

## 1 Introduction I

**Definition 1** (Symmetric function). A function  $\varphi(x_1, \dots, x_n)$  is called symmetric if

$$\varphi(x_1, \dots, x_n) = \varphi(x_{\omega(1)}, \dots, x_{\omega(n)})$$

for all  $\omega \in S_n$ .

**Definition 2** (Elementary symmetric polynomial).

$$\begin{aligned}\sigma_1 &= \sigma_1(x_1, \dots, x_n) = x_1 + \dots + x_n \\ \sigma_2 &= \sigma_2(x_1, \dots, x_n) = x_1x_2 + \dots + x_1x_n + x_2x_3 + \dots + x_{n-1}x_n \\ &\dots \\ \sigma_k &= \sigma_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} \dots x_{i_k} \\ &\dots \\ \sigma_n &= \sigma_n(x_1, \dots, x_n) = \prod_{i=1}^n x_i\end{aligned}$$

**Theorem 1.1.** For any symmetric function  $\psi(x_1, \dots, x_n)$ , there exists a unique polynomial  $P(t_1, \dots, t_n)$  such that  $\psi(x_1, \dots, x_n) = P(\sigma_1, \dots, \sigma_n)$ .

**Vieta formulae:**

$$\begin{aligned}x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n &= (x - x_1)(x - x_2) \dots (x - x_n) \\ &= x^n - \sigma_1x^{n-1} + \sigma_2x^{n-2} + \dots + (-1)^n \sigma_n\end{aligned}$$

**Corollary 1.** The discriminant  $D$  of  $f \in R[x]$ , where  $R$  is a ring and  $f = x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ , is a polynomial in  $a_1, \dots, a_n$  and coefficients from  $R$  (i.e.  $D \in R[a_1, \dots, a_n]$ ).

**Note:** Any cubic equation can be converted to a depressed cubic by

$$x^3 + Ax^2 + Bx + c = \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q.$$

**Vieta's method:** Using the trigonometric formula  $\cos 3\varphi = 4\cos^3 \varphi - 3\cos \varphi$ , we can solve certain cubic equations. For example, consider  $4x^3 - 3x = -\frac{1}{2}$ . Let  $x = \cos \varphi$ . Then

$$\begin{aligned}\cos 3\varphi = -\frac{1}{2} &\iff 3\varphi = \pm \frac{2\pi}{3} + 2\pi k \quad \text{for } k \in \mathbb{Z} \\ &\iff \varphi = \pm \frac{2\pi}{9} + 2\pi k \\ &\iff x \in \left\{ \cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{8\pi}{9} \right\}.\end{aligned}$$

In general, we can use this method to solve  $4x^3 - 3x = a \implies x = \cos \varphi$ ,  $\cos 3\varphi$  and  $\cos : \mathbb{C} \rightarrow \mathbb{C}$  is now a complex function. For  $x^3 + px + q = 0$ , set  $x = ky$  such that  $\frac{k^3}{pk} = \frac{-4}{3} \implies k = \pm \frac{\sqrt{-4p}}{3}$ .

**Definition 3** (Ferrari resolvent). Let  $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$  be a quartic polynomial over a field  $K$  of characteristic not 2. We define the Ferrari resolvent of  $f$  to be the associated cubic resolvent polynomial  $R(z) \in K[z]$  given by

$$R(z) = z^3 - 2bz^2 + (b^2 - 4d + ac)z - c^2 - a^2d + 4bd.$$

Solving the resolvent allows one to reduce solving  $f$  to solving a system of quadratics.

**Lagrange's method:** Suppose  $f(x) = x^3 + px + q$  is a depressed cubic with roots  $x_1, x_2, x_3$ . Lagrange's method finds expressions involving the roots that take only a few values under permutation, then uses symmetry to connect them to the coefficients.

For instance, define

$$y_1 = x_1 + \zeta x_2 + \zeta^2 x_3,$$

where  $\zeta = e^{2\pi i/3}$  is a primitive cube root of unity. Then define

$$y_2 = x_1 + \zeta^2 x_2 + \zeta x_3.$$

These expressions are not symmetric, but they only take a few values when the  $x_i$ 's are permuted. In particular,  $y_1^3$  and  $y_2^3$  are symmetric functions of the roots and thus can be written as polynomials in  $p$  and  $q$ .

Since the roots  $x_i$  are related to  $y_1$  and  $y_2$ , we can use symmetric combinations such as

$$x = \frac{1}{3}(y_1 + y_2)$$

to recover the original roots of  $f(x)$ .