

**Exercise 5.1.** Which of the following field extensions are normal? Justify your answers.

1.  $\mathbb{Q}(i) : \mathbb{Q}$

*Solution.* Normal. By theorem, we know that any finite extension  $L : K$  is normal  $\iff L$  is a splitting field extension for some non-constant  $f \in K[t]$ . Hence, since  $\mathbb{Q}(i)$  is the splitting field for  $t^2 + 1$  over  $\mathbb{Q}$ , the extension  $\mathbb{Q}(i) : \mathbb{Q}$  is normal.  $\square$

2.  $\mathbb{Q}(2^{1/4}) : \mathbb{Q}$

*Solution.* Not normal. By definition, an extension  $L : K$  is normal if  $\forall \alpha \in L$ , the minimum polynomial of  $\alpha$  over  $K$ ,  $\mu_\alpha^K(t)$ , splits over  $L[t]$ . Obviously,  $\sqrt[4]{2} \in \mathbb{Q}(2^{1/4})$  by construction. However, notice that for  $\alpha = \sqrt[4]{2}$ ,

$$\begin{aligned}\mu_\alpha^{\mathbb{Q}}(t) &= t^4 - 2 \\ &= (t^2 + \sqrt{2})(t^2 - \sqrt{2}) \\ &= (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2}),\end{aligned}$$

but the linear factors  $(t + i\sqrt[4]{2})$  and  $(t - i\sqrt[4]{2})$  are not in  $\mathbb{Q}(2^{1/4})[t]$ . Hence, the extension  $\mathbb{Q}(2^{1/4}) : \mathbb{Q}$  is not normal by definition.  $\square$

3.  $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$

*Solution.* Normal. Consider the polynomial  $f(t) = (t^4 - 2)(t^2 - 1) \in \mathbb{Q}(2^{1/4}, i)[t]$ . Then,

$$f(t) = (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2})(t + i)(t - i),$$

whence  $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$  is a splitting field extension for  $f$ . By applying the same theorem as in part 1, this extension is normal.  $\square$

4.  $\mathbb{Q}(2^{1/4}, i, \sqrt{5}) : \mathbb{Q}$

*Solution.* Normal. Consider the polynomial  $f(t) = (t^4 - 2)(t^2 - 1)(t^2 - 5) \in \mathbb{Q}(2^{1/4}, i, \sqrt{5})[t]$ . Then,

$$f(t) = (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2})(t + i)(t - i)(t - \sqrt{5})(t + \sqrt{5}),$$

whence  $\mathbb{Q}(2^{1/4}, i, \sqrt{5}) : \mathbb{Q}$  is a splitting field extension for  $f$ . By applying the same theorem as in part 1, this extension is normal.  $\square$

5.  $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$

*Solution.* Normal. Consider the polynomial  $f(t) = (t^2 - 3)(t^3 - 3)$ . Then,

$$f(t) = (t + \sqrt{3})(t - \sqrt{3})(t - \sqrt[3]{3})(t - \varepsilon_3 \sqrt[3]{3})(t - \varepsilon_3^2 \sqrt[3]{3}),$$

where  $\varepsilon_3 = \exp\left(\frac{2\pi}{3}i\right)$ . Notice,

$$\begin{aligned}\varepsilon_3 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) & \varepsilon_3^2 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} & &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \in \mathbb{Q}(3^{1/3}, i, \sqrt{3}) \\ &= \frac{1}{2}(-1 + i\sqrt{3}) \in \mathbb{Q}(3^{1/3}, i, \sqrt{3}) & &= \frac{1}{2}(-1 - i\sqrt{3}) \in \mathbb{Q}(3^{1/3}, i, \sqrt{3}).\end{aligned}$$

Thus  $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$  is a splitting field extension for  $f$ , whence must be normal by the same theorem as part 1.  $\square$

**Exercise 5.2.** Let  $\psi : L \rightarrow M$  be a homomorphism, suppose that  $L$  is algebraically closed. Prove that  $\psi(L)$  is algebraically closed.

*Solution.* Let  $g \in \psi(L)[t]$  be some irreducible polynomial over  $\psi(L)$ . Then, we have some  $f \in L[t]$  such that  $g = \psi f$  with  $\deg g = \deg f$ . Now, assume ad absurdum that  $g$  has a degree greater than 1. Then  $\deg f > 1$ . By algebraic closure of  $L$ , any irreducible polynomials must be linear. Since  $\deg f \neq 1$ ,  $f$  must be reducible and thus  $f = h\ell$  for some  $h, \ell \in L[t]$  such that  $\deg h \geq 1$  and  $\deg \ell \geq 1$ . Since  $\psi$  must preserve operations, this implies that  $\exists \hat{h}, \hat{\ell} \in \psi(L)$  such that  $g = \hat{h}\hat{\ell}$ , where  $\deg \hat{h} \geq 1$  and  $\deg \hat{\ell} \geq 1$ . However, this contradicts the fact that  $g$  is irreducible, so our assumption that  $\deg g > 1$  must be false and hence  $\deg g = 1$ . Therefore,  $\psi(L)$  is algebraically closed.  $\square$

**Exercise 5.3.** Let  $L : K$  be a field extension. Then  $\overline{K}$  is isomorphic to  $\overline{L}$ . In addition, if  $K \subset L \subseteq \overline{L}$ , then  $\overline{K} = \overline{L}$ .

*Solution.* Let  $\varphi_1 : K \rightarrow L$  and  $\varphi_2 : L \rightarrow \overline{L}$  be the monomorphisms corresponding to the field extensions  $L : K$  and  $\overline{L} : L$ , respectively. Then  $\overline{L} : K$  is the field extension relative to the composition  $\varphi_2 \circ \varphi_1$ . By algebraic closure of  $\overline{L}$ , it must be an algebraic closure for  $K$ . By theorem from lecture, we know that any two algebraic closures for the same field must be isomorphic to one another. Thus,  $\overline{L} \cong \overline{K}$ .

Assume ad absurdum that we have some algebraic closure  $\overline{K}$  of  $K$  such that  $|\overline{K}| < |\overline{L}|$ . By definition of algebraic closure, we know that  $\overline{K}$  and  $\overline{L}$  are both algebraic extensions of  $K$ , so  $K \subseteq \overline{K}$  and  $K \subseteq \overline{L}$ . Let  $\psi : K \hookrightarrow \overline{L}$  be the monomorphism corresponding to the extension  $\overline{L} : K$ . By theorem, since  $\overline{K} : K$  is an algebraic extension, there exists an extension of  $\psi$  to another mono from  $\overline{K} \rightarrow \overline{L}$ . Hence  $\overline{L} : \overline{K}$  is an algebraic extension with degree greater than 1, since  $\overline{K}$  is smaller than  $\overline{L}$ . However, this contradicts the algebraic closure of  $\overline{K}$ , since the only algebraic extension of an algebraically closed field is itself. Thus,  $\overline{K} = \overline{L}$ .  $\square$

**Exercise 5.4.** Let  $K - L$  be a normal extension,  $K \subseteq L \subseteq \overline{K}$ . Then for any  $K$ -homomorphism  $\tau : L \rightarrow \overline{K}$  one has  $\tau(L) = L$ .

*Solution.* Suppose we have some  $K$ -homomorphism  $\tau : L \rightarrow \overline{K}$ , and let  $\ell \in L$ . By definition of normal extension,  $K - L$  must be algebraic, whence  $\mu_\ell^K$  exists. Since  $\tau$  fixes all elements of  $K$  and  $\mu_\ell^K$  is a polynomial with coefficients in  $K$ , we can see that  $\tau(\mu_\ell^K(\ell)) = \mu_\ell^K(\tau(\ell)) = 0$ . By theorem, the normality of  $L : K$  implies that all algebraic conjugates of  $\ell$  are in  $L$ . Thus, we have that  $\tau(\ell) \in L$ . Since  $\ell$  is an arbitrary element of  $L$ , this implies that  $\tau(L) \subseteq L$ . By theorem, since  $L$  extends  $K$  and  $\tau : L \rightarrow L$  is a  $K$ -homomorphism, we have that  $\tau$  is an automorphism of  $L$ . Thus  $\tau(L) = L$ .  $\square$

**Exercise 5.5.** Put  $K = \mathbb{F}_2(t)$  and consider  $L = K(t^{1/3})$ . Prove that the extension  $L : K$  is algebraic but not normal.

*Solution.* Obviously since  $K(t^{1/3}) : K$  is a finite field extension, it is algebraic. Suppose  $x \in \overline{K}$  solves the equation  $x^3 - t = 0$ . Then,  $x = t^{1/3} \implies \left(\frac{x}{t^{1/3}}\right)^3 = 1 \implies x = yt^{1/3}$  such that  $y^3 = 1$ . Then we have  $y^3 - 1 = (y - 1)(y^2 + y + 1) = 0$ , so either  $y = 1$  or  $y$  is a root of  $y^2 + y + 1 = 0$ . Notice that  $y^2 + y + 1$  is irreducible over  $K$ , since neither 0 nor 1 are roots. If we suppose there was some  $z \in L \setminus K$  such that  $z^2 + z + 1 = 0$ , then that implies that there exists some  $f \in K[t]$  with  $f(t^{1/3}) = 0$ . However,  $t$  is obviously transcendental over  $K$ , forcing a contradiction. Thus, the only solution to the cubic  $x^3 - t$  in  $L$  is  $x = 1$ , whence the minimum polynomial for  $t^{1/3} \in L$  does not split over  $L$ . Therefore  $L : K$  does not meet the requirements to be a normal field extension.  $\square$