1 Solvability by radicals and Galois theory II

Lemma 1.1. Let p be prime and $G \leqslant S_p$ such that G acts transitivley on $\{1, \dots, p\}$. Then G contains a cycle of order p.

Theorem 1.2. Let $\operatorname{char} K = 0$ and $f \in K[t] \setminus K$. Then $\operatorname{Gal}_K(f)$ is soluble $\implies$ f is SBR.

Lemma 1.3 (Wooley 14.8)**.** Let $\operatorname{char} K = 0$, and suppose that $L : K$ is a cyclic extension of degree n. Suppose also that K contains a primitive n-th root of 1. Then there exists $\theta \in K$ having the property that $t^n - \theta$ is irreducible over K, and $L : K$ is a splitting field for $t^n - \theta$. Further, if β is a root of $t^n - \theta$ over L, then $L = K(\beta)$.

Theorem 1.4 (Abel-Galois)**.** Let $\operatorname{char} K = 0$ and $f \in K[t]$ be irreducible over K with $\deg f = p$. Then following are equivalent

1. f is SBR over K;
2. $\operatorname{Gal}_K(f)$ is conjugated to a subgroup of $\operatorname{Aff}(\mathbb{F}_p)$;
3. for the splitting field L of f, one has $L = K(\alpha_i, \alpha_j)$ where α_i, α_j are any two destinct roots of f.

Lemma 1.5. Let $\{\text{Id.}\} \neq N \trianglelefteq G \leqslant S_p$ for p prime. If G is a transitive group, then N is a transitive group.