PURDUE UNIVERSITY

Department of Mathematics

GALOIS THEORY HONORS, MA 45401

09 May 2025 120 minutes

This paper contains **SIX** *questions worth a total of 175 points.*

Final examination

Calculators, textbooks, notes and cribsheets are **not** *permitted in this examination.*

Do not turn over until instructed.

1 (5+5+5+5+5+5=30) Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with “T”, and those which are false with “F”.

(a) The cardinality of the Galois group of any finite extension $K - L$ does not exceed $[L : K]$.
(b) Every finite normal extension is Galois.
(c) It is possible to construct by ruler and compass the number $2^{1/4} + 2^{1/3}$.
(d) Any subgroup of a soluble group is soluble.
(e) Any cyclotomic polynomial has integer coefficients.
(f) If a group acts transitively on the roots $f \in K[t]$, then f is irreducible over K.

2 (5+5+10+15=35) *a*) Formulate the Tower Law.

b) Is it true that the degree of the product of two algebraic $\alpha, \beta \in \mathbb{R}$ does not exceed the product of its degrees? Justify your answer.

c) Prove that $\sqrt{7}$ does not belong to $\mathbb{Q}(3^{1/3}, \varepsilon_3)$, where ε_3 is a primitive root of unity of order 3.

d) Compute algebraic conjugates of $i3^{1/6}$ over $\mathbb{Q}$, then over $\mathbb{Q}(i\sqrt{3})$ and, finally, over $\mathbb{Q}(3^{1/3})$.

3 (5+10+10+15=40) *a*) Let $L : K$ be an extension, $G = \mathrm{Aut}_K L$ and H be a subgroup of G. Define the fixed field of H.

b) Formulate the Fundamental Theorem of Galois theory.

c) Let L be the splitting field of $t^3 - a \in \mathbb{Q}$, where a is a positive integer and a is not a cube. Find all of the subfields of L.

d) Draw the lattice of subfields and corresponding lattice of subgroups of $\mathrm{Gal}_{\mathbb{Q}}(L)$.

4 (5+5+5+5+15=35) *a*) Define what it means for a polynomial $f \in K[t]$ to be solvable by radicals.

b) Formulate the criterion for solvability by radicals of $f \in K[t]$ in terms of $\mathrm{Gal}_K(f)$.

c) Is it true that any equation $f(t) = 0$, where $f \in \mathbb{Q}[t]$ is an irreducible polynomial of degree at least five, is not solvable by radicals?

d) Is the polynomial $t^6 - 10t^2 + 1$ solvable by radicals over $\mathbb{Q}$?

e) Is the polynomial $t^5 - 9t^4 + 3$ solvable by radicals over $\mathbb{Q}$?

5 (5+5=10) *a*) Define what it means for a group $\mathbf{G}$ to be soluble.

b) Is $\mathrm{Gal}_{\mathbb{Q}}(t^n - a)$ soluble?

6 (5+20=25) *a*) Let $\mathbb{F}_p \subseteq K$ be a field, $\mathrm{char}(K) = p > 0$. Define the Frobenius automorphism Φ and show that Φ is a linear map over $\mathbb{F}_p$.

b) Find $\mathrm{Gal}_{\mathbb{F}_2(t^3)}(\mathbb{F}_4(t))$.