1 extra

Proposition 1. Suppose that K and L are fields and that $\varphi : K \to L$ is a homomorphism. With t and y denoting indeterminates, extend the homomorphism φ to the mapping $\psi : K[t] \to L[y]$ by defining

$$\psi(a_0 + a_1 t + \cdots + a_n t^n) = \varphi(a_0) + \varphi(a_1)y + \cdots + \varphi(a_n)y^n.$$

Then $\psi : K[t] \to L[y]$ is an injective homomorphism. Also, when $\varphi : K \to L$ is surjective, then $\psi : K[t] \to L[y]$ is surjective and maps irreducible polynomials in $K[t]$ to irreducible polynomials in $L[y]$.

Proposition 2. Suppose $L : K$ is a field extension with $K \subseteq L$, and $\alpha \in L$. Then E_α is a ring homomorphism.

Proposition 3. Let $L : K$ be a field extension with $K \subseteq L$, and suppose that $\alpha \in L$ is algebraic over K. Then

$$I = \ker(E_\alpha) = f \in K[t] : f(\alpha) = 0$$

is a nonzero ideal of $K[t]$, and there is a unique monic polynomial $\mu_\alpha^K \in K[t]$ that generates I.

Proposition 4. Let $L : K$ be a field extension with $K \subseteq L$. Let $A \subseteq L$ and

$$\mathcal{C} = \{C \subseteq A : C \text{ is a finite set}\}.$$

Then $K(A) = \cup_{C \in \mathcal{C}} K(C)$. Further, when $[K(C) : K] < \infty$ for all $C \in \mathcal{C}$, then $K(A) : K$ is an algebraic extension.

Proposition 5. Let $L : K$ be a field extension with $K \subseteq L$, and suppose that $\alpha \in L$. Then

$$K[\alpha] = \left\{c_0 + c_1\alpha + \cdots + c_d\alpha^d : d \in \mathbb{Z}_{\leq 0},\ c_0, \ldots, c_d \in K\right\}$$

and

$$K(\alpha) = \{f/g : f, g \in K[\alpha], g \neq 0\}.$$

Proposition 6. Let $L : K$ be a field extension with $K \subseteq L$, and suppose that $\alpha \in L$. Then α is algebraic over K if and only if $[K(\alpha) : K] < \infty$.

1.1 Review of finite fields and tests for irreducibility

Definition 1 (Characteristic)**.** Let K be a field with additive identity 0_K and multiplicative identity 1_K. When $n \in \mathbb{N}$, we write $n \cdot 1_K$ to denote $1_K + \ldots + 1_K$ (as an n-fold sum). We define the *characteristic* of K, denoted by $\operatorname{char} K$, to be the smallest positive integer m with the property that $m \cdot 1_K = 0_K$; if no such integer m exists, we define the characteristic of K to be 0.

Proposition 7. Let K be a field with $\operatorname{char} K > 0$. Then $\operatorname{char} K$ is equal to a prime number p, and then for all $x \in K$ one has $p \cdot x = 0$.

Theorem 1.1 (Localisation principle)**.** Let R be an integral domain, and let I be a prime ideal of R. Define $\varphi : R[X] \to (R/I)[X]$ by putting

$$\varphi(a_0 + a_1 X + \cdots + a_n X^n) = \overline{a}_0 + \overline{a}_1 X + \cdots + \overline{a}_n X^n,$$

where $\overline{a}_j = a_j + I$. Then φ is a surjective homomorphism. Moreover, if $f \in R[X]$ is primitive with leading coefficient not in I, then f is irreducible in $R[X]$ whenever $\varphi(f)$ is irreducible in $(R/I)[X]$.

Note: Proposition 3.1 tells us that when $f \in K[t]$ and $\sigma \in \operatorname{Gal}(L : K)$, the mapping σ permutes the roots of f that lie in L.

Theorem 1.2. Suppose that $L : K$ is an algebraic extension, and $\sigma : L \to L$ is a K-homomorphism. Then σ is an automorphism of L.

Theorem 1.3. If $L : K$ is a finite extension, then $|\operatorname{Gal}(L : K)| \leq [L : K]$.

Corollary 1. Suppose that $L : F$ and $L : F'$ are finite extensions with $F \subseteq L\ tand F' \subseteq L$, and further that $\psi : F \to F'$ is an isomorphism. Then there are at most $[L : F]$ ways to extend ψ to a homomorphism from L into L.

Corollary 2. Let $L : K$ be a finite extension with $K \subseteq L$. Suppose that $\alpha_1, \ldots, \alpha_n \in L$ and put $L = K(\alpha_1, \ldots, \alpha_n)$. Let $K_0 = K$, and for $1 \leq i \leq n$, let $K_i = K_{i-1}(\alpha_i)$. Then every automorphism $\tau \in \mathrm{Gal}(L : K)$ corresponds to a sequence of homomorphisms $\sigma_1, \ldots, \sigma_n$, such that $\sigma_0 : K \to L$ is the inclusion map, one has $\sigma_n = \tau$, and for $1 \leq i \leq n$, the map $\sigma_i : K_i \to L$ is a homomorphism extending $\sigma_{i-1} : K_{i-1} \to L$.

2 Algebraic closures

2.1 The definition of an algebraic closure, and Zorn's Lemma

2.2 The existence of an algebraic closure

Corollary 3. When K is a field, the field $\overline{K}$ is a maximal algebraic extension of K.

2.3 Properties of algebraic closures

Corollary 4. Suppose that $\overline{K}$ is an algebraic closure of K, and assume that $K \subseteq \overline{K}$. Take $\alpha \in \overline{K}$ and suppose that $\sigma : K \to \overline{K}$ is a homomorphism. Then the number of distinct roots of μ_α^K in $\overline{K}$ is equal to the number of distinct roots of $\sigma(\mu_\alpha^K)$ in $\overline{K}$.

Proposition 8. Suppose that L and M are fields such that L is algebraically closed, and $\psi : L \to M$ is a homomorphism. Then $\psi(L)$ is algebraically closed.

Proposition 9. If $L : K$ is an algebraic extension, then $\overline{L}$ is an algebraic closure of K, and hence $\overline{L} \cong \overline{K}$. If in addition $K \subseteq L \subseteq \overline{L}$, then we can take $\overline{K} = \overline{L}$.

3 Splitting field extensions

Definition 2 (Splitting field, splitting field extension)**.** Suppose that $L : K$ is a field extension relative to the embedding $\varphi : K \to L$, and $f \in K[t] \setminus K$.

(i) We say that f *splits over* L if $\varphi(f) = \lambda(t - \alpha_1) \cdots (t - \alpha_n)$, for some $\lambda \in \varphi(K)$ and $\alpha_1, \ldots, \alpha_n \in L$.

(ii) Suppose that f splits over L, and let M be a field with $\varphi(K) \subseteq M \subseteq L$. We say that $M : K$ *is a splitting field extension for* f if M is the smallest subfield of L containing $\varphi(K)$ over which f splits.

(iii) More generally, suppose that $S \subseteq K[t] \setminus K$ has the property that every $f \in S$ splits over L. Let M be a field with $\varphi(K) \subseteq M \subseteq L$. We say that *M:K is a splitting field extension for S* if M is the smallest subfield of L containing $\varphi(K)$ over which every polynomial $f \in S$ splits.

Proposition 10. Suppose that $L : K$ is a splitting field extension for the polynomial $f \in K[t] \setminus K$ with associated embedding $\varphi : K \to L$. Let $\alpha_1, \ldots, \alpha_n \in L$ be the roots of $\varphi(f)$. Then $L = \varphi(K)(\alpha_1, \ldots, \alpha_n)$.

Proposition 11. Suppose that $L : K$ is a splitting field extension for the polynomial $f \in K[t] \setminus K$. Then $[L : K] \leq (\deg f)!$

Proposition 12. Given $S \subseteq K[t] \setminus K$, there exists a splitting field extension $L : K$ for S, and $L : K$ is an algebraic extension. More explicitly, suppose that $\overline{K}$ is an algebraic closure of K, and that $\overline{K} : K$ is an extension relative to the embedding $\varphi : \overline{K} \to K$. Let

$$A = \left\{\alpha \in \overline{K} : \alpha \text{ is a root of } \varphi(f), \text{ for some } f \in S\right\}.$$

Put $K' = \varphi(K)$. Then $K'(A) : K$ is a splitting field extension for S.

Theorem 3.1. Let $f \in K[t] \setminus K$, and suppose that $L : K$ and $M : K$ are splitting field extensions for f. Then $L \cong M$, and thus $[L : K] = [M : K]$.

Theorem 3.2. Suppose that $S \subseteq K[t] \setminus K$, and suppose that $L : K$ and $M : K$ are splitting field extensions for S. Then $L \cong M$ and $[L : K] = [M : K]$.

4 Normal extensions and composita

4.1 Normal extensions and splitting field extensions

Proposition 13. Suppose that $L : M : K$ is a tower of field extensions and $L : K$ is a normal extension. Then $L : M$ is also a normal extension.

4.2 Normal closures

Theorem 4.1. Suppose that $M : L : K$ is a tower of field extensions such that $M : K$ is normal. Assume that $K \subseteq L \subseteq M$. Then the following are equivalent:

(i) the field extension $L : K$ is normal;

(ii) any K-homomorphism of L into M is an automorphism of L;

(iii) whenever $\sigma : M \to M$ is a K-automorphism, then $\sigma(L) \subseteq L$.

4.3 Composita of field extensions

Proposition 14. Suppose that $E : K$ and $F : K$ are finite extensions such that K, E and F are contained in a field L. Then $EF : K$ is a finite extension.

Theorem 4.2. Let $E : K$ and $F : K$ be finite extensions such that K, E and F are contained in a field L.

(a) When $E : K$ is normal, then $EF : F$ is normal.

(b) When $E : K$ and $F : K$ are both normal, then $EF : K$ and $E \cap F : K$ are normal.

4.4 Normal closures (non-examinable)

5 Separability

Theorem 5.1. Suppose that $L : M : K$ is a tower of algebraic extensions. Then $L : K$ is separable if and only if $L : M$ and $M : K$ are both separable.

Theorem 5.2. Suppose tht $E : K$ and $F : K$ are finite extensions with $E \subseteq L$ and $F \subseteq L$, where L is a field.

(a) When $E : K$ is separable, then so too is $EF : F$;

(b) When $E : K$ and $F : K$ are both separable, then so too are $EF : K$ and $E \cap F : K$.

6 Inseparable polynomials, differentiation, and the Frobenius map

6.1 Inseparable polynomials and differentiation

6.2 The Frobenius map

7 The Primitive Element Theorem

8 Fixed fields and Galois extensions

Theorem 8.1. Suppose that $L : K$ is an algebraic extension. Then $L : K$ is Galois if and only if $K = \mathrm{Fix}_L(\mathrm{Gal}(L : K))$.

Theorem 8.2. Suppose that L is a field and G is a finite subgroup of $\mathrm{Aut}(L)$, and put $K = \mathrm{Fix}_L(G)$. Then $L : K$ is a finite Galois extension with $[L : K] = |\mathrm{Gal}(L : K)|$, and furthermore $G = \mathrm{Gal}(L : K)$.

Theorem 8.3. Suppose that $L : K$ is a finite extension. Then, if $L : K$ is a Galois extension, one has $|\mathrm{Gal}(L : K)| = [L : K]$ and $K = \mathrm{Fix}_L(\mathrm{Gal}(L : K))$. If $L : K$ is not Galois, meanwhile, one has $|\mathrm{Gal}(L : K)| < [L : K]$ and K is a proper subfield of $\mathrm{Fix}_L(\mathrm{Gal}(L : K))$.

Proposition 15. Suppose that $L : K$ is a Galois extension, and further that $L : M : K$ is a tower of field extensions. Then $L : M$ is a Galois extension.

9 The main theorems of Galois theory

9.1 The Fundamental Theorem

Definition 25. Suppose that $L : K$ is a field extension. When G is a subgroup of $\mathrm{Aut}(L)$, we write $\phi(G)$ for $\mathrm{Fix}_L(G)$, and when $L : M : K_0$ is a tower of field extensions with $K_0 = \phi(\mathrm{Gal}(L : K))$, we write $\gamma(M)$ for $\mathrm{Gal}(L : M)$.

Theorem 9.1 (The Fundamental Theorem of Galois Theory)**.** Suppose that $L : K$ is a finite extension, let $G = \mathrm{Gal}(L : K)$, and put $K_0 = \phi(G)$. Then one has the following:

(a) the map ϕ is a bijection from the set of subgroups of G onto the set of fields M intermediate between L and K_0, and γ is the inverse map;

(b) if $H \leqslant G$, then $H \trianglelefteq G$ if and only if $\phi(H) : K_0$ is a normal extension;

(c) if $H \trianglelefteq G$, one has $\mathrm{Gal}(\phi(H) : K_0) \cong G/H$. In particular, if $\sigma \in G$, one has $\sigma|_{\phi(H)} \in \mathrm{Gal}(\phi(H) : K_0)$, and the map $\sigma \mapsto \sigma|_{\phi(H)}$ is a homomorphism of G onto $\mathrm{Gal}(\phi(H) : K_0)$ with kernel H.

Definition 26 (Galois group of polynomial)**.** When $f \in K[t]$ and $L : K$ is a splitting field extension for f, we define the *Galois group of the polynomial f over K* to be $\mathrm{Gal}_K(f) = \mathrm{Gal}(L : K)$.

9.2 Non-examinable: consequences for composita and intersections

10 Finite fields

11 Solvability and solubility

Definition 27 (Soluble group)**.** A finite group G is *soluble* if there is a series of groups

$$\{\mathrm{id}\} = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G,$$

with the property that $G_i \trianglelefteq G_{i+1}$ and G_{i+1}/G_i is abelian $(0 \leq i < n)$.

Theorem 11.1. Let K be a field of characteristic 0. Then $f \in K[t]$ is solvable by radicals if and only if $\mathrm{Gal}_K(f)$ is soluble.

Lemma 11.2. Suppose $\mathrm{char}\, K = 0$ and $L : K$ is a radical extension. Then there exists an extension $N : L$ such that $N : K$ is normal and radical.

Definition 28 (Cyclic extension)**.** The extension $L : K$ is *cyclic* if $L : K$ is a Galois extension and $\mathrm{Gal}(L : K)$ is a cyclic group.

Lemma 11.3. Suppose that $\mathrm{char}\, K = 0$ and let p be a prime number. Also, let $L : K$ be a splitting field extension for $t^p - 1$. Then $\mathrm{Gal}(L : K)$ is cyclic, and hence $L : K$ is a cyclic extension.

Lemma 11.4. Let $\mathrm{char}\, K = 0$ and suppose that n is an integer such that $t^n - 1$ splits over K. Let $L : K$ be a splitting field extension for $t^n - a$, for some $a \in K$. Then $\mathrm{Gal}(L : K)$ is abelian.

Theorem 11.5. Let $\mathrm{char}\, K = 0$ and suppose that $L : K$ is Galois. Suppose that there is an extension $M : L$ with the property that $M : K$ is radical. Then $\mathrm{Gal}(L : K)$ is soluble.

Corollary 5. Suppose that $\mathrm{char}\, K = 0$. Then $\mathrm{Gal}_K(f)$ is soluble whenever $f \in K[t]$ is soluble by radicals.

Corollary 6. There exist quintic polynomials in $\mathbb{Q}[t]$ with insoluble Galois groups, such as $f(t) = t^5 - 4t + 2$, and which are not solvable by radicals.

Lemma 11.6. Let $\mathrm{char}\, K = 0$, and suppose that $L : K$ is a cyclic extension of degree n. Suppose also that K contains a primitive n-th root of 1. Then there exists $\theta \in K$ such that $t^n - \theta$ is irreducible over K, and $L : K$ is a splitting field for $t^n - \theta$. Further, if β is a root of $t^n - \theta$ over L, then $L = K(\beta)$.

Theorem 11.7. Let $\mathrm{char}\, K = 0$, and suppose that $f \in K[t] \setminus K$. Then f is solvable by radicals whenever $\mathrm{Gal}_K(f)$ is soluble.