Exercise 10.1. Let $K, E, F \subseteq L$ be fields, $E : K, F : K$ be finite extensions. Prove

(a) if $E : K$ is separable, then $EF : F$ is separable;

(b) if $E : K$ and $F : K$ are both separable, then $EF : K$ and $E \cap F : K$ are both separable;

(c) if $E : K$ is Galois, then $EF : F$ is Galois;

(d) if $E : K$ and $F : K$ are both Galois, then $EF : K$ and $E \cap F : K$ are both Galois.

(a) *Solution.* Suppose $E : K$ is separable. We are given that $E : K$ and $F : K$ are finite, so we can write $E = K(\alpha_1, \ldots, \alpha_n)$ and $F = K(\beta_1, \ldots, \beta_m)$ for $\alpha_i \in E$ and $\beta_j \in F$. Then the composite field EF becomes

$$\begin{aligned} EF &= K(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m) \\ &= F(\alpha_1, \ldots, \alpha_n). \end{aligned}$$

Since $E : K$ is finite it is also algebraic, hence the minimum polynomial for each element of E is well defined over K, and similarly for $EF : F$. For any $b \in F$, the minimal polynomial over F is $x - b$, which has distinct roots, so b is separable over F. Hence it is enough to show that $\alpha_1, \ldots, \alpha_n$ is separable over F.

We have that μ_α^K is separable by hypothesis for all $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$. Then $\mu_\alpha^K(x) \in K[x] \subseteq F[x]$ so μ_α^F divides μ_α^K and thus μ_α^F is thus also separable, whence $EF : F$ is separable. ☐

(b) *Solution.* Suppose $E : K$ and $F : K$ are both separable. Similarly to part (a), we can write

$$EF = K(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m),$$

for $\alpha_i \in E$ and $\beta_j \in F$. By definition, a is separable over K for all $a \in E$, and similarly for $b \in F$. Then each $\alpha_1, \ldots, \alpha_n \in E,\ \beta_1, \ldots, \beta_m \in F$ is separable over K. By theorem an extension $K(\gamma_1, \ldots, \gamma_k) : K$ is separable iff each γ_i is separable over K. Thus $EF : K$ is separable. Furthermore, we know $E : K$ is separable and $E \cap F \subseteq E$, so $E \cap F : K$ is separable by definition. ☐

(c) *Solution.* Suppose $E : K$ is Galois. Then $E : K$ is normal and separable by definition. Since $E : K$ and $F : K$ are both finite and $E : K$ is normal, we have by lemma that $EF : F$ is normal and by part (a), $EF : F$ is separable. Thus $EF : F$ is Galois. ☐

(d) *Solution.* Suppose $E : K$ and $F : K$ are both Galois. Then $E : K$ and $F : K$ are both normal and separable by definition. Since $E : K$ and $F : K$ are both finite and normal, we have by lemma that $EF : K$ and $E \cap F : K$ are both normal and by part (b), $EF : K$ and $E \cap F : K$ are both separable. Thus $EF : K$ and $E \cap F : K$ are both Galois. ☐

Exercise 10.2. (a) Find the splitting field L of the polynomial $f(t) = t^4 - 4t^2 + 5$.

(b) Prove that $[L : \mathbb{Q}]$ is either 4 or 8.

(c) Find 10 intermediate fields of the extension $L : \mathbb{Q}$ and their degrees.

(d) (for enthusiasts) Draw the lattice of subfields and corresponding lattice of subgroups of $\mathrm{Gal}_{\mathbb{Q}}(f)$.

(a) *Solution.* Notice if we set $t^4 - 4t^2 + 5 = 0$, then we can subtract 1 to see $t^4 - 4t^2 + 4 = (t^2 - 2)^2 = -1$. Hence $t^2 - 2 = \pm i$ and $t \in \{\pm\sqrt{2 \pm i}\}$. We note that if $w = \sqrt{a + bi}$ then $w^2 = a + bi$ and $\overline{w^2} = \overline{w}^2 = a - bi$, whence $\overline{w} = \sqrt{a - bi}$. That is, the square roots of complex conjugates are themselves complex conjugates. So it is enough to construct L by adjoining $\sqrt{2 + i}$ to $\mathbb{Q}$ and thus $L = \mathbb{Q}(\sqrt{2 + i})$. ☐

(b) *Solution.* Set $x = \sqrt{2+i}$. Then

$$\begin{aligned} x^2 &= 2+i \\ x^2 - 2 &= i \\ x^4 - 4x + 4 &= -1 \\ x^4 - 4x + 5 &= 0 \end{aligned}$$

Hence the minimum polynomial for $\sqrt{2+i}$ is $\mu^{\mathbb{Q}}_{\sqrt{2+i}}(x) = x^4 - 4x + 5 = f(x)$ and $[L : \mathbb{Q}] = 4$. $\square$

(c) *Solution.* $\square$

(d) *Solution.* $\square$

Exercise 10.3. Draw the lattice of subfields and corresponding lattice of subgroups of $Gal_{\mathbb{Q}}(t^6 + 3)$.
Hint: Use the calculations (and the notation, if you like) from Lecture 18.

Solution. $\square$