

# 1 Separability

**Definition 1** (Separable). *Let  $K$  be a field.*

- (i) *An irreducible polynomial  $f \in K[t]$  is separable over  $K$  if it has no multiple roots, meaning that  $f = \lambda(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_d)$ , where  $\alpha_1, \dots, \alpha_d \in \overline{K}$  are distinct.*
- (ii) *A non-zero polynomial  $f \in K[t]$  is separable over  $K$  if its irreducible factors in  $K[t]$  are separable over  $K$ .*
- (iii) *When  $L : K$  is a field extension, we say that  $\alpha \in L$  is separable over  $K$  when  $\alpha$  is algebraic over  $K$  and  $\mu_\alpha^K$  is separable.*
- (iv) *An algebraic extension  $L : K$  is a separable extension if every  $\alpha \in L$  is separable over  $K$ .*

**Lemma 1.1.** *Suppose that  $L : M : K$  is a tower of algebraic field extensions. Assume that  $K \subseteq M \subseteq L \subseteq \overline{K}$ , and suppose that  $f \in K[t] \setminus K$  satisfies the property that  $f$  is separable over  $K$ . If  $g \in M[t] \setminus M$  has the property that  $g \mid f$ , then  $g$  is separable over  $M$ . Thus, if  $\alpha \in L$  is separable over  $K$  then  $\alpha$  is separable over  $M$ , and if  $L : K$  is separable then so is  $L : M$ .*

**Lemma 1.2.** *Suppose that  $L : M$  is an algebraic field extension. Let  $\alpha \in L$  and  $\sigma : M \rightarrow \overline{M}$  be a homomorphism. Then  $\sigma(m_\alpha(M))$  is separable over  $\sigma(M)$  if and only if  $m_\alpha(M)$  is separable over  $M$ .*

**Theorem 1.3.** *Let  $L : K$  be a finite extension with  $K \subseteq L \subseteq \overline{K}$ , whence  $L = K(\alpha_1, \dots, \alpha_n)$  for some  $\alpha_1, \dots, \alpha_n \in L$ . Put  $K_0 = K$ , and for  $1 \leq i \leq n$ , set  $K_i = K_{i-1}(\alpha_i)$ . Finally, let  $\sigma_0 : K \rightarrow \overline{K}$  be the inclusion map.*

- (i) *If  $\alpha_i$  is separable over  $K_{i-1}$  for  $1 \leq i \leq n$ , then there are  $[L : K]$  ways to extend  $\sigma_0$  to a homomorphism  $\tau : L \rightarrow \overline{K}$ .*
- (ii) *If  $\alpha_i$  is not separable over  $K_{i-1}$  for some  $i$  with  $1 \leq i \leq n$ , then there are fewer than  $[L : K]$  ways to extend  $\sigma_0$  to a homomorphism  $\tau : L \rightarrow \overline{K}$ .*

**Theorem 1.4.** *Let  $L : K$  be a finite extension with  $L = K(\alpha_1, \dots, \alpha_n)$ . Set  $K_0 = K$ , and for  $1 \leq i \leq n$ , inductively define  $K_i$  by putting  $K_i = K_{i-1}(\alpha_i)$ . Then the following are equivalent:*

- (i) *the element  $\alpha_i$  is separable over  $K_{i-1}$  for  $1 \leq i \leq n$ ;*
- (ii) *the element  $\alpha_i$  is separable over  $K$  for  $1 \leq i \leq n$ ;*
- (iii) *the extension  $L : K$  is separable.*

**Corollary 1.5.** *Suppose that  $L : K$  is a finite extension. If  $L : K$  is a separable extension, then the number of  $K$ -homomorphism  $\sigma : L \rightarrow \overline{K}$  is  $[L : K]$ , and otherwise the number is smaller than  $[L : K]$ .*

**Corollary 1.6.** *Suppose that  $f \in K[t] \setminus K$  and that  $L : K$  is a splitting field extension for  $f$ . Then  $L : K$  is a separable extension if and only if  $f$  is separable over  $K$ . More generally, suppose that  $L : K$  is a splitting field extension for  $S \subseteq K[t] \setminus K$ . Then  $L : K$  is a separable extension if and only if each  $f \in S$  is separable over  $K$ .*