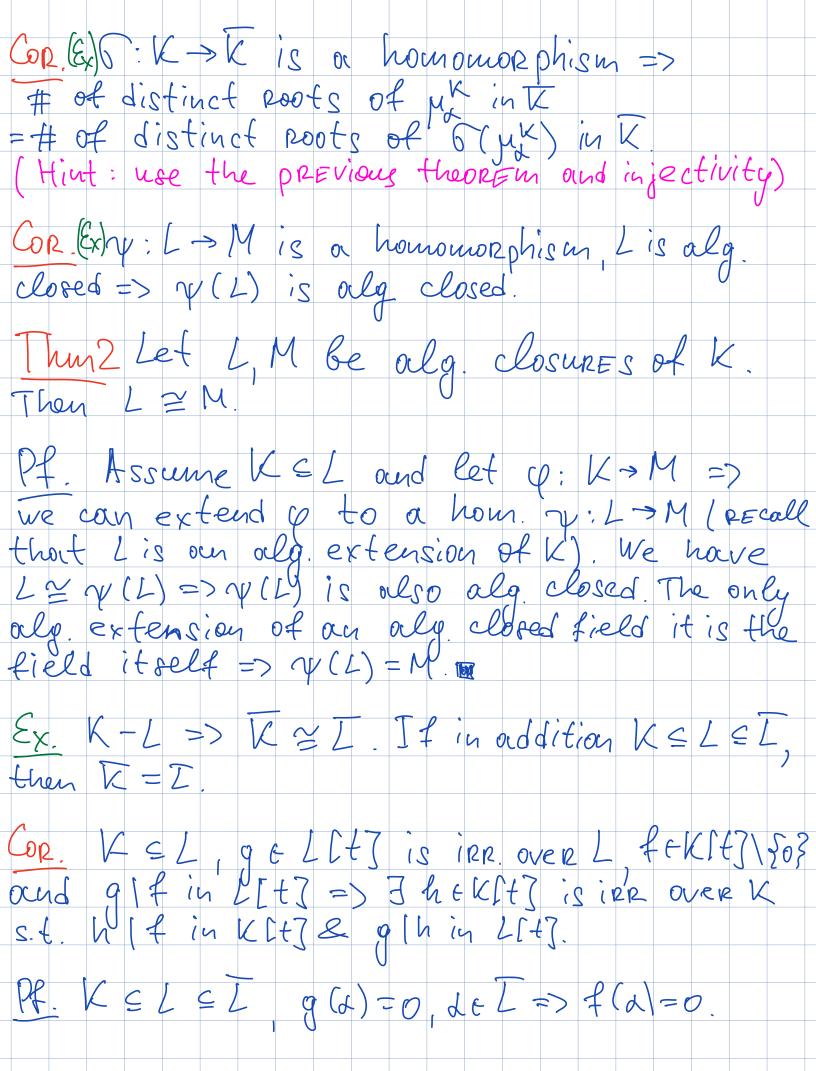
Algebraic closures II (Lecture 10) Thin 1 K-L orlaebraic extension Q: K > K is or homomorphism => 3 extension of Q to or homomorphism: L > K. Pf. S = E(F,y): K & F & L & y: F > K is or homom. extending  $\varphi$ ?  $\neq \emptyset$ Let (F1, 7/1) = (IF2, 7/2) C=> IF, CIF & 7/2 extends If  $C = ((F; \gamma))_{i \in I}$  is a chain => consider  $F = UF; \subseteq L => F$  is a subfield of L & y: It > K is defined ons y(d) = y; (d), de IF, ( \( \gamma' \) (\( \alpha \)) = \( \gamma' \) (\( \alpha \) = \( \gamma' \) (\( \alpha \)) = \( \gamma' \) (\( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \) (\( \gamma' \)) = \( \gamma' \) (\( \gamma' \)) (\( \gamma' \) (\( \gamma' \)) (\( \gamma' well-defined). Thus (F, y) +S=> (F, y) is an upper bound for C => By Zoen's lemma S conforing a maximal element (M, m) If M= L, then we art done, it not, their 3d+L/M => L is orly over K (and hence M). We know
that K is orly closed => 3 BEK S.t.

m(M) = M(M) By the last theorem of the

previous lecture we have an extension of M

J: M(a) = K (i.e. V is a homomorphism). It gives us or contradiction with the maximality of (M,m).



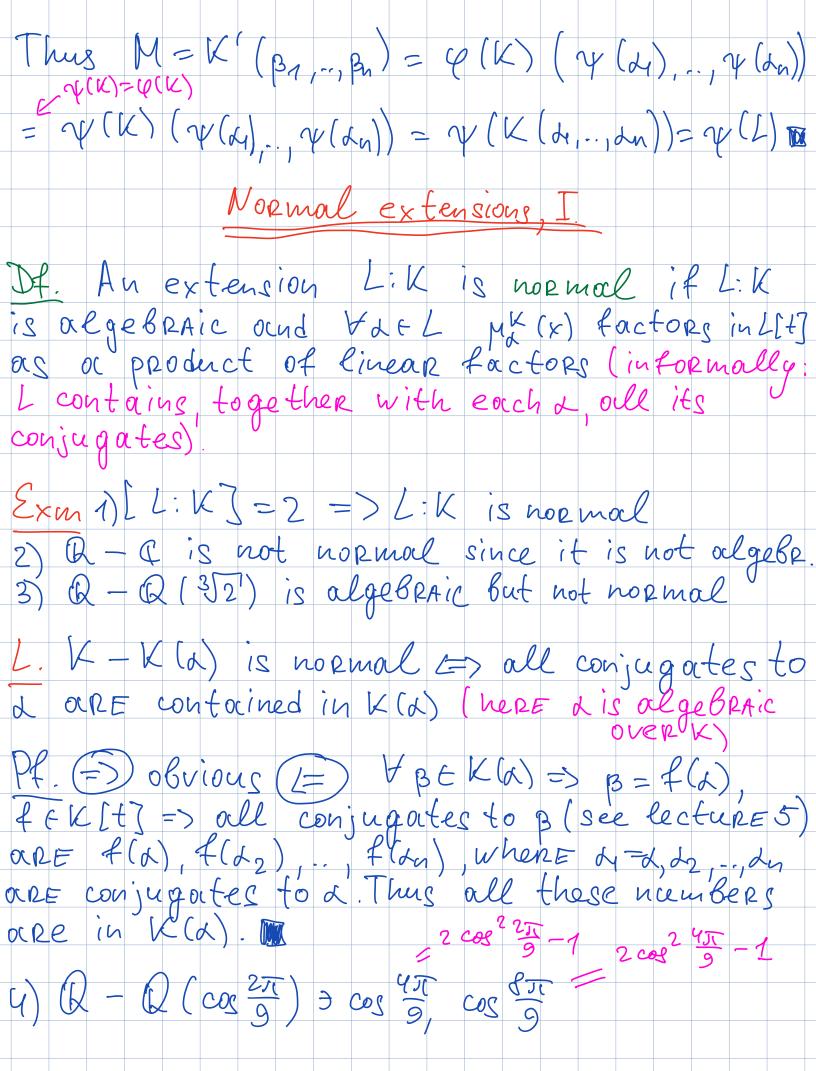
Thus I is orly over K. Put h= µx => h[f]
We have h(x) = 0 & m; (x) = 0 => µ² | h (in K[t])
But g(x)=0 and g f L[t] is irr. => or = c p;
=> g(h) in L[t]. Ihm3 If # EK[t] / and L: K, M: K ERE splitting field extensions for f. Then L=M (in porrticular [L:K]=[M:K]). Pf. Let  $K \subseteq L$ ,  $L_1$ ,...,  $L_n \in L$  be roots of f in  $L = K(L_1,...,L_n)$  and  $\varphi: K \to M$ . Finally let  $K' = \varphi(K)$ ,  $f' = \varphi(f)$ , f = c  $\Pi(t-L_i)$ ,  $c \in K$ . Let M G M => M: M, M: K are alg. extensions

=> M: K is oilso on oilg. extension => M = K

(see above). Further  $\varphi$ : K > M  $\subseteq$  M and L: K

is oilgebraic => By Thm. 1  $\varphi$  can be extended

to or hom.  $\varphi$ : L > M. Let  $\beta$ : =  $\varphi$  ( $\alpha$ :). Then  $f' = \varphi(f) = \gamma(f) = \gamma(f) \left[ \frac{\gamma}{(t - \gamma(x_i))} = \varphi(x_i) \right] \left[ \frac{\gamma}{(t - \beta_i)} \right]$ => f'splits over K'(B1,...,B1). Since M[t]
is or UFD & f'splits over M => B. EM.
We howe K' = M => K'(B1,...,B1) & M =>
M = K'(B1,...,B1) ( recall that M is a splitting
tield ext. of f => M must be minimal)



Ihm K: L is or finite ext. Then Lis normaltes Lis or splitting field ext. for some ff K[t] K Pf. Any finite extension L is K (M, -, dn)

for some orlgebraic dn, -, dn + L Consider

f = [7 MK . If L is normal, then f splits over

L => K(di, dn) = K (B1, -, Br), where B, orrer roots

of f. Thus L is or splitting field of f Now let L= L(L,-, dn), where f=c [7(t-2;) EK [t] Take any g EK [t,-, for ourd an element g(L, dn). Then all its conjugates belong to the set 9 g (doin, -, doin) 30 + Sn EL. Indeed orsh of (x-g (x64), ..., docn)) EK[t] (we used this orgunant several times) and this polynomial hows a root g (x1,..., don)