1 Introduction I

Definition 1.1 (Symmetric function)**.** A function $\varphi(x_1, \dots, x_n)$ is called *symmetric* if

$$\varphi(x_1, \dots, x_n) = \varphi(x_{\omega(1)}, \dots, x_{\omega(n)})$$

for all $\omega \in S_n$.

Definition 1.2 (Elementary symmetric polynomial)**.**

$$\begin{aligned}
\sigma_1 = \sigma_1(x_1, \dots, x_n) &= x_1 + \cdots + x_n \\
\sigma_2 = \sigma_2(x_1, \dots, x_n) &= x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\
&\vdots \\
\sigma_k = \sigma_k(x_1, \dots, x_n) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \\
&\vdots \\
\sigma_n = \sigma_n(x_1, \dots, x_n) &= \prod_{i=1}^{n} x_i
\end{aligned}$$

Theorem 1.3. For any symmetric function $\psi(x_1, \dots, x_n)$, there exists a unique polynomial $P(t_1, \dots, t_n)$ such that $\psi(x_1, \dots, x_n) = P(\sigma_1, \dots, \sigma_n)$.

Definition 1.4 (Vieta formulae)**.** Suppose $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ has roots $r_1, \dots, r_n$. Then,

$$\begin{aligned}
r_1 + r_2 + \cdots + r_n &= -a_{n-1} \\
\sum_{1 \le i < j \le n} r_i r_j &= a_{n-2} \\
&\vdots \\
\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{i_1} r_{i_2} \cdots r_{i_k} &= (-1)^k a_{n-k} \\
&\vdots \\
r_1 r_2 \cdots r_n &= (-1)^n a_0
\end{aligned}$$

Corollary 1.5. The discriminant D of $f \in R[x]$, where R is a ring and $f = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, is a polynomial in $a_1, \dots, a_n$ and coefficients from R (i.e. $D \in R[a_1, \dots, a_n]$).

Note: Any cubic equation can be converted to a depressed cubic by

$$x^3 + Ax^2 + Bx + c = \left(x + \frac{A}{3}\right)^3 + p\left(x + \frac{A}{3}\right) + q.$$

Theorem 1.6 (Vieta's method)**.** Using the trigonometric identity $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$, we can solve certain cubic equations. For example, consider $4x^3 - 3x = -\frac{1}{2}$. Let $x = \cos\varphi$. Then

$$\begin{aligned}
\cos 3\varphi = -\frac{1}{2} &\iff 3\varphi = \pm\frac{2\pi}{3} + 2\pi k \quad \text{for } k \in \mathbb{Z} \\
&\iff \varphi = \pm\frac{2\pi}{9} + 2\pi k \\
&\iff x \in \left\{\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}\right\}.
\end{aligned}$$

In general, we can use this method to solve $4x^3 - 3x = a \implies x = \cos\varphi,\ \cos 3\varphi$ and $\cos : \mathbb{C} \to \mathbb{C}$ is now a complex function. For $x^3 + px + q = 0$, set $x = ky$ such that $\frac{k^3}{pk} = \frac{-4}{3} \implies k = \pm\frac{\sqrt{-4p}}{3}$.

Definition 1.7 (Ferrari's resolvent). Let $f(x) = x^4 + ax^2 + bx + c$, and assume $b^2 - 4ac \neq 0$. Consider a parameter y. Then

$$\begin{aligned} f(x) &= \left(x^2 + \frac{y}{2}\right)^2 + (a-y)x^2 + bx + c - \frac{y^2}{4} \\ \implies D &= b^2 - 4(a-y)\left(c - \frac{y^2}{4} = 0\right) \end{aligned}$$

and hence we obtain *Ferrari' resolvent*:

$$y^3 - ay^2 - 4cy + 4ac - b^2 = 0.$$

Solving the resolvent allows one to reduce solving f to solving a system of quadratics.