Problem Set 1: Math 454 Spring 2017
Due Thursday January 19

January 16, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Let (μ, e) be a group structure on G.

(a) If $g_0 \in G$ satisfies $g_0 \cdot g = g \cdot g_0 = g$ for all $g \in G$, prove that $g_0 = e$.

(b) Prove that if $h_1, h_2 \in G$ are inverses of g (i.e., $h_1 g = g h_1 = e$ and $h_2 g = g h_2 = e$), then $h_1 = h_2$.

Problem 2. Prove that if $\psi \colon G \to H$ is a homomorphism of groups, then the following holds:

(a) $\psi(e_G) = e_H$ where $e_G \in G$ and $e_H \in H$ are the identity elements.

(b) $\psi(g^{-1}) = (\psi(g))^{-1}$.

(c) Prove that $\mathrm{Ad}_g \colon G \to G$ defined by $\mathrm{Ad}_g(h) = g^{-1} h g$ is a homomorphism.

Problem 3. Let G be a group.

(a) Prove that $H \subseteq G$ is a subgroup if and only if $e \in H$ and for each $h_1, h_2 \in H$, we have $h_1 \cdot h_2^{-1} \in H$.

(b) Prove that $\{e\}$ is a subgroup of G.

(c) Prove that G is a subgroup of G.

(d) Given a subset $S \subseteq G$, let $G_S \subseteq G$ be the subset of all elements of G of the form

$$s_1 \cdot s_2 \cdot \ldots \cdot s_\ell$$

where for each $i = 1, \ldots, \ell$, either $s_i \in S$ or $s_i^{-1} \in S$. Prove that G_S is a subgroup of G and $S \subseteq G_S$.

(e) Given $S \subseteq G$ and G_S as in (d), prove that if $H \leq G$ is any subgroup of G with $S \subseteq H$, then $G_S \subseteq H$. In particular,

$$G_S = \bigcap_{\substack{H \leq G, \\ S \subseteq H}} H.$$

Problem 4. Let G be a group and H be a subgroup of G.

(a) Define the partial relation $\sim_H$ on G by $g_1 \sim_H g_2$ if and only if $g_1 \cdot g_2^{-1} \in H$. Prove that $\sim_H$ is an equivalence relation on G.

(b) Prove that $g_1 \sim_H g_2$ if and only if $g_1 H = g_2 H$.

(c) Prove that if $g_1, g_2 \in G$, then either $g_1 H = g_2 H$ or $g_1 H \cap g_2 H = \emptyset$. [Hint: Use (a), (b)].

(d) Prove that H is normal in G if and only if $gH = Hg$ for all $g \in G$.

Problem 5. Let $\varphi\colon G \times X \to X$ be a group action on X and let $\mathrm{Aut}_{\mathrm{set}}(X)$ denote the set of bijection function $\lambda\colon X \to X$.

(a) Prove that $\mathrm{Aut}_{\mathrm{set}}(X)$ is a group where the identity element is given by the function $\mathrm{Id}_X\colon X \to X$ defined by $\mathrm{Id}_X(x) = x$ and the binary operation on $\mathrm{Aut}_{\mathrm{set}}(X)$ is composition of functions.

(b) For each $g \in G$, define the function $\varphi_g\colon X \to X$ by $\varphi_g(x) = \varphi(g,x)$. Prove that $\varphi_g \in \mathrm{Aut}_{\mathrm{set}}(X)$.

(c) Define the function $\Phi\colon G \to \mathrm{Aut}_{\mathrm{set}}(X)$ by $\Phi(g) = \varphi_g$. Prove that Φ is a homomorphism.

(d) Prove that if $\Phi\colon G \to \mathrm{Sym}(X)$ is a homomorphism, then the function $\varphi\colon G \times X \to X$ given by $\varphi(g,x) = \Phi(g)(x)$ is a group action of G on X.

Problem 6. Let G be a group and $H \leq G$.

(a) Prove that $N_G(H)$ and $C_G(H)$ are subgroups of G.

(b) Prove that $H \leq N_G(H)$.

(c) Prove that H is normal in $N_G(H)$.

(d) Prove that if $H \lhd K \leq G$ for some subgroup K of G, then $K \leq N_G(H)$.

(e) Prove that $C_G(H) \leq N_G(H)$.

(f) Prove that $C_G(H)$ is normal in $N_G(H)$.

Problem 7. Let G be a group with lower central series $\{G_i\}$ and derived series $\left\{G^i\right\}$.

(a) Prove that G_i is normal in G for all $i \geq 0$.

(b) Prove that $G_{i+1} \subseteq G_i$ for all $i \geq 0$.

(c) Prove that G^i is normal in G for all $i \geq 0$.

(d) Prove that $G^{i+1} \subseteq G^i$ for all $i \geq 0$.

(e) Prove that $G^i \subseteq G_i$ for all $i \geq 0$.

(f) Prove that $G^1 = G_1$.

Problem 8. Let G be a group. Prove the following:

(a) If $H \leq K \leq G$, then $[G:H] = [G:K][K:H]$.

(b) If $H \leq G$, then
$$\frac{|G|}{|H|} = [G:H].$$

(c) If $H, K \leq G$, then $H \cap K \leq G$.

(d) If $H, K \leq G$, then $[G : H \cap K] \leq [G:H][G:K]$.

(e) If $H \lhd G$ and $K \leq G$, then $[HK:H] = [K : H \cap K]$.

Problem 9. Let G be a group and $H \leq G$ a subgroup.

(a) Prove that if G is nilpotent of step size j, then H is nilpotent of step size at most j.

(b) Use (i) to prove that if G is abelian, then H is abelian.

(c) Prove that if G is solvable of step size j, then H is solvable of step size at most j.

(d) Prove that if G is cyclic, then H is cyclic.