MA 454: Honors Galois Theory Notes

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Lecture 1

1 Introduction

1.1 Quadratic polynomials

Example 1.1 (n=3).

Definition 1.1 (Symmetric function). Let $\phi(x_1, \ldots, x_n)$ be a function. Then ϕ is *symmetric* if $\forall$ permutations $\omega \in S_n$, $\phi(x_1, \ldots, x_n) = \phi(x_{\omega(1)}, \ldots, x_{\omega(n)})$

Definition 1.2 (Elementary symmetric functino in $x_1, \ldots, x_n$).

$$\begin{aligned}
\sigma_1 &= \sigma_1(x_1, \ldots, x_n) = x_1 + \ldots + x_n \\
\sigma_2 &= \sigma_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\
\sigma_n &= \sigma_n(x_1, \ldots, x_n) = \sum_{i_1 < \ldots < i_k} x_{i_1} \cdots x_{i_k} \quad (\# \text{ of terms is } \binom{n}{k})
\end{aligned}$$

Theorem 1.1.

1. For $\forall$ symmetric function ϕ $\exists!$ polynomial $P(t_1, \ldots, t_n)$ such that $\phi(x_1, \ldots, x_n) = P(\sigma_1, \ldots, \sigma_n)$
2. Moreover, if ϕ is a polynomial with coefficients in a ring R ($\phi \in R[x_1, \ldots, x_n]$) then $P \in R[x_1, \ldots, x_n]$

Theorem 1.2 (Vieta Formula).

$$\begin{aligned}
x^n + a_1x^{n-1} + \ldots + a_n &= (x - x_1) \cdots (x - x_n) \\
&= x^n - \sigma_1(x_1, \ldots, x_n)x^{n-1} + \sigma_2(x_1, \ldots, x_n)x^{n-2} + \cdots + (-1)^n\sigma_n(x_1, \ldots, x_n)
\end{aligned}$$

Corollary 1.2.1. The discriminant $D = P(a_1, \ldots, a_n)$ is a polynomial

1.2 Cubic polynomials

If $ax^3 + bx^2 + cx + d = 0$, then one solution is

$$\begin{aligned}
x = &\sqrt[3]{-\frac{1}{2}\left(\frac{2b^3-9abc+27a^2d}{27a^3}\right) + \sqrt{\left(\frac{1}{2}\left(\frac{2b^3-9abc+27a^2d}{27a^3}\right)\right)^2 + \left(\frac{3ac-b^2}{9a^2}\right)^3}} \\
&+ \sqrt[3]{-\frac{1}{2}\left(\frac{2b^3-9abc+27a^2d}{27a^3}\right) - \sqrt{\left(\frac{1}{2}\left(\frac{2b^3-9abc+27a^2d}{27a^3}\right)\right)^2 + \left(\frac{3ac-b^2}{9a^2}\right)^3}}
\end{aligned}$$

$$\begin{aligned}
x^3 + Ax^2 + Bx + C &= \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q \\
\implies &x^3 + px + q = 0
\end{aligned}$$

$$\underbrace{(a+b)}_{x}^3 = 3ab(a+b) + a^3 + b^3$$
$$x^3 - 3abx - a^3 - b^3 = 0, \quad x_1 = a_b$$

$$x_1 + x_2 + x_3 = 0 \implies x_2 + x_3 = -a - b$$
$$x_1x_2 + x_1x_3 + x_2x_3 = a^3 + b^3 \implies x_2x_3 = \frac{a^3+b^3}{x_1} = \frac{a^3+b^3}{a+b} = a^2 - ab + b^2$$

Theorem 1.3 (Inverse Vieta Theorem)**.**

Example 1.2 (Root of unity)**.** ε

Example 1.3. What about $x^3 + px + q = 0$?

1.3 Quadric Method

Let $f(x) = x^4 + ax^2 + bx + c = 0$.

1. If $b = 0$, it is simply a quadratic equation.
2. If $x^4 - g^2(x) = 0 \implies x^2 = g(x), x^2 = -g(x)$

$$f(x) = \left(x^2 + \frac{y}{2}\right)^2 + (a-y)x^2 + bx + c - \left(\frac{y^2}{4}\right)$$
$$D = b^2 - 4(a-y)(c - \frac{y^2}{4}) = 0$$

Definition 1.3 (Ferrari's Resolvent)**.** $y^3 - ay^2 - 4cy + 4ac - b^2 = 0$

$$\begin{aligned} g(x) &= Ax + B \\ 0 = f(x) &= \left(x^2 + \frac{y}{2}\right)^2 - (Ax+B)^2 \\ &= \left(x^2 + \frac{y}{2} - Ax - B\right)\left(x^2 + \frac{y}{2} + Ax + B\right) \end{aligned}$$

$$\begin{aligned} x_1 + x_2 = A;\ x_1x_2 &= \frac{y}{2} - B \\ x_3 + x_4 = -A;\ x_3x_4 &= \frac{y}{2} + B \end{aligned}$$

$$\begin{aligned} x_1x_2 + x_3x_4 &= y_1 \\ x_1x_3 + x_2x_4 &= y_2 \\ x_1x_4 + x_2x_3 &= y_3 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$$

Suppose we have some quadric equation $f(x) = x^4 + ax^2 + bx + c$. Then we have unknown roots x_1, x_2, x_3, and x_4.

Claim 1. *y_1, y_2, y_3 are roots of a cubic equation*

$y_1 + y_2 + y_3 = \sigma_2(x_1, x_2, x_3, x_4) = a$
$\sigma_2(y_1, y_2, y_3) = \phi(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$

Example 1.4. Consider the polynomial $\phi(x_1, x_2, x_3, x_4) = x_1 + x_2 - x_3 - x_4$

$$\begin{cases} z_1 = (x_1 + x_2 - x_3 - x_4)^2 \\ z_2 = (x_1 - x_2 + x_3 - x_4)^2 \\ z_3 = (x_1 - x_2 - x_3 + x_4)^2 \end{cases}$$