Problem Set 3: Math 454 Spring 2017
Due Thursday February 2

January 30, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Let F be a commutative ring with identity. Prove that the following are equivalent:

(a) The only two ideals in F are F and the trivial ideal $\{0_F\}$.

(b) F is a field.

Problem 2. Let F be a field.

(a) Prove that $F[t]$ is a commutative ring with identity.

(b) Prove that the subset of constant polynomials of $F[t]$ is a subring. Moreover, prove this set is a field and this field is isomorphic to F.

(c) Prove that the group of units of $F[t]$ is $F^\times$, viewed as the group of units of the field of constant polynomials.

Problem 3. Prove that if E/F is an extension of fields and S_1, S_2 are both maximal, algebraically independent subsets of E, then $|S_1| = |S_2|$. In particular, the transcendence degree of E is well defined.

Problem 4. Let E/F be an extension of fields with $F \leq E_1, E_2 \leq E$. Prove there exists an F–basis $\mathscr{B}$ for E_1E_2 given as follows. Let $\mathscr{B}_1$ be an F–basis for $E_1 \cap E_2$. Let $\mathscr{B}_2$ be an $(E_1 \cap E_2)$–basis for E_1 and let $\mathscr{B}_3$ be an $(E_1 \cap E_2)$–basis for E_2. Prove that

$$\mathscr{B} = \{uv \,:\, u \in \mathscr{B}_1,\ v \in \mathscr{B}_2\} \cup \{uw \,:\, u \in \mathscr{B}_1,\ w \in \mathscr{B}_3\}$$

is an F–basis for E_1E_2.

Problem 5. Let E/F, E_1/F, and E_2/F be an extensions such that $E_1, E_2 \leq E$ and E_1/F and E_2/F are finite.

(a) Prove that the composite E_1E_2/F is a finite extension.

(b) Prove that $E_1 \cap E_2$ is an extension of F.

(c) Prove that $[E_1E_2 : E_1] = [E_2 : E_1 \cap E_2]$ and $[E_1E_2 : E_2] = [E_1 : E_1 \cap E_2]$.

(d) Prove that $[E_1E_2 : F] = \frac{[E_1:F][E_2:F]}{[E_1 \cap E_2:F]}$.

Problem 6. Let R, R' be integral domains and $\psi\colon R \to R'$ an isomorphism of rings. Prove that there exists an isomorphism of fields $\widetilde{\psi}\colon \operatorname{Frac}(R) \to \operatorname{Frac}(R')$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[d, "\psi_R"'] & R' \arrow[d, "\psi_{R'}"] \\ \operatorname{Frac}(R) \arrow[r, "\widetilde{\psi}"'] & \operatorname{Frac}(R') \end{tikzcd}$$

commutes.

Problem 7. Prove that $L\colon E \to \operatorname{Hom}_E(E,E)$ given by $L(\beta) = L_\beta$ is an isomorphism of F–algebras and the restriction of L to $E^\times$ gives an isomorphism of groups $L\colon E^\times \to \operatorname{Aut}_{E\text{-vec}}(E)$.