Problem Set 2: Math 454 Spring 2017
Due Thursday January 26

January 24, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Let G be a group and $H \leq G$.

(a) Prove that $\mathrm{Core}_G(H)$ is a normal subgroup of G.

(b) Prove that if $H_0 \lhd G$ and $H_0 \subseteq H$, then $H_0 \subseteq \mathrm{Core}_G(H)$. In particular, $\mathrm{Core}_G(H)$ is the largest normal subgroup of G that is contained in H.

(c) Prove that

$$\overline{H} = \bigcap_{\substack{N \lhd G, \\ H \subseteq N}} N$$

is a normal subgroup of G that contains H.

(d) Prove that if $N \lhd G$ and $H \subseteq N$, then $\overline{H} \subseteq N$. In particular, $\overline{H}$ is the smallest normal subgroup of G that contains H.

Problem 2. Let G be a group and $H \leq G$.

(a) Prove that $\mathrm{Ad}_{G,H} \colon N_G(H) \to \mathrm{Aut}(H)$ is a homomorphism and $\ker \mathrm{Ad}_{G,H} = C_G(H)$.

(b) Prove that $N_G(H)/C_G(H) \leq \mathrm{Aut}(H)$.

Problem 3. Let G be a group.

(a) Prove that $\{(g, \mathrm{Id}_G) \ : \ g \in G\}$ is a subgroup of $G \rtimes \mathrm{Aut}(G)$.

(b) Prove that G is isomorphic to the subgroup from (a).

(c) Prove that the subgroup from (a) is normal in $G \rtimes \mathrm{Aut}(G)$.

Problem 4. For each integer $m \in \mathbf{N}$, we define $m\mathbf{Z} = \{mn \ : \ n \in \mathbf{Z}\}$.

(a) Prove that $m\mathbf{Z}$ is an ideal in $\mathbf{Z}$.

(b) Prove that $||m\mathbf{Z}|| = m$.

Problem 5. Let R be a commutative ring with identity and $\mathfrak{a}_1, \mathfrak{a}_2 \lhd R$ be ideals.

(a) Prove that $\mathfrak{a}_1 \cap \mathfrak{a}_2$ is an ideal.

(b) Prove that

$$\mathfrak{a}_1 + \mathfrak{a}_2 \stackrel{\text{def}}{=} \{a_1 + a_2 \,:\, a_1 \in \mathfrak{a}_1,\ a_2 \in \mathfrak{a}_2\}$$

is an ideal.

(c) Prove that

$$\mathfrak{a}_1\mathfrak{a}_2 \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{n} a_{i,1}a_{i,2} \,:\, a_{i,1} \in \mathfrak{a}_1,\ a_{i,2} \in \mathfrak{a}_2 \text{ for all } i = 1, \dots, n \right\}$$

is an ideal.

Problem 6. For $m \in \mathbf{N}$, let $m\mathbf{Z} \lhd \mathbf{Z}$ be the ideal generated by m.

(a) Prove that $m_1\mathbf{Z} + m_2\mathbf{Z} = \gcd(m_1, m_2)\mathbf{Z}$ where $\gcd(m_1, m_2)$ is the greatest common divisor of m_1, m_2.

(b) Prove that $m_1\mathbf{Z} \cap m_2\mathbf{Z} = \operatorname{lcm}(m_1, m_2)$ where $\operatorname{lcm}(m_1, m_2)$ is the least common multiple of m_1, m_2.

(c) Prove that $m\mathbf{Z}$ is a prime ideal if and only if m is a prime. [Hint: Division algorithm]

(d) Prove that $m\mathbf{Z}$ is a prime ideal if and only if $m\mathbf{Z}$ is a maximal ideal. [Hint: Greatest common divisors]

(e) Prove that every non-zero ideal $\mathfrak{a}$ of $\mathbf{Z}$ is principal. [Hint: Greatest common divisors]

Problem 7. Let R be a commutative ring with identity. Prove that R is an integral domain if and only if whenever $ab = 0$ for some $a, b \in R$, either $a = 0$ or $b = 0$.