f(u + v, -u + v, x_3, \dots, x_n) := f_*(u, v, x_3, \dots, x_n) \,.$$

Putting $x_1 = x_2$, we see that $f(x_1, x_1, x_3, \dots, x_n) = 0$ and hence $f_*(0, v, x_3, \dots, x_n) = 0$. Thus f_* is divided by $u = (x_1 - x_2)/2$ as required.