

1 Galois Groups I

Definition 1 (Galois group of polynomial). *Let $L = K(\alpha_1, \dots, \alpha_n)$ and let $P(\alpha_1, \dots, \alpha_n)$ where $P \in K[\alpha_1, \dots, \alpha_n]$ is an element of L . Then we define*

$$\text{Gal}_K(f) = \{ \sigma \in S_n \mid \forall P \in K[\alpha_1, \dots, \alpha_n], \text{ if } P(\alpha_1, \dots, \alpha_n) = 0 \text{ then } P(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = 0 \}$$

Lemma 1.1. $\text{Gal}_K(f) \leq S_n$

Lemma 1.2. *If $K_1 : K$, then $\text{Gal}_{K_1}(f) \leq \text{Gal}_K(f)$.*

Definition 2. *Let $L : K$ be a field extension. Then*

$$\text{Gal}_K(L) = \text{Gal}(L : K) = \{ \varphi \in \text{Aut}(L) : \varphi \text{ is a } K\text{-homomorphism} \}$$

Lemma 1.3. *Suppose that $M : K$ is a normal extension. Then:*

- (a) *for any $\sigma \in \text{Gal}(M : K)$ and $\alpha \in M$, we have $\mu_{\sigma(\alpha)}^K = \mu_\alpha^K$;*
- (b) *for any $\alpha, \beta \in M$ with $\mu_\alpha^K = \mu_\beta^K$, there exists $\tau \in \text{Gal}(M : K)$ having the property that $\tau(\alpha) = \beta$.*