

1 Fixed Fields

Definition 1 (Fixed field). Let $L : K$ be a field extension and $G \leq \text{Aut}(L)$. Then the fixed field of G is

$$\text{Fix}_L(G) = L^G = \{\alpha \in L : g\alpha = \alpha \ \forall g \in G\}$$

Definition 2 (Galois Extension). Let $L : K$ be a field extension. Then $L : K$ is a Galois extension if it is normal and separable.

Lemma 1.1. Let $K, M \subseteq L$ be fields and $G, H \leq \text{Aut}(L)$. Then

- 1) if $K \subseteq M$, then $\text{Gal}(L : K) \geq \text{Gal}(L : M)$;
- 2) if $G \leq H$, then $\text{Fix}_L(G) \subseteq \text{Fix}_L(H)$;
- 3) $K \subseteq \text{Fix}_L(\text{Gal}(L : K))$;
- 4) $G \leq \text{Gal}(L : \text{Fix}_L(G))$;
- 5) $\text{Gal}(L : K) = \text{Gal}(L : \text{Fix}_L(\text{Gal}(L : K)))$;
- 6) $\text{Fix}_L(G) = \text{Fix}_L(\text{Gal}(L : \text{Fix}_L(G)))$.

Theorem 1.2. Let $L : K$ be algebraic. Then $L : K$ is Galois $\iff K = \text{Fix}_L(\text{Gal}_K(L))$

Theorem 1.3. Suppose that L is a field, $G \leq \text{Aut}(L)$ such that $|G| < \infty$, and put $K = \text{Fix}_L(G)$. Then $L : K$ is a finite Galois extension with $[L : K] = |\text{Gal}(L : K)|$, and furthermore $G = \text{Gal}_K(L)$.

Theorem 1.4. Let $L : K$ be finite.

1. If $L : K$ is a Galois extension, then $|\text{Gal}(L : K)| = [L : K]$ and $K = \text{Fix}_L(\text{Gal}(L : K))$.
2. If $L : K$ is not Galois, then $|\text{Gal}(L : K)| < [L : K]$ and K is a proper subfield of $\text{Fix}_L(\text{Gal}(L : K))$.

Corollary 1. Let $L : M : K$ be a tower such that $L : K$ is Galois. Then $L : M$ is Galois.

Proposition 1. Let $f \in K[t] \setminus K$ be separable. Then $\text{Gal}_K(f) \leq A_n \iff \sqrt{D} \in K$