

1 Separability

Definition 1 (Separable). Let K be a field.

- (i) An irreducible polynomial $f \in K[t]$ is separable over K if it has no multiple roots, meaning that $f = \lambda(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_d)$, where $\alpha_1, \dots, \alpha_d \in \overline{K}$ are distinct.
- (ii) A non-zero polynomial $f \in K[t]$ is separable over K if its irreducible factors in $K[t]$ are separable over K .
- (iii) When $L : K$ is a field extension, we say that $\alpha \in L$ is separable over K when α is algebraic over K and μ_α^K is separable.
- (iv) An algebraic extension $L : K$ is a separable extension if every $\alpha \in L$ is separable over K .

Lemma 1.1. Suppose that $L : M : K$ is a tower of algebraic field extensions. Assume that $K \subseteq M \subseteq L \subseteq \overline{K}$, and suppose that $f \in K[t] \setminus K$ satisfies the property that f is separable over K . If $g \in M[t] \setminus M$ has the property that $g \mid f$, then g is separable over M . Thus, if $\alpha \in L$ is separable over K then α is separable over M , and if $L : K$ is separable then so is $L : M$.

Lemma 1.2. 1. If $L : M$ is an algebraic field extension, $\alpha \in L$ and $\sigma : M \rightarrow \overline{M}$ is a homomorphism, then $\sigma(\mu_\alpha^M)$ is separable over $\sigma(M) \iff \mu_\alpha^M$ is separable over M .

2. If $L : K$ is a splitting field extension for $f \in K[t]$ and f is separable over K , then $L : K$ is separable.

Theorem 1.3. Let $L : K$ be a finite extension with $K \subseteq L \subseteq \overline{K}$, whence $L = K(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n \in L$. Put $K_0 = K$, and for $1 \leq i \leq n$, set $K_i = K_{i-1}(\alpha_i)$. Finally, let $\sigma_0 : K \rightarrow \overline{K}$ be the inclusion map.

- (i) If α_i is separable over K_{i-1} for $1 \leq i \leq n$, then there are $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \rightarrow \overline{K}$.
- (ii) If α_i is not separable over K_{i-1} for some i with $1 \leq i \leq n$, then there are fewer than $[L : K]$ ways to extend σ_0 to a homomorphism $\tau : L \rightarrow \overline{K}$.

Theorem 1.4. Let $L : K$ be a finite extension with $L = K(\alpha_1, \dots, \alpha_n)$. Set $K_0 = K$, and for $1 \leq i \leq n$, inductively define K_i by putting $K_i = K_{i-1}(\alpha_i)$. Then the following are equivalent:

- (i) the element α_i is separable over K_{i-1} for $1 \leq i \leq n$;
- (ii) the element α_i is separable over K for $1 \leq i \leq n$;
- (iii) the extension $L : K$ is separable.

Corollary 1. Suppose that $L : K$ is a finite extension. If $L : K$ is a separable extension, then the number of K -homomorphism $\sigma : L \rightarrow \overline{K}$ is $[L : K]$, and otherwise the number is smaller than $[L : K]$.

Corollary 2. Suppose that $f \in K[t] \setminus K$ and that $L : K$ is a splitting field extension for f . Then $L : K$ is a separable extension $\iff f$ is separable over K . More generally, suppose that $L : K$ is a splitting field extension for $S \subseteq K[t] \setminus K$. Then $L : K$ is a separable extension \iff each $f \in S$ is separable over K .