Problem 1. Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with "T", and those which may be false with "F".

(a) There is a field isomorphism $\varphi : \mathbb{Q}(\sqrt{-5}) \to \mathbb{Q}(\sqrt{5})$.

(b) There is a homomorphism of finite fields $\psi : \mathbb{F}_3 \to \mathbb{F}_{37}$.

(c) If $L : K$ is a field extension, and $\alpha, \beta \in L$ are distinct elements with the same minimal polynomial over K, then $K(\alpha)$ and $K(\beta)$ are isomorphic fields.

(d) It is *not* possible to construct, using compass and straightedge in the usual way, a length whose 14^{th} power is twice a given length.

(e) The polynomial $x^{36} + x^{35} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

(f) If K is a field and α is an element of an extension field L of K, then every element of $K(\alpha)$ can be expressed as a polynomial in α with coefficients in K.

Problem 2.

(a) For $j = 1$ and 2, let $L_j : K_j$ be a field extension relative to the embedding $\varphi_j : K_j \to L_j$. Suppose that $\sigma : K_1 \to K_2$ and $\tau : L_1 \to L_2$ are isomorphisms. Define what is meant by the statement that τ *extends* σ.

(b) Let $L : M : K$ be a tower of field extensions with $K \subseteq M \subseteq L$. Define what is meant by the statement that $\sigma : M \to L$ *is a K-homomorphism.*

(c) Suppose that $L : K$ is a field extension. Define what is meant by the *degree* of $L : K$.

(d) Suppose that $L : K$ is a field extension with $K \subseteq L$, and α is algebraic over K. Define what is meant by the *minimal polynomial* of α over K.

Problem 3. Let $L : K$ be a field extension. Suppose that $\alpha \in L$ is algebraic over K and $\beta \in L$ is transcendental over K. Suppose also that $\alpha \notin K$. Show that $K(\alpha, \beta) : K$ is not a simple field extension.

Problem 4. Let θ denote the real number $\sqrt{3 + 3\sqrt[3]{6}}$, and write $L = \mathbb{Q}(\theta)$.

(a) Calculate the minimal polynomial of θ over $\mathbb{Q}$, and hence determine the degree of the field extension $L : \mathbb{Q}$.

(b) Let $f \in \mathbb{Q}[t]$ be a monic polynomial of degree 4. Suppose $\alpha \in L$ satisfies $f(\alpha) = 0$. Is it possible that f is irreducible over $\mathbb{Q}$? Justify your answer.

(c) Suppose β and γ are elements in $\mathbb{C}$ having the property that both $\beta + \gamma$ and $\beta\gamma$ are algebraic over $\mathbb{Q}$. Prove that β and γ are algebraic over $\mathbb{Q}$.

Problem 5. Let $L : \mathbb{Q}$ be an algebraic extension with $\mathbb{Q} \subseteq L$, and consider a homomorphism of fields $\varphi : L \to L$.

(a) By considering $\varphi(\mathbb{Z})$, or otherwise, show that φ is a $\mathbb{Q}$-homomorphism.

(b) Suppose that $\alpha \in L$. Show that the minimal polynomial of α over $\mathbb{Q}$ has $\varphi^n(\alpha)$ as a root, for each non-negative integer n, where φ^n denotes the n-fold composition of φ.

(c) Let $\alpha \in L$. Show that there is a positive integer d with the property that $\varphi^d(\alpha) = \alpha$. Moreover, putting $\beta = \alpha + \varphi(\alpha) + \cdots + \varphi^{d-1}(\alpha)$, with d taken to be the smallest such non-negative integer, show that φ is a $\mathbb{Q}(\beta)$-homomorphism of L.

Problem 6. With t an indeterminate, let $f \in \mathbb{Z}[t]$ be a polynomial of degree $n \geq 1$, and put $K = \mathbb{Q}(f)$.

(a) Find a polynomial $F \in K[X]$ with $F(t) = 0$, and deduce that $\mathbb{Q}(t) : K$ is algebraic of degree at most n.

(b) Let $g \in \mathbb{Z}[t]$ be a polynomial distinct from f. By considering μ_g^K, or otherwise, show there exists a non-zero polynomial $H(X, Y) \in \mathbb{Z}[X, Y]$ with $H(f(t), g(t)) = 0$.