Exercise 11.1. Let $G = \mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number. Construct a subnormal series G_j of subgroups of G such that $|G_{j-1}/G_j| = p$.

Solution. Obviously G is a p-group by definition. We know all p-groups are soluble, so it has a finite subnormal series for which G_{j-1}/G_j is cyclic by definition of soluble. Additionally, since G is abelian every subgroup is normal. For $0 \leq j \leq n$ set

$$G_j \coloneqq p^j\mathbb{Z}/p^n\mathbb{Z} \quad \subseteq \quad G = \mathbb{Z}/p^n\mathbb{Z}.$$

Because $p^j\mathbb{Z} \supseteq p^n\mathbb{Z}$, the quotient $p^j\mathbb{Z}/p^n\mathbb{Z}$ is a subgroup of G. The orders of the successive terms is thus

$$|G_j| = \left|\frac{p^j\mathbb{Z}}{p^n\mathbb{Z}}\right| = p^{n-j}, \qquad 0 \leq j \leq n,$$

and the orders of the quotients of consecutive terms is

$$\left|\frac{G_{j-1}}{G_j}\right| = \frac{|G_{j-1}|}{|G_j|} = \frac{p^{\,n-(j-1)}}{p^{\,n-j}} = p.$$

Thus every quotient G_{j-1}/G_j is cyclic of order p. □

Exercise 11.2.1. Let G be a group. Prove that G' is a normal subgroup of G such that G/G' is abelian.

Solution. By definition, we have that $G' = [G, G] = \{[x, y] : x, y \in G\}$. Let $g \in G$ be arbitrary and consider any generator $[x, y] \in G'$. Then

$$\begin{aligned}
g[a,b]g^{-1} &= ga^{-1}b^{-1}abg^{-1} \\
&= \left(ga^{-1}g^{-1}\right)\left(gb^{-1}g^{-1}\right)\left(gag^{-1}\right)\left(gbg^{-1}\right) \\
&= \left(gag^{-1}\right)^{-1}\left(gbg^{-1}\right)^{-1}\left(gag^{-1}\right)\left(gbg^{-1}\right) \\
&= [gag^{-1}, gbg^{-1}] \in G'.
\end{aligned}$$

Thus $gG'g^{-1} \subseteq G$ and hence $G' \trianglelefteq G$.

Now, consider $aG', bG' \in G/G'$. Then

$$aG'bG' = abG' \quad \text{and} \quad bG'aG' = baG'.$$

Notice that $ba[a,b] = baa^{-1}b^{-1}ab = ab$. But $[a,b] \in \ker(G/G')$ and thus $ba[a,b]G' = baG' = abG'$. □

Exercise 11.2.2. Prove that if N is any normal subgroup of G such that G/N is abelian, then $G' \leqslant N$.

Solution. Suppose $N \trianglelefteq G$ such that G/N is abelian. Obviously $\varphi : G \mapsto G/N$ is a well defined epimorphism. We know for any epimorphism from a group to a subgroup, the image of the derived group is exactly the derived subgroup. Thus $\varphi(G') = (G/N)'$. By hypothesis G/N is abelian so $(G/N) = \{\text{Id.}\} = \varphi(G')$. Thus $G' \subseteq \ker\varphi = N$ and hence $G' \leqslant N$. □

Exercise 11.3. Let $\mathbb{F}$ be a field and

$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F} \right\}$$

be the Heisenberg group. Prove that H is soluble.

Solution. Suppose we let

$$X(a) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y(b) = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } (a, b, c \in \mathbb{F}).$$

Note that $[X(a), Z(c)] = Y(ac)$. So the derived subgroup of $Y(b)$ is

$$H' = \langle Y(b) : b \in \mathbb{F} \rangle = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in \mathbb{F} \right\} = Z(H).$$

By definition of center, H' is abelian. For any $A, B \in H(F)$,

$$ABH' = BAH'$$

because the factor $AB(BA)^{-1} = [A, B] \in H'$. Thus H/H' is abelian.

Next, we have that $H'' = \{I_3\} = \{\text{Id.}\}$. Hence the derived series terminates and H is soluble by definition. $\square$

Exercise 11.4. Prove that A_n, $n \geq 3$ is generated by 3-cycles.

Solution. We know that A_n is generated by the set S of products of transpositions. Let $S = \{\text{Id.}\} \sqcup \mathcal{B} \sqcup \mathcal{T}$ where $\mathcal{B}$ is the set of products of transpositions, and $\mathcal{T}$ is the set of 3-cycles. The case for $n = 3$ is trivial, as $A_3 = \{\{\text{Id.}\}, (1\,2\,3), (1\,3\,2)\}$. Assume $n \geq 4$. Since every element of A_n can be decomposed as a product of an even number of transpositions by definition, we can pair up transpositions and it is enough to show that each pair can be written as 3-cycles. Consider some $\sigma\tau \in \mathcal{B}$ such that $\sigma \neq \tau$. The only 2 cases are if σ and τ overlap, or if they don't. Suppose they overlap. Then without loss of generality $\sigma = (a\,b)$ and $\tau = (a\,c)$ for some $a < b < c$ so obviously $\sigma\tau = (a\,b\,c)$. Now, suppose they are disjoint. Then without loss of generality $\sigma = (a\,b)$ and $\tau = (c\,d)$ for distinct a, b, c, d. Notice that $(a\,b)(c\,d) = (a\,c\,b)(a\,c\,d)$. Hence A_n can be generated by 3-cycles for $n \geq 4$. $\square$

Exercise 11.5. Let G be a group. Find G' for
a) $G = S_3$ *b)* $G = A_4$ *c)* $G = S_4$ (use the previous question).

Solution. (a) Consider the epimorphism $\varphi : S_3 \to \mathbb{Z}_2$, where $\sigma \mapsto 0$ if σ is even, and $\sigma \mapsto 1$ if σ is odd. Then we have $\ker\varphi = A_3$. Since any commutator is necessarily even, we have $S_3' \subseteq A_3$. Notice that any 3-cycle $(a\,b\,c)$ can be written as a commutator $[(a\,c), (a\,b)]$, hence $A_3 \subseteq S_3'$. Thus $S_3' = A_3$.

(b) By exercise 11.4, we know A_4 can be generated by 3-cycles. A 3-cycle γ satisfies the relation $\gamma^3 = \text{Id.}$, so for any 3-cycles $\alpha, \widehat{\alpha} \in A_4$, we have $[\alpha, \widehat{\alpha}] = \alpha^2\widehat{\alpha}^2\alpha\widehat{\alpha}$. If we define $\alpha = (\alpha_1\,\alpha_2\,\alpha_3)$ and $\widehat{\alpha} = (\alpha_1\,\alpha_3\,\alpha_4)$, and we can quickly check that $[\alpha, \widehat{\alpha}] = (\alpha_1\,\alpha_4)(\alpha_2\,\alpha_3)$. Conjugation of this permutation yields $(\alpha_1\,\alpha_2)(\alpha_3\,\alpha_4)$ and $(\alpha_1\,\alpha_3)(\alpha_2\,\alpha_4)$. These elements together form the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $|A_4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 3$ so it must be isomorphic to $\mathbb{Z}_3$ and is hence abelian. Thus $A_4' = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(c) Using a similar mapping as in part (a), we can see that $\varphi : S_4 \to \mathbb{Z}_2$ has $\ker \varphi = A_4$. Then by the same logic as before, the kernel bounds the derived group below since the image of the map is abelian. Thus $S_4' \subseteq A_4$. Recall that A_4 can be generated by 3 cycles. Then we notice that $(1\,2\,3) = [(1\,3), (1\,2)]$ as before, hence $A_4 \subseteq S_4'$. Thus $S_4' = A_4$.

$\square$