MA 45401-H01: Galois Theory Honors
Definitions and Results

Prof. Ilya Shkredov
Transcribed by Josh Park

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Contents

1 Introduction I

Definition 1 (Symmetric function)**.** A function $\varphi(x_1, \ldots, x_n)$ is called symmetric if

$$\varphi(x_1, \ldots, x_n) = \varphi(x_{\omega(1)}, \ldots, x_{\omega(n)})$$

for all $\omega \in S_n$.

Definition 2 (Elementary symmetric polynomial)**.**

$$\begin{aligned}
\sigma_1 = \sigma_1(x_1, \ldots, x_n) &= x_1 + \cdots + x_n \\
\sigma_2 = \sigma_2(x_1, \ldots, x_n) &= x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\
&\cdots \\
\sigma_k = \sigma_k(x_1, \ldots, x_n) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \\
&\cdots \\
\sigma_n = \sigma_n(x_1, \ldots, x_n) &= \prod_{i=1}^{n} x_i
\end{aligned}$$

Theorem 1.1. *For any symmetric function $\psi(x_1, \ldots, x_n)$, there exists a unique polynomial $P(t_1, \ldots, t_n)$ such that $\psi(x_1, \ldots, x_n) = P(\sigma_1, \ldots, \sigma_n)$.*

Vieta formulae:

$$\begin{aligned}
x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n &= (x - x_1)(x - x_2)\cdots(x - x_n) \\
&= x^n - \sigma_1x^{n-1} + \sigma_2x^{n-2} + \cdots + (-1)^n\sigma_n
\end{aligned}$$

Corollary 1. *The discriminant D of $f \in R[x]$, where R is a ring and $f = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, is a polynomial in $a_1, \ldots, a_n$ and coefficients from R (i.e. $D \in R[a_1, \ldots, a_n]$).*

Note: Any cubic equation can be converted to a depressed cubic by

$$x^3 + Ax^2 + Bx + c = \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q.$$

Vieta's method: Using the trigonometric formula $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$, we can solve certain cubic equations. For example, consider $4x^3 - 3x = -\frac{1}{2}$. Let $x = \cos\varphi$. Then

$$\begin{aligned}
\cos 3\varphi = -\frac{1}{2} &\iff 3\varphi = \pm\frac{2\pi}{3} + 2\pi k \quad \text{for } k \in \mathbb{Z} \\
&\iff \varphi = \pm\frac{2\pi}{9} + 2\pi k \\
&\iff x \in \left\{\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}\right\}.
\end{aligned}$$

In general, we can use this method to solve $4x^3 - 3x = a \implies x = \cos\varphi,\ \cos 3\varphi$ and $\cos : \mathbb{C} \to \mathbb{C}$ is now a complex function. For $x^3 + px + q = 0$, set $x = ky$ such that $\frac{k^3}{pk} = \frac{-4}{3} \implies k = \pm\frac{\sqrt{-4p}}{3}$.

Definition 3 (Ferrari resolvent)**.** Let $f(x) = x^4 + ax^3 + bx^2 + cx + d \in K[x]$ be a quartic polynomial over a field K of characteristic not 2. We define the Ferrari resolvent of f to be the associated cubic resolvent polynomial $R(z) \in K[z]$ given by

$$R(z) = z^3 - 2bz^2 + (b^2 - 4d + ac)z - c^2 - a^2d + 4bd.$$

Solving the resolvent allows one to reduce solving f to solving a system of quadratics.

Lagrange's method: Suppose $f(x) = x^3 + px + q$ is a depressed cubic with roots x_1, x_2, x_3. Lagrange's method finds expressions involving the roots that take only a few values under permutation, then uses symmetry to connect them to the coefficients.

For instance, define

$$y_1 = x_1 + \zeta x_2 + \zeta^2 x_3,$$

where $\zeta = e^{2\pi i/3}$ is a primitive cube root of unity. Then define

$$y_2 = x_1 + \zeta^2 x_2 + \zeta x_3.$$

These expressions are not symmetric, but they only take a few values when the x_i's are permuted. In particular, y_1^3 and y_2^3 are symmetric functions of the roots and thus can be written as polynomials in p and q.

Since the roots x_i are related to y_1 and y_2, we can use symmetric combinations such as

$$x = \frac{1}{3}(y_1 + y_2)$$

to recover the original roots of $f(x)$.

2 Introduction II

Theorem 2.1 (Lagrange)**.** *Let $\varphi = \varphi(x_1, \ldots, x_n)$ and*

$$\mathrm{orb}(\varphi) = \left\{ \varphi^\omega = \varphi(x_{\omega(1)}, \ldots, x_{\omega(n)}) \mid \omega \in S_n \right\}.$$

Then $y_1, \ldots, y_k$ are roots of some polynomial with degree $\leq k$ whose coefficients depend on elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ in a polynomial way.

Theorem 2.2 (Lagrange)**.** *Let $\varphi, \psi \in K[x_1, \ldots, x_n]$ and $G_\varphi = \{\omega \in S_n \mid \varphi^\omega = \varphi\} \leq G_\psi$. Then $\psi = R(\varphi)$ where R is a rational function whose coefficients are symmetric functions on $x_1, \ldots, x_n$.*

Definition 4 (Group action)**.** Let G be a group and X be a set. The (left) group action of G on X is the map $\cdot : G \times X \to X$ such that

1. $e_G \cdot x = x, \quad \forall x \in X$
2. $g \cdot (h \cdot x) = (g \cdot h) \cdot x, \quad \forall x \in X, \forall g, h \in G$

Definition 5 (Orbit)**.** Let G be a group, X be a set, and $x \in X$. Then we define the orbit of x, $G \cdot x = \mathrm{orb}(x)$, as $\{g \cdot x \mid g \in G\}$. Moreover, $\mathrm{orb}(x) \subseteq X$.

Definition 6 (Stabilizer)**.** Let G be a group, X be a set, and $x \in X$. Then we define the stabilizer of x, $\mathrm{stab}(x)$, as $\{g \in G \mid g \cdot x = g\}$. Moreover, $\mathrm{stab}(x) \leq G$.

Theorem 2.3. *Let G be a finite group that acts on X. Then for all $x \in X$, $|\mathrm{orb}(x)| \cdot |\mathrm{stab}(x)| = |G|$.*

Definition 7 (Polynomial ring)**.** Let R be a commutative ring. Then the ring of polynomials with coefficients in R is

$$R[t] = \left\{ \sum_{i=0}^{n} c_i t^i : n \in \mathbb{Z}_+, c_i \in R \right\}$$

3 Field Extensions I

Definition 8 (Integral domain)**.** Let R be a commutative ring. Then R is an integral domain if $ab = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$.

Definition 9 (Euclidean domain)**.** Let R be an integral domain. Then R is a Euclidean domain if there exists some function $f : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ such that for all $a, b_{\not\equiv 0} \in R$, there exist elements $q, r \in R$ such that $a = qb + r$ where $r = 0$ or $f(r) < f(b)$.

Theorem 3.1 (Bézout's Identity)**.** *Let R be a Euclidean domain. For $a, b \in R$, there exists $\alpha, \beta \in R$ such that* $\gcd(a, b) = \alpha a + \beta b$

Definition 10 (Irreducible)**.** Let F be a field, and $f \in F[t] \setminus F$. Then f is irreducible if $\nexists g, h \in F[t] \setminus F$ of strictly smaller degree such that $f = gh$.

Definition 11 (Unique factorization domain)**.** Let R be an integral domain. Then R is a unique factorization domain (UFD) if for irreducible $p_i \in R$, any nonzero $x \in R$ can be written uniquely (up to ordering) as $x = p_1 p_2 \cdots p_k, \quad k \geq 1$.

Fact: If R is an Euclidean domain, then R is a UFD (and PID)

Corollary 2. *Let $f \in \mathbb{F}[t]$ be a monic polynomial with $\deg f \geq 1$. Then we can write $f = f_1 f_2 \cdots f_k$ uniquely (up to ordering) for irreducible monic polynomials f_j.*

Definition 12. Let R be a UFD. When $a_0, \ldots, a_n \in R$ are not all 0, we can generalize the greatest common divisor of $a_0, \ldots, a_n$ (written $\gcd(a_0, \ldots, a_n)$) any element $c \in R$ satisfying

(i) $c \,\big|\, a_i$ $(0 \leq i \leq n)$, and

(ii) if $d \,\big|\, a_i$ $(0 \leq i \leq n)$, then $d \,\big|\, c$.

When $f = \sum_{j=0}^{d} a_j x^j \in R[x]$ is a non-zero polynomial, we define a content of f to be any $\gcd(a_0, \ldots, a_d)$ and $\gcd(f) = \gcd(a_0, \ldots, a_d)$. We say that $f \in R[X]$ is primitive if $f \neq 0$ and the content of f is divisible only by units of R.

Lemma 3.2 (Gauss)**.** $\gcd(fg) = \gcd f \cdot \gcd g$

Corollary 3. *$f \in \mathbb{Z}[t]$ is irreducible $\Longleftrightarrow f$ is irreducible over $\mathbb{Q}[t]$*

Corollary 4. *If R is a UFD with field of fractions Q and $f \in R[X]$ with $\deg f > 0$, then f is irreducible in $R[X] \Longleftrightarrow f$ is irreducible in Q.*

Theorem 3.3 (Eisenstein's Criterion)**.** *Let R be a UFD with field of fractions Q and let $f = a_0 + a_1 X + \ldots + a_n X^n \in R[X]$ with $\gcd(f) = 1$. Suppose there exists an irreducible element $p \in R$ such that*

(i) $p \,\big|\, a_i$ for $0 \leq i < n$,

(ii) $p^2 \nmid a_0$, and

(iii) $p \nmid a_n$,

then f is irreducible in $R[X]$ (and hence also in $Q[X]$).

Definition 13 (Field extension)**.** Let L and K be fields. Then L is an extension of K if there exists a homomorphism $\varphi : K \to L$. Then we write $L : K$ or L/K, $\varphi(K) \cong K$ and identify $\varphi(K)$ with K.

Fact: Suppose that L is a field extension of K with associated embedding $\varphi : K \to L$. Then L forms a vector space over K, under the operations

$$\text{(vector addition) } \psi : L \times L \to L \quad \text{given by} \quad (v_1, v_2) \mapsto v_1 + v_2$$
$$\text{(scalar multiplication) } \tau : K \times L \to L \quad \text{given by} \quad (k, v) \mapsto \varphi(k)v.$$

Definition 14 (Degree, finite extension)**.** Let $L : K$. Then the degree of $L : K$ is $[L : K] = \dim L$ over K as a vector space. We say that $L : K$ is a finite extension if $[L : K] < \infty$.

Definition 15 (Tower, intermediate field)**.** We say that $M : L : K$ is a tower of field extensions if $M : L$ and $L : K$ are field extensions, and in this case we say that L is an intermediate field (relative to the extension $M : K$)

Theorem 3.4 (The Tower Law)**.** *Suppose that $M : L : K$ is a tower of field extensions. Then $M : K$ is a field extension, and $[M : K] = [M : L][L : K]$.*

Corollary 5. *Suppose that $L : K$ is a field extension for which $[L : K]$ is a prime number. Then whenever $L : M : K$ is a tower of field extensions with $K \subseteq M \subseteq L$, one has either $M = L$ or $M = K$.*

4 Field Extensions II

Definition 16 (Smallest subring/subfield)**.** Let $L : K$ with $K \subseteq L$.

(i) When $\alpha \in L$, we denote by $K[\alpha]$ the smallest subring of L containing K and α, and by $K(\alpha)$ the smallest subfield of L containing K and α;

(ii) More generally, when $A \subseteq L$, we denote by $K[A]$ the smallest subring of L containing K and A, and by $K(A)$ the smallest subfield of L containing K and A.

Then

$$\begin{aligned} K[\alpha] &= \left\{ \sum_{i=0}^{d} c_i \alpha^i : d \in \mathbb{Z}_{\leq 0},\ c_0, \dots, c_d \in K \right\} \\ K(\alpha) &= \{ f/g : f, g \in K[\alpha], g \neq 0 \} . \end{aligned}$$

Definition 17 (Algebraic/transcendental element)**.** Suppose that $L : K$ is a field extension with $K \subseteq L$ and $\alpha \in L$.

(i) We say α is algebraic over K if $\exists f_{\not\equiv 0} \in K[t]$ such that $f(\alpha) = 0$.

(ii) If α is not algebraic over K, then we say α is transcendental over K.

(iii) When every element of L is algebraic over K, we say that L is algebraic over K.

Definition 18 (Evaluation map)**.** Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\alpha \in L$. We define the evaluation map $E_\alpha : K[t] \to L$ by putting $E_\alpha(f) = f(\alpha)$ for each $f \in K[t]$.

Definition 19 (Minimal polynomial)**.** Suppose that $L : K$ is a field extension with $K \subseteq L$, and suppose that $\alpha \in L$ is algebraic over K. Then the minimal polynomial of α over K is the unique monic polynomial μ_α^K having the property that $\ker(E_\alpha) = (\mu_\alpha^K)$.

Lemma 4.1. *1. μ_α^K is irreducible over K;*

2. If $f \in K[t]$ such that $f(\alpha) = 0$, then $\mu_\alpha^K \mid f$;

3. If $f \in K[t]$ such that $f(\alpha) = 0$ and f is irreducible over K, then $\exists k \in K$ such that $f = k\mu_\alpha^K$.

Theorem 4.2. *Let $L : K$ with $K \subseteq L$, and suppose that $\alpha \in L$ is algebraic over K.*

(i) $K[\alpha]$ is a field, and $K[\alpha] = K(\alpha)$;

(ii) If $n = \deg \mu_\alpha^K$, then $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ is a basis for $K(\alpha)$ over K ($\implies [K(\alpha) : K] = \deg \mu_\alpha^K$).

Theorem 4.3 (Rational Root Theorem)**.** *Let $\frac{p}{q}$ be a root of $f = a_0 t^n + \cdots + a_{n-1} t^{n-1} + a_n$, for $a_j \in \mathbb{Z}$, where p and q are coprime. Then $p \mid a_n$ and $q \mid a_0$.*

Note: If α is transcendental over K, then $K(\alpha) \cong K(x)$ (where x is a formal variable).

Corollary 6. *Let $L : K$ with $K \subseteq L$, and suppose that $\alpha \in L$ is algebraic over K. Then every element of $K(\alpha)$ is algebraic over K.*

Corollary 7. *Let $L : K$ with $K \subseteq L$. Then $[L : K] < \infty \iff L = K(\alpha_1, \dots, \alpha_n)$ for $\alpha_j \in L$.*

Theorem 4.4. *Let $L : K$ be a field extension, and define*

$$L^{\text{alg}} = \{\alpha \in L : \alpha \text{ is algebraic over } K\}.$$

Then L^{alg} is a subfield of L.

5 Algebraic Conjugates

Lemma 5.1. *Let $\mathbb{F}$ be a field with $f \in \mathbb{F}[t]$ irreducible. Then $\mathbb{F}[t] \,/\, (f)$ is a field.*

Corollary 8. *If $L : K$ with $\alpha \in L$ algebraic over K, then $K[t] \,/\, (\mu_\alpha^K)$ is a field.*

Theorem 5.2. *Let K be a field, and suppose that $f \in K[t]$ is irreducible. Then there exists a field extension $L : K$, with associated embedding $\varphi : K[t] \to L[y]$, having the property that L contains a root of $\varphi(f)$.*

Definition 20 (Algebraic conjugate)**.** Suppose α algebraic over K and μ_α^K factors as a product of linear polynomials over a field $L \supseteq K$:

$$\mu_\alpha^K(x) = (x - \alpha_1) \cdots (x - \alpha_n), \quad \alpha_1, \ldots, \alpha_n \in L.$$

Then $\alpha_1, \ldots, \alpha_n$ are <u>algebraic conjugates</u> of α.

Lemma 5.3. *Let $(x-\alpha_1)\cdots(x-\alpha_n) \in K[x]$ and $f(\overline{y}, x_1, \ldots, x_n) \in K[\overline{y}, x_1, \ldots, x_n]$ be symmetric polynomial in $x_1, \ldots, x_n$. Then $f(\overline{y}, x_1, \ldots, x_n) \in K[\overline{y}]$.*

Theorem 5.4. *Let α be algebraic over K with algebraic conjugates $\alpha = \alpha_1, \ldots, \alpha_n$. Then for all $f \in K[x]$, the conjugates of $f(\alpha)$ are exactly $f(\alpha_1), \ldots, f(\alpha_n)$.*

6 Ruler and Compass Constructions

7 Cyclotomic Polynomials

Theorem 7.1. *For prime p, we have $x^p - 1 = (x-1)(x^{p-1} + \cdots + 1)$ and $\mu_{\varepsilon_p}^{\mathbb{Q}} = x^{p-1} + \cdots + 1$.*

Definition 21 (n^{th} cyclotomic polynomial)**.**

$$\Phi_n(x) = \prod_{\substack{\varepsilon \in \sqrt[n]{1} \\ |\varepsilon| = n}} (x - \varepsilon) = \frac{x^n - 1}{\prod\limits_{d \mid n, d < n} \Phi_d(x)}$$

Theorem 7.2. *Φ_n is irreducible over $\mathbb{Q}$.*

Corollary 9. (a) $[\mathbb{Q}(\exp\left(\frac{2\pi i}{n}\right)) : \mathbb{Q}] = \varphi(n)$ *(where φ is Euler's totient function);*

(b) $[\mathbb{Q}(\cos\left(\frac{2\pi}{n}\right)) : \mathbb{Q}] = \frac{1}{2}\varphi(n)$. *Furthermore, all algebraic conjugates of* $\cos\frac{2\pi}{n}$ *are* $\cos\frac{2\pi k}{n}$ *for* $\gcd(k, n) = 1$.

(c) *Let $c = \frac{a+bi}{a-bi} \in \sqrt[\infty]{1}$, where $a, b \in \mathbb{Z}$. Then $c \in \{\pm i, \pm 1\}$*

Lemma 7.3. *Let $\mathbb{F}$ be a finite field. Then $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ is a cyclic group.*

8 Splitting Fields, Abel-Ruffini

Definition 22 (Splitting field)**.** Let $L : K$ with embedding $\varphi : K \to L$ and $f \in K[t] \setminus K$. We say <u>f splits over L</u> if $\varphi(f) = c \prod\limits_{j=1}^{n} (x - \alpha_j)$ for $\alpha_j \in L$ and $c \in \varphi(K)$. We say that $M : K$ is a <u>splitting field extension</u> for f if f splits over L, $\varphi(K) \subseteq M \subseteq L$, and M is the smallest subfield of L containing $\varphi(K)$ over which f splits.

Lemma 8.1. *Let $L : K$ be a splitting field extension for $f \in K[t]$ relative to the embedding $\varphi : K \to L$, and let $\alpha_j \in L$ be roots of $\varphi(f)$. Then $L = \varphi(K)(\alpha_1, \ldots, \alpha_n)$.*

Lemma 8.2. *Let $L : K$ be a splitting field extension for $f \in K[t] \setminus K$. Then $[L : K] \leq (\deg f)!$.*

Lemma 8.3. *Let $L : K$ and $M : K$ be splitting field extensions for $f \in K[t] \setminus K$. Then $L \cong M$ (in particular, $[L : K] = [M : K]$).*

Definition 23 (Radical, radical extension, solvability by radicals)**.** Let $L : K$ and $\beta \in L$. We say that β is radical over K when $\beta^n \in K$ for some $n \in \mathbb{N}$ (so $\beta = \alpha^{1/n}$ for some $\alpha \in K$ and some $n \in \mathbb{N}$). We say that $L : K$ is an extension by radicals when there is a tower of field extensions $L = L_r : L_{r-1} : \cdots : L_0 = K$ such that $L_i = L_{i-1}(\beta_i)$ with β_i radical over L_{i-1} (for $1 \leq i \leq r$). We say $f \in K[t]$ is solvable by radicals if there is a radical extension of K over which f splits.

Theorem 8.4 (Abel-Ruffini)**.** *Let $K = \mathbb{C}(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are formal variables. Let $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n \in K[x]$ be the generic polynomial of degree $n \geq 5$ over K. Then $f(x)$ is not solvable by radicals.*

9 Algebraic Closure I

Definition 24 (Algebraically closed field, algebraic closure)**.** Let M be a field.

(i) We say that M is algebraically closed if every non-constant polynomial $f \in M[t]$ has a root in M.

(ii) We say that M is an algebraic closure of K if $M : K$ is an algebraic field extension having the property that M is algebraically closed.

Lemma 9.1. *Let M be a field. The following are equivalent:*

(i) The field M is algebraically closed;

(ii) every non-constant polynomial $f \in M[t]$ factors in $M[t]$ as a product of linear factors;

(iii) every irreducible polynomial in $M[t]$ has degree 1;

(iv) the only algebraic extension of M containing M is itself.

Definition 25 (Chain)**.** Suppose that X is a nonempty, partially ordered set with $\leq$ denoting the partial ordering. A chain C in X is a collection of elements $\{a_i\}_{i \in I}$ of X having the property that for every $i, j \in I$, either $a_i \leq a_j$ or $a_j \leq a_i$.

Zorn's Lemma: Suppose that X is a nonempty, partially ordered set with $\leq$ the partial ordering. If every non-empty chain C in X has an upper bound in X, then X has at least one maximal element m (i.e. $b \in X$ with $m \leq b \implies b = m$).

Corollary 10. *Any proper ideal A of a commutative ring R is contained in a maximal ideal.*

Lemma 9.2. *Let K be a field. Then there exists an algebraic extension $E : K$, with $K \subseteq E$, having the property that E contains a root of every irreducible $f \in K[t]$, and hence also every $g \in K[t] \setminus K$.*

Theorem 9.3 (Existence of Algebraic Closures)**.** *Suppose that K is a field. Then there exists an algebraic extension $\overline{K}$ of K having the property that $\overline{K}$ is algebraically closed.*

Definition 26 (Extension of field homomorphism, isomorphic field extensions)**.** For $i = 1$ and 2, let $L_i : K_i$ be a field extension relative to the embedding $\varphi_i : K_i \to L_i$. Suppose that $\sigma : K_1 \to K_2$ and $\tau : L_1 \to L_2$ are isomorphisms. We say that τ extends σ if $\tau \circ \varphi_1 = \varphi_2 \circ \sigma$. In such circumstances, we say that $L_1 : K_1$ and $L_2 : K_2$ are isomorphic field extensions.

$$\begin{tikzcd} L_1 \arrow[r, "\tau"] & L_2 \\ K_1 \arrow[u, "\varphi_1"] \arrow[r, "\sigma"] \arrow[ur, dashed] & K_2 \arrow[u, "\varphi_2"'] \end{tikzcd}$$

When $\sigma : K_1 \to K_2$ and $\tau : L_1 \to L_2$ are homomorphisms (instead of isomorphisms), then τ extends σ as a homomorphism of fields when the isomorphism $\tau : L_1 \to L_1' = \tau(L_1)$ extends the isomorphism $\sigma : K_1 \to K_1' = \sigma(K_1)$.

Definition 27 (K-homomorphism)**.** Let $L : K$ be a field extension relative to the embedding $\varphi : K \to L$, and let M be a subfield of L containing $\varphi(K)$. Then, when $\sigma : M \to L$ is a homomorphism, we say that σ is a K-homomorphism if σ leaves $\varphi(K)$ pointwise fixed, which is to say that for all $\alpha \in \varphi(K)$, one has $\sigma(\alpha) = \alpha$.

Lemma 9.4. *Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\tau : L \to L$ is a K-homomorphism. Suppose that $f \in K[t]$ has the property that $\deg f \geq 1$, and additionally that $\alpha \in L$.*

(i) if $f(\alpha) = 0$, one has $f(\tau(\alpha)) = 0$;

(ii) if τ is a K-automorphism of L, then $f(\alpha) = 0 \iff f(\tau(\alpha)) = 0$.

Theorem 9.5. *Let $\sigma : K_1 \to K_2$ be a field isomorphism. Suppose that L_i is a field with $K_i \subseteq L_i$ $(i = 1, 2)$. Suppose also that $\alpha \in L_1$ is algebraic over K_1, and that $\beta \in L_2$ is algebraic over K_2. Then we can extend σ to an isomorphism $\tau : K_1(\alpha) \to K_2(\beta)$ in such a manner that $\tau(\alpha) = \beta$ if and only if $m_\beta(K_2) = \sigma(m_\alpha(K_1))$.*

$$\begin{tikzcd} K_2 \arrow[r, "\varphi_2"] \arrow[d, "\sigma"] & K_2(\beta) \arrow[r, hook, "\iota_2"] \arrow[d, "\tau"] & L_2 \\ K_1 \arrow[r, "\varphi_1"] & K_1(\alpha) \arrow[r, hook, "\iota_1"] & L_1 \end{tikzcd}$$

Note: When $\tau : K_1(\alpha) \to K_2(\beta)$ is a homomorphism, and τ extends the homomorphism $\sigma : K_1 \to K_2$, then τ is completely determined by σ and the value of $\tau(\alpha)$.

Corollary 11. *Let $L : M$ be a field extension with $M \subseteq L$. Suppose that $\sigma : M \to L$ is a homomorphism, and $\alpha \in L$ is algebraic over M. Then the number of ways we can extend σ to a homomorphism $\tau : M(\alpha) \to L$ is equal to the number of distinct roots of $\sigma(m_\alpha(M))$ that lie in L.*

10 Algebraic Closure II

Theorem 10.1. *Let $L : K$ be an algebraic extension with $K \subseteq L$ and $\varphi : K \to \overline{K}$ be a homomorphism. Then there exists an extension of φ to a homomorphism $\psi : L \to \overline{K}$.*

Theorem 10.2. *If L and M are both algebraic closures of K, then $L \cong M$.*

Corollary 12. *Let $L : K$ be an extension with $K \subseteq L$. Suppose that $g \in L[t]$ is irreducible over L, and that $g \,\big|\, f$ in $L[t]$, where $f \in K[t] \setminus \{0\}$. The g divides a factor of f that is irreducible over K.*

Thus, there exists an irreducible $h \in K[t]$ having the property that $h \,\big|\, f$ in $K[t]$, and $g \,\big|\, h$ in $L[t]$.

Definition 28 (Normal extension)**.** The extension $L : K$ is normal if it is algebraic, and every irreducible polynomial $f \in K[t]$ either splits over L or has no root in L.

Theorem 10.3. *$K(\alpha) : K$ is normal $\iff$all conjugates of α are contained in $K(\alpha)$.*

Theorem 10.4. *A finite extension $L : K$ is normal $\iff$$L$ is a splitting field extension for some $f \in K[t] \setminus K$.*

11 Galois Groups I

Definition 29 (Galois group of polynomial)**.** Let $L = K(\alpha_1, \ldots, \alpha_n)$ and let $P(\alpha_1, \ldots, \alpha_n)$ where $P \in K[\alpha_1, \ldots, \alpha_n]$ is an element of L. Then we define

$$\mathrm{Gal}_K(f) = \{\sigma \in S_n \mid \forall P \in K[\alpha_1, \ldots, \alpha_n], \text{ if } P(\alpha_1, \ldots, \alpha_n) = 0 \text{ then } P(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})\}$$

Lemma 11.1. *1.* $\mathrm{Gal}_K(f) \leq S_n$*;*

2. If $K_1 : K$*, then* $\mathrm{Gal}_{K_1}(f) \leq \mathrm{Gal}_K(f)$*.*

Definition 30. Let $L : K$ be a field extension. Then

$$\mathrm{Gal}_K(L) = \mathrm{Gal}(L : K) = \{\varphi \in \mathrm{Aut}(L) : \varphi \text{ is a K-homomorphism}\}$$

Definition 31 (Galois automorphism on splitting field)**.** Let $\sigma \in \mathrm{Gal}_K f$ where L is a splitting field for f over K, and define $\underline{\widehat{\sigma} \in \mathrm{Aut}_K(L)}$ such that $\widehat{\sigma}(P(\alpha_1, \ldots, \alpha_n)) = P(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$.

Lemma 11.2. *The map* $\psi(\sigma) = \widehat{\sigma}$ *is a group isomorphism.*

Theorem 11.3. *If* $L : K$ *is an algebraic extension and* $\sigma : L \to L$ *is a* K*-homomorphism, then* $\sigma \in \mathrm{Aut}(L)$

Lemma 11.4. *Suppose that* $M : K$ *is a normal extension. Then:*

(a) for any $\sigma \in \mathrm{Gal}(M : K)$ *and* $\alpha \in M$*, we have* $\mu^K_{\sigma(\alpha)} = \mu^K_\alpha$*;*

(b) for any $\alpha, \beta \in M$ *with* $\mu^K_\alpha = \mu^K_\beta$*, there exists* $\tau \in \mathrm{Gal}(M : K)$ *having the property that* $\tau(\alpha) = \beta$*.*

12 Galois Groups II

Lemma 12.1. *Suppose that* $L : K$ *is a normal extension with* $K \subseteq L \subseteq \overline{K}$*. Then for any* K*-homomorphism* $\tau : L \to \overline{K}$*, we have* $\tau(L) = L$*.*

Lemma 12.2. *For* $n \geq 2$*,* S_n *is generated by*

1. *transpositions* $(i\, j)$*;*
2. *transpositions* $(1\, i)$*;*
3. *adjacent transpositions* $(1\,2), (2\,3), \ldots, (n-1,\, n)$*;*
4. $(1\,2)$ *and* $(1\,2 \ldots n)$*;*
5. $(1\,2)$ *and* $(2\,3 \ldots n)$*;*
6. $(i\, j)$ *and* $(i \ldots i_p)$ *where p is prime.*

Lemma 12.3. *Let* $(i_1 \ldots i_k) \in S_n$*. Then for all* $\sigma \in S_n$*, one has* $\sigma(i_1 \ldots i_k)\sigma^{-1} = (\sigma(i_1) \ldots \sigma(i_k))$*.*

Note: $|Gal_K(f)| = [L : K]$ where $L : K$ is a splitting field extension for f.

13 Galois Groups III

Theorem 13.1 (Kronecker)**.** *Let* $p \geq 3$ *be a prime and* $f \in \mathbb{Q}[x]$ *be irreducible over* $\mathbb{Q}$ *with* $\deg f = p$*. If the equation* $f(x) = 0$ *is solvable by radicals, then the number of real roots of* f *is 1 or* p*.*

Lemma 13.2. *Let* p *be prime and* $G \leq S_p$ *such that* G *acts transitively on* $\{1, \ldots, p\}$*. Then* G *contains a cycle of order* p*.*

Theorem 13.3. *If* $L : K$ *is a finite extension, then* $|\mathrm{Gal}_K(L)| \leq [L : K]$*.*

14 Separability

Definition 32 (Separable)**.** Let K be a field.

(i) An irreducible polynomial $f \in K[t]$ is separable over K if it has no multiple roots, meaning that $f = \lambda(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_d)$, where $\alpha_1, \dots, \alpha_d \in \overline{K}$ are distinct.

(ii) A non-zero polynomial $f \in K[t]$ is separable over K if its irreducible factors in $K[t]$ are separable over K.

(iii) When $L : K$ is a field extension, we say that $\alpha \in L$ is separable over K when α is algebraic over K and μ_α^K is separable.

(iv) An algebraic extension $L : K$ is a separable extension if every $\alpha \in L$ is separable over K.

Lemma 14.1. *Suppose that $L : M : K$ is a tower of algebraic field extensions. Assume that $K \subseteq M \subseteq L \subseteq \overline{K}$, and suppose that $f \in K[t] \setminus K$ satisfies the property that f is separable over K. If $g \in M[t] \setminus M$ has the property that $g \,\big|\, f$, then g is separable over M. Thus, if $\alpha \in L$ is separable over K then α is separable over M, and if $L : K$ is separable then so is $L : M$.*

Lemma 14.2. *1. If $L : M$ is an algebraic field extension, $\alpha \in L$ and $\sigma : M \to \overline{M}$ is a homomorphism, then $\sigma(\mu_\alpha^M)$ is separable over $\sigma(M) \iff \mu_\alpha^M$ is separable over M.*

2. If $L : K$ is a splitting field extension for $f \in K[t]$ and f is separable over K, then $L : K$ is separable.

Theorem 14.3. *Let $L : K$ be a finite extension with $K \subseteq L \subseteq \overline{K}$, whence $L = K(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n \in L$. Put $K_0 = K$, and for $1 \leq i \leq n$, set $K_i = K_{i-1}(\alpha_i)$. Finally, let $\sigma_0 : K \to \overline{K}$ be the inclusion map.*

(i) If α_i is separable over K_{i-1} for $1 \leq i \leq n$, then there are $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \to \overline{K}$.

(ii) If α_i is not separable over K_{i-1} for some i with $1 \leq i \leq n$, then there are fewer than $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \to \overline{K}$.

Theorem 14.4. *Let $L : K$ be a finite extension with $L = K(\alpha_1, \dots, \alpha_n)$. Set $K_0 = K$, and for $1 \leq i \leq n$, inductively define K_i by putting $K_i = K_{i-1}(\alpha_i)$. Then the following are equivalent:*

(i) the element α_i is separable over K_{i-1} for $1 \leq i \leq n$;

(ii) the element α_i is separable over K for $1 \leq i \leq n$;

(iii) the extension $L : K$ is separable.

Corollary 13. *Suppose that $L : K$ is a finite extension. If $L : K$ is a separable extension, then the number of K-homomorphism $\sigma : L \to \overline{K}$ is $[L : K]$, and otherwise the number is smaller than $[L : K]$.*

Corollary 14. *Suppose that $f \in K[t] \setminus K$ and that $L : K$ is a splitting field extension for f. Then $L : K$ is a separable extension $\iff f$ is separable over K. More generally, suppose that $L : K$ is a splitting field extension for $S \subseteq K[t] \setminus K$. Then $L : K$ is a separable extension $\iff$ each $f \in S$ is separable over K.*

15 The Primitive Element Theorem

Definition 33 (Simple extension)**.** Suppose $L : K$ is a field extension relative to the embedding $\varphi : K \to L$. We say that $L : K$ is a simple extension if there is some $\gamma \in L$ having the property that $L = \varphi(K)(\gamma)$.

Theorem 15.1 (The Primitive Element Theorem)**.** *If $L : K$ be a finite, separable extension with $K \subseteq L$, then $L : K$ is a simple extension.*

Corollary 15. *Suppose that $L : K$ is an algebraic, separable extension, and suppose that for every $\alpha \in L$, the polynomial μ_α^K has degree at most n over K. Then $[L : K] \leq n$.*

Fact: Let $L : K$ be a normal extension and let $\deg(\mu_\alpha^K) \leq n$ for all $\alpha \in L$. Then $[L : K] \leq n$.

Corollary 16. *If $f \in K[t]$ is irreducible over K, then $\mathrm{Gal}_K(f)$ acts transitively on the roots of f.*

16 Galois Fields I

Definition 34 (Formal derivative)**.** We define the derivative operator $\mathcal{D} : K[t] \to K[t]$ by

$$\mathcal{D}\left(\sum_{k=0}^{n} a_k t^k\right) = \sum_{k=1}^{n} k a_k t^{k-1}.$$

Theorem 16.1. *Let $f \in K[t] \setminus K$, and let $L : K$ be a splitting field extension for f with $K \subseteq L$. Then the following are equivalent:*

(i) f has a repeated root over L;

(ii) There exists $\alpha \in L$ such that $f(\alpha) = 0 = (\mathcal{D}f)(\alpha)$;

(iii) There exists $g \in K[t]$ with $\deg g \geq 1$ such that $g \,\big|\, f$ and $g \,\big|\, \mathcal{D}f$.

Definition 35 (Inseparable)**.** A polynomial $f \in K[t]$ is inseparable over K if f is not separable over K, meaning that f has an irreducible factor $g \in K[t]$ having the property that g has fewer than $\deg g$ distinct roots in K.

Theorem 16.2. *Suppose $f \in K[t]$ is irreducible over K. Then f is inseparable over $K \iff \mathrm{char}(K) = p > 0$ and $f \in K[t^p]$.*

Definition 36 (Frobenius map)**.** Suppose that $\mathrm{char}(K) = p > 0$. The Frobenius map $\varphi : K \to K$ is defined by $\varphi(\alpha) = \alpha^p$.

Theorem 16.3. *Suppose that $\mathrm{char}(K) = p > 0$, and put $F = \{c \cdot 1_K : c \in \mathbb{Z}\}$. Then F is a subfield (called the prime subfield) of K, and $F \cong \mathbb{Z}/p\mathbb{Z}$.*

Definition 37 (Fixed field)**.** Let $L : K$ be a field extension and $G \leq \mathrm{Aut}(L)$. We define the fixed field of G as

$$\mathrm{Fix}_L(G) = \{\alpha \in L : \sigma(\alpha) = \alpha \text{ for all } \sigma \in G\}.$$

Theorem 16.4. *Suppose that $\mathrm{char}(K) = p > 0$, and let F be the prime subfield of K. Let $\varphi : K \to K$ denote the Frobenius map. Then φ is an injective homomorphism, and $\mathrm{Fix}_\varphi(K) = F$.*

Corollary 17. *Suppose that $\mathrm{char}(K) = p > 0$ and K is algebraic over its prime subfield. Then the Frobenius map is an automorphism of K.*

Corollary 18. *Suppose that $\mathrm{char}(K) = p > 0$ and K is algebraic over its prime subfield. Then all polynomials in $K[t]$ are separable over K.*

Corollary 19 (**)**.** *Suppose that $\mathrm{char}(K) = 0$. Then all polynomials in $K[t]$ are separable over K.*

Theorem 16.5. *Suppose that $\mathrm{char}(K) = p > 0$. Let*

$$f(t) = g(t^p) = a_0 + a_1 t^p + \cdots + a_{n-1} t^{(n-1)p} + t^{np}$$

be a non-constant monic polynomial over K. Then $f(t)$ is irreducible in $K[t]$ if and only if $g(t)$ is irreducible in $K[t]$ and not all the coefficients a_i are p-th powers in K.

17 Galois Fields II

Theorem 17.1. *Let p be a prime, and let $q = p^n$ for some $n \in \mathbb{N}$. Then:*

(a) There exists a field $\mathbb{F}_q$ of order q, and this field is unique up to isomorphism.

(b) All elements of $\mathbb{F}_q$ satisfy the equation $t^q = t$, and hence $\mathbb{F}_q : \mathbb{F}_p$ is a splitting field extension for $t^q - t$.

(c) There is a unique copy of $\mathbb{F}_q$ inside any algebraically closed field containing $\mathbb{F}_p$.

Theorem 17.2. *Let p be a prime, and suppose that $q = p^n$ for some $n \in \mathbb{N}$. Then:*

(a) $\mathrm{Gal}(\mathbb{F}_q : \mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$*;*

(b) The field $\mathbb{F}_q$ contains a subfield of order p^d if and only if $d \,\big|\, n$. When $d \,\big|\, n$, moreover, there is a unique subfield of $\mathbb{F}_q$ of order p^d.

Definition 38 (Norm, Trace)**.** Let p be a prime and let $\alpha \in F_q$ where $q = p^n$ for some $n \in \mathbb{N}$. Then we define

$$\begin{aligned}\mathrm{Tr}(\alpha) &= \alpha + \alpha^p + \cdots + \alpha^{p^{n-1}} \\ &= \alpha + \varphi(\alpha) + \cdots + \varphi^{n-1}(\alpha)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Norm}(\alpha) &= \alpha \cdot \alpha^p \cdots \alpha^{p^{n-1}} = \alpha^{\frac{p^n-1}{p-1}} \\ &= \alpha \cdot \varphi(\alpha) \cdots \varphi^{n-1}(\alpha)\end{aligned}$$

Lemma 17.3. *Let p be a prime and let $\alpha \in F_q$ where $q = p^n$ for some $n \in \mathbb{N}$.*

1. *For all $\alpha \in \mathbb{F}_q$, one has* $\mathrm{Tr}(\alpha), \mathrm{Norm}(\alpha) \in \mathbb{F}_p$*;*
2. *If $p \neq 2$, then $\exists \alpha_1$ such that* $\mathrm{Tr}(\alpha_1) \neq 0$ *and $\exists \alpha_2 (\neq 0)$ such that* $\mathrm{Norm}(\alpha_2) \neq 1$.

18 Further Examples, Fixed Fields

19 Further Examples, Fixed Fields

Definition 39 (Fixed field)**.** Let $L : K$ be a field extension and $G \leq \mathrm{Aut}(L)$. Then the fixed field of G is

$$\mathrm{Fix}_L(G) = L^G = \{\alpha \in L : g\alpha = \alpha \ \forall g \in G\}$$

Definition 40 (Galois Extension)**.** Let $L : K$ be a field extension. Then $L : K$ is a Galois extension if it is normal and separable.

Lemma 19.1. *Let $K, M \subseteq L$ be fields and $G, H \leq \mathrm{Aut}(L)$. Then*

1) if $K \subseteq M$, then $\mathrm{Gal}(L : K) \geqslant \mathrm{Gal}(L : M)$*;*

2) if $G \leq H$, then $\mathrm{Fix}_L(G) \supseteq \mathrm{Fix}_L(H)$*;*

3) $K \subseteq \mathrm{Fix}_L(\mathrm{Gal}(L : K))$*;*

4) $G \leq \mathrm{Gal}(L : \mathrm{Fix}_L(G))$*;*

5) $\mathrm{Gal}(L : K) = \mathrm{Gal}(L : \mathrm{Fix}_L(\mathrm{Gal}(L : K)))$*;*

6) $\mathrm{Fix}_L(G) = \mathrm{Fix}_L(\mathrm{Gal}(L : \mathrm{Fix}_L(G)))$.

Theorem 19.2. *Let $L : K$ be algebraic. Then $L : K$ is Galois $\iff K = \mathrm{Fix}_L(\mathrm{Gal}_K(L))$*

Theorem 19.3. *Suppose that L is a field, $G \leq \mathrm{Aut}(L)$ such that $|G| < \infty$, and put $K = \mathrm{Fix}_L(G)$. Then $L : K$ is a finite Galois extension with $[L : K] = |\mathrm{Gal}(L : K)|$, and furthermore $G = \mathrm{Gal}_K(L)$.*

Theorem 19.4. *Let $L : K$ be finite.*

1. *If $L : K$ is a Galois extension, then $|\mathrm{Gal}(L : K)| = [L : K]$ and $K = \mathrm{Fix}_L(\mathrm{Gal}(L : K))$.*
2. *If $L : K$ is not Galois, then $|\mathrm{Gal}(L : K)| < [L : K]$ and K is a proper subfield of $\mathrm{Fix}_L(\mathrm{Gal}(L : K))$.*

Corollary 20. *Let $L : M : K$ be a tower such that $L : K$ is Galois. Then $L : M$ is Galois.*

Proposition 1. *Let $f \in K[t] \setminus K$ be separable. Then $\mathrm{Gal}_K(f) \leq A_n \iff \sqrt{D} \in K$*

20 Fundamental Theorem of Galois Theory I

Let $L : K$ be any extension and let $G \subseteq \mathrm{Aut}(L)$. Let $\mathcal{I}(K, L)$ be the collection of all intermediate fields of $L : K$ (i.e. all fields M with $K \subseteq M \subseteq L$), and let $\mathcal{S}(G)$ be the family of all subgroups of G.

We have the sets:

$$\mathcal{I}(K, L) \quad \text{and} \quad \mathcal{S}(G).$$

For any $M \in \mathcal{I}(K, L)$, we consider $\mathrm{Gal}(L : M) \subseteq \mathrm{Aut}(L)$. For any $H \in \mathcal{S}(G)$, we consider $\mathrm{Fix}_L(H) = \{\, \alpha \in L : h(\alpha) = \alpha \ \forall h \in H\}$.

Galois Correspondence Claim. There is a one-to-one correspondence between

$$\mathcal{I}(K, L) \quad \text{and} \quad \mathcal{S}(G),$$

where $G = \mathrm{Gal}(L : K)$. In other words,

$$M \longleftrightarrow \mathrm{Gal}(L : M) \quad \text{and} \quad H \longleftrightarrow \mathrm{Fix}_L(H).$$

1. If $\widetilde{L} : K$ is a Galois extension (see Theorem 2 of the previous lecture), then $L : K$ is also a Galois extension. By Corollary of Theorem 4, if $K \subset M \subset L$ is a tower and $L : K$ is Galois, then $L : M$ is Galois as well.

2. Using the primitive element theorem, we know $L = K(\alpha)$ for some $\alpha \in L$. Consider the orbit $H \cdot \alpha$ and the polynomial

$$\mu_{\alpha,K}(t) \ = \ \prod_{\beta \in H \cdot \alpha} (t - \beta).$$

Since $\alpha \in L$, we see $\mu_{\alpha,K}(t) \in K[t]$. The orbit of α under the action of H yields $\deg(\mu_{\alpha,K}(t)) = |H \cdot \alpha|$.

3. We also know that

$$K\big(\alpha\big)^H = \big\{\gamma \in K(\alpha) : h(\gamma) = \gamma \ \forall h \in H\big\}.$$

Hence

$$\big|\,\mathrm{Gal}\big(L : K\big)\big| \ = \ \big|H \cdot \alpha\big|.$$

Further Example: Cyclotomic Extensions

For a prime p, consider $\zeta_p = e^{2\pi i/p}$. The extension $\mathbb{Q}(\zeta_p) : \mathbb{Q}$ is Galois with

$$\mathrm{Gal}\big(\mathbb{Q}(\zeta_p) : \mathbb{Q}\big) \cong (\mathbb{Z}/p\mathbb{Z})^\times,$$

where the non-trivial subgroup is generated by the complex conjugation or by Frobenius automorphisms.

Subgroups and Fixed Fields. For example, with $p = 5$, we have $\zeta_5 = e^{2\pi i/5}$. The only non-trivial subgroup of the Galois group corresponds to complex conjugation. Hence the fixed field of this subgroup is $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$, etc.

Conclusion. This establishes a Galois correspondence between the intermediate fields $\mathcal{I}(K, L)$ and the subgroups $\mathcal{S}(\mathrm{Gal}(L : K))$. In particular, if M is an intermediate field, then $\mathrm{Gal}(L : M)$ is the subgroup of $\mathrm{Gal}(L : K)$ that fixes M; and if H is a subgroup of $\mathrm{Gal}(L : K)$, then $\mathrm{Fix}_L(H)$ is the intermediate field fixed by H.