

1 The Primitive Element Theorem

Definition 1 (Simple extension). Suppose $L : K$ is a field extension relative to the embedding $\varphi : K \rightarrow L$. We say that $L : K$ is a simple extension if there is some $\gamma \in L$ having the property that $L = \varphi(K)(\gamma)$.

Theorem 1.1 (The Primitive Element Theorem). If $L : K$ be a finite, separable extension with $K \subseteq L$, then $L : K$ is a simple extension.

Corollary 1.2. Suppose that $L : K$ is an algebraic, separable extension, and suppose that for every $\alpha \in L$, the polynomial μ_α^K has degree at most n over K . Then $[L : K] \leq n$.

Fact: Let $L : K$ be a normal extension and let $\deg(\mu_\alpha^K) \leq n$ for all $\alpha \in L$. Then $[L : K] \leq n$.

Corollary 1.3. If $f \in K[t]$ is irreducible over K , then $\text{Gal}_K(f)$ acts transitively on the roots of f .