# 1   Soluble Groups I

**Definition 1** (Soluble group)**.** A group $G$ is <u>soluble</u> if there exists a finite series of subgroups

$$\{Id.\} = G_n \leqslant G_{n-1} \leqslant \cdots \leqslant G_0 = G$$

such that

1. $G_j \triangleleft G_{j-1} \quad \forall 1 \leq j \leq n$ and

2. $G_{j-1}/G_j$ is cyclic $\forall 1 \leq j \leq n$.

**Exercise 1.** The Heisenberg group $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is soluble.

**Definition 2** (Simple group)**.** A group $G$ is <u>simple</u> if $G$ has no non-triival normal subgroups.

**Lemma 1.1.** *For $n \geq 5$ the group $A_n$ is simple (and hence not soluble).*

**Exercise 2.** $V_4$ is the only non-trivial subgroup of $A_4$.

**Lemma 1.2.** *Let $G$ be a group with $G \trianglelefteq G$ and $A \leqslant G$. Then*

1. *$(A \cap H) \trianglelefteq A$ and $A/(A \cap H) \cong (HA)/H$*

2. *if $H \subseteq A$ and $A \trianglelefteq G$, then $H \trianglelefteq A$, $(A/H) \trianglelefteq (G/H)$ and $(G/H)/(A/H) \cong G/A$.*

**Theorem 1.3.**     *1. If $G$ is a soluble group with $A \leqslant G$, then $A$ is soluble.*

2. *Let $H \trianglelefteq G$. Then $G$ is soluble $\iff H$ and $G/H$ are soluble.*

**Corollary 1.** *$S_n$ is not soluble for $n \geq 5$.*