1 Composita and further comments on FTGT

Definition 1 (Compositum)**.** Let K_1 and K_2 be fields contained in some field L. The compositum of K_1 and K_2 in L (or the composite field), denoted by K_1K_2, is the smallest subfield of L containing both K_1 and K_2.

Lemma 1.1. *Let $K, E, F \subseteq L$. Then*

1. *If $E : K$ and $F : K$ both finite, then $[EF : K] < \infty$;*
2. *If $E : K$ and $F : K$ both normal, then $E \cap F : K$ normal;*
3. *If $E : K$ and $F : K$ both finite and $E : K$ normal, then $EF : F$ normal;*
4. *If $E : K$ and $F : K$ both finite and normal, then $EF : K$ and $E \cap F : K$ both normal;*
5. *If $E : K$ and $F : K$ both normal, then $EF : E \cap F$ is normal.*

Definition 2 (Subnormal series)**.** Suppose $\operatorname{char} K = 0$, $\sqrt[\infty]{1} \subset K$, and $K - L$ is a radical Galois extension. That is, we have that

$$K = K_0 - K_1 - K_2 - \cdots - K_m = L,$$

for $K_j = K_{j-1}(r_j), r_j^{n_j} \in K_{j-1}$

$$\operatorname{Gal}_K L = G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots \geqslant G_m = \{Id.\}$$

By assumption $\sqrt[\infty]{1} \subset K \implies K_j : K_{j-1}$ is a normal extension, so $G_j \trianglelefteq G_{j-1}$ and we have

$$\operatorname{Gal}_K L = G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_m = \{Id.\}\,.$$

This is called a subnormal series.

Definition 3 (Soluble group)**.** A group G is soluble if there exists a finite series of subgroups

$$\{Id.\} = G = G_0 \leqslant G_1 \leqslant G_2 \leqslant \cdots \leqslant G_m = G$$