

1 Algebraic Closure I

Definition 1 (Algebraically closed field, algebraic closure). Let M be a field.

- (i) We say that M is algebraically closed if every non-constant polynomial $f \in M[t]$ has a root in M .
- (ii) We say that M is an algebraic closure of K if $M : K$ is an algebraic field extension having the property that M is algebraically closed.

Lemma 1.1. *Let M be a field. The following are equivalent:*

- (i) *The field M is algebraically closed;*
- (ii) *every non-constant polynomial $f \in M[t]$ factors in $M[t]$ as a product of linear factors;*
- (iii) *every irreducible polynomial in $M[t]$ has degree 1;*
- (iv) *the only algebraic extension of M containing M is itself.*

Definition 2 (Chain). Suppose that X is a nonempty, partially ordered set with \leq denoting the partial ordering. A chain C in X is a collection of elements $\{a_i\}_{i \in I}$ of X having the property that for every $i, j \in I$, either $a_i \leq a_j$ or $a_j \leq a_i$.

Zorn's Lemma: Suppose that X is a nonempty, partially ordered set with \leq the partial ordering. If every non-empty chain C in X has an upper bound in X , then X has at least one maximal element m (i.e. $b \in X$ with $m \leq b \implies b = m$).

Corollary 1. *Any proper ideal A of a commutative ring R is contained in a maximal ideal.*

Lemma 1.2. *Let K be a field. Then there exists an algebraic extension $E : K$, with $K \subseteq E$, having the property that E contains a root of every irreducible $f \in K[t]$, and hence also every $g \in K[t] \setminus K$.*

Theorem 1.3 (Existence of Algebraic Closures). *Suppose that K is a field. Then there exists an algebraic extension \bar{K} of K having the property that \bar{K} is algebraically closed.*

Definition 3 (Extension of field homomorphism, isomorphic field extensions). For $i = 1$ and 2 , let $L_i : K_i$ be a field extension relative to the embedding $\varphi_i : K_i \rightarrow L_i$. Suppose that $\sigma : K_1 \rightarrow K_2$ and $\tau : L_1 \rightarrow L_2$ are isomorphisms. We say that τ extends σ if $\tau \circ \varphi_1 = \varphi_2 \circ \sigma$. In such circumstances, we say that $L_1 : K_1$ and $L_2 : K_2$ are isomorphic field extensions.

$$\begin{array}{ccc}
 L_1 & \xrightarrow{\tau} & L_2 \\
 \varphi_1 \uparrow & \nearrow & \uparrow \varphi_2 \\
 K_1 & \xrightarrow{\sigma} & K_2
 \end{array}$$

When $\sigma : K_1 \rightarrow K_2$ and $\tau : L_1 \rightarrow L_2$ are homomorphisms (instead of isomorphisms), then τ extends σ as a homomorphism of fields when the isomorphism $\tau : L_1 \rightarrow L'_1 = \tau(L_1)$ extends the isomorphism $\sigma : K_1 \rightarrow K'_1 = \sigma(K_1)$.

Definition 4 (K -homomorphism). Let $L : K$ be a field extension relative to the embedding $\varphi : K \rightarrow L$, and let M be a subfield of L containing $\varphi(K)$. Then, when $\sigma : M \rightarrow L$ is a homomorphism, we say that σ is a K -homomorphism if σ leaves $\varphi(K)$ pointwise fixed, which is to say that for all $\alpha \in \varphi(K)$, one has $\sigma(\alpha) = \alpha$.

Lemma 1.4. *Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\tau : L \rightarrow L$ is a K -homomorphism. Suppose that $f \in K[t]$ has the property that $\deg f \geq 1$, and additionally that $\alpha \in L$.*

- (i) *if $f(\alpha) = 0$, one has $f(\tau(\alpha)) = 0$;*
- (ii) *if τ is a K -automorphism of L , then $f(\alpha) = 0 \iff f(\tau(\alpha)) = 0$.*

Theorem 1.5. *Let $\sigma : K_1 \rightarrow K_2$ be a field isomorphism. Suppose that L_i is a field with $K_i \subseteq L_i$ ($i = 1, 2$). Suppose also that $\alpha \in L_1$ is algebraic over K_1 , and that $\beta \in L_2$ is algebraic over K_2 . Then we can extend σ to an isomorphism $\tau : K_1(\alpha) \rightarrow K_2(\beta)$ in such a manner that $\tau(\alpha) = \beta$ if and only if $m_\beta(K_2) = \sigma(m_\alpha(K_1))$.*

$$\begin{array}{ccccc}
K_2 & \xrightarrow{\varphi_2} & K_2(\beta) & \xhookrightarrow{\iota_2} & L_2 \\
\downarrow \sigma & & \downarrow \tau & & \\
K_1 & \xrightarrow{\varphi_1} & K_1(\alpha) & \xhookrightarrow{\iota_1} & L_1
\end{array}$$

Note: When $\tau : K_1(\alpha) \rightarrow K_2(\beta)$ is a homomorphism, and τ extends the homomorphism $\sigma : K_1 \rightarrow K_2$, then τ is completely determined by σ and the value of $\tau(\alpha)$.

Corollary 2. *Let $L : M$ be a field extension with $M \subseteq L$. Suppose that $\sigma : M \rightarrow L$ is a homomorphism, and $\alpha \in L$ is algebraic over M . Then the number of ways we can extend σ to a homomorphism $\tau : M(\alpha) \rightarrow L$ is equal to the number of distinct roots of $\sigma(m_\alpha(M))$ that lie in L .*