1 Galois Groups I

Definition 1 (Galois group of polynomial)**.** *Let $L = K(\alpha_1, \ldots, \alpha_n)$ and let $P(\alpha_1, \ldots, \alpha_n)$ where $P \in K[\alpha_1, \ldots, \alpha_n]$ is an element of L. Then we define*

$$\mathrm{Gal}_K(f) = \left\{\sigma \in S_n \mid \forall P \in K[\alpha_1, \ldots, \alpha_n],\ \textit{if } P(\alpha_1, \ldots, \alpha_n) = 0 \textit{ then } P(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})\right\}$$

Lemma 1.1. $\mathrm{Gal}_K(f) \leq S_n$

Lemma 1.2. *If $K_1 : K$, then $\mathrm{Gal}_{K_1}(f) \leq \mathrm{Gal}_K(f)$.*

Definition 2. *Let $L : K$ be a field extension. Then*

$$\mathrm{Gal}_K(L) = \mathrm{Gal}(L : K) = \{\varphi \in \mathrm{Aut}(L) : \varphi \textit{ is a } K\textit{-homomorphism}\}$$

Lemma 1.3. *Suppose that $M : K$ is a normal extension. Then:*

(a) for any $\sigma \in \mathrm{Gal}(M : K)$ and $\alpha \in M$, we have $\mu^K_{\sigma(\alpha)} = \mu^K_\alpha$;

(b) for any $\alpha, \beta \in M$ with $\mu^K_\alpha = \mu^K_\beta$, there exists $\tau \in \mathrm{Gal}(M : K)$ having the property that $\tau(\alpha) = \beta$.