1 Final remarks I

Definition 1 (Sylvester matrix)**.** Let $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ be two polynomials in $\mathbb{K}[x]$. The *Sylvester matrix* of f and g, denoted $S(f,g)$, is the $(m+n) \times (m+n)$ matrix whose first n rows are the coefficients of f shifted right, and whose last m rows are the coefficients of g shifted right. Concretely,

$$S(f,g) = \begin{pmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & b_{n-1} & \cdots & b_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_0 \end{pmatrix}.$$

Definition 2 (Resultant)**.** The *resultant* of f and g is

$$R(f,g) = \det\left(S(f,g)\right).$$

Equivalently, if $\alpha_1, \ldots, \alpha_m$ are the roots of f in an algebraic closure of $\mathbb{K}$, then

$$R(f,g) = a_m^n \prod_{i=1}^{m} g(\alpha_i).$$

Theorem 1.1. Let α_i be roots of f and β_j be roots of g. Then

$$\begin{aligned} R(f,g) &= a_0^m b_0^n \prod_i (\alpha_i - \beta_j) \\ &= a_0^m \prod_i g(\alpha_i) = b_0^n \prod_i f(\beta_i) \end{aligned}$$

Corollary 1.

1. $R(f,g) = (-1)^{\deg f \cdot \deg g} R(g,f)$
2. If $f = gq + r \implies R(f,g) = b_0^{\deg f - \deg R} R(r,g)$
3. $R(f,gh) = R(f,g)R(f,h)$

Corollary 2. Let $f(t) = a_0t^n + \cdots + a_n,\ a_0 \neq 0$. Then $R(f,f') = (-1)^{\frac{n(n-1)}{2}} \prod_{i<j}(\alpha_i - \alpha_j)^2$