Problem 1. True or False [3+3+3+3+3+3=18 points]

(a) Let $f \in \mathbb{Z}[t]$ be a polynomial, every root of which has multiplicity 2024. Then f is not separable over $\mathbb{Q}$.

(b) If $L : K$ is an algebraic extension of fields with $K \subseteq L$, then the algebraic closure $\overline{L}$ of L is isomorphic to the algebraic closure $\overline{K}$ of K.

(c) Every algebraic extension of $\mathbb{Q}$ is separable.

(d) Suppose that K and L are fields with $K \subseteq L$, and L is algebraically closed. Then the field extension $L : K$ is normal.

(e) Suppose that $L : M$ and $M : K$ are field extensions with $L : K$ normal. Then $L : M$ is a normal field extension.

(f) Let $f \in \mathbb{Z}[x]$ be a polynomial having prime degree p, and let θ be any root of f in a splitting field extension for f over $\mathbb{Q}$. Then $[\mathbb{Q}(\theta) : \mathbb{Q}] = p$.

Problem 2. [3+3+3+3=12 points]

(a) Define what it means for a field extension $L : K$ to be a splitting field extension.

(b) Define what it means for a field extension $L : K$ to be normal.

(c) Let $L : K$ be a field extension. Define what it means for an element $\alpha \in L$ to be separable over K.

(d) Define what it means for a field extension $L : K$ to be separable.

Problem 3. [8+8+8=24 points]

This question concerns the polynomial $f(t) = t^4 - (t+1)^2 \in \mathbb{Q}[t]$.

(a) Find a splitting field extension $L : \mathbb{Q}$ for f, justifying your answer.

(b) Determine the degree of your splitting field extension $L : \mathbb{Q}$, justifying your answer.

(c) Determine the subgroup of S_4 to which $\mathrm{Gal}(L : \mathbb{Q})$ is isomorphic.

Problem 4. [14 points]
Suppose that $L : K$ is a splitting field extension for the polynomial $f \in K[t] \setminus K$. Prove that $[L : K]$ divides $(\deg f)!$.

Problem 5. [7+7=14 points]

(a) Suppose that M is an algebraically closed field. Show that all polynomials in $M[t]$ are separable.

(b) Suppose that p is a prime number and t is an indeterminate, and let $L = \overline{\mathbb{F}}_p(t)$, where $\overline{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$. Are all polynomials in $L[X]$ separable? Justify your answer.

Problem 6. [8+8=16 points]

Throughout, let f denote the polynomial $t^5 - 9t - 3 \in \mathbb{Q}[t]$, let L be a splitting field for f over $\mathbb{Q}$, and let M be a field with $\mathbb{Q} \subsetneq M \subsetneq L$ (that is, a field strictly intermediate between $\mathbb{Q}$ and L).

(a) Show that, for any $\sigma \in \mathrm{Gal}(L : \mathbb{Q})$, and for any $\alpha \in M$, the polynomial $\sigma(m_\alpha(\mathbb{Q}))$ is monic and irreducible over $\mathbb{Q}$. Here $m_\alpha(\mathbb{Q})$ denotes the minimal polynomial of α over $\mathbb{Q}$.

(b) Suppose that $M : \mathbb{Q}$ is normal and that f factors as a product of monic irreducibles $f_1, \ldots, f_r$ (of positive degree) over $M[t]$. Show that $\deg(f_i) = \deg(f_1)$ for each i.

(c) Show that if $M : \mathbb{Q}$ is normal, then f remains irreducible over M.