1 Ruler and Compass Constructions

Definition 1 (Constructible points/angles)**.** Let $P_0 = (0,0)$ and $P_1 = (1,0)$, and let $\mathcal{S}_n = (P_0, \ldots, P_n)$. Then P_{n+1} is a constructible point if it is the intersection of either

1. two lines containing points in $\mathcal{S}_n$;
2. two circles with centers in $\mathcal{S}_n$;
3. a circle and line with center and endpoints in $\mathcal{S}_n$.

Similarly, an angle θ is constructible if for some $a \in \mathbb{R}$, there exists some constructible point x such that $x^2 = a^2$.

Lemma 1.1. If n-gon constructible, then $2n$-gon is constructible.

Lemma 1.2. If a, b, c constructible (or polyquadratic), then $a \pm b$, $\frac{ab}{c}$, and $\sqrt{ab}$ constructible.

Fact 1. If m-gon and n-gon are constructible for coprime m, n, then mn-gon is contructible.

Fact 2. If $p \geq$ prime, then p^k-gon constructible for $k \in \mathbb{N}$.

Theorem 1.3 (Gauss)**.**

$$\cos\frac{2\pi}{17} = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}}}{16}$$

Corollary 1. The 17-gon is constructible.

Corollary 2. If $a \in \mathbb{R}$ is constructible, then $[\mathbb{Q}(a) : \mathbb{Q}] = 2^n$ for some $n \geq$

Corollary 3. Given a cube C_1 with volume V_1, it is impossible to construct a cube C_2 with volume $2V_2$ by ruler and compass. That is, the volume of a cube can not be duplicated by ruler and compass.

Corollary 4. An arbitrary angle cannot be trisected by ruler and compass.

Theorem 1.4 (Gauss-Wantzel)**.** A regular n-gon is constructible $\iff n = 2^r p_1 p_2 \cdots p_s$ for $r \in \mathbb{Z}_{\geq 0}$ and Fermat primes $p_j = 2^{(2^k)} + 1$ for $k \in \mathbb{Z}_{\geq 0}$.