PURDUE UNIVERSITY

Department of Mathematics

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Homework 8 (Mar 14 – Apr 4)

1 (5+5+5) Let $K \subseteq L$ be a splitting field extension for some $f \in K[t] \setminus K$. Then the following are equivalent:
(*i*) f has a repeated root over L;
(*ii*) $\exists \alpha \in L$ s.t. $0 = f(\alpha) = (\mathcal{D}f)(\alpha)$;
(*iii*) $\exists g \in K[t]$, $\deg g \geq 1$ s.t. g divides both f and $\mathcal{D}f$.

2 (5) Let K be a field, $\mathrm{char}(K) = p > 0$ and $f \in K[t^p]$ is an irreducible polynomial over K. Prove that f is inseparable.

3 (10) Let K be a field, $\mathrm{char}(K) = p > 0$ and $f \in K[t^p]$ is an irreducible polynomial over K. Prove that there is $g \in K[t]$ and a non-negative n such that $f(t) = g(t^{p^n})$ and g is an irreducible and separable polynomial.

4 (10) Prove that $\prod_{\alpha \in \mathbb{F}_q^*} \alpha = -1$.

5 (5+5+5+5) a) Let $\alpha \in \mathbb{F}_q$ and $\alpha = \beta - \beta^p$ for some $\beta \in \mathbb{F}_q$. Prove that $\mathrm{Tr}(\alpha) = 0$.

b) Let $\alpha \in \mathbb{F}_q$ and $\alpha = \gamma^{1-p}$ for some nonzero $\gamma \in \mathbb{F}_q$. Prove that $\mathrm{Norm}(\alpha) = 1$.

c) Let $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$. Prove that $\mathrm{Tr}(\alpha) = n\alpha$.

d) Let $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$. Prove that $\mathrm{Norm}(\alpha) = \alpha^n$.

6 The midterm exam will be on Thursday the 27th!