Part I

Introduction to Probability

1 Definitions

1. Experiment - A well defined procedure (rolling a die, flipping a coin)
2. Sample Space - The set of all possible outcomes for particular experiment
3. Event - A subset of the sample space, subset of outcomes possible

Example: Rolling a six-sided fair die

- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$
- Possible Events:

  A = "you roll an even number" A = {2, 4, 6}
  P(A) = 3/6 = 1/2

  B = "you roll a # less than 5" $\rightarrow B = \{1, 2, 3, 4\}$

  $P(B) = 4/6 = 2/3$

2 Principles

1. $0 \leq P(event) \leq 1$
2. $\sum P(outcome) = 1$ The sum of all the outcomes must equal 1, Something has to happen

Example: Suppose a 4 sided die has sides A, B, C, and D. You roll it twice

1. Find the sample space
   $S = \{AA, AB, AC, AD, BA, BB, BC, BD, CA, CB, CC, CD, DA, DB, DC, DD\}$
2. Find the following probabilities:

   P(you roll 2 As) = $\frac{1}{16}$

   P(1^{st} roll is A) = $\frac{1}{4}$

   P(at least one roll is A) = $\frac{7}{16}$

   P(roll at least on A or at least one B) = $\frac{12}{16} = \frac{3}{4}$

Part II
Probability Rules and Independence

3 Rules

- Addition Rule: P(A or $\bigcup$ B) = P(A) + P(B) - P(A and $\bigcap$ B)

  We subtract the intersection (P $\bigcap$ B)because A and B are already counted. Think of a Ven diagram

  - P(you roll at least one A or at least one B)
    P(you roll at least one A) + P (you roll at least one B)
    AA, AB, AC, AD, BA, CA, DA +BA, BB, BC, BD, AB, CB, DB
  - P(you roll at least one A and at least one B)
  - P (you roll at least one A or at least on B)

- Complement Rule: P(A^c) = 1 - P(A)
  X^C = All outcomes that do not satisfy X
  Ex: P(you roll at least on A or at least one B) = 1- P(no A and no B)

4 Independence

Definition: Events A and B are independent if $P(A) = P(A)P(B)$

Example: Flip a fair coin twice. Then $S = \{HH, HT, TH, TT\}$

- A = 1^{st} toss is heads $\rightarrow A = \{HH, HT\}$ P(A) = 1/2
- B = 2^{nd} toss is heads $\rightarrow B = \{HH, TH\}$ P(B) = 1/2
- C = both tosses are heads $\rightarrow C = \{HH\}$ P(C) = 1/4

Then:

- A and B are independent: $P(A \bigcap B) = \frac{1}{4} = P(A)P(B) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
- A and C are not indepndent: $P(A \bigcap C) = \frac{1}{4} \neq P(A)P(C) = \frac{1}{2} * \frac{1}{4}$

Part III

Random Variables and Mass Density Functions

Definition: A random variable, X, is a function that assigns a real number to each element of the sample space. The mass density function of X is the function defined by $p(k) = P(x = k)$

Example: Flip a fair coin twice. Let X be the # of heads.

$$HH \to 2 \qquad HT \to 1 \qquad TH \to 1 \qquad TT \to 0$$

The range of X is {0, 1, 2}. The mass density function of X is:

- $p(0) = \frac{1}{4} = P(x = 0)$
- $p(1) = \frac{2}{4} = P(x = 1)$
- $p(2) = \frac{1}{4} = P(x = 2)$

Example: Roll two six-sided dice. Let Y equal the sum of all the rolls. Find the mass density function of Y.

- The range of Y is {2, 3, 4, ..., 12}

  $p(2) = \frac{1}{36} = p(12)$
  $p(3) = \frac{2}{36} = p(11)$
  $p(4) = \frac{3}{36} = p(10)$
  $p(5) = \frac{4}{36} = p(9)$
  $p(6) = \frac{5}{36} = p(8)$
  $p(7) = \frac{6}{36}$

  $\sum_{k=2}^{12} p(k) = 1$

Part IV

Expected Value

Supposed a group of students has the following 7 quiz scores:

3, 5, 8, 8, 8, 10, 10

The average quiz score is:

$\frac{3+5+8+8+8+10+10}{7} = \frac{52}{7}$

Alternatively, we could find the average score by using a weighted average:

$3(\frac{1}{7}) + 5(\frac{1}{7}) + 8(\frac{3}{7} + 10(\frac{2}{7}) = \frac{52}{7})$

Definition: Suppose random variable X takes on values $x_1, x_2, ..., x_n$ Then the expected value of X is

$$E[X] = \sum_{k=1}^{n} x_k P(X = x_k) = \sum_{k=1}^{n} x_k p(x_k)$$

Example: Flip a fair coin 3 times. be the # of heads. Find $E[X]$

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

- range of X is $\{0, 1, 2, 3\}$
- mass density function:

  $p(0) = \frac{1}{8}$

  $p(1) = \frac{3}{8}$

  $p(2) = \frac{3}{8}$

  $p(3) = \frac{1}{8}$

$$\begin{aligned} E[X] &= (0) + p(0) + (1) + p(1) + (2)p(2) + (3)p(3) \\ &= (1)\frac{3}{8} + (2)\frac{3}{8} + (3)\frac{1}{8} \\ &= \frac{1}{8}(3 + 6 + 3) = \frac{12}{8} = \frac{3}{2} \end{aligned}$$

Part V

Expected Value Examples

Example: Flip a fair coin 3 times. Let$Y =$ of heads multiplied by the of tails, Find $E[Y]$

- range of $Y = \{0, 2\}$

- mass density function

  $p(0) = 2/8 = 1/4$

  $p(2) = 6/8 = 3/4$

$E[Y] = 0p(0) + 2p(2) = 2(\frac{3}{4} = \frac{3}{2})$

Example: Supposed you play a game where you roll two six-sided dice. If the sum of the rolls is > 9, you win two dollars. Otherwise you lose c. Find the c so that this is a fair game. Let X be your winnings.

- The range of $X = \{2, -c\}$ Either win two dollars are lose c dollars

  $p(2) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{6}$

  $p(-c) = \frac{5}{6}$ Complement rule

$E[X] = 2p(2) + (-c)p(c) = 2(\frac{1}{6}) - c(\frac{5}{6})$ $c = \$\frac{2}{5}$

Example: Supposed you keep flipping a pair of coints until you see HH. Let X be the # of times you need to toss the pair for this to occur

- What is the range of X?

  $\{1, 2, 3, 4, ...\} \rightarrow$ infinitely many values of X

- Find the mass density function

  $p(1) = \frac{1}{4}$

  $p(2) = \frac{3}{4}\frac{1}{4}$

  $p(2) = \frac{3}{4}\frac{3}{4}\frac{1}{4} = \frac{3}{4}^2(\frac{1}{4})$

  $p(k) = (\frac{3}{4})^{k-1}(\frac{1}{4})$

We know $\sum_{k=1}^{\infty} p(k) = 1$ (From probability principles) This means

$$\sum_{k=1}^{\infty}(\frac{3}{4})^{k-1}(\frac{1}{4}) = 1.$$

How would we know this without the context of probability?