**Exercise 8.1.** Let  $K \subseteq L$  be a splitting field extension for some  $f \in K[t] \setminus K$ . Then the following are equivalent:

- (i) f has a repeated root over L;
- (ii)  $\exists \alpha \in L \text{ s.t. } 0 = f(\alpha) = (\mathcal{D}f)(\alpha);$
- (iii)  $\exists g \in K[t]$ ,  $\deg g \ge 1$  s.t. g divides both f and  $\mathcal{D}f$ .

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Exercise 8.2. Let K be a field,  $\operatorname{char}(K) = p > 0$  and  $f \in K[t^p]$  is an irreducible polynomial over K. Prove that f is inseparable.

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Exercise 8.3. Let K be a field,  $\operatorname{char}(K) = p > 0$  and  $f \in K[t^p]$  is an irreducible polynomial over K. Prove that there is  $g \in K[t]$  and a non-negative n such that  $f(t) = g(t^{p^n})$  and g is an irreducible and separable polynomial.

 $\Box$ 

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Exercise 8.4. Prove that  $\prod_{\alpha \in \mathbb{F}_q^*} \alpha = -1$ 

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Exercise 8.5.1. Let  $\alpha \in \mathbb{F}_q$  and  $\alpha = \beta - \beta^p$  for some  $\beta \in \mathbb{F}_q$ . Prove that  $\text{Tr}(\alpha) = 0$ .

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Exercise 8.5.2. Let  $\alpha \in \mathbb{F}_q$  and  $\alpha = \gamma^{1-p}$  for some nonzero  $\gamma \in \mathbb{F}_q$ . Prove that  $Norm(\alpha) = 1$ .

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**Exercise 8.5.3.** Let  $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$ . Prove that  $\mathrm{Tr}(\alpha) = n\alpha$ .

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Exercise 8.5.4. Let  $\alpha \in \mathbb{F}_p \subseteq \mathbb{F}_{p^n}$ . Prove that  $\text{Norm}(\alpha) = \alpha^n$ .