Josh Park
MA454
HW 1
Shkredov

1.1) $x^3 - 3x + 1 = 0$

$$x^3 - 3x = -1 \qquad (1)$$

Let $x = 2\cos\theta$. Then

$$8\cos^3\theta - 6\cos\theta = -1$$

By formula, $4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

$$\Rightarrow 2\cos 3\theta = -1$$

$$\cos 3\theta = -\tfrac{1}{2}$$

$$3\theta = \tfrac{2}{3}\pi + 2k\pi,\ \tfrac{4}{3}\pi + 2k\pi \qquad (k \in \mathbb{Z})$$

$$\Rightarrow \theta = \tfrac{2}{9}\pi + \tfrac{2}{3}k\pi,\ \tfrac{4}{9}\pi + \tfrac{2}{3}k\pi$$

$$\Rightarrow \theta_1 = \tfrac{2}{9}\pi \quad \theta_2 = \tfrac{4}{9}\pi \quad \theta_3 = \tfrac{8}{9}\pi$$

$$\Rightarrow \boxed{x_1 = 2\cos\tfrac{2\pi}{9} \quad x_2 = 2\cos\tfrac{4\pi}{9} \quad x_3 = 2\cos\tfrac{8\pi}{9}} \quad \square$$

1.2) $f(x) := x^3 - 3x\sqrt[3]{2} - 3 = 0$

$$\left.\begin{array}{l} A^3 + B^3 = 3 \\ AB = \sqrt[3]{2} \end{array}\right\} \Rightarrow \begin{array}{l} A^3 = 2 \\ B^3 = 1 \end{array} \Rightarrow \begin{array}{l} A = \sqrt[3]{2} \\ B = 1 \end{array} \Rightarrow AB = \sqrt[3]{2}$$

Let $x := A + B = 1 + \sqrt[3]{2}$

$$x^3 = (1 + \sqrt[3]{2})^3 = A^3 + B^3 + 3AB(A+B) = 3 + 3\cdot\sqrt[3]{2} + 3\sqrt[3]{2}^2$$

$$\Rightarrow f(x) = x^3 - 3x\sqrt[3]{2} - 3 = [3 + 3\cdot\sqrt[3]{2} + 3\sqrt[3]{2}^2] - 3\sqrt[3]{2}(1 + \sqrt[3]{2}) - 3$$

$$= \cancel{3} + \cancel{3\cdot\sqrt[3]{2}} + \cancel{3\sqrt[3]{2}^2} - \cancel{3\sqrt[3]{2}} - \cancel{3\sqrt[3]{2}^2} - \cancel{3}$$

$$= 0$$

$\Rightarrow x_1$ is a root of f. Let $\omega = \exp(\tfrac{2}{3}\pi i)$

$$\left.\begin{array}{ll} A_1 = 1 & B_1 = \sqrt[3]{2} \\ A_2 = \omega & B_2 = \sqrt[3]{2}\,\omega^2 \\ A_3 = \omega^2 & B_2 = \sqrt[3]{2}\,\omega \end{array}\right\} = AB = \sqrt[3]{2}\ \forall A, B \Rightarrow \boxed{\begin{array}{l} x_1 = 1 + \sqrt[3]{2} \\ x_2 = \omega + \sqrt[3]{2}\,\omega^2 \\ x_3 = \omega^2 + \sqrt[3]{2}\,\omega \end{array}} \quad \square$$

2) $f(x) = x^3 + ax^2 + bx + c = 0$

Want $x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1}$

By Viete, (a) $x_1 + x_2 + x_3 = -a$

(b) $x_1x_2 + x_2x_3 + x_1x_3 = b$

(c) $x_1x_2x_3 = -c$

$$(\bar{a}+\bar{b}+\bar{c})^2 = \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + 2(\bar{a}\bar{b} + \bar{b}\bar{c} + \bar{a}\bar{c})$$

$$\Rightarrow \bar{a}^2 + \bar{b}^2 + \bar{c}^2 = \underbrace{(\bar{a}+\bar{b}+\bar{c})}^2 - 2\underbrace{(\bar{a}\bar{b}+\bar{b}\bar{c}+\bar{a}\bar{c})}$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 = (-a)^2 - 2b$$

$$= a^2 - 2b$$

$$x_1^{-1} + x_2^{-1} + x_3^{-1} = \frac{x_1x_2 + x_2x_3 + x_1x_3}{x_1x_2x_3} = \frac{b}{-c}$$

$$\boxed{\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1} = a^2 - 2b - \frac{b}{c}}$$ $\square$

3) $\overset{=f}{f}(x_1, x_2, x_3, x_4, x_5) = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$

WTS $\text{stab}(f) \cong D_5$

$\text{stab}(f) = \{g \in S_5 \mid g: \mathbb{Z}_5 \to \mathbb{Z}_5,\ gx = \pm x + b \text{ where } b \in \mathbb{Z}_5\}$

$= \{\phi \in S_5 \mid \phi(f) = f\}$