Exercise 4.1. For each of the following polynomials, construct a splitting field L over $\mathbb{Q}$ and compute the degree $[L : \mathbb{Q}]$

1. $t^4 + 7t^2 + 12$

   Solution. We notice $f(t) = t^4 + 7t^2 + 12 = (t^2 + 3)(t^2 + 4)$, so let $g(t) = t^2 + 3$ and $h(t) = t^2 + 4$. We have that $h = (t - 2i)(t + 2i)$, and by the rational root test h is irreducible over $\mathbb{Q}$. Then $h = \mu_{2i}^{\mathbb{Q}}$ and $\mathbb{Q}(i) = M : \mathbb{Q}$ is the splitting field extension for h with degree 2. Next, we have $g = (t - i\sqrt{3})(t + i\sqrt{3})$, and by Eisenstein's criterion with $p = 3$, g is irreducible. Let $L : M$ be the splitting field extension for g. We already have that $i \in M$, so $L = M(\sqrt{3})$ and $[L : M] = 2$. Thus, $L = \mathbb{Q}(i, \sqrt{3}) : \mathbb{Q}$ is the splitting field extension for f and by the Tower Law, $[L : \mathbb{Q}] = [L : M][M : \mathbb{Q}] = 2 \cdot 2 = 4$. $\square$

2. $t^4 + t^2 + 12$

   Solution. We notice $f(t) = t^4 + t^2 + 12 = (t^2 - 3)(t^2 + 4)$, so let $g(t) = t^2 - 3$ and $h(t) = t^2 + 4$. These are the same polynomials as in part 1, but this time the roots of g do not have an imaginary factor. However, we note that this did not have any impact on our argument in part 1, whence $L = \mathbb{Q}(i, \sqrt{3}) : \mathbb{Q}$ is again the splitting field extension for f and $[L : \mathbb{Q}] = 4$. $\square$

3. $t^{2n} - 2^n$, where $n = 3, 4$.

   Solution. $(n = 3)$ We have that $f(t) = t^6 - 2^3$. Then, $t = (2^3)^{1/6}$ and $t = \sqrt{2} \cdot \varepsilon_6^k$, where $k \in \mathbb{Z}_6$ and $\varepsilon_6 = \exp(i\frac{2\pi}{6}) = \exp(i\frac{\pi}{3})$. We know the minimum polynomial of ε_6 over $\mathbb{Q}$ is the sixth cyclotomic polynomial Φ_6, which has degree $\phi(6) = \phi(2)\phi(3) = 2$. Thus $[\mathbb{Q}(\varepsilon_6) : \mathbb{Q}] = 2$. Now, $\sqrt{2} \notin \mathbb{Q}(\varepsilon_6)$, so $L = \mathbb{Q}(\sqrt{2}, \varepsilon_6) = \mathbb{Q}(\varepsilon_6)(\sqrt{2})$. Trivially, the minimum polynomial of $\sqrt{2}$ has degree 2, hence $[L : \mathbb{Q}(\varepsilon_6)] = 2$ and by the Tower Law, $[L : \mathbb{Q}] = [L : \mathbb{Q}(\varepsilon_6)][\mathbb{Q}(\varepsilon_6) : \mathbb{Q}] = 2 \cdot 2 = 4$.

   $(n = 4)$ We have that $f(t) = t^8 - 2^4$. Then, $t = (2^4)^{1/8}$ and $t = \sqrt{2} \cdot \varepsilon_8^k$, where $k \in \mathbb{Z}_8$ and $\varepsilon_8 = \exp(i\frac{2\pi}{8}) = \exp(i\frac{\pi}{4})$. We know the minimum polynomial of ε_8 over $\mathbb{Q}$ is the eighth cyclotomic polynomial Φ_8, which has degree $\phi(8) = 2^3 - 2^2 = 4$. Thus $[\mathbb{Q}(\varepsilon_8) : \mathbb{Q}] = 4$. Now, notice $\varepsilon_8 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ and $\varepsilon_8^{-1} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. So, $\varepsilon_8 + \varepsilon_8^{-1} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \in \mathbb{Q}(\varepsilon_8)$. Thus $L = \mathbb{Q}(\varepsilon_8)$ is the splitting field for f over $\mathbb{Q}$ and $[L : \mathbb{Q}] = 4$. $\square$

4. $t^{14} - 1$

   Solution. Obviously, $f(t) = t^{14} - 1$ has root ε_{14}^k for $k \in \mathbb{Z}_{14}$, so $L = \mathbb{Q}(\varepsilon_{14})$ The minimum polynomial for ε_{14} is Φ_{14}, whence $[L : \mathbb{Q}] = \phi(14) = \phi(7)\phi(2) = 6$. $\square$

Exercise 4.2. Let $K - L - M$ be a field extension and $K - L$, $L - M$ are algebraic extensions. Prove that $K - M$ is also an algebraic extension.

Solution. Suppose $k \in K$ with $\mu_k^L = x^n + \ell_{n-1}x^{n-1} + \cdots + \ell_0$ for $\ell_i \in L$. Then by definition, k is algebraic over $M(\ell_0, \ldots, \ell_{n-1})$. By theorem, we know that for some field extension $F_1 : F_2$, $\alpha \in F_1$ is algebraic over F_2 $\iff [F_2(\alpha) : F_2] < \infty$. Thus, we have that

$$[M(\ell_0, \ldots, \ell_{n-1})(k) : M(\ell_0, \ldots, \ell_{n-1})] = [\underbrace{M(\ell_0, \ldots, \ell_{n-1}, k)}_{M''} : \underbrace{M(\ell_0, \ldots, \ell_{n-1})}_{M'}] < \infty.$$

Using a corollary from lecture, we also know that $[M' : M] < \infty$. Then by the Tower Law, we have $[M'' : M] = [M'' : M'][M' : M] < \infty$. From a result in homework 2, any finite extension is necessarily algebraic. Thus, k is algebraic over M for arbitrary $k \in K \implies K - M$ is algebraic. $\square$

Exercise 4.3. Let α be transcendental over a field $K \subset \mathbb{C}$. Show that $K(\alpha)$ is not algebraically closed (hint: consider the polynomial $t^2 - \alpha$).

Solution. Consider the polynomial $t^2 - \alpha \in K(\alpha)$. Assume ad absurdum that f is reducible over $K(\alpha)$. Then, there is some $\beta \in K(\alpha)$ such that $\beta^2 = \alpha$. By definition of $K(\alpha)$, we have that $\beta = \frac{g(\alpha)}{h(\alpha)}$ for some $g(t), h(t) \in K[t]$ such that $h(\alpha) \neq 0$. Thus, $\beta^2 = \frac{g(\alpha)^2}{h(\alpha)^2} = \alpha$ and $g(\alpha)^2 - \alpha h(\alpha)^2 = 0$.

Claim 1. *$g(x)^2 - xh(x)^2$ is a nontrivial polynomial in $K[x]$*

Proof. Assume ad absurdum that $g(x)^2 - xh(x) \equiv 0$ (the trivial polynomial). Now, let $m = \deg(g)$ and $n = \deg(h)$. By definition, m and n must be integers. Obviously, $\deg(g^2) = 2m$ and $\deg(h^2) = 2n$, so $\deg(xh^2) = 2n+1$. If we let $g(x)^2 - xh(x)^2 = 0$, then $g(x)^2 = xh(x)^2$. However, $\deg(g(x)^2) = \deg(xh(x)^2) \iff 2m = 2n+1$, but $2m$ is even and $2n+1$ is odd, an obvious contradiction. Thus $g(x)^2 - xh(x)^2 \in K[x]$ must be nontrivial. ■

Notice that the claim above contradicts that α is transcendental over K by definition. Thus $K(\alpha)$ is not algebraically closed. □

Exercise 4.4. Let $L : K$ be a splitting field extension for a non-constant polynomial $f \in K[t]$. Prove that $[L : K]$ divides $(\deg f)!$ (hint: at the very end look at some binomial coefficients).

Solution. We prove this statement by induction on $\deg f = n$. The base case, $\deg f = n = 1$ is trivial, there is nothing to prove. Now assume we have shown $[L : K] \,\big|\, (\deg f)!$ for all $1 \leq \deg f < n$. We have two cases, one in which f is reducible and one in which it is not.

Case 1 (f is irreducible). Let α be a root of f. Then f factors as $(t - \alpha)g$ for some polynomial $g \in K(\alpha)[t]$ such that $\deg g = n - 1$. Furthermore, we can see that $L : K(\alpha)$ is a splitting field extension for g. By our inductive hypothesis, $[L : K(\alpha)]$ divides $(n-1)!$. By lemma, the irreducibility of f implies $f = \lambda \mu_\alpha^K$ for some constant $\lambda \in K$, so $\deg f = \deg \mu_\alpha^K = n$ and $[K(\alpha) : K] = n$. Then by using the Tower Law, we can see that $[L : K(\alpha)][K(\alpha) : K] = [L : K]$ divides $n(n-1)! = n!$. ■

Case 2 (f is reducible). Let $f = gh$ for polynomials $g, h \in K[t]$, $M : K$ be a splitting field extension for g, and $L : M$ be a splitting field extension for h. Then we can say $\deg g = d$ for some $1 \leq d < n$ by def degree, hence $\deg h = n - d$. By our inductive hypothesis, $[M : K]$ divides $d!$ and $[L : M]$ divides $(n-d)!$. Then by use of the Tower Law again, we can easily see that $[L : M][M : K] = [L : K]$ divides $d!(n-d)!$. Notice that $\frac{n!}{d!(n-d)!}$ is exactly the binomial coefficient $\binom{n}{d}$, which we know to be an integer for integers n, d. Thus we can say that $d!(n-d)!$ divides $n!$ so finally we can conclude that, $[L : M]$ divides $n!$. ■

Since our inductive hypothesis holds in both cases for reducible and irreducible f, we may conclude that it holds for all $\deg f \geq 1$ and hence any non-constant polynomial $f \in K[t]$. □