Problem Set 7: Math 454 Spring 2017
Due Thursday March 30

March 28, 2017

Do the problems below. Please write neatly, especially your name! Show all your work and justify all your steps. Write in complete, coherent sentences. I expect and openly encourage you to collaborate on this problem set. I will insist that you list your collaborators on the handed in solutions (list them on the top of your first page).

Problem 1. Given a pair of fields F, F' and an isomorphism of fields $\psi \colon F \to F'$, we define $\psi_* \colon F[t] \to F'[t]$ by

$$\psi_*\left(\sum_{i=0}^{m} \alpha_i t^i\right) = \sum_{i=0}^{m} \psi(\alpha_i) t^i.$$

(a) Prove that ψ_* is an isomorphism of F–algebras.

(b) Identifying $F, F' \leq F[t], F'[t]$ as the field of constant functions, prove that the restriction of ψ_* to F is equal to ψ. That is

$$\begin{tikzcd} F[t] \arrow[r, "\psi_*"] & F'[t] \\ F \arrow[u, "\iota_F"] \arrow[r, "\psi"'] & F' \arrow[u, "\iota_{F'}"'] \end{tikzcd}$$

is a commutative diagram where $\iota_F, \iota_{F'}$ are isomorphisms with the fields of constant polynomials.

(c) Prove that if $P \in F[t]$ is irreducible, then $\psi_*(P) \in F'[t]$ is irreducible.

(d) Given $P \in F[t]$ with $Q = \psi_*(P)$, prove that there exists a ring isomorphism $\overline{\psi} \colon F[t]/\langle P \rangle \to F'[t]/\langle Q \rangle$ such that the diagram

$$\begin{tikzcd} F[t] \arrow[r, "\psi_*"] \arrow[d, "\psi_P"'] & F'[t] \arrow[d, "\psi_Q"] \\ F[t]/\langle P \rangle \arrow[r, "\overline{\psi}"'] & F'[t]/\langle Q \rangle \end{tikzcd}$$

where ψ_P and ψ_Q are the canonical homomorphisms (i.e. quotient homomorphisms).

(e) Let $P \in F[t]$ be irreducible, $Q = \psi_*(P) \in F'[t]$, $E = F[t]/\langle P \rangle$ and $E' = F'[t]/\langle Q \rangle$. Prove that there exists a field isomorphism $\widetilde{\psi} \colon E \to E'$ such that $\widetilde{\psi}$ restricted to $F \leq E$ is ψ. In particular, we have the commutative diagram

$$\begin{tikzcd} & F[t] \arrow[r, "\psi_*"] \arrow[d, "\psi_P"] & F'[t] \arrow[d, "\psi_Q"'] & \\ & E \arrow[r, "\overline{\psi}"'] & E' & \\ F \arrow[uur, "\iota_F"] \arrow[ur] \arrow[rrr, "\psi"'] & & & F' \arrow[uul, "\iota_{F'}"'] \arrow[ul] \end{tikzcd}$$

Problem 2. Prove that if $\{H_i\}$ is an arbitrary collection of subgroups in $\mathscr{L}_{\text{sub}}^{\text{closed}}(E/F)$, then $\cap H_i \in \mathscr{L}_{\text{sub}}^{\text{closed}}(E/F)$.

Problem 3. Prove that if $\{K_i\}$ is an arbitrary collection of subfields in $\mathscr{L}_{\text{int}}^{\text{closed}}(E/F)$, then $\cap K_i \in \mathscr{L}_{\text{int}}^{\text{closed}}(E/F)$.