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Homework 11 (Apr 18 – Apr 25)

1 (5) Let $G = \mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number. Construct a subnormal series G_j of subgroups of G such that $|G_{j-1}/G_j| = p$.

2 (5+5) *a*) Let G be a group. Prove that G' is a normal subgroup of G such that G/G' is abelian.
b) Prove that if N is any normal subgroup of G such that G/N is abelian, then $G' \leq N$.

3 (10) Let $\mathbb{F}$ be a field and

$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{F} \right\} \tag{1}$$

be the Heisenberg group. Prove that H is soluble.

4 (15) Prove that A_n, $n \geq 3$ is generated by 3–cycles.

5 (5+5+5) Let G be a group. Find G' for
a) $G = S_3$ *b*) $G = A_4$ *c*) $G = S_4$ (use the previous question).

Solutions

General remark. If there is a typo in any task, then the maximum score will be awarded for that task.

1 (5) Let $G = \mathbb{Z}/p^n\mathbb{Z}$, where p is a prime number. Construct a subnormal series G_j of subgroups of G such that $|G_{j-1}/G_j| = p$.

Solution. Put $G_j = \langle p^j \rangle$, $j = 0, 1, \dots, n$. Then $G_0 = G$, $G_n = \{0\}$ and $G_{j-1}/G_j \cong \mathbb{Z}/p\mathbb{Z}$.

2 (5+5) *a*) Let G be a group. Prove that G' is a normal subgroup of G such that G/G' is abelian.
b) Prove that if N is any normal subgroup of G such that G/N is abelian, then $G' \leq N$.

Solution. We know that for any automorphism φ of G one has $\varphi(G') = G'$ (see lectures). Taking any inner automorphism, we derive that G' is a normal subgroup of G.

Now suppose that N is any normal subgroup of G such that G/N is abelian. Take $x, y \in G/N$. Since G/N is abelian, it follows that

$$xN \cdot yN = xyN = yxN = yN \cdot xN$$

and this takes place iff $x^{-1}y^{-1}xy \in N$. Thus $G' \subseteq N$. Also, this argument shows that G/G' is abelian.

3 (10) Let $\mathbb{F}$ be a field and

$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \ : \ a, b, c \in \mathbb{F} \right\} \tag{2}$$

be the Heisenberg group. Prove that H is soluble.

Solution. By $[a, b, c]$ denote the matrix above. In this notation we have $[a, b, c] * [a', b', c'] = [a + a', b + b' + ac', c + c']$. Thus $H' = [0, x, 0]$, $x \in \mathbb{F}$ and this is an abelian group.

4 (15) Prove that A_n, $n \geq 3$ is generated by 3–cycles.

Solution. Clearly, e is a product of 3–cycles, e.g. $(123)(132)$. Further any element of A_n is a product of even number of transpositions, so it is enough to show that for any transpositions τ_1, τ_2 its product $\pi = \tau_1\tau_2$ is a product of 3–cycles. If $\tau_1 = \tau_2$, then $\pi = e$ and we are done. If τ_1 and τ_2 are disjoint, say, $\tau_1 = (12)$, $\tau_2 = (34)$, then put $\pi = (123)(234)$. Finally, if, say, $\tau_1 = (12)$ and $\tau_2 = (13)$, then $\pi = (12)(13) = (312) = (123)(123)$.

5 (5+5+5) Let G be a group. Find G' for
a) $G = S_3$ *b*) $G = A_4$ *c*) $G = S_4$ (use the previous question).

Solution. *a*) By Question 2, we see that $G' \leq A_3$ (since A_3 is normal in G and G/G' is clearly abelian). Thus G' either A_3 or e. But if $x = (12)$, $y = (13)$, then one can check that $[x, y] = (231) = (123) \neq e$. Thus $G' = A_3$.

b) Again, $G' \leq V_4$ since V_4 is normal in G and G/V_4 is abelian. By the same argument as in *a*), we see that $|G'| \geq |\langle (231) \rangle| = 3$. Thus $G' = V_4$.

c) Again $G' \leq A_4$ and by *a*) we see that G' contains all 3–cycles. Thus by Question 4 we obtain that $G' = A_4$.