1 Soluble Groups II

Theorem 1.1 (Theorem - Definition)**.** Let G be a group. Then the following are equivalent:

0. G is a (finite) soluble group;
1. There exists some $n \in \mathbb{Z}^+$ such that $G^{(n)} = \{e\}$;
2. There exists a normal series

$$\{Id.\} = G_n \leqslant G_{n-1} \leqslant \cdots \leqslant G_1 \leqslant G_0 = G$$

   such that $G_j \lhd G$ and all quotients G_{j-1}/G_j are abelian;
3. There exists a subnormal series such that quotients G_{j-1}/G_j are abelian.

Definition 1 (Derived group)**.** Let G be a group. Then the *derivative of* G is $G' = \langle [x, y] : x, y \in G\rangle = [G, G]$ where $[x, y] = xyx^{-1}y^{-1}$ is the *commutator* of x and y, and $(G')' = G''$.

Exercise 1. $G' \trianglelefteq G$, G/G' abelian

Exercise 2. G' is a minimal normal subgroup of G such that G/G' is abelian.

Definition 2. The *derived series* of G is $G^{(n)} = \left(G^{(n-1)}\right)'$ and $\{\text{Id.}\} = G^{(n)} \lhd G^{(n-1)} \lhd \cdots \lhd G' \lhd G$ (not to be confused with $G_{n+1} = [G_n, G]$, the *lower central series*).

Lemma 1.2. Let $\varphi : G \mapsto H$ be an epimorphism. Then $\varphi(G') = H'$.

Definition 3 (Composition series)**.** Let G be a group. Then a *composition series* of G is a subnormal series of finite length

$$\{Id.\} = G_0 \lhd G_1 \lhd \cdots \lhd G_{\ell-1} \lhd G_\ell = G$$

such that G_j/G_{j-1} is a simple group for all j.

Theorem 1.3 (Jordan-Hölder)**.** Any 2 composition series of some group G are equivalent up to permutation and isomorphism.

Theorem 1.4. Let K be a field with $\operatorname{char} K \neq 2$ and let $f \in K[t]$ be a separable polynomial with splitting field L. Then $f = 0$ is solvable by quadratic radicals $\iff [L : K] = 2^t$.