Exercise 5.1. Which of the following field extensions are normal? Justify your answers.

1. $\mathbb{Q}(i) : \mathbb{Q}$

   Solution. Normal. By theorem, we know that any finite extension $L : K$ is normal $\iff$ L is a splitting field extension for some non-constant $f \in K[t]$. Hence, since $\mathbb{Q}(i)$ is the splitting field for $t^2 + 1$ over $\mathbb{Q}$, the extension $\mathbb{Q}(i) : \mathbb{Q}$ is normal. ☐

2. $\mathbb{Q}(2^{1/4}) : \mathbb{Q}$

   Solution. Not normal. By definition, an extension $L : K$ is normal if $\forall \alpha \in L$, the minimum polynomial of α over K, $\mu_\alpha^K(t)$, splits over $L[t]$. Obviously, $\sqrt[4]{2} \in \mathbb{Q}(2^{1/4})$ by construction. However, notice that for $\alpha = \sqrt[4]{2}$,

$$\begin{aligned}\mu_\alpha^{\mathbb{Q}}(t) &= t^4 - 2\\ &= (t^2 + \sqrt{2})(t^2 - \sqrt{2})\\ &= (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2}),\end{aligned}$$

   but the linear factors $(t + i\sqrt[4]{2})$ and $(t - i\sqrt[4]{2})$ are not in $\mathbb{Q}(2^{1/4})[t]$. Hence, the extension $\mathbb{Q}(2^{1/4}) : \mathbb{Q}$ is not normal by definition. ☐

3. $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$

   Solution. Normal. Consider the polynomial $f(t) = (t^4 - 2)(t^2 - 1) \in \mathbb{Q}(2^{1/4}, i)[t]$. Then,

$$f(t) = (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2})(t + i)(t - i),$$

   whence $\mathbb{Q}(2^{1/4}, i) : \mathbb{Q}$ is a splitting field extension for f. By applying the same theorem as in part 1, this extension is normal. ☐

4. $\mathbb{Q}(2^{1/4}, i, \sqrt{5}) : \mathbb{Q}$

   Solution. Normal. Consider the polynomial $f(t) = (t^4 - 2)(t^2 - 1)(t^2 - 5) \in \mathbb{Q}(2^{1/4}, i, \sqrt{5})[t]$. Then,

$$f(t) = (t + i\sqrt[4]{2})(t - i\sqrt[4]{2})(t + \sqrt[4]{2})(t - \sqrt[4]{2})(t + i)(t - i)(t - \sqrt{5})(t + \sqrt{5}),$$

   whence $\mathbb{Q}(2^{1/4}, i, \sqrt{5}) : \mathbb{Q}$ is a splitting field extension for f. By applying the same theorem as in part 1, this extension is normal. ☐

5. $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$

   Solution. Normal. Consider the polynomial $f(t) = (t^2 - 3)(t^3 - 3)$. Then,

$$f(t) = (t + \sqrt{3})(t - \sqrt{3})(t - \sqrt[3]{3})(t - \varepsilon_3\sqrt[3]{3})(t - \varepsilon_3^2\sqrt[3]{3}),$$

   where $\varepsilon_3 = \exp\left(\frac{2\pi}{3}i\right)$. Notice,

$$\begin{aligned}\varepsilon_3 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}\\ &= \frac{1}{2}\left(-1 + i\sqrt{3}\right) \in \mathbb{Q}(3^{1/3}, i, \sqrt{3})\end{aligned} \qquad \begin{aligned}\varepsilon_3^2 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \in \mathbb{Q}(3^{1/3}, i, \sqrt{3})\\ &= \frac{1}{2}\left(-1 - i\sqrt{3}\right) \in \mathbb{Q}(3^{1/3}, i, \sqrt{3}).\end{aligned}$$

   Thus $\mathbb{Q}(3^{1/3}, i, \sqrt{3}) : \mathbb{Q}$ is a splitting field extension for f, whence must be normal by the same theorem as part 1. ☐

Exercise 5.2. Let $\psi : L \to M$ be a homomorphism, suppose that L is algebraically closed. Prove that $\psi(L)$ is algebraically closed.

Solution. Let $g \in \psi(L)[t]$ be some irreducible polynomial over $\psi(L)$. Then, we have some $f \in L[t]$ such that $g = \psi f$ with $\deg g = \deg f$. Now, assume ad absurdum that g has a degree greater than 1. Then $\deg f > 1$. By algebraic closure of L, any irreducible polynomials must be linear. Since $\deg f \neq 1$, f must be reducible and thus $f = h\ell$ for some $h, \ell \in L[t]$ such that $\deg h \geq 1$ and $\deg \ell \geq 1$. Since ψ must preserve operations, this implies that $\exists \hat{h}, \hat{\ell} \in \psi(L)$ such that $g = \hat{h}\hat{\ell}$, where $\deg \hat{h} \geq 1$ and $\deg \hat{\ell} \geq 1$. However, this contradicts the fact that g is irreducible, so our assumption that $\deg g > 1$ must be false and hence $\deg g = 1$. Therefore, $\psi(L)$ is algebraically closed. □

Exercise 5.3. Let $L : K$ be a field extension. Then $\overline{K}$ is isomorphic to $\overline{L}$. In addition, if $K \subset L \subseteq \overline{L}$, then $\overline{K} = \overline{L}$.

Solution. Let $\varphi_1 : K \to L$ and $\varphi_2 : L \to \overline{L}$ be the monomorphisms corresponding to the field extensions $L : K$ and $\overline{L} : L$, respectively. Then $\overline{L} : K$ is the field extension relative to the composition $\varphi_2 \circ \varphi_1$. By algebraic closure of $\overline{L}$, it must be an algebraic closure for K. By theorem from lecture, we know that any two algebraic closures for the same field must be isomorphic to one another. Thus, $\overline{L} \cong \overline{K}$.

Assume ad absurdum that we have some algebraic closure $\overline{K}$ of K such that $\left|\overline{K}\right| < \left|\overline{L}\right|$. By definition of algebraic closure, we know that $\overline{K}$ and $\overline{L}$ are both algebraic extensions of K, so $K \subseteq \overline{K}$ and $K \subseteq \overline{L}$. Let $\psi : K \hookrightarrow \overline{L}$ be the monomorphism corresponding to the extension $\overline{L} : K$. By theorem, since $\overline{K} : K$ is an algebraic extension, there exists an extension of ψ to another mono from $\overline{K} \to \overline{L}$. Hence $\overline{L} : \overline{K}$ is an algebraic extension with degree greater than 1, since $\overline{K}$ is smaller than $\overline{L}$. However, this contradicts the algebraic closure of $\overline{K}$, since the only algebraic extension of an algebraically closed field is itself. Thus, $\overline{K} = \overline{L}$. □

Exercise 5.4. Let $K - L$ be a normal extension, $K \subseteq L \subseteq \overline{K}$. Then for any K-homomorphism $\tau : L \to \overline{K}$ one has $\tau(L) = L$.

Solution. Suppose we have some K-homomorphism $\tau : L \to \overline{K}$, and let $\ell \in L$. By definition of normal extension, $K - L$ must be algebraic, whence μ_ℓ^K exists. Since τ fixes all elements of K and μ_ℓ^K is a polynomial with coefficients in K, we can see that $\tau(\mu_\ell^K(\ell)) = \mu_\ell^K(\tau(\ell)) = 0$. By theorem, the normality of $L : K$ implies that all algebraic conjugates of ℓ are in L. Thus, we have that $\tau(\ell) \in L$. Since ℓ is an arbitrary element of L, this implies that $\tau(L) \subseteq L$. By theorem, since L extends K and $\tau : L \to L$ is a K-homomorphism, we have that τ is an automorphism of L. Thus $\tau(L) = L$. □

Exercise 5.5. Put $K = \mathbb{F}_2(t)$ and consider $L = K(t^{1/3})$. Prove that the extension $L : K$ is algebraic but not normal.

Solution. Obviously since $K(t^{1/3}) : K$ is a finite field extension, it is algebraic. Suppose $x \in \overline{K}$ solves the equation $x^3 - t = 0$. Then, $x = t^{1/3} \implies \left(\frac{x}{t^{1/3}}\right)^3 = 1 \implies x = yt^{1/3}$ such that $y^3 = 1$. Then we have $y^3 - 1 = (y-1)(y^2+y+1) = 0$, so either $y = 1$ or y is a root of $y^2 + y + 1 = 0$. Notice that $y^2 + y + 1$ is irreducible over K, since neither 0 nor 1 are roots. If we suppose there was some $z \in L \setminus K$ such that $z^2 + z + 1 = 0$, then that implies that there exists some $f \in K[t]$ with $f(t^{1/3}) = 0$. However, t is obviously transcendental over K, forcing a contradiction. Thus, the only solution to the cubic $x^3 - t$ in L is $x = 1$, whence the minimum polynomial for $t^{1/3} \in L$ does not split over L. Therefore $L : K$ does not meet the requirements to be a normal field extension. □