

# 1 Algebraic Closure I

**Definition 1** (Algebraically closed field, algebraic closure). Let  $M$  be a field.

- (i) We say that  $M$  is *algebraically closed* if every non-constant polynomial  $f \in M[t]$  has a root in  $M$ .
- (ii) We say that  $M$  is an algebraic closure of  $K$  if  $M : K$  is an algebraic field extension such that  $M$  is algebraically closed.

**Lemma 1.1.** Let  $M$  be a field. The following are equivalent:

- (i) The field  $M$  is algebraically closed;
- (ii) every non-constant polynomial  $f \in M[t]$  factors in  $M[t]$  as a product of linear factors;
- (iii) every irreducible polynomial in  $M[t]$  has degree 1;
- (iv) the only algebraic extension of  $M$  containing  $M$  is itself.

**Definition 2** (Chain). Suppose that  $X$  is a nonempty, partially ordered set with  $\leq$  denoting the partial ordering. A *chain*  $C$  in  $X$  is a collection of elements  $\{a_i\}_{i \in I}$  of  $X$  such that for every  $i, j \in I$ , either  $a_i \leq a_j$  or  $a_j \leq a_i$ .

**Zorn's Lemma:** Suppose that  $X$  is a nonempty, partially ordered set with  $\leq$  the partial ordering. If every non-empty chain  $C$  in  $X$  has an upper bound in  $X$ , then  $X$  has at least one maximal element  $m$  (i.e.  $b \in X$  with  $m \leq b \implies b = m$ ).

**Corollary 1.** Any proper ideal  $A$  of a commutative ring  $R$  is contained in a maximal ideal.

**Lemma 1.2.** Let  $K$  be a field. Then there exists an algebraic extension  $E : K$ , with  $K \subseteq E$ , such that  $E$  contains a root of every irreducible  $f \in K[t]$ , and hence also every  $g \in K[t] \setminus K$ .

**Theorem 1.3** (Existence of Algebraic Closures). Suppose that  $K$  is a field. Then there exists an algebraic extension  $\overline{K}$  of  $K$  such that  $\overline{K}$  is algebraically closed.

**Definition 3** (Extension of field homomorphism, isomorphic field extensions). For  $i = 1$  and  $2$ , let  $L_i : K_i$  be a field extension relative to the embedding  $\varphi_i : K_i \rightarrow L_i$ . Suppose that  $\sigma : K_1 \rightarrow K_2$  and  $\tau : L_1 \rightarrow L_2$  are isomorphisms. We say that  $\tau$  *extends*  $\sigma$  if  $\tau \circ \varphi_1 = \varphi_2 \circ \sigma$ . In such circumstances, we say that  $L_1 : K_1$  and  $L_2 : K_2$  are *isomorphic field extensions*.

$$\begin{array}{ccc}
 L_1 & \xrightarrow{\tau} & L_2 \\
 \varphi_1 \uparrow & \nearrow \tau & \uparrow \varphi_2 \\
 K_1 & \xrightarrow{\sigma} & K_2
 \end{array}$$

When  $\sigma : K_1 \rightarrow K_2$  and  $\tau : L_1 \rightarrow L_2$  are homomorphisms (instead of isomorphisms), then  $\tau$  *extends*  $\sigma$  as a *homomorphism of fields* when the isomorphism  $\tau : L_1 \rightarrow L'_1 = \tau(L_1)$  extends the isomorphism  $\sigma : K_1 \rightarrow K'_1 = \sigma(K_1)$ .

**Definition 4** ( $K$ -homomorphism). Let  $L : K$  be a field extension relative to the embedding  $\varphi : K \rightarrow L$ , and let  $M$  be a subfield of  $L$  containing  $\varphi(K)$ . Then, when  $\sigma : M \rightarrow L$  is a homomorphism, we say that  $\sigma$  is a  *$K$ -homomorphism* if  $\sigma$  leaves  $\varphi(K)$  pointwise fixed, which is to say that for all  $\alpha \in \varphi(K)$ , one has  $\sigma(\alpha) = \alpha$ .

**Lemma 1.4.** Suppose that  $L : K$  is a field extension with  $K \subseteq L$ , and that  $\tau : L \rightarrow L$  is a  $K$ -homomorphism. Suppose that  $f \in K[t]$  has the property that  $\deg f \geq 1$ , and additionally that  $\alpha \in L$ .

- (i) if  $f(\alpha) = 0$ , one has  $f(\tau(\alpha)) = 0$ ;
- (ii) if  $\tau$  is a  $K$ -automorphism of  $L$ , then  $f(\alpha) = 0 \iff f(\tau(\alpha)) = 0$ .

**Theorem 1.5.** Let  $\sigma : K_1 \rightarrow K_2$  be a field isomorphism. Suppose that  $L_i$  is a field with  $K_i \subseteq L_i$  ( $i = 1, 2$ ). Suppose also that  $\alpha \in L_1$  is algebraic over  $K_1$ , and that  $\beta \in L_2$  is algebraic over  $K_2$ . Then we can extend  $\sigma$  to an isomorphism  $\tau : K_1(\alpha) \rightarrow K_2(\beta)$  in such a manner that  $\tau(\alpha) = \beta$  if and only if  $m_\beta(K_2) = \sigma(m_\alpha(K_1))$ .

$$\begin{array}{ccccc}
K_2 & \xrightarrow{\varphi_2} & K_2(\beta) & \xhookrightarrow{\iota_2} & L_2 \\
\downarrow \sigma & & \downarrow \tau & & \\
K_1 & \xrightarrow{\varphi_1} & K_1(\alpha) & \xhookrightarrow{\iota_1} & L_1
\end{array}$$

**Note:** When  $\tau : K_1(\alpha) \rightarrow K_2(\beta)$  is a homomorphism, and  $\tau$  extends the homomorphism  $\sigma : K_1 \rightarrow K_2$ , then  $\tau$  is completely determined by  $\sigma$  and the value of  $\tau(\alpha)$ .

**Corollary 2.** Let  $L : M$  be a field extension with  $M \subseteq L$ . Suppose that  $\sigma : M \rightarrow L$  is a homomorphism, and  $\alpha \in L$  is algebraic over  $M$ . Then the number of ways we can extend  $\sigma$  to a homomorphism  $\tau : M(\alpha) \rightarrow L$  is equal to the number of distinct roots of  $\sigma(m_\alpha(M))$  that lie in  $L$ .