# 1 Ruler and Compass Constructions

**Definition 1** (Constructible points/angles)**.** Let $P_0 = (0,0)$ and $P_1 = (1,0)$, and let $\mathcal{S}_n = (P_0, \dots, P_n)$. Then $P_{n+1}$ is a constructible point if it is the intersection of either

1. two lines containing points in $\mathcal{S}_n$;
2. two circles with centers in $\mathcal{S}_n$;
3. a circle and line with center and endpoints in $\mathcal{S}_n$.

Similarly, an angle $\theta$ is constructible if for some $a \in \mathbb{R}$, there exists some constructible point $x$ such that $x^2 = a^2$.

**Lemma 1.1.** If $n$-gon constructible, then $2n$-gon is constructible.

**Lemma 1.2.** If $a, b, c$ constructible (or polyquadratic), then $a \pm b$, $\frac{ab}{c}$, and $\sqrt{ab}$ constructible.

**Fact 1.** If $m$-gon and $n$-gon are constructible for coprime $m, n$, then $mn$-gon is contructible.

**Fact 2.** If $p \geq$ prime, then $p^k$-gon constructible for $k \in \mathbb{N}$.

**Theorem 1.3** (Gauss)**.**

$$\cos\frac{2\pi}{17} = \frac{-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+2\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}}}{16}$$

**Corollary 1.** The 17-gon is constructible.

**Corollary 2.** If $a \in \mathbb{R}$ is constructible, then $[\mathbb{Q}(a) : \mathbb{Q}] = 2^n$ for some $n \geq$

**Corollary 3.** Given a cube $C_1$ with volume $V_1$, it is impossible to construct a cube $C_2$ with volume $2V_2$ by ruler and compass. That is, the volume of a cube can not be duplicated by ruler and compass.

**Corollary 4.** An arbitrary angle cannot be trisected by ruler and compass.

**Theorem 1.4** (Gauss-Wantzel)**.** A regular $n$-gon is constructible $\iff n = 2^r \prod_{i=1}^{s} p_i$ for $r \in \mathbb{Z}_{\geq 0}$ and Fermat primes $p_i = 2^{(2^k)} + 1$ for $k \in \mathbb{Z}_{\geq 0}$.

TODO: define $V_n$

**Lemma 1.5.** For all integers $d$ and all $d$-periods $\theta$, $V_d = \mathbb{Q}(\theta)$.