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Homework 7 (Mar 7 – Mar 14)

1 (10) Let $K = \mathbb{Q}$, $M = \mathbb{Q}(2^{1/3})$ and $L = \mathbb{Q}(2^{1/3}, \sqrt{3}, i)$. Prove that $L : K$ and $L : M$ are normal but $M : K$ is not normal.

2 (10+5) *a*) Let $K - L$ be algebraic, $\alpha \in L$ and $\sigma : K \to \overline{K}$ be a homomorphism. Prove that μ_α^K is separable over K iff $\sigma(\mu_\alpha^K)$ is separable over $\sigma(K)$.

b) Let $L : K$ be a splitting filed for $f \in K[t]$. Prove that if f is separable, then $L : K$ is separable.

3 (10) Let $L : K$ be a splitting field extension for a polynomial $f \in K[t]$. Then $L : K$ is separable iff f is separable over K.

4 (15) Let $K - M - L$ be an algebraic extension. Prove that $K - L$ is separable iff $K - M$ and $M - L$ are separable.

Solutions

General remark. If there is a typo in any task, then the maximum score will be awarded for that task.

1 (10) Let $K = \mathbb{Q}$, $M = \mathbb{Q}(2^{1/3})$ and $L = \mathbb{Q}(2^{1/3}, \sqrt{3}, i)$. Prove that $L : K$ and $L : M$ are normal but $M : K$ is not normal.

Solution. Obviously, $M : K$ is not normal (since $\varepsilon_3 \notin M$). Let us show that $L : K$ is a normal extension (it implies that $L : M$ is normal). Consider the polynomial $(t^3-2)(t^2-3)(t^2+1)$. Then all roots of this polynomial are $\pm i, \pm\sqrt{3}, 2^{1/3}\varepsilon_3^k$, where $k = 0, 1, 2$. We have $\varepsilon_3 = (1 + i\sqrt{3})/2 \in \mathbb{Q}(i, \sqrt{3})$. Hence the splitting field of this polynomial is $\mathbb{Q}(2^{1/3}, i, \sqrt{3})$ and all splitting fields are normal.

2 (10+5) *a*) Let $K - L$ be algebraic, $\alpha \in L$ and $\sigma : K \to \overline{K}$ be a homomorphism. Prove that μ_α^K is separable over K iff $\sigma(\mu_\alpha^K)$ is separable over $\sigma(K)$.

b) Let $L : K$ be a splitting filed for $f \in K[t]$. Prove that if f is separable, then $L : K$ is separable.

Solution. *a*) We can extend σ to $\sigma : \overline{K} \to \overline{K}$. Let $\mu_\alpha^K(t) = \prod_j (t - \alpha_j)^{r_j}$, where $\alpha_j \in \overline{K}$ and $r_j \in \mathbb{Z}^+$. Then $\sigma(\mu_\alpha^K(t)) = \prod_j (t - \sigma(\alpha_j))^{r_j}$ and all $\sigma(\alpha_j)$ are distinct (recall that σ is injective). Thus μ_α^K has multiple roots iff $\sigma(\mu_\alpha^K)$ has multiple roots.

b) We have $L = K(\alpha_1, \ldots, \alpha_n)$, where α_j are roots of f. Clearly, each $\mu_{\alpha_i}^K$ divides f and since f is separable, we see that all α_i are separable. Applying Theorem 1' of Lecture 14, we see that $L : K$ is separable.

3 (10) Let $L : K$ be a splitting field extension for a polynomial $f \in K[t]$. Then $L : K$ is separable iff f is separable over K.

Solution. If f is separable, then by Question 2.b we know that $L : K$ is separable. Now, let $L : K$ is separable. Then any root of f is a separable and by Theorem 1' of Lecture 14, we see that $L : K$ is separable.

4 (15) Let $K - M - L$ be an algebraic extension. Prove that $K - L$ is separable iff $K - M$ and $M - L$ are separable.

Solution. If $K - L$ is separable, then we know (see Lecture 14) that $M - L$ is separable and automatically $K - M$ is separable.

Now suppose that $K - M$ and $M - L$ are both separable and take an arbitrary $\alpha \in L$. Then α is separable over M and therefore μ_α^M is separable. Adjoint all coefficients of μ_α^M to K and obtain a subfield $M' \subseteq M$ such that $\mu_\alpha^{M'} = \mu_\alpha^M$ (in particular, $\mu_\alpha^{M'}$ is separable over M'). Thus as $M' : K$ is finite and separable (by assumption $M : K$ is separable) and hence by the primitive element theorem that there exists $\beta \in M'$ such that $M' = K(\beta)$. Hence using Theorem 1' of Lecture 14, we see that the extension $M'(\alpha) : K = K(\alpha, \beta) : K$ is separable. Then $\alpha \in K(\alpha, \beta)$ is separable over K.