Problem Set # 6

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1

1.a

1.a.1 Find the following probabilities

- $P(A) = \frac{5}{20}$
- $P(B) = \frac{10}{20}$
- $P(A \bigcap B) = \frac{2}{20} = \frac{1}{10}$

1.a.2 Are A and B independent

$P(A \bigcap B) = \frac{1}{10} \neq \frac{5}{20} * \frac{10}{20}$ ∴ No, they are not independent

1.b Roll 3 times

1.b.1 Mass Density function

$$(DDD), 3*(DDN), 3*(DNN), (NNN)$$
$$P(X) = \frac{4}{20}$$
$$p(0) = (\frac{16}{20})^3 = (\frac{4}{5})^3 = \frac{64}{125}$$
$$3*p(1) = \frac{4}{20}(\frac{4}{5})^2 = \frac{1}{5}(\frac{4}{5})^2 = \frac{1}{5} * \frac{16}{12} = \frac{16}{125} * 3 = \frac{48}{25}$$
$$3*p(2) = (\frac{1}{5})^2 * \frac{4}{5} = \frac{1}{25} * \frac{4}{5} = \frac{4}{125} * 3 = \frac{12}{125}$$
$$p(3) = (\frac{1}{5})^3 = \frac{1}{125}$$

1.b.2 Expected value

$$0 * \frac{64}{125} + 1 * \frac{48}{125} + 2 * \frac{12}{125} + 3 * \frac{1}{125}$$
$$\frac{48}{125} + \frac{24}{125} + \frac{3}{125} = \frac{75}{125} = \frac{3}{5}$$

2

2.a

$$\sum_{k=1}^{n} x_{kp(x_k)}$$
$$p(Y = -2 \bigcup -1) = \frac{1}{2}$$
$$\frac{1}{10} + a = \frac{1}{2}$$
$$a = \frac{4}{10}$$
$$-2(\frac{1}{10}) - \frac{4}{10} + 0(b) + 1(\frac{3}{10}) + 2(c) = 0$$
$$-\frac{2}{10} - \frac{4}{10} + 0(b) + \frac{3}{10} + 2(c) = 0$$
$$-\frac{6}{10} + \frac{3}{10} + 2(c) = 0$$
$$-\frac{3}{10} + 2(c) = 0$$
$$c = \frac{3}{20}$$
$$b = 1 - \frac{4}{10} - \frac{3}{10} - \frac{1}{10} - \frac{3}{20}$$
$$b = 1 - \frac{8}{10} - \frac{3}{20}$$
$$b = 1 - \frac{16}{20} - \frac{3}{20}$$
$$b = 1 - \frac{19}{20} = \frac{1}{20}$$

2.b Find $E[Y^2]$

$$4(\frac{1}{10}) + \frac{4}{10} + 0 + 1(\frac{3}{10}) + 4(\frac{3}{20}) = 0$$
$$\frac{4}{10} + \frac{4}{10} + \frac{3}{10} + \frac{3}{5}$$
$$\frac{11}{10} + \frac{6}{10} = \frac{17}{10}$$

3

3.a

3.a.1

$$p(1) = \frac{c}{1}$$
$$p(2) = \frac{c}{2}$$
$$p(3) = \frac{c}{3}$$
$$p(4) = \frac{c}{4}$$
$$p(5) = \frac{c}{5}$$
$$p(6) = \frac{c}{6}$$

3.a.2 Roll an even

$$\frac{c}{2} + \frac{c}{4} + \frac{c}{6}$$
$$\frac{6c}{12} + \frac{3c}{12} + \frac{2c}{12} = \frac{11c}{12}$$

3.a.3 $\frac{1}{X} < \frac{1}{5}$

$$S = \{\frac{1}{6}\}$$
$$p(6) = \frac{c}{6}$$

3.a.4 $P(X^2 - 4 < 5)$

$$X = 1, 2$$
$$P(1 \bigcup 2) = \frac{c}{1} + \frac{c}{2}$$
$$P(1 \bigcup 2) = \frac{3c}{2}$$

3.a.5 $E[X]$

$$\begin{aligned} 1 * p(1) + 2 * p(2) + \dots \\ 1 * \frac{c}{1} + \cancel{2} * \frac{c}{\cancel{2}} + \dots \\ c + c + c + c + c + c = 6c \end{aligned}$$

3.b Roll twice, $P(2X \leq 2Y - 5)$

$$\begin{aligned} p(6)p(3) + p(6)p(2) + p(6)p(1) + p(4)p(1) + p(5)p(1) + p(5)p(2) \\ (\frac{c}{6} * \frac{c}{3}) + (\frac{c}{6} * \frac{c}{2}) + (\frac{c}{6} * c) + (\frac{c}{4} * c) + (\frac{c}{5} * c) + (\frac{c}{5} * \frac{c}{2}) \\ (\frac{c^2}{18}) + (\frac{c^2}{12}) + (\frac{c^2}{6}) + (\frac{c^2}{4}) + (\frac{c^2}{5}) + (\frac{c^2}{10}) \\ \frac{77c^2}{90} \end{aligned}$$

3.c Find the value of c

$$\begin{aligned} c(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}) = 1 \\ c(\frac{49}{20}) = 1 \\ c = \frac{20}{49} \end{aligned}$$

4

4.a a_1

$$\frac{1}{2(1)+4} = \frac{1}{6}$$

4.b $\sum_{k=1}^{\infty} a_k$

$$\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$$
$$1 - \frac{1}{2} = \frac{1}{2}$$

4.c $\lim_{k\to\infty}$

$$\lim_{k\to\infty} a_k = 0 \text{ because } \sum a_k \text{ exists}$$

4.d $\sum_{k=101}^{200} a_k$

$$\sum_{101}^{200} a_k = S_{200} - S_{100}$$
$$\sum_{101}^{200} a_k = 1 - \frac{200}{404} + 1 - \frac{100}{204}$$

4.e a_{100}

$$1 - \frac{100}{200+4} = 1 - \frac{100}{204}$$

5

5.a

$$\sum_{k=1}^{\infty}(9^{\frac{1}{k}} - 9^{\frac{1}{k+2}})$$

$$\begin{aligned}
S_1 &= 9^1 - 9^{\frac{1}{3}} \\
S_2 &= (9^1 - 9^{\frac{1}{3}}) + (9^{\frac{1}{2}} - 9^{\frac{1}{4}}) \\
S_3 &= (9^1 - \cancel{9^{\frac{1}{3}}}) + (9^{\frac{1}{2}} - 9^{\frac{1}{4}}) + (\cancel{9^{\frac{1}{3}}} - 9^{\frac{1}{5}}) \\
S_3 &= (9^1) + (3 - 9^{\frac{1}{4}}) + (-9^{\frac{1}{5}}) \\
S_n &= 9 + 3 - 9^{\frac{1}{n+1}} - 9^{\frac{1}{n+2}} \\
S_n &= 12 - 9^{\frac{1}{n+1}} - 9^{\frac{1}{n+2}} \\
\lim_{n\to\infty} S_n &= 12 - 9^0 - 9^0 = 10
\end{aligned}$$

5.b $\sum_{k=1}^{\infty} e^{\sin(\frac{1}{k})}$

$$\lim_{k\to\infty} e^{\sin(0)}$$

$\lim_{k\to\infty} e^0 = 1 \neq 0 \therefore$ Divergent because of limit test

5.c $\sum_{k=1}^{\infty} \frac{1+e^{-k}}{4e^{-k}+3}$

$$\lim_{k\to\infty} \frac{1+e^{-k}}{4e^{-k}+3}$$

$\lim_{k\to\infty} \frac{1}{4+3} = \frac{1}{7} \neq 0 \therefore$ Divergent because of limit test

5.d $\sum_{k=1}^{\infty}(\cos(\frac{\pi}{k}) - \cos(\frac{\pi}{k+2}))$

$$S_1 = \cos(\frac{\pi}{1}) - \cos(\frac{\pi}{3})$$
$$S_2 = (\cos(\frac{\pi}{1}) - \cos(\frac{\pi}{3})) + (\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{4}))$$
$$S_3 = (\cos(\frac{\pi}{1}) - \cancel{\cos(\frac{\pi}{3}})) + (\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{4})) + (\cancel{\cos(\frac{\pi}{3}}) - \cos(\frac{\pi}{5}))$$
$$S_3 = (\cos(\frac{\pi}{1})) + (\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{4})) - \cos(\frac{\pi}{5}))$$
$$S_n = -1 + 0 - \cos(\frac{\pi}{n+1}) - \cos(\frac{\pi}{n+2})$$
$$S_n = -1 - \cos(\frac{\pi}{n+1}) - \cos(\frac{\pi}{n+2})$$
$$\lim_{n\to\infty} = -1 - \cos(0) - \cos(0) = -1 - 1 - 1 = -3$$