1 Soluble Groups I

Definition 1 (Soluble group)**.** A group G is *soluble* if there exists a finite series of subgroups

$$\{Id.\} = G_n \leqslant G_{n-1} \leqslant \cdots \leqslant G_0 = G$$

such that

1. $G_j \lhd G_{j-1}$ $\forall 1 \leq j \leq n$ and
2. G_{j-1}/G_j is cyclic $\forall 1 \leq j \leq n$.

Exercise 1. The Heisenberg group $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is soluble.

Definition 2 (Simple group)**.** A group G is *simple* if G has no non-triival normal subgroups.

Lemma 1.1. For $n \geq 5$ the group A_n is simple (and hence not soluble).

Exercise 2. V_4 is the only non-trivial subgroup of A_4.

Lemma 1.2. Let G be a group with $G \trianglelefteq G$ and $A \leqslant G$. Then

1. $(A \cap H) \trianglelefteq A$ and $A/(A \cap H) \cong (HA)/H$
2. if $H \subseteq A$ and $A \trianglelefteq G$, then $H \trianglelefteq A$, $(A/H) \trianglelefteq (G/H)$ and $(G/H)/(A/H) \cong G/A$.

Theorem 1.3. 1. If G is a soluble group with $A \leqslant G$, then A is soluble.

2. Let $H \trianglelefteq G$. Then G is soluble $\iff$ H and G/H are soluble.

Corollary 1. S_n is not soluble for $n \geq 5$.

Corollary 2. All p-groups are soluble (i.e. groups G such that $|G| = p^n$ for some prime p)