Problem Set 8: Math 454 Spring 2017
Due Thursday April 6

March 28, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Let E/F be an extension and $H \leq \mathrm{Aut}_{F-\mathrm{alg}}(E)$ be a subgroup. For $\alpha \in E$, define

$$\mathrm{Eval}_{H,\alpha} \colon H \longrightarrow E$$

to be $\mathrm{Eval}_{H,\beta}(\sigma) = \sigma(\beta)$. Prove that $L_H \colon E \to \mathrm{Fun}(H,E)$ defined by $L_H(\beta) = \mathrm{Eval}_{H,\beta}$ is an K_H–linear function.

Problem 2. Let E/F be an extension of fields and $H \leq \mathrm{Aut}_{F-\mathrm{alg}}(E)$ be a finite subgroup.

(a) Prove that if $\lambda \in E$, then

$$\alpha_\lambda \overset{\mathrm{def}}{=} \sum_{\sigma \in H} \sigma(\lambda)$$

is invariant under H. That is, for each $\sigma' \in H$, we have $\sigma'(\alpha_\lambda) = \alpha_\lambda$.

(b) Prove that if $\lambda \in E$ and $\lambda \neq 0$, then there exists $\lambda' \in E$ such that

$$\sum_{\sigma \in H} \sigma(\lambda'\lambda) \neq 0.$$

Problem 3. Let E/F be a finite extension with $F \leq K \leq E$.

(a) Prove that $[E:F]_s$ divides $[E:F]$.

(b) Prove that $[E:F]_s = [K:F]_s[E:K]_s$.

Problem 4. Let K_1/F, K_2/K_1, and E/K_2 be algebraic extensions with K_2/K_1 is finite.

(a) Prove that

$$\left|\mathrm{Hom}_{K_1-\mathrm{alg}}(K_2,E)\right| \leq [K_2:K_1].$$

(b) Prove that

$$\left|\mathrm{Hom}_{K_1-\mathrm{alg}}(K_2,E)\right| = [K_2:K_1]_s.$$

Problem 5. Let E/F be a separable extension and $F \leq K \leq E$. Prove that E/K and K/F are separable extensions.

Problem 6. Prove that if F is a field with $\mathrm{char}(F) = p$ with $p \neq 0$, then F if perfect if and only if for every $\alpha \in F$, there exists $\beta \in F$ such that $\beta^p = \alpha$.

Problem 7. Prove that if E/F is a normal extension and $F \leq K \leq E$, then E/K is normal.

Problem 8. Prove that if E/F is algebraic, then $\{e\} \in \mathscr{L}_{\text{sub}}^{\text{closed}}(E/F)$.

Problem 9. Let E/F be Galois and K/F an extension. Prove that EK/K is Galois.

Problem 10. Let E/F be a normal extension, $\beta \in E$, and $\mathscr{O}_\beta = \left\{\sigma(\beta) \;:\; \sigma \in \mathrm{Aut}_{F-\mathrm{alg}}(E)\right\}$. Define

$$Q(t) = \prod_{\beta' \in \mathscr{O}_\beta} (t - \beta') = \sum_{i=0}^{|\mathscr{O}_\beta|} \alpha_i t^i.$$

Prove that for each $\sigma \in \mathrm{Aut}_{F-\mathrm{alg}}(E)$ and each α_i, we have $\sigma(\alpha_i) = \alpha_i$. Deduce that $\alpha_i \in F$ for each i and so $Q \in F[t]$.