Algebraic closures II (Lecture 10)

Thm 1 $K-L$ algebraic extension, $\varphi: K \to \overline{K}$ is a homomorphism $\Rightarrow \exists$ extension of φ to a homomorphism $: L \to \overline{K}$.

Pf. $S = \{(\mathbb{F}, \psi) : K \subseteq \mathbb{F} \subseteq L \ \& \ \psi: \mathbb{F} \to \overline{K}$ is a homom. extending $\varphi\} \neq \emptyset$

Let $(\mathbb{F}_1, \psi_1) \leq (\mathbb{F}_2, \psi_2) \overset{\text{def}}{\Longleftrightarrow} \mathbb{F}_1 \subseteq \mathbb{F}_2$ & ψ_2 extends ψ_1

If $C = ((\mathbb{F}_i, \psi_i))_{i \in I}$ is a chain $\Rightarrow$ consider $\mathbb{F} = \bigcup_{i \in I} \mathbb{F}_i \subseteq L \Rightarrow \mathbb{F}$ is a subfield of L &

$\psi: \mathbb{F} \to \overline{K}$ is defined as $\psi(\alpha) = \psi_i(\alpha)$, $\alpha \in \mathbb{F}$, $i \in I$ ($\psi_i(\alpha) = \psi_j(\alpha)$ for $\alpha \in \mathbb{F}_i \cap \mathbb{F}_j \Rightarrow \psi$ is well-defined). Thus $(\mathbb{F}, \psi) \in S \Rightarrow (\mathbb{F}, \psi)$ is an upper bound for $C \Rightarrow$ by Zorn's lemma S contains a maximal element (M, m). If $M = L$, then we are done, if not, then $\exists \alpha \in L \setminus M$ $\Rightarrow \alpha$ is alg. over K (and hence M). We know that $\overline{K}$ is alg. closed $\Rightarrow \exists \beta \in \overline{K}$ s.t. $m(\mu_\alpha^M) = \mu_\beta^{m(M)}$. By the last theorem of the previous lecture we have an extension of m $\nu: M(\alpha) \to \overline{K}$ (i.e. ν is a homomorphism). It gives us a contradiction with the maximality of (M, m). ∎

Cor. (Ex) $\sigma : K \to \overline{K}$ is a homomorphism $\Rightarrow$
# of distinct roots of μ_α^K in $\overline{K}$
= # of distinct roots of $\sigma(\mu_\alpha^K)$ in $\overline{K}$.
(Hint: use the previous theorem and injectivity)

Cor. (Ex) $\psi : L \to M$ is a homomorphism, L is alg. closed $\Rightarrow$ $\psi(L)$ is alg closed.

Thm2 Let L, M be alg. closures of K. Then $L \cong M$.

Pf. Assume $K \subseteq L$ and let $\varphi : K \to M$ $\Rightarrow$ we can extend φ to a hom. $\psi : L \to M$ (recall that L is an alg. extension of K). We have $L \cong \psi(L) \Rightarrow \psi(L)$ is also alg. closed. The only alg. extension of an alg. closed field it is the field itself $\Rightarrow \psi(L) = M$. ∎

Ex. $K - L \Rightarrow \overline{K} \cong \overline{L}$. If in addition $K \subseteq L \subseteq \overline{L}$, then $\overline{K} = \overline{L}$.

Cor. $K \subseteq L$, $g \in L[t]$ is irr. over L, $f \in K[t] \setminus \{0\}$ and $g \mid f$ in $L[t]$ $\Rightarrow \exists\, h \in K[t]$ is irr over K s.t. $h \mid f$ in $K[t]$ & $g \mid h$ in $L[t]$.

Pf. $K \subseteq L \subseteq \overline{L}$, $g(\alpha) = 0$, $\alpha \in \overline{L} \Rightarrow f(\alpha) = 0$.

Thus α is alg. over K. Put $h = \mu_\alpha^K \Rightarrow h \mid f$
We have $h(\alpha) = 0$ & $\mu_\alpha^L(\alpha) = 0 \Rightarrow \mu_\alpha^L \mid h$ (in $K[t]$)
But $g(\alpha) = 0$ and $g \in L[t]$ is irr. $\Rightarrow g = c\,\mu_\alpha^L$
$\Rightarrow g \mid h$ in $L[t]$. ∎

Thm3 If $f \in K[t] \setminus K$, and $L:K$, $M:K$ are splitting field extensions for f. Then $L \cong M$ (in particular $[L:K] = [M:K]$).

Pf. Let $K \subseteq L$, $\alpha_1, \dots, \alpha_n \in L$ be roots of f in L, $L = K(\alpha_1, \dots, \alpha_n)$ and $\varphi : K \to M$. Finally, let $K' = \varphi(K)$, $f' = \varphi(f)$, $f = c \prod_{j=1}^{n} (t - \alpha_j)$, $c \in K$.

Let $M \subseteq \overline{M} \Rightarrow \overline{M}:M$, $M:K$ are alg. extensions $\Rightarrow \overline{M}:K$ is also an alg. extension $\Rightarrow \overline{M} = \overline{K}$ (see above). Further $\varphi : K \to M \subseteq \overline{M}$ and $L:K$ is algebraic $\Rightarrow$ by Thm. 1 φ can be extended to a hom. $\psi : L \to \overline{M}$. Let $\beta_j = \psi(\alpha_j)$. Then

$$f' = \varphi(f) = \psi(f) = \psi(c) \prod_{j=1}^{n} (t - \psi(\alpha_j)) = \varphi(c) \prod_{j=1}^{n} (t - \beta_j)$$

$\Rightarrow f'$ splits over $K'(\beta_1, \dots, \beta_n)$. Since $\overline{M}[t]$ is a UFD & f' splits over $M \Rightarrow \beta_j \in M$.
We have $K' \subseteq M \Rightarrow K'(\beta_1, \dots, \beta_n) \subseteq M \Rightarrow$
$M = K'(\beta_1, \dots, \beta_n)$ (recall that M is a splitting field ext. of $f \Rightarrow M$ must be minimal).

Thus $M = K'(\beta_1, \dots, \beta_n) = \varphi(K)(\psi(\alpha_1), \dots, \psi(\alpha_n))$

$\psi(K) = \varphi(K)$

$= \psi(K)(\psi(\alpha_1), \dots, \psi(\alpha_n)) = \psi(K(\alpha_1, \dots, \alpha_n)) = \psi(L)$ ∎

Normal extensions, I

Df. An extension $L:K$ is **normal** if $L:K$ is algebraic and $\forall \alpha \in L$ $\mu_\alpha^K(x)$ factors in $L[t]$ as a product of linear factors (informally: L contains, together with each α, all its conjugates).

Exm 1) $[L:K] = 2 \Rightarrow L:K$ is normal

2) $\mathbb{R} - \mathbb{C}$ is not normal since it is not algebr.

3) $\mathbb{Q} - \mathbb{Q}(\sqrt[3]{2})$ is algebraic but not normal

L. $K - K(\alpha)$ is normal $\Leftrightarrow$ all conjugates to α are contained in $K(\alpha)$ (here α is algebraic over K)

Pf. $(\Rightarrow)$ obvious $(\Leftarrow)$ $\forall \beta \in K(\alpha) \Rightarrow \beta = f(\alpha)$, $f \in K[t] \Rightarrow$ all conjugates to β (see lecture 5) are $f(\alpha), f(\alpha_2), \dots, f(\alpha_n)$, where $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n$ are conjugates to α. Thus all these numbers are in $K(\alpha)$. ∎

4) $\mathbb{Q} - \mathbb{Q}\left(\cos\frac{2\pi}{9}\right) \ni \cos\frac{4\pi}{9} = 2\cos^2\frac{2\pi}{9} - 1$, $\cos\frac{8\pi}{9} = 2\cos^2\frac{4\pi}{9} - 1$

Thm $K:L$ is a finite ext. Then L is normal $\iff$ L is a splitting field ext. for some $f \in K[t] \setminus K$.

Pf. Any finite extension L is $K(\alpha_1, .., \alpha_n)$ for some algebraic $\alpha_1, .., \alpha_n \in L$. Consider $f = \prod \mu_{\alpha_j}^K$. If L is normal, then f splits over $L \Rightarrow K(\alpha_1, .., \alpha_n) = K(\beta_1, .., \beta_r)$, where β_j are roots of f. Thus L is a splitting field of f.

Now let $L = K(\alpha_1, .., \alpha_n)$, where $f = c\prod(t - \alpha_j) \in K[t]$. Take any $g \in K[t_1, .., t_n]$ and an element $g(\alpha_1, .., \alpha_n)$. Then all its conjugates belong to the set $\{g(\alpha_{\sigma(1)}, .., \alpha_{\sigma(n)})\}_{\sigma \in S_n} \in L$. Indeed

$$\prod_{\sigma \in S_n} (x - g(\alpha_{\sigma(1)}, .., \alpha_{\sigma(n)})) \in K[t]$$ (we used this argument several times)

and this polynomial has a root $g(\alpha_1, .., \alpha_n)$. ∎