Algebraic closures I (Lecture 9)

Df. Let $\mathbb{F}$ be a field. Then $\mathbb{F}$ is *algebraically closed* if $\forall$ non-constant $f \in M[t]$ has a root in $\mathbb{F}$. We say that $\mathbb{F}$ is an *algebraic closure* of K if $\mathbb{F}:K$ is an algebraic field extension and $\mathbb{F}$ is algebraically closed.

L. $\mathbb{F}$ is algebraically closed iff
1) $\forall f \in \mathbb{F}[t]$, $f \neq$ const $\Rightarrow$ f factors in $\mathbb{F}[t]$ as a product of linear factors;
2) $\forall$ irreducible pol. in $\mathbb{F}[t]$ has degree 1;
3) the only algebraic extension of $\mathbb{F}$ containing $\mathbb{F}$ is $\mathbb{F}$ itself.

Df. Let $\emptyset \neq X$ be a partially ordered set $X = (X, \leq)$ ($\forall a: a \leq a$; $a \leq b$, $b \leq a \Rightarrow a = b$; $a \leq b$ & $b \leq c \Rightarrow a \leq c$). A chain C in X is a collection $\{x_i\}_{i \in I}$ s.t. $\forall i, j \in I$, either $x_i \leq x_j$ OR $x_j \leq x_i$

Zorn's Lemma Let $\emptyset \neq X = (X, \leq)$ be a partially ordered set s.t. $\forall$ chain C in X has an upper bound in X. Then X has at least one maximal element m (i.e. if $b \in X$ with $m \leq b$, then $b = m$).

Cor. $\forall$ proper ideal A of a commutative ring R is contained in a maximal ideal.

Pf. Let $S = \{$ all proper ideals of R that contain $A\}$ $\Rightarrow \emptyset \neq S$. Let $C = \{I_j\}_{j\in J}$ be a chain in S $\Rightarrow I := \bigcup_{j\in J} I_j \supseteq A$. It is easy to see that I is an ideal and $1 \notin I \Rightarrow I$ is a proper ideal. Finally, $\forall j \in J$ one has $I_j \leq I \Rightarrow$ I is an upper bound for C. By Zorn's lemma $\exists$ a maximal element $M \in S$. So, $A \leq M \lneq R$. Now if $\exists$ ideal $\hat{I}$ s.t. $M \lneq \hat{I} \leq R$, then either $\hat{I} \in S$ (and this is a contradiction with maximality of M), or else $\hat{I} = R$. Thus M is a maximal ideal. ■

L. Let K be a field. Then $\exists$ an algebraic extension $L:K$, $K \leq L$ s.t. L contains a root of $\forall$ irreducible $f \in K[t]$ and hence $\forall\, g \in K[t] \setminus K$.

Pf. Let $\{m_j\}_{j\in J}$ be the set of all irr. pol. over K and consider $R = K[\{t_j\}_{j\in J}]$. Also, let I be the ideal of R generated by $\{m_j(t_j)\}_{j\in J}$. Let us check that $I \neq R$.

If not, then $1 = \sum_{s\in S} u_s\, m_s(t_j)$ (1), where $u_j \in R$ and the summation is taken over a finite set S. As $|T| < \infty$ we can construct (see Lecture 5) an extension $\mathbb{F}:K$ s.t. $\forall s \in S$, the polynomial m_s has a root $\alpha_s \in \mathbb{F}$. Consider the homomorphism $\varphi : R \to \mathbb{F}$ s.t. $\varphi|_K$ is the identity map on K and

$$\varphi(t_s) = \begin{cases} \alpha_s, & s \in S \\ 0, & s \in \mathcal{J} \setminus S. \end{cases}$$

Then by (1) we have

$$1 = \varphi(1) = \sum_{s \in S} \varphi(u_j)\, \varphi(m_s)(\alpha_s) = 0.$$

Thus $I \neq R$. By corollary above there is a maximal ideal of R, say M s.t. $I \subseteq M$. Put $L = R/M$. Then $L:K$ is a field extension relative to the canonical embedding $\varphi : K \to L$, i.e. $\varphi(k) = k + M$. As always we identify k with $\varphi(k)$. Take any irr. pol. m. Then $m = m_j$, $j \in \mathcal{J}$. Let $\sigma : R \to L$, $\sigma(r) = r + M$. Put $\alpha_j = t_j + M$. We have $\sigma(t_j) = \alpha_j$. Thus

$$\varphi(m_j)(\alpha_j) = \sigma(m_j(t_j)) = m_j(t_j) + M = \overset{0 \text{ in } L}{\overset{\parallel}{0}} + M$$

since $m_j(t_j) \in I \subseteq M$ and hence $\forall$ irr. pol. has a root in L. Finally, each α_j is algebraic over $K \Rightarrow L = \varphi(K)[\{\alpha_j\}_{j\in J}]$ $\Rightarrow L$ is an algebraic extension of K. ∎

Thm (existence of algebraic closure)
Let $\mathbb{F}$ be a field. Then $\exists$ an algebraic extension $\overline{\mathbb{F}}$ of $\mathbb{F}$ s.t. $\overline{\mathbb{F}}$ is algebraically closed.

Pf. $\mathbb{F} = L_0 - L_1 - \dots - L_n -$, where L_j is an algebraic extension of L_{j-1} obtained by the lemma above. Thus L_j contains a root of any $f \in L_{j-1}[t] \setminus L_j$. Put $\overline{\mathbb{F}} = \bigcup_{j\in\mathbb{N}} L_j$. Since L_j are algebraic over L_{j-1}, it follows that L_j is algebraic over $\mathbb{F} \Rightarrow \overline{\mathbb{F}}$ is algebraic over $\mathbb{F}$ (it requres the following simple fact (exercise)
$K - L - M$, L is an alg. over K, M is an alg. over L $\Rightarrow M$ is an alg. over K). Now take any $f \in \overline{\mathbb{F}}[t] \setminus \overline{\mathbb{F}} \Rightarrow f = \sum_{j=1}^{n} c_j t^j$ the number of distinct c_j is finite $\Rightarrow \exists j \in \mathbb{N}$ s.t. $f \in L_{j-1}[t]$. Clearly, $f \notin L_{j-1} \Rightarrow$ by the lemma f has a root in $L_j \subseteq \overline{\mathbb{F}} \Rightarrow \overline{\mathbb{F}}$ is algebraically closed ∎

Df.
$$\begin{array}{ccc} L_1 & \xrightarrow{\tau} & L_2 \\ \uparrow{\scriptstyle\varphi_1} & & \uparrow{\scriptstyle\varphi_2} \\ K_1 & \xrightarrow{\sigma} & K_2 \end{array}$$
Here $\varphi_i : K_i \to L_i$ are fields extensions and σ, τ are isomorphisms

We say that τ extends σ if

$$\tau \circ \varphi_1 = \varphi_2 \circ \sigma$$

and we say that $L_1 : K_1$ and $L_2 : K_2$ are isomorphic field extensions.

If we assume that $K_i \subseteq L_i$ (i.e. $\varphi_1, \varphi_2 \equiv 1$), then the above commutative diagram implies that $\tau|_{K_1} = \sigma$. Thus, τ does indeed extend σ.

<u>Df</u> (we need this definition later) Let $\varphi : K \to L$ be a field extension and $\varphi(K) \subseteq M \subseteq L$ be a subfield of L. A homomorphism $\sigma : M \to L$ is a K-homomorphism if $\forall \alpha \in \varphi(K)$, one has $\sigma(\alpha) = \alpha$.

L. $K : L$, $\tau : L \to L$ is a K-homomorphism. Then $\forall f \in K[t]$ $\deg f \geq 1$ and $\forall \alpha \in L$ one has 1) $f(\alpha) = 0 \Rightarrow f(\tau(\alpha)) = 0$ 2) if τ is K-automorphism, then $f(\alpha) = 0 \Leftrightarrow f(\tau(\alpha)) = 0$.

<u>Thm</u>

$$\begin{tikzcd} K_2 \arrow[r, "\varphi_2"] & K_2(\beta) \subseteq L_2 \\ K_1 \arrow[u, "\sigma"] \arrow[r, "\varphi_1"] & K_1(\alpha) \subseteq L_1 \arrow[u, "\tau"'] \end{tikzcd}$$

$K_i \subseteq L_i$,
$\alpha \in L_1$ is alg. over K_1
$\beta \in L_2$ is alg. over K_2

Suppose that $\sigma : K_1 \to K_2$ is a field isomorphism.

Then σ can be extended to an isomorphism $\tau : K_1(\alpha) \to K_2(\beta)$ s.t. $\tau(\alpha) = \beta \iff \mu_\beta^{K_2} = \sigma(\mu_\alpha^{K_1})$.
(of course τ is determined by σ & $\tau(\alpha)$).

Pf. Let $f_1 := \mu_\alpha^{K_1} = \sum_{j=1}^{d} c_j t^j$, $c_j \in K \Rightarrow$ if $\tau(\alpha) = \beta$, then
$0 = \tau(\mu_\alpha^{K_1}(\alpha)) = \sum_j \tau(c_j)\tau(\alpha)^j = \underbrace{\sum \sigma(c_j)\beta^j}_{=: f_2}$
$\Rightarrow \beta$ is a root of $\sigma(\mu_\alpha^{K_1}) \Rightarrow \sigma(\mu_\alpha^{K_1}) = \mu_\beta^{K_2}$
(recall that our polynomials are monic).

Now let β is a root of f_2 (coincides with $\mu_\beta^{K_2}$). Since f_1, f_2 are irreducible polynomials we can consider

$\psi_1 : K_1[t]/(f_1) \to K_1(\alpha)$, where $\psi_1(g + (f_1)) = g(\alpha)$

$\psi_2 : K_2[t]/(f_2) \to K_2(\beta)$, where $\psi_2(h + (f_2)) = h(\beta)$

$\Rightarrow \psi_1, \psi_2$ are isomorphisms (exercise: check)
Put $\varphi : K_2[t] \to K_2[t]/(f_2)$, where $\varphi(q) = q + (f_2)$
$\Rightarrow \varphi$ is a surjective homomorphism.

Consider $\varphi \circ \sigma : K_1[t] \to K_2[t]/(f_2) \Rightarrow$ this is a surjective hom. We have

$\operatorname{Ker}(\varphi \circ \sigma) = \{ g \in K_1[t] : \sigma(g) + (f_2) = 0 + (f_2) \}$

$= \{ g : \sigma(g) = f_2 h_2, \text{ where } h_2 \in K_2[t] \}$
$= \{ \sigma^{-1}(f_2 h_2) : h_2 \in K_2[t] \}$

Recall that $\sigma(f_1) = f_2$ and $\sigma(K_1[t]) = K_2[t]$
$\Rightarrow \operatorname{Ker}(\varphi \circ \sigma) = (f_1) \Rightarrow$ by the Fundamental Homomorphism Theorem the map

$\omega : K_1[t]/(f_1) \to K_2[t]/(f_2)$, $\omega(g + (f_1)) = \sigma(g) + (f_2)$

is an isomorphism $\Rightarrow \tau := \psi_2 \circ \omega \circ \psi_1^{-1}$ is an isomorphism (as a composition of some isomorphisms), we have $\tau : K_1(\alpha) \to K_2(\beta)$.

Finally, $\tau(\alpha) = \psi_2 \circ \omega \circ \psi_1^{-1}(\alpha) = \psi_2 \circ \omega(t + (f_1))$
$= \psi_2(\sigma(t) + (f_2)) = \psi_2(t + (f_2)) = \beta$
(recall that $\sigma : K_1[t] \to K_2[t]$ is an isomorphism)

and if $k_1 \in K_1$, then
$\tau(k_1) = \psi_2 \circ \omega \circ \psi_1^{-1}(k_1) = \psi_2 \circ \omega(k_1 + (f_1))$
$= \psi_2(\sigma(k_1) + (f_2)) = \sigma(k_1) \Rightarrow \tau$ extends σ and $\tau(\alpha) = \beta$. $\blacksquare$

Cor. $M \subseteq L$ be a field extension, $\sigma : M \to L$ be a homomorphism, and $\alpha \in L$ is alg. over M. Then the number of ways we can extend σ to a hom. $\tau : M(\alpha) \to L$ is equal to the number of distinct roots of $\sigma(\mu_\alpha^M)$ that lie in L.

Indeed, we have the following picture

$$\begin{tikzcd} L \arrow[r] & L \\ M \arrow[u, "\sigma"] \arrow[r] & M(\alpha) \arrow[u, "\tau"] \end{tikzcd}$$

$(= L(\alpha) = L(\alpha_j)$, where $\alpha_j \in L$ is a root of $\sigma(\mu_\alpha^M))$

If τ extends σ then $\tau(\mu_\alpha^M(\alpha)) = \mu_\alpha^M(\tau(\alpha))$ $\Rightarrow \tau(\alpha) = \beta$ is another root of $\sigma(\mu_\alpha^M)$ (for simplicity, we assume that $M \subseteq L$). We want to have $\tau : M(\alpha) \to L \Rightarrow \tau(\alpha) = \beta$ must be in L. After that repeat the proof of the theorem.