

# MA 454: Honors Galois Theory Notes

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## Lecture 1

### 1 Introduction

#### 1.1 Quadratic polynomials

**Example 1.1** (n=3).

**Definition 1.1** (Symmetric function). Let  $\phi(x_1, \dots, x_n)$  be a function. Then  $\phi$  is *symmetric* if  $\forall$  permutations  $\omega \in S_n$   $\phi(x_1, \dots, x_n) = \phi(x_{\omega(1)}, \dots, x_{\omega(n)})$

**Definition 1.2** (Elementary symmetric function in  $x_1, \dots, x_n$ ).

$$\begin{aligned}\sigma_1 &= \sigma_1(x_1, \dots, x_n) = x_1 + \dots + x_n \\ \sigma_2 &= \sigma_2(x_1, \dots, x_n) = x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots + x_{n-1}x_n \\ \sigma_n &= \sigma_n(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k} \quad \# \text{ of terms is } \binom{n}{k}\end{aligned}$$

**Theorem 1.1.**

1. For  $\forall$  symmetric function  $\phi$   $\exists!$  polynomial  $P(t_1, \dots, t_n)$  such that  $\phi(x_1, \dots, x_n) = P(\sigma_1, \dots, \sigma_n)$
2. Moreover, if  $\phi$  is a polynomial with coefficients in a ring  $R$  ( $\phi \in R[x_1, \dots, x_n]$ ) then  $P \in R[\sigma_1, \dots, \sigma_n]$

**Theorem 1.2** (Vieta Formula).

$$\begin{aligned}x^n + a_1x^{n-1} + \dots + a_n &= (x - x_1) \dots (x - x_n) \\ &= x^n - \sigma_1(x_1, \dots, x_n)x^{n-1} + \sigma_2(x_1, \dots, x_n)x^{n-2} + \dots + (-1)^n \sigma_n(x_1, \dots, x_n)\end{aligned}$$

**Corollary 1.2.1.** The discriminant  $D = P(a_1, \dots, a_n)$  is a polynomial

#### 1.2 Cubic polynomials

If  $ax^3 + bx^2 + cx + d = 0$ , then one solution is

*cubic formula*

$$\begin{aligned}x^3 + Ax^2 + Bx + C &= \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q \\ \implies x^3 + px + q &= 0\end{aligned}$$

$$\begin{aligned}\underbrace{(a+b)^3}_x &= 3ab(a+b) + a^3 + b^3 \\ x^3 - 3abx - a^3 - b^3 &= 0, \quad x_1 = a_b\end{aligned}$$

$$x_1 + x_2 + x_3 = 0 \implies x_2 + x_3 = -a - b$$

$$x_1x_2 + x_1x_3 + x_2x_3 = a^3 + b^3 \implies x_2x_3 = \frac{a^3 + b^3}{x_1} = \frac{a^3 + b^3}{a + b} = a^2 - ab + b^2$$

**Theorem 1.3** (Inverse Vieta Theorem).

**Example 1.2** (Root of unity).  $\varepsilon$

**Example 1.3.** What about  $x^3 + px + q = 0$ ?

### 1.3 Quadric Method

Let  $f(x) = x^4 + ax^2 + bx + c = 0$ .

1. If  $b = 0$ , it is simply a quadratic equation.
2. If  $x^4 - g^2(x) = 0 \implies x^2 = g(x), x^2 = -g(x)$

$$f(x) = \left(x^2 + \frac{y}{2}\right)^2 + (a - y)x^2 + bx + c - \left(\frac{y^2}{4}\right)$$

$$D = b^2 - 4(a - y)\left(c - \frac{y^2}{4}\right) = 0$$

**Definition 1.3** (Ferrari's Resolvent).  $y^3 - ay^2 - 4cy + 4ac - b^2 = 0$

$$g(x) = Ax + B$$

$$0 = f(x) = \left(x^2 + \frac{y}{2}\right)^2 - (Ax + B)^2$$

$$= \left(x^2 + \frac{y}{2} - Ax - B\right) \left(x^2 + \frac{y}{2} + Ax + B\right)$$

$$x_1 + x_2 = A; \quad x_1x_2 = \frac{y}{2} - B$$

$$x_3 + x_4 = -A; \quad x_3x_4 = \frac{y}{2} + B$$

$$x_1x_2 + x_3x_4 = y_1$$

$$x_1x_3 + x_2x_4 = y_2$$

$$x_1x_4 + x_2x_3 = y_3$$

$$x_1 + x_2 + x_3 + x_4 = 0$$

Suppose we have some quadric equation  $f(x) = x^4 + ax^2 + bx + c$ . Then we have unknown roots  $x_1, x_2, x_3$ , and  $x_4$ .

**Claim 1.**  $y_1, y_2, y_3$  are roots of a cubic equation

$$\begin{aligned}y_1 + y_2 + y_3 &= \sigma_2(x_1, x_2, x_3, x_4) = a \\ \sigma_2(y_1, y_2, y_3) &= \phi(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4\end{aligned}$$

**Example 1.4.** Consider the polynomial  $\phi(x_1, x_2, x_3, x_4) = x_1 + x_2 - x_3 - x_4$

$$\begin{cases} z_1 = (x_1 + x_2 - x_3 - x_4)^2 \\ z_2 = (x_1 - x_2 + x_3 - x_4)^2 \\ z_3 = (x_1 - x_2 - x_3 + x_4)^2 \end{cases}$$