Part I

The Alternating Series Test

1 The Alternating Series Test

Definition: An alternating series is a series with terms that alternate between positive and negative (every other term)

Examples:

1.a

$$\sum_{k=1}^{\infty} \frac{\boxed{(-1)^{k-1}}}{k}$$

1.b

$$\sum_{k=1}^{\infty} \frac{\boxed{(-1)^{k}k}}{k^3+1}$$

1.c

$$\sum_{k=1}^{\infty} \frac{\boxed{(-1)^{k+1}}}{2^k+1}$$

Odd indexed Terms are negative (1.b) Even indexed Terms are negative (1.a and 1.c)

For series with positive terms, S_n is increasing.

`week7graph.jpg`

This is not the case with alternating series

2 Example: Alternating Harmonix Series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + (-\frac{1}{2}) = \frac{1}{2} \\
S_3 &= 1 + (-\frac{1}{2}) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\
S_4 &= 1 + (-\frac{1}{2}) = \frac{1}{2} + \frac{1}{3} + (-\frac{1}{4}) + \frac{14}{24}
\end{aligned}$$

./week7graph2.png

Because the 1^st is positive
$S_1 > S_{>S_5>\dots}$
$S_2 < S_4 < S_6 < \dots$

Alternating Series: Consider the alternating series $\sum_{k=1}^{\infty} a_k$ where

$$\begin{aligned}
a_k &= (-1)^k b_k \text{ or} \\
&= (-1)^{k-1} b_k \\
&= (-1)^{k+1} b_k
\end{aligned}$$

If both

1. $\lim_{k\to\infty} b_k = 0$ AND
2. $b_{k+1} < b_k$ (decreasing)

Then the alternating series converges

Part II
Alternating Series Test Examples

Examples:

2.a

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

(i) $\lim_{k\to\infty} \frac{1}{k} = 0$
(ii) $\frac{1}{k+1} < \frac{1}{k}$
So The series converges by Alternating Series Test (AST)

2.b

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k+4}$$

$$\begin{aligned}
&(i) \lim_{k\to\infty} \frac{\sqrt{k}}{k+4} \\
&(ii) \frac{\sqrt{k+1}}{(k+1)+4} < \frac{\sqrt{k}}{k+4} \\
&= \frac{k+1}{(k+5)^2} < \frac{k}{(k+4)^2} \\
&= (k+1)(k^2+8k+16) < k(k^2+10k+25) \\
&= \cancel{k^3} + 8k^2 + 16k + k^2 + 8k + 16 < \cancel{k^3} + 10k^2 + 25k \\
&= 16 < k^2 + k
\end{aligned}$$

This holds for the $k \geq 4$, This suffices. We just have to prove that it holds true for all values of k after some fixed value.
$\boxed{\text{This converges by AST}}$

2.c $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+3}$

$$(i) \lim_{k\to\infty} \frac{k}{k+3} = 1 \neq 0$$

The series $\boxed{\text{diverges}}$ by the Test for Divergence

2.d

Does the series $\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{4^2} - \frac{1}{4^3} + \dots$ converge?

$$(i) \lim_{k\to\infty} b_k = 0 \checkmark$$

but

$$\cancel{(ii)} b_k + 1 < b_k \text{ X}$$

Can't use AST
This can be written as
$\sum_{k=2}^{\infty} \frac{1}{k^2} - \sum_{k=2}^{\infty} \frac{1}{k^3}$
Both of these converge (p-series, $p > 1$) So the original series converges

Part III
Absolute vs. Conditional Convergence

Example:

$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ This series has negative terms, but is not alternating

3 Absolute Convergence Theorem

if $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Important: Not an if and only if statement

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{ converges, but}$$
$$\sum_{k=1}^{\infty} |\frac{(-1^{k-1})}{k}| = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

3.a $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$

$$\sum_{k=1}^{\infty} |\frac{sin(k)}{k^2}| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ which converges}$$

So $\sum_{k=1}^{\infty} |\frac{\sin(k)}{k^2}|$
So $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ $\boxed{\text{converges}}$ by Abs. cmv. thm.