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Homework 5 (Feb 21 – Feb 28)

1 (5+5+5+10+15) Which of the following field extensions are normal? Justify your answers.
1) $\mathbb{Q}(i) : \mathbb{Q}$
2) $\mathbb{Q}(2^{1/4}) : \mathbb{Q}$
3) $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$
4) $\mathbb{Q}(2^{1/4}, i, \sqrt{5}) : \mathbb{Q}$
5) $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$.

2 (15) Let $\psi : L \to M$ be a homomorphism, suppose that L is algebraically closed. Prove that $\psi(L)$ is algebraically closed.

3 (20) Let $L : K$ be an algebraic field extension. Then $\overline{K}$ is isomorphic to $\overline{L}$. In addition, if $K \subset L \subseteq \overline{L}$, then one may take $\overline{K} = \overline{L}$.

4 (15) Let $K - L$ be a normal extension, $K \subseteq L \subseteq \overline{K}$. Then for any K–homomorphism $\tau : L \to \overline{K}$ one has $\tau(L) = L$.

5 (25) Put $K = \mathbb{F}_2(t)$ and consider $L = K(t^{1/3})$. Prove that the extension $L : K$ is algebraic but not normal.

Solutions

General remark. If there is a typo in any task, then the maximum score will be awarded for that task.

1 (5+5+5+10+15) Which of the following field extensions are normal? Justify your answers.
1) $\mathbb{Q}(i) : \mathbb{Q}$
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3) $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$
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5) $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$.

Solution. 1) Normal: $\pm i \in \mathbb{Q}(i)$.

2) Not normal: $i \notin \mathbb{Q}(2^{1/4})$.

3) Normal: all roots of $t^4 - 2$ belong to $\mathbb{Q}(2^{1/4}, i)$.

4) Normal. Consider the polynomial $(t^2 - 5)(t^4 - 2)$. Then its splitting field is exactly $\mathbb{Q}(2^{1/4}, i, \sqrt{5})$ and all splitting fields are normal.

5) Normal. Consider the polynomial $(t^3 - 3)(t^2 - 3)(t^2 + 1)$. Then all roots of this polynomial are $\pm i, \pm\sqrt{3}, 3^{1/3}\varepsilon_3^k$, where $k = 0, 1, 2$. We have $\varepsilon_3 = (1 + i\sqrt{3})/2 \in \mathbb{Q}(i, \sqrt{3})$. Hence the splitting field of this polynomial is $\mathbb{Q}(3^{1/3}, i, \sqrt{3})$ and all splitting fields are normal.

2 (15) Let $\psi : L \to M$ be a homomorphism, suppose that L is algebraically closed. Prove that $\psi(L)$ is algebraically closed.

Solution. We need to show that any irreducible $f \in \psi(L)[t]$ has degree 1. If not, then $f = gh$, where $f = \psi(f_0)$, $g = \psi(g_0)$, $h = \psi(h_0)$ and $f_0, g_0, h_0 \in L[t]$. We have $\deg(f_0) = \deg(f) > 1$ and therefore f_0 is reducible since L is algebraically closed. But then $f_0 = g'h'$ for some $g', h' \in L[t]$ and hence $f = \psi(f_0) = \psi(g')\psi(h')$. Thus f is not irreducible and this is a contradiction.

3 (20) Let $L : K$ be an algebraic field extension. Then $\overline{K}$ is isomorphic to $\overline{L}$. In addition, if $K \subset L \subseteq \overline{L}$, then one may take $\overline{K} = \overline{L}$.

Solution. We have $K - L - \overline{L}$ and hence $\overline{L}$ is an algebraic closure of K. But we know (see lectures) that two algebraic closures are isomorphic. Thus $\overline{K}$ is isomorphic to $\overline{L}$.

Now let $K \subset L \subseteq \overline{L}$. Then the previous argument shows that $\overline{L}$ is an algebraic closure of K and both $\overline{K}, \overline{L}$ contain K. The inclusion mapping $\varphi : K \to \overline{L}$ can be extended to a homomorphism from $\overline{K}$ to $\overline{L}$ (see lectures). If $\overline{K} = \overline{L}$, then we are done, if not then $[\overline{L} : \overline{K}] > 1$ but this is a contradiction with the assumption that $\overline{K}$ is algebraically closed.

Remark. The second part of Exercise 3 was not formulated correctly, so to get a full mark it is enough to solve only the the first part.

4 (15) Let $K - L$ be a normal extension, $K \subseteq L \subseteq \overline{K}$. Then for any K–homomorphism $\tau : L \to \overline{K}$ one has $\tau(L) = L$.

Solution. Take any $\alpha \in L$. Then $0 = \mu_\alpha^K(\alpha) = \mu_\alpha^K(\tau(\alpha))$ and since $L : K$ is a normal extension, we see that $\tau(\alpha) \in L$. Thus $\tau(L) \subseteq L$ and hence τ is K–homomorphism of L. By the first theorem of Lecture 11 τ is an automorphism.

5 (25) Put $K = \mathbb{F}_2(t)$ and consider $L = K(t^{1/3})$. Prove that the extension $L : K$ is algebraic but not normal.

Solution. Clearly, $t^{1/3}$ is an algebraic number (in $\overline{K}$, say) as it is a root of $F(X) = X^3 - t \in K[X]$. Thus $L : K$ is algebraic extension. To obtain that $L : K$ is not normal it is enough to show that $F(X)$ has no other linear factors than $X - t^{1/3}$ over L (notice that $F(X) \neq (X - t^{1/3})^3$). Let $F(\alpha) = 0$, where $\alpha \in L$ and $\alpha \neq t^{1/3}$. Writing α as $\beta t^{1/3}$, where

$\beta \in L$ one has $\beta^3 = 1$ and $\beta \neq 1$. Thus $\beta^2+\beta+1 = 0$ and hence either $\beta \in \mathbb{F}_2$ (it is easy to check that this is impossible) or $\beta \in L \setminus \mathbb{F}_2$. In the last case we can write $\beta = h(t^{1/3})$, where $h \in \mathbb{F}_2[X]$ and therefore $h^2(t^{1/3}) + h(t^{1/3}) + 1 = 0$. In other words, $t^{1/3}$ satisfies an algebraic equation over $\mathbb{F}_2$. But then t is also algebraic over $\mathbb{F}_2$ and this a contradiction (recall that t is transcendental over $\mathbb{F}_2$).