Problem 1. True or False

(a) There is a field isomorphism $\varphi : \mathbb{Q}(\sqrt{-5}) \to \mathbb{Q}(\sqrt{5})$.

(b) There is a homomorphism of finite fields $\psi : \mathbb{F}_3 \to \mathbb{F}_{37}$.

(c) If $L : K$ is a field extension, and $\alpha, \beta \in L$ are distinct elements with the same minimal polynomial over K, then $K(\alpha)$ and $K(\beta)$ are isomorphic fields.

(d) It is not possible to construct, using compass and straightedge in the usual way, a length whose 14th power is twice a given length.

(e) The polynomial $x^{36} + x^{35} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

(f) If K is a field and α is an element of an extension field L of K, then every element of $K(\alpha)$ can be expressed as a polynomial in α with coefficients in K.

Problem 2.

(a) Let $\varphi_1 : K_1 \to L_1$, $\varphi_2 : K_2 \to L_2$ be embeddings and $\sigma : K_1 \to K_2$, $\tau : L_1 \to L_2$ isomorphisms. Define what it means for τ to *extend* σ.

(b) Let $L : M : K$ be a tower of field extensions. Define what it means for $\sigma : M \to L$ to be a *K-homomorphism*.

(c) Define what is meant by the *degree* of a field extension $L : K$.

(d) Define what is meant by the *minimal polynomial* of an algebraic element $\alpha \in L$ over K.

Problem 3. Let $L : K$ be a field extension. Suppose that $\alpha \in L$ is algebraic over K and $\beta \in L$ is transcendental over K. Suppose also that $\alpha \notin K$. Show that $K(\alpha, \beta) : K$ is not a simple field extension.

Problem 4. Let $\theta = \sqrt{3 + 3\sqrt[3]{6}}$, and write $L = \mathbb{Q}(\theta)$.

(a) Calculate the minimal polynomial of θ over $\mathbb{Q}$, and hence determine the degree of $L : \mathbb{Q}$.

(b) Let $f \in \mathbb{Q}[t]$ be a monic polynomial of degree 4. Suppose $\alpha \in L$ satisfies $f(\alpha) = 0$. Is it possible for f to be irreducible over $\mathbb{Q}$? Justify your answer.

(c) Suppose $\beta, \gamma \in \mathbb{C}$ with $\beta + \gamma \in \mathbb{Q}^{\mathrm{alg}}$ and $\beta\gamma \in \mathbb{Q}^{\mathrm{alg}}$. Prove that β and γ are algebraic over $\mathbb{Q}$.

Problem 5. Let $L : \mathbb{Q}$ be an algebraic extension, and let $\varphi : L \to L$ be a field homomorphism.

(a) Show that φ is a $\mathbb{Q}$-homomorphism.

(b) Suppose $\alpha \in L$. Show that the minimal polynomial of α over $\mathbb{Q}$ has $\varphi^n(\alpha)$ as a root for each $n \geq 0$.

(c) Let $\alpha \in L$. Show that there is a positive integer d with the property that $\varphi^d(\alpha) = \alpha$. Moreover, putting $\beta = \alpha + \varphi(\alpha) + \cdots + \varphi^{d-1}(\alpha)$, with d taken to be the smallest such non-negative integer, show that φ is a $\mathbb{Q}(\beta)$-homomorphism of L.

Problem 6. With t an indeterminate, let $f \in \mathbb{Z}[t]$ be a polynomial of degree $n \geq 1$, and put $K = \mathbb{Q}(f)$.

(a) Find a polynomial $F \in K[X]$ with $F(t) = 0$, and deduce that $\mathbb{Q}(t) : K$ is algebraic of degree at most n.

(b) Let $g \in \mathbb{Z}[t]$, $g \neq f$. Show there exists a non-zero polynomial $H(X, Y) \in \mathbb{Z}[X, Y]$ with $H(f(t), g(t)) = 0$.