1 Composita and further comments

Remark 1. Let A, B be sets. Then $A \cap B$ can be expressed using only the operation $\subseteq$. Notice $A \cap B \subseteq A, B$ and $A \cap B$ is the maximal set with this property:

$$\forall C \text{ such that } C \subseteq A, B \quad \implies \quad C \subseteq A \cap B.$$

Let $H_1, H_2 \leqslant G$. Then $H_1 \cap H_2 \leqslant G$ is the *maximal* subgroup contained in both H_1 and H_2. Hence by the Galois correspondence we have $L^{H_1 \cap H_2}$ is the *minimal* subfield of L containing both L^{H_1} and L^{H_2}.

Definition 1 (Compositum)**.** Let K_1 and K_2 be fields contained in some field L. The compositum of K_1 and K_2 in L (or the composite field), denoted by $K_1 K_2$, is the smallest subfield of L containing both K_1 and K_2.

Lemma 1.1. *Let $K, E, F \subseteq L$. Then*

1. $E : K,\ F : K$ *finite* $\implies$ $EF : K$ *finite;*
2. $E : K,\ F : K$ *normal* $\implies$ $E \cap F : K$ *normal;*
3. $E : K,\ F : K$ *finite and* $E : K$ *normal* $\implies$ $EF : F$ *normal;*
4. $E : K,\ F : K$ *finite and normal* $\implies$ $EF : K,\ E \cap F : K$ *normal;*
5. $E : K,\ F : K$ *normal* $\implies$ $EF : E \cap F$ *normal.*

Definition 2 (Subnormal series)**.** Suppose $\operatorname{char} K = 0$, $\sqrt[\infty]{1} \subset K$, and $K - L$ is a radical Galois extension. Then,

$$K = K_0 - K_1 - K_2 - \cdots - K_m = L,$$

for $K_j = K_{j-1}(r_j), r_j^{n_j} \in K_{j-1}$. Then

$$\operatorname{Gal}_K L = G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots \geqslant G_m = \{Id.\}.$$

By assumption $\sqrt[\infty]{1} \subset K \implies K_j : K_{j-1}$ is a normal extension, so $G_j \trianglelefteq G_{j-1}$ and we have

$$\operatorname{Gal}_K L = G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_m = \{Id.\}.$$

This is called a subnormal series.

Definition 3 (Soluble group)**.** A group G is soluble if there exists a finite series of subgroups

$$\{Id.\} = G = G_0 \leqslant G_1 \leqslant G_2 \leqslant \cdots \leqslant G_m = G$$

such that

1. $G_j \lhd G_{j-1} \quad \forall 1 \leq j \leq n$ and
2. G_{j-1}/G_j is cyclic $\forall 1 \leq j \leq n$.