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MA454
HW 1
Shkredov

1.1) $x^3-3x+1=0$

$$x^3-3x=-1 \qquad (1)$$

Let $x=2\cos\theta$. Then

$$8\cos^3\theta-6\cos\theta=-1$$

By formula, $4\cos^3\theta-3\cos\theta=\cos 3\theta$.

$\Rightarrow 2\cos 3\theta=-1$

$\cos 3\theta=-\frac{1}{2}$

$3\theta=\frac{2}{3}\pi+2k\pi,\ \frac{4}{3}\pi+2k\pi \qquad (k\in\mathbb{Z})$

$\Rightarrow \theta=\frac{2}{9}\pi+\frac{2}{3}k\pi,\ \frac{4}{9}\pi+\frac{2}{3}k\pi$

$\Rightarrow \theta_1=\frac{2}{9}\pi \quad \theta_2=\frac{4}{9}\pi \quad \theta_3=\frac{8}{9}\pi$

$\Rightarrow \boxed{x_1=2\cos\frac{2\pi}{9} \quad x_2=2\cos\frac{4\pi}{9} \quad x_3=2\cos\frac{8\pi}{9}}$ ∎

1.2) $f(x):=x^3-3x\cdot\sqrt[3]{2}-3=0$

$\left.\begin{matrix}A^3+B^3=3\\ AB=\sqrt[3]{2}\end{matrix}\right\}\Rightarrow \begin{matrix}A^3=2\\ B^3=1\end{matrix} \Rightarrow \begin{matrix}A=\sqrt[3]{2}\\ B=1\end{matrix} \Rightarrow AB=\sqrt[3]{2}$

Let $x=A+B=1+\sqrt[3]{2}$

$x^3=(1+\sqrt[3]{2})^3=A^3+B^3+3AB(A+B)=3+3\cdot\sqrt[3]{2}+3\sqrt[3]{2}^2$

$\Rightarrow f(x)=x^3-3x\sqrt[3]{2}-3=[3+3\cdot\sqrt[3]{2}+3\sqrt[3]{2}^2]-3\sqrt[3]{2}(1+\sqrt[3]{2})-3$

$=\cancel{3}+\cancel{3\cdot\sqrt[3]{2}}+\cancel{3\sqrt[3]{2}^2}-\cancel{3\sqrt[3]{2}}-\cancel{3\sqrt[3]{2}^2}\cancel{-3}$

$=0$

$\Rightarrow x_1$ is a root of f. Let $\omega=\exp(\frac{2}{3}\pi i)$

$\left.\begin{matrix}A_1=1 & B_1=\sqrt[3]{2}\\ A_2=\omega & B_2=\sqrt[3]{2}\omega^2\\ A_3=\omega^2 & B_2=\sqrt[3]{2}\omega\end{matrix}\right\} = AB=\sqrt[3]{2}\ \forall A,B \Rightarrow \begin{matrix}x_1=1+\sqrt[3]{2}\\ x_2=\omega+\sqrt[3]{2}\omega^2\\ x_3=\omega^2+\sqrt[3]{2}\omega\end{matrix}$ ∎

2) $f(x) = x^3 + ax^2 + bx + c = 0$

Want $x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1}$

By Viete, (i) $x_1 + x_2 + x_3 = -a$

(ii) $x_1x_2 + x_2x_3 + x_1x_3 = b$

(iii) $x_1x_2x_3 = -c$

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2 = \underbrace{(\alpha+\beta+\gamma)}^2 - 2(\underbrace{\alpha\beta+\beta\gamma+\alpha\gamma})$$

$$\Rightarrow x_1^2+x_2^2+x_3^2 = (-a)^2 - 2b$$

$$= a^2 - 2b$$

$$x_1^{-1}+x_2^{-1}+x_3^{-1} = \frac{x_1x_2+x_2x_3+x_1x_3}{x_1x_2x_3} = \frac{b}{-c}$$

$$\boxed{\Rightarrow x_1^2+x_2^2+x_3^2+x_1^{-1}+x_2^{-1}+x_3^{-1} = a^2-2b-\frac{b}{c}}$$ ⊠

3) $f = f(x_1, x_2, x_3, x_4, x_5) = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$

WTS $\text{stab}(f) = D_5 = \langle r, s \mid r^5 = s^2 = e, srs^{-1} = r^{-1} \rangle$

Notice that

$r = (1\,2\,3\,4\,5)$: $f \to x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 + x_1x_2 = f$

$s = (23)(14)$: $f \to x_4x_3 + x_3x_2 + x_2x_1 + x_1x_5 + x_5x_4 = f$

So $r, s \in \text{stab}(f)$ and trivially $|r| = 5$ and $|s| = 2$.

By def group action, $r(sf) = (rs)f$.

Then $s \in \text{stab}(f) \Rightarrow r(sf) = rf = f \Rightarrow rs \in \text{stab}(f)$.

Also $srs^{-1} = srs = (23)(14)(12345)(23)(14) = (23451) = r^{-1}$

Thus by def D_5 $\langle r, s \rangle \cong D_5 \leq \text{stab}(f)$

Assume $\sigma \in stab(f)$.

$x_1x_2 \to x_2x_3 \to x_3x_4 \to x_4x_5 \to x_5x_1$

The monomials of f form a 5-cycle in which a rotation preserves direction and reflection reverses it.

$\Rightarrow \sigma$ must be rot. or ref. to preserve the 5-cycle structure of f.

$\Rightarrow \sigma \in D_5 \Rightarrow stab(f) \leq D_5$

$\Rightarrow stab(f) = D_5$ $\blacksquare$

4.1) By def, $F = \sum_{h \in H} h \cdot f$.

Pick some $\hat{h} \in H$.

$$\Rightarrow \hat{h} \cdot F = \hat{h} \cdot \sum_{h \in H} (h \cdot f) = \sum_{h \in H} (\hat{h} \cdot (h \cdot f)).$$

Def gp action $\Rightarrow \hat{h} \cdot (h \cdot f) = (\hat{h} \cdot h) \cdot f$.

By closure of H, $\hat{h} \cdot h \in H$. Thus

$$\sum_{h \in H} ((\hat{h} \cdot h) \cdot f) = \sum_{k \in \sigma H} k \cdot f = F$$

$\therefore \hat{h} \in H \Rightarrow \hat{h} \cdot F = F$ $\blacksquare$

4.2) ($\Rightarrow$) Followsfrom part 1.

($\Leftarrow$) Suppose wehave some perm $\sigma \in S_n$.

Assume $\sigma F = F$. By part 1,

$$\sigma \cdot F = \sigma \cdot \sum_{h \in H} h \cdot f = \sum_{k \in \sigma H} k \cdot f$$

Note that since every exponent is unique, the monomial $\varphi \cdot f$ is unique for each $\varphi \in S_n$.

That is, given another perm $\mu \in S_n$,

$$\sigma \cdot f = \mu \cdot f \iff \sigma = \mu.$$

So the sums $\sum_{k \in \sigma H} k \cdot f$ and $\sum_{h \in H} h \cdot f$ are equal

$\iff \sigma H = H$.

It is a basic property of cosets that

$$\sigma H = H \iff \sigma \in H.$$

Thus $\sigma F = F \Rightarrow \sigma \in H$. $\blacksquare$

5.1) Supp. $f(x_1, \dots, x_n)$ and $g(x_1, \dots, x_n)$ are skew-symmetric.

WLOG, let $\varphi(x_1, \dots, x_n) := \frac{f(\dots)}{g(\dots)}$.

Then for any perm $\sigma \in S_n$,

$$\sigma\varphi = \frac{f(x_{\sigma(1)}, \dots, x_{\sigma(n)})}{g(x_{\sigma(1)}, \dots, x_{\sigma(n)})} = \frac{-f}{-g} = \varphi$$

$\Rightarrow \varphi$ is symm by def symm.

5.2) Consider some transposition $\tau = (i\ j)$.

Notice the only difference between Δ and $\tau\Delta$ is the term $(x_i - x_j)$ in Δ and $(x_j - x_i) = -(x_i - x_j)$ in $\tau\Delta$.

Thus transposing Δ changes it by a factor of -1, exactly the def. of skew symm.

Thus Δ is skew symm. $\blacksquare$

5.3) Assuming the problem should say

skew

3) Prove that any symmetric polynomial f is a product of Δ and another symmetric polynomial g.

Notice for some skew polynomial $f(x_1, \dots, x_n)$, transposing x_i and x_j multiplies it by -1.

If $x_i = x_j$, transposing by some τ does nothing
$\Rightarrow f = \tau f$, but def skew $\Rightarrow f = -f \Rightarrow f = 0$.

Thus by the factor theorem, $(x_i - x_j)$ divides f for $i<j$. Thus for skew f, $\Delta = \prod_{i<j} (x_i - x_j)$ divides f. By the division algorithm

$$f = \Delta g + h \text{ for some polynom. } g, h$$

but we know $q = 0$. Thus $f = \Delta g$.

Now, for some perm σ, we know $\sigma f = -f$ and

$$\sigma f = \sigma(\Delta g) = (\sigma \Delta)(\sigma g) = -\Delta(\sigma g)$$

so $-f = -\Delta(\sigma g) \Rightarrow f = \Delta(\sigma g)$.

Thus $\Delta g = f = \Delta(\sigma g) \Rightarrow g = \sigma g$.

Thus g is symmetric by def symmetric. $\blacksquare$