PURDUE UNIVERSITY
Department of Mathematics

GALOIS THEORY HONORS, MA 45401

Homework 1 (Jan 16 – Jan 24).

1 (10+10) 1) Using Vieta's trigonometric method, solve $x^3 - 3x + 1 = 0$.
2) Applying the cube of sum formula, solve $x^3 - 3 \cdot 2^{1/3} x - 3 = 0$.

2 (10) Let x_1, x_2, x_3 be the roots of the cubic $x^3 + ax^2 + bx + c = 0$. Compute $x_1^2 + x_2^2 + x_3^2 + x_1^{-1} + x_2^{-1} + x_3^{-1}$.

3 (10) Prove that the stabilizer of the polynomial $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$ is D_5, that is the subgroup of permutations $g \in S_5$ of the form $g : \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ and $gx = \pm x + b$, where $b \in \mathbb{Z}/5\mathbb{Z}$.

4 (5+5) Let $H \le S_n$ be a subgroup and K be a field. Take any $f \in K[x_1, \dots, x_n]$ and form

$$F = F(f) = \sum_{h \in H} f(x_{h(1)}, \dots, x_{h(n)}) := \sum_{h \in H} h \cdot f \,,$$

where $h \cdot f$ and the natural action of S_n on $K[x_1, \dots, x_n]$ (i.e. $(h \cdot f)(x_1, \dots, x_n) := f(x_{h(1)}, \dots, x_{h(n)})$).
1) Prove that for any $h \in H$ one has $h \cdot F = F$.
2) Take $f = x_1 x_2^2 \dots x_n^n$ and prove that $h \cdot F = F$ iff $h \in H$.
3) *(for enthusiasts, does not affect the rating)* Is the second part true for any f?

5 (5+5+15) A complex polynomial $f(x_1, \dots, x_n)$ is called *skew–symmetric* if $h \cdot f = -f$ for any transposition h.
1) Prove that the ratio of any skew–symmetric polynomials is a symmetric rational function.
2) Let $D = D(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)^2$ be the discriminant and $\Delta = \Delta(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)$, $\Delta^2 = D$. Prove that Δ is a skew–symmetric polynomial.
3) Prove that any symmetric polynomial f is a product of Δ and another symmetric polynomial g.