# Problem Set 6: Math 454 Spring 2017
# Due Thursday February 23

February 7, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

**Problem 1.** Let $F$ be a field.

(a) Prove that if $P \in F[t]$ and $\deg(P) = 1$, then $P$ is irreducible.

(b) Prove that if $P \in F[t]$ and $\alpha \in F$ such that $P(\alpha) = 0$, then $P(t) = Q(t)(t-\alpha)$ for some $Q \in F[t]$.

(c) Prove that if $P \in F[t]$ and $2 \leq \deg(P) \leq 3$, then $P$ is irreducible over $F$ if and only if $P(\alpha) \neq 0$ for all $\alpha \in F$.

(d) Show that (b) is false if $\deg(P) > 3$.

**Problem 2.** Let $P_1(t) = t^2 - 2$, $P_2(t) = t^2 - 8$, and $P_3(t) = t^2 - 3$.

(a) Prove that $P_1, P_2, P_3$ are irreducible over $\mathbf{Q}$.

(b) Prove that $\mathbf{Q}(\sqrt{2})$ is a splitting field for $P_1, P_2$.

(c) Prove that $P_3$ is irreducible over $\mathbf{Q}(\sqrt{2})$.

**Problem 3.** Let $F$ be a field and $P \in F[t]$ with $\deg(P) = 2$ and $P(t) = at^2 + bt + c$ for $a, b, c \in F$.

(a) Prove that $P$ is irreducible over $F$ if and only if $b^2 - 4ac$ is not a square in $F$. That is, if $b^2 - 4ac \neq \alpha^2$ for some $\alpha \in F$.

(b) Prove that if $P_1 \in F[t]$ is irreducible with $\deg(P_1) = 2$ and $\Delta(P_1) = \alpha^2 \Delta(P)$ for some $\alpha \in F$, then $P_1$ splits over an extension $E/F$ if and only if $P$ splits over $E/F$.

(c) Prove that if $E/F$ is degree two, then there exists $\alpha \in F$ such that $\alpha$ is not a square in $F$ and $E = F(\sqrt{\alpha})$.

**Problem 4.** Let $P_1, P_2 \in F[t]$ be irreducible polynomials with $\deg(P_1) = \deg(P_2) = 2$.

(a) Prove that $P_i$ splits over $F(\sqrt{\Delta(P_i)})$ and $[F(\sqrt{\Delta(P_i)}) : F] = 2$.

(b) Prove that $P = P_1 P_2$ has a splitting field $E/F$ with $[E : F] = 2$ if and only if there exists an extension $E'/F$ with $[E' : F] = 2$ such that $E'$ is a splitting field for both $P_1, P_2$.

(c) Prove that $P_2$ splits over the splitting field of $P_1$ if and only if $\sqrt{\Delta(P_2)}$ is a square in $F(\sqrt{\Delta(P_1)})$.

**Problem 5.** Prove that $\mathrm{Aut}_{F\text{–alg}}(E)$ is a group with group operation given by composition. Specifically, prove that the composition of $F$–algebra automorphisms of $E$ is an $F$–algebra automorphism of $E$ and that the inverse function of an $F$–algebra automorphism of $E$ is also an $F$–algebra automorphism.

**Problem 6.** Let $E/F$ be an extension, $F \leq K \leq E$, and $H_K = \left\{\sigma \in \mathrm{Aut}_{F\text{–alg}}(E) \;:\; \sigma(\beta) = \beta \text{ for all } \beta \in K\right\}$.

(a) Prove that if $\sigma \in H_K$, then $\sigma$ is a $K$–algebra automorphism of $E$. In particular, we have a group homomorphism $\psi\colon H_K \to \mathrm{Aut}_{K\text{–alg}}(E)$.

(b) Prove that $\psi$ is injective.

(c) Prove that $\psi$ is surjective.