Problem Set 9: Math 454 Spring 2017
Due Thursday April 13

March 28, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Prove that if $P,Q \in F[t]$ have $\gcd(P,Q) = 1$, then for any finite extension E/F, we have $\gcd(P,Q) = 1$ where $P,Q \in E[t]$.

Problem 2. Find $\beta \in E$ such that $E = F(\beta)$ for the examples below:

(i) $F = \mathbf{Q}$ and $E = \mathbf{Q}(\sqrt{2}, \sqrt{3})$.

(ii) $F = \mathbf{Q}(\sqrt{2})$ and $E = F(\sqrt{3}, \sqrt{11})$.

Problem 3. Let $n \in \mathbf{N}$ and F be a field. If $\zeta_n \in F$ is an nth root of unity, prove that ζ_n^ℓ is an nth root of unity for any $\ell = 1, \dots, n$.

Problem 4. Prove that if $m,n \in \mathbf{N}$ and m divides n, then $t^m - 1$ divides $t^n - 1$. Deduce that if ζ_m is an mth root of unity, then ζ_m is an nth root of unity for any n such that m divides n.

Problem 5. Prove that if $n \in \mathbf{N}$ and $\mathrm{char}(F) = 0$, then there exists an nth root of unity ζ_n that is not an mth root of unity for any m that divides n. We call any nth root of unity with this property a primitive root of unity.

Problem 6. Let $n \in N$, $\mathrm{char}(F) = 0$, and ζ_n be a primitive nth root of unity. Prove that if ζ is an nth root of unity, then $\zeta = \zeta_n^\ell$ for some $\ell = 1, \dots, n$. Deduce that $\mathrm{Roots}(t^n - 1) = \{\zeta_n, \zeta_n^2, \dots, \zeta_n^n\}$.

Problem 7. Let F be a field with $\mathrm{char}(F) = 0$. Prove that if $\zeta_n \in F$ is an nth root of unity, then $\zeta_n \in \overline{\mathbf{Q}}$. In particular, when studying roots of unity over characteristic zero fields, it suffices to use $F = \mathbf{Q}$.

Problem 8. Let $F = \mathbf{Q}$, $P \in \mathbf{Q}[t]$, and F_P be the splitting field for P.

(a) Prove that if $P = t^3 - 1$, then $\mathrm{Gal}(F_P/\mathbf{Q})$ is a cyclic group of order three. [Hint: Prove that if ζ_3 is a primitive 3rd root of unity, then $F_P = \mathbf{Q}(\zeta_3)$ and $\zeta_3 \to \zeta_3^2$ generates $\mathrm{Gal}(F_P/\mathbf{Q})$]

(b) Prove that if $P = t^3 - 2$, then $\mathrm{Gal}(F_P/\mathbf{Q}) = \mathrm{Sym}(3)$. [Hint: It is enough to prove that $[F_P : \mathbf{Q}] = 6$]

(c) List the subgroups of $\mathrm{Sym}(3)$ and the associate subfields of F_P in (ii) under the Galois correspondence.