Problem Set 3: Math 454 Spring 2017 Due Thursday February 2

January 30, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Let F be a commutative ring with identity. Prove that the following are equivalent:

- (a) The only two ideals in F are F and the trivial ideal $\{0_F\}$.
- (b) F is a field.

Problem 2. Let F be a field.

- (a) Prove that F[t] is a commutative ring with identity.
- (b) Prove that the subset of constant polynomials of F[t] is a subring. Moreover, prove this set is a field and this field is isomorphic to F.
- (c) Prove that the group of units of F[t] is F^{\times} , viewed as the group of units of the field of constant polynomials.

Problem 3. Prove that if E/F is an extension of fields and S_1, S_2 are both maximal, algebraically independent subsets of E, then $|S_1| = |S_2|$. In particular, the transcendence degree of E is well defined.

Problem 4. Let E/F be an extension of fields with $F \le E_1, E_2 \le E$. Prove there exists an F-basis \mathscr{B} for E_1E_2 given as follows. Let \mathscr{B}_1 be an F-basis for $E_1 \cap E_2$. Let \mathscr{B}_2 be an $(E_1 \cap E_2)$ -basis for E_1 and let \mathscr{B}_3 be an $(E_1 \cap E_2)$ -basis for E_2 . Prove that

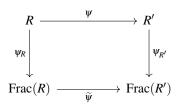
$$\mathscr{B} = \{uv : u \in \mathscr{B}_1, v \in \mathscr{B}_2\} \cup \{uw : u \in \mathscr{B}_1, w \in \mathscr{B}_3\}$$

is an F-basis for E_1E_2 .

Problem 5. Let E/F, E_1/F , and E_2/F be an extensions such that $E_1, E_2 \le E$ and E_1/F and E_2/F are finite.

- (a) Prove that the composite E_1E_2/F is a finite extension.
- (b) Prove that $E_1 \cap E_2$ is an extension of F.
- (c) Prove that $[E_1E_2:E_1] = [E_2:E_1 \cap E_2]$ and $[E_1E_2:E_2] = [E_1:E_1 \cap E_2]$.
- (d) Prove that $[E_1E_2:F] = \frac{[E_1:F][E_2:F]}{[E_1\cap E_2:F]}$.

Problem 6. Let R, R' be integral domains and $\psi: R \to R'$ an isomorphism of rings. Prove that there exists an isomorphism of fields $\widetilde{\psi}$: $\operatorname{Frac}(R) \to \operatorname{Frac}(R')$ such that the diagram



commutes.

Problem 7. Prove that $L: E \to \operatorname{Hom}_E(E,E)$ given by $L(\beta) = L_\beta$ is an isomorphism of F-algebras and the restriction of L to E^\times gives an isomorphism of groups $L: E^\times \to \operatorname{Aut}_{E-\mathrm{vec}}(E)$.