

Part I

The Alternating Series Test

1 The Alternating Series Test

Definition: An alternating series is a series with terms that alternate between positive and negative (every other term)

Examples:

1.a

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

1.b

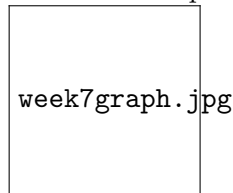
$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^3 + 1}$$

1.c

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k + 1}$$

Odd indexed Terms are negative (1.b) Even indexed Terms are negative (1.a and 1.c)

For series with positive terms, S_n is increasing.



This is not the case with alternating series

2 Example: Alternating Harmonic Series

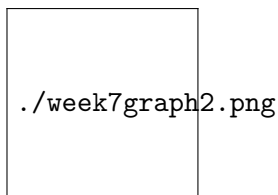
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

$$S_1 = 1$$

$$S_2 = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$S_3 = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$S_4 = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{3} + \left(-\frac{1}{4}\right) + \frac{14}{24}$$



Because the 1st is positive

$$S_1 > S_3 > S_5 > \dots$$

$$S_2 < S_4 < S_6 < \dots$$

Alternating Series: Consider the alternating series $\sum_{k=1}^{\infty} a_k$ where

$$a_k = (-1)^k b_k \text{ or}$$

$$= (-1)^{k-1} b_k$$

$$= (-1)^{k+1} b_k$$

If both

$$1. \lim_{k \rightarrow \infty} b_k = 0 \text{ AND}$$

$$2. b_{k+1} < b_k \text{ (decreasing)}$$

Then the alternating series converges

Part II

Alternating Series Test Examples

Examples:

2.a

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

(i) $\lim_{k \rightarrow \infty} \frac{1}{k} = 0$

(ii) $\frac{1}{k+1} < \frac{1}{k}$

So The series converges by Alternating Series Test (AST)

2.b

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k+4}$$

(i) $\lim_{k \rightarrow \infty} \frac{\sqrt{k}}{k+4}$

(ii) $\frac{\sqrt{k+1}}{(k+1)+4} < \frac{\sqrt{k}}{k+4}$

$$= \frac{k+1}{(k+5)^2} < \frac{k}{(k+4)^2}$$

$$= (k+1)(k^2+8k+16) < k(k^2+10k+25)$$

$$= k^3+8k^2+16k+k^2+8k+16 < k^3+10k^2+25k$$

$$= 16 < k^2+k$$

This holds for the $k \geq 4$, This suffices. We just have to prove that it holds true for all values of k after some fixed value.

This converges by AST

2.c $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+3}$

(i) $\lim_{k \rightarrow \infty} \frac{k}{k+3} = 1 \neq 0$

The series diverges by the Test for Divergence

2.d

Does the series $\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{4^2} - \frac{1}{4^3} + \dots$ converge?

(i) $\lim_{k \rightarrow \infty} b_k = 0$ ✓

but

~~(ii)~~ $b_k + 1 < b_k$ X

Can't use AST

This can be written as

$\sum_{k=2}^{\infty} \frac{1}{k^2} - \sum_{k=2}^{\infty} \frac{1}{k^3}$

Both of these converge (p-series, $p > 1$) So the original series converges

Part III

Absolute vs. Conditional Convergence

Example:

$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ This series has negative terms, but is not alternating

3 Absolute Convergence Theorem

if $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Important: Not an if and only if statement

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{ converges, but}$$

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k-1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

3.a $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ which converges}$$

So $\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right|$

So $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ converges by Abs. cmv. thm.