Analysis of Popular Bet-Sizing Strategies for the Game of Baccarat

B_t is the streak progression of consecutive bets on banker, starting at $B_0 = 0$, whereby

$$B_t := \begin{cases} 0.95\mathbb{1}_{\{B_{t-1}\leq 0\}} + (B_{t-1} + 0.95)\,\mathbb{1}_{\{B_{t-1}>0\}} & \text{w.p.} \sim 0.4585, \text{ (banker)} \\ B_{t-1} & \text{w.p.} \sim 0.0953, \text{ (tie)} \\ (B_{t-1} - 1)\,\mathbb{1}_{\{B_{t-1}\leq 0\}} - \mathbb{1}_{\{B_{t-1}>0\}} & \text{w.p.} \sim 0.4462. \text{ (player)} \end{cases}$$

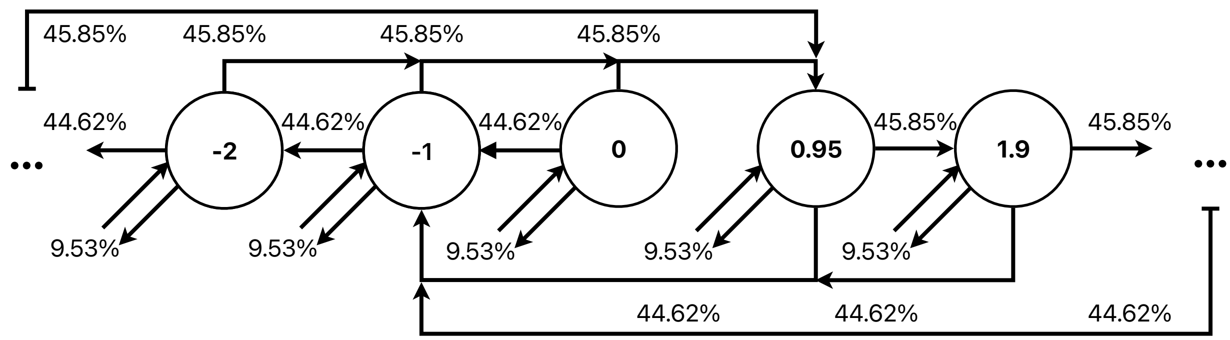


Figure 1: The above Markov chain depicts the transition probabilities and the discrete states B_t traverses as the game is played. Note that state 0 is transient and is where the chain initiates. All other states are positive recurrent.

The gambler must always bet on banker as it is the optimal wager (expected return on banker is $0.4585 \times 0.95 + 0.0953 \times 0 - 0.4462 \times 1 = -0.010625$). Equivalently, the house edge is just 1.0625%. In Even Money Baccarat, the house edge on the banker increases to around 1.46%, therefore making the player bet the optimal wager. To manage bankroll, gambler should keep winnings in a separate pile in order to keep track of the amount of bets placed. This will ensure that in the long run, the gambler's loss converges to the product of the initial bankroll and the house edge.

V_T is the bankroll for the gambler at time T, where $V_0 : V_0 > 0$ is the initial surplus and $K : 0 < K < V_0$ is the lowest betting unit. According to the strategy,

$$
\begin{aligned}
V_T &:= V_0 + K \sum_{t \le T} \left\lfloor \mathrm{Fib}\left(1 + \max\left\{0, B_{t-1}\right\}\right) \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right) \\
&= V_0 + K \sum_{t \le T} \left\lfloor \frac{\varphi}{\sqrt{5}} \max\left\{1, \varphi^{B_{t-1}}\right\} \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right), \quad \forall t \in \mathbb{N}.
\end{aligned}
$$

$\mathrm{Fib}(\cdot)$ is the analytic continuation of the Fibonacci sequence, as defined by,

$$
\mathrm{Fib}(x) := \frac{\varphi^x - \varphi^{-x} \cos(\pi x)}{\sqrt{5}}, \quad \forall x \in \mathbb{R}.
$$

$$
\begin{aligned}
V_T &:= V_0 + K \sum_{t \le T} \left\lfloor \mathrm{N}\left(\max\left\{0, B_{t-1}\right\}\right) \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right) \\
&= V_0 + K \sum_{t \le T} \left\lfloor \eta \max\left\{1, \lambda^{B_{t-1}}\right\} \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right), \quad \forall t \in \mathbb{N}.
\end{aligned}
$$

$\mathrm{N}(\cdot)$ is the analytic continuation of Narayana's Cows, as defined by,

$$
\mathrm{N}(x) := \eta \lambda^x, \quad \forall x \in \mathbb{R}.
$$

Martingale strategy can be stated as,

$$
\begin{aligned}
V_T &:= V_0 + K \sum_{t \le T} \left\lfloor 2^{\max\{0, B_{t-1}\} - 1} \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right) \\
&= V_0 + K \sum_{t \le T} \left\lfloor \max\left\{1, 2^{B_{t-1} - 1}\right\} \right\rceil \left(0.95 \mathbb{1}_{\{\Delta B_t > 0\}} - \mathbb{1}_{\{\Delta B_t < 0\}}\right), \quad \forall t \in \mathbb{N}.
\end{aligned}
$$

Exact constants used in this paper:

$$
\begin{aligned}
\varphi &= \frac{1 + \sqrt{5}}{2} \approx 1.618034, \\
\eta &= \frac{1}{3} \left(1 + \frac{4}{\sqrt{31}} \cosh\left(\frac{1}{3} \cosh^{-1}\left(\frac{\sqrt{31}}{2}\right)\right)\right) \approx 0.611492,
\end{aligned}
$$

$$\lambda = \frac{1}{3}\left(1 + 2\cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{29}{2}\right)\right)\right) \approx 1.465571.$$

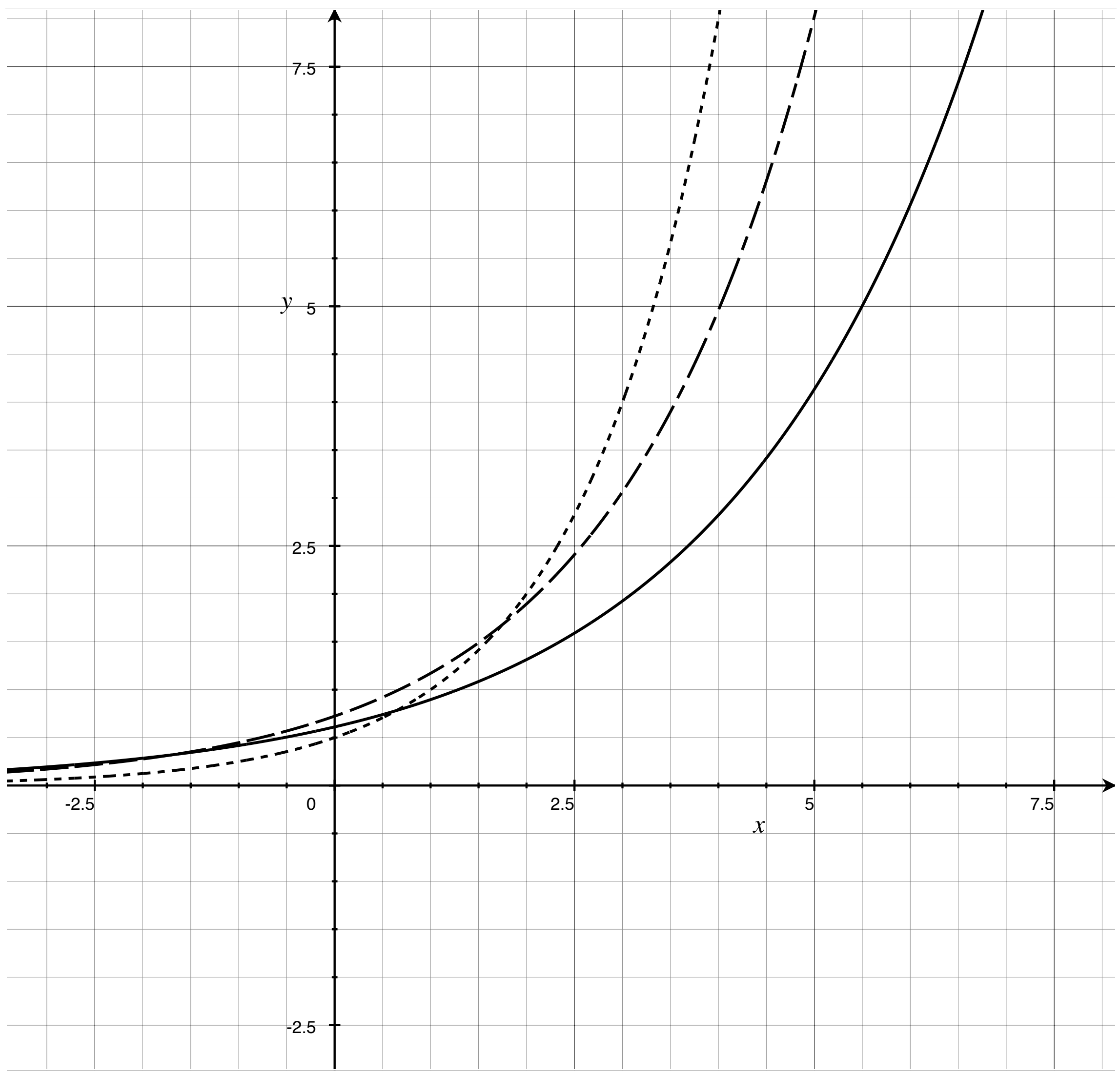


Figure 2: If we suppose that the wagers under the famous Martingale strategy can be modelled by $f(x) = 2^{x-1}$ (assuming bets are rounded to the nearest integer), then it is easy to see that the Fibonacci strategy is far more conservative, as the rate of increase in wager sizes under the Martingale progression is far greater.

We can calculate the probability of consecutive hands without a loss (win; if betting on player) via the geometric series,

$$\begin{aligned}\Pr(\text{``}\textit{streak of } 1\text{''}) &= \Pr(\text{``}\textit{hand wins}\text{''}) \sum_{j=1}^{\infty} \Pr(\text{``}\textit{hand ties}\text{''})^{j-1} \\ &= \frac{\Pr(\text{``}\textit{hand wins}\text{''})}{1 - \Pr(\text{``}\textit{hand ties}\text{''})}.\end{aligned}$$

Note that wagers are pushed, when hand is a tie. Likewise for streaks greater than 1,

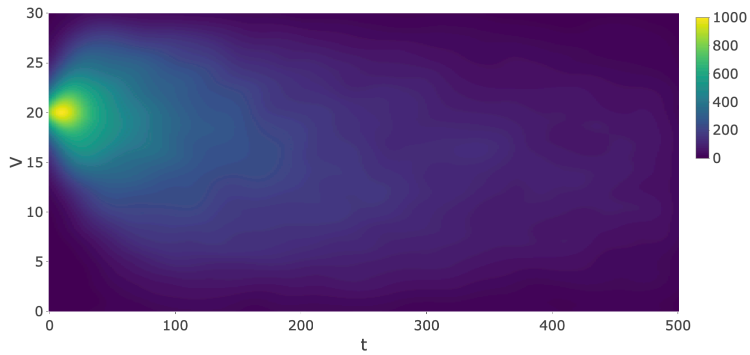
$$\Pr(\text{``}\textit{streak of } N\text{''}) = \left(\frac{\Pr(\text{``}\textit{hand wins}\text{''})}{1 - \Pr(\text{``}\textit{hand ties}\text{''})}\right)^{N}.$$

Streak	**Martingale**		**Fibonacci**		**Narayana**		**Flat betting**	
	Wager	**Total payoff (b)**	**Wager**	**Total payoff (b)**	**Wager**	**Total payoff (b)**	**Wager**	**Total payoff (b/p)**
1	1	-0.05	1	-0.05	1	-0.05	1	-0.05/-0.05
2	1	-0.1	1	-0.1	1	+0.9	1	+0.9/-1.05
3	2	-0.2	2	+0.8	1	+0.85	1	+1.85/-2.05
4	4	+0.6	3	+1.65	2	+1.75	1	+2.8/-3.05
5	7	+1.25	5	+4.4	3	+3.6	1	+3.75/-4.05
6	13	+0.6	7	+7.05	4	+6.4	1	+4.7/-5.05
7	26	+1.3	11	+10.5	5	+8.15	1	+5.65/-6.05
8	50	+1.8	18	+17.6	8	+12.75	1	+6.6/-7.05
9	97	+3.95	28	+28.2	11	+18.2	1	+7.55/-8.05
10	187	+6.6	44	+44	16	+26.4	1	+8.5/-9.05
11	362	+13.5	70	+69.5	23	+38.25	1	+9.45/-10.05
12	699	+25.55	111	+110.95	33	+54.6	1	+10.4/-11.05

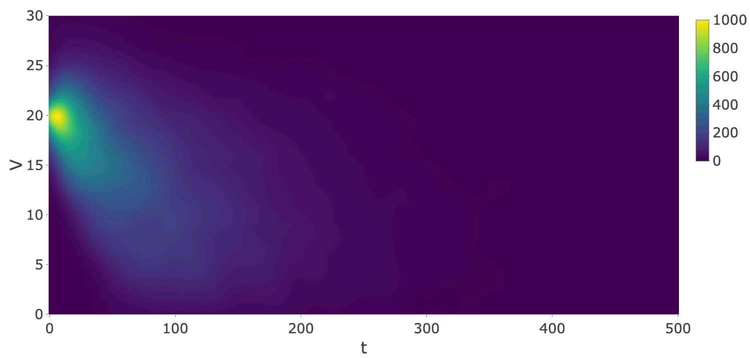
Table 1: Probability of occurrence and total payoffs on consecutive banker/player hands. Strategy is played out and position is closed when streak is lost on the $(n+1)$th hand.

Streak	**Probability (b/p)**	**Total payoff**			
		Martingale (b)	**Fibonacci (b)**	**Narayana (b)**	**Flat betting (b/p)**
1	50.680%/49.320%	+0.95	+0.95	+0.95	+0.95/-1
2	25.684%/24.325%	+1.9	+1.9	+1.9	+1.9/-2
3	13.017%/11.997%	+3.8	+3.8	+2.85	+2.85/-3
4	6.597%/5.917%	+7.6	+6.65	+4.75	+3.8/-4
5	3.343%/2.918%	+14.25	+11.4	+7.6	+4.75/-5
6	1.694%/1.439%	+26.6	+18.05	+11.4	+5.7/-6
7	0.859%/0.710%	+51.3	+28.5	+16.15	+6.65/-7
8	0.435%/0.350%	+98.8	+45.6	+23.75	+7.6/-8
9	0.221%/0.173%	+190.95	+72.2	+34.2	+8.55/-9
10	0.112%/0.085%	+368.6	+114	+49.4	+9.5/-10
11	0.057%/0.042%	+712.5	+180.5	+71.25	+10.45/-11
12	0.029%/0.021%	+1376.55	+285.95	+102.6	+11.4/-12

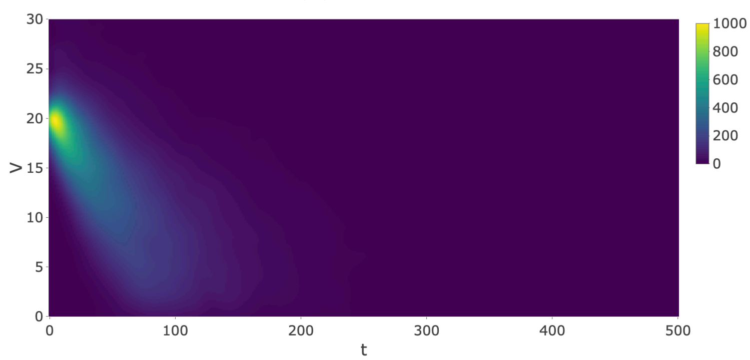
Table 2: Payoff for when the gambler ends their game on the nth hand.



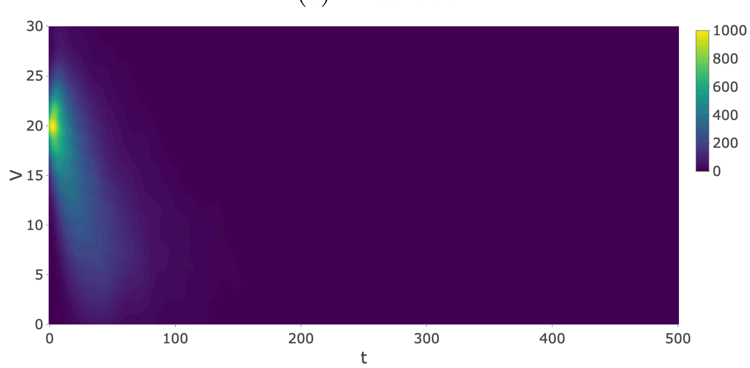
(a) Flat betting



(b) Narayana



(c) Fibonacci



(d) Martingale

Figure 3: 1000 Monte Carlo simulations of a gamblers' bankroll over 500 hands, with $V_0 = 20$ and $K = 1$. Runs are terminated when the gamblers' bankroll increases by 50% or is completely depleted.

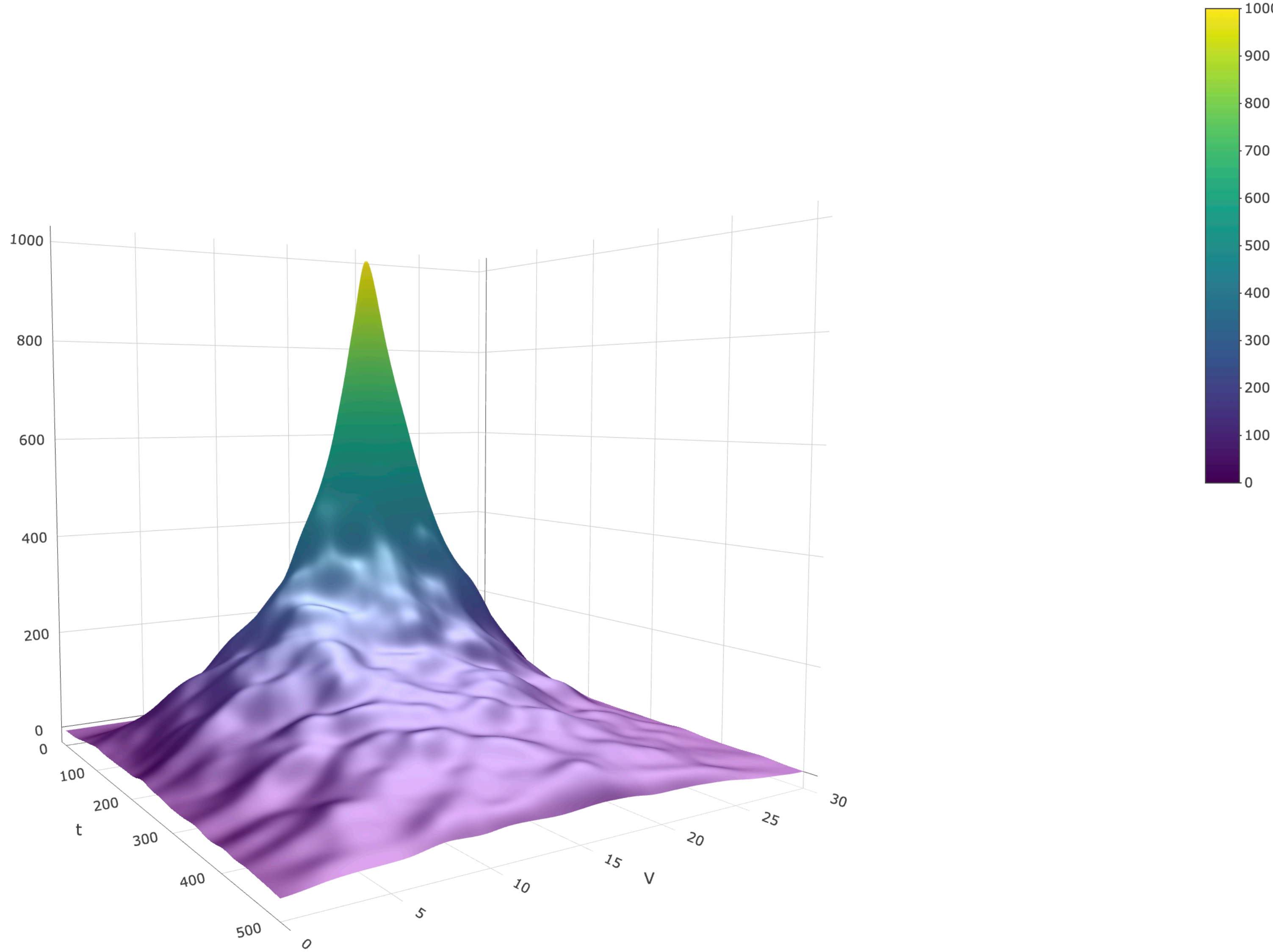


Figure 4: Flat betting

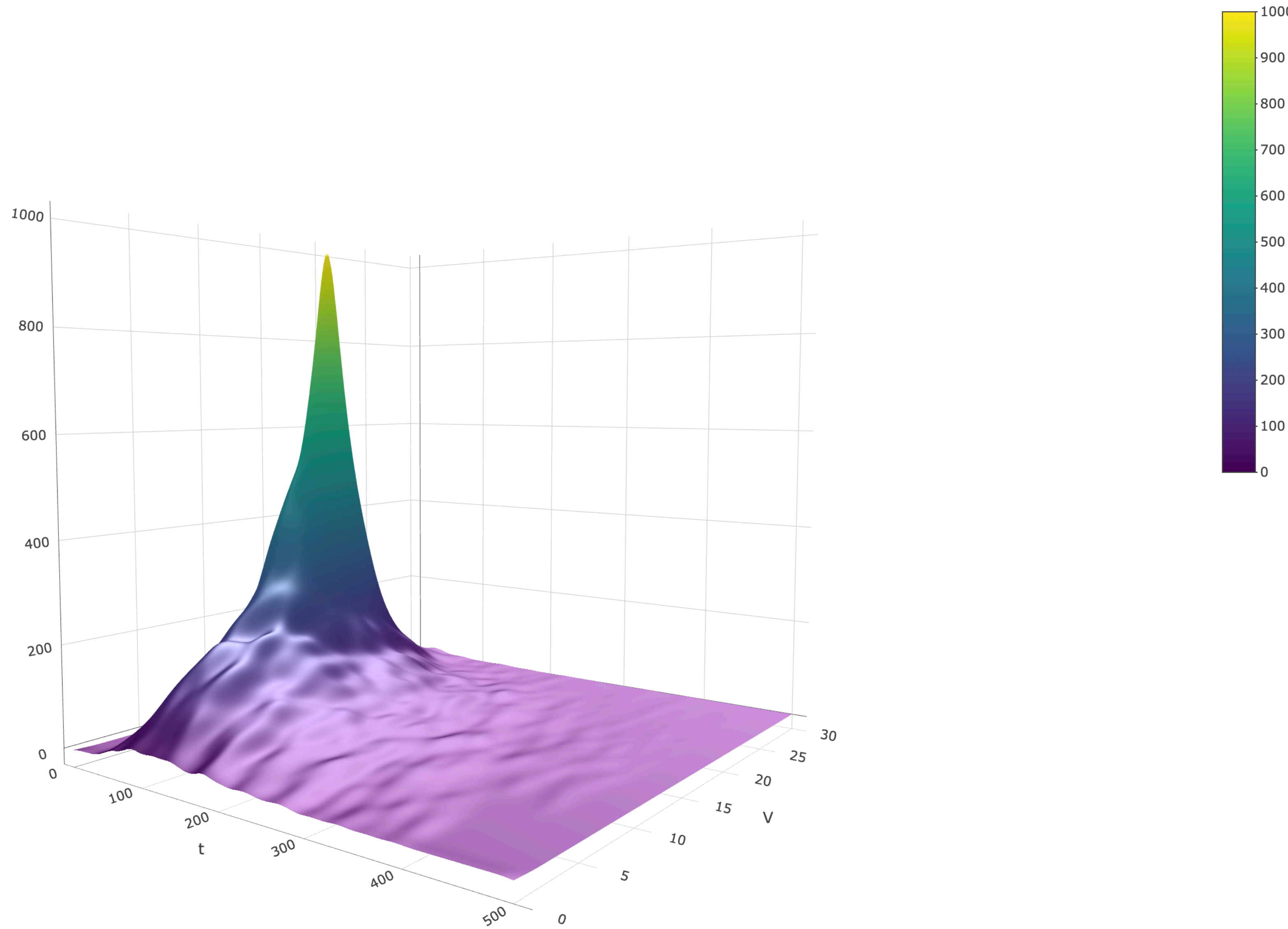


Figure 5: Narayana

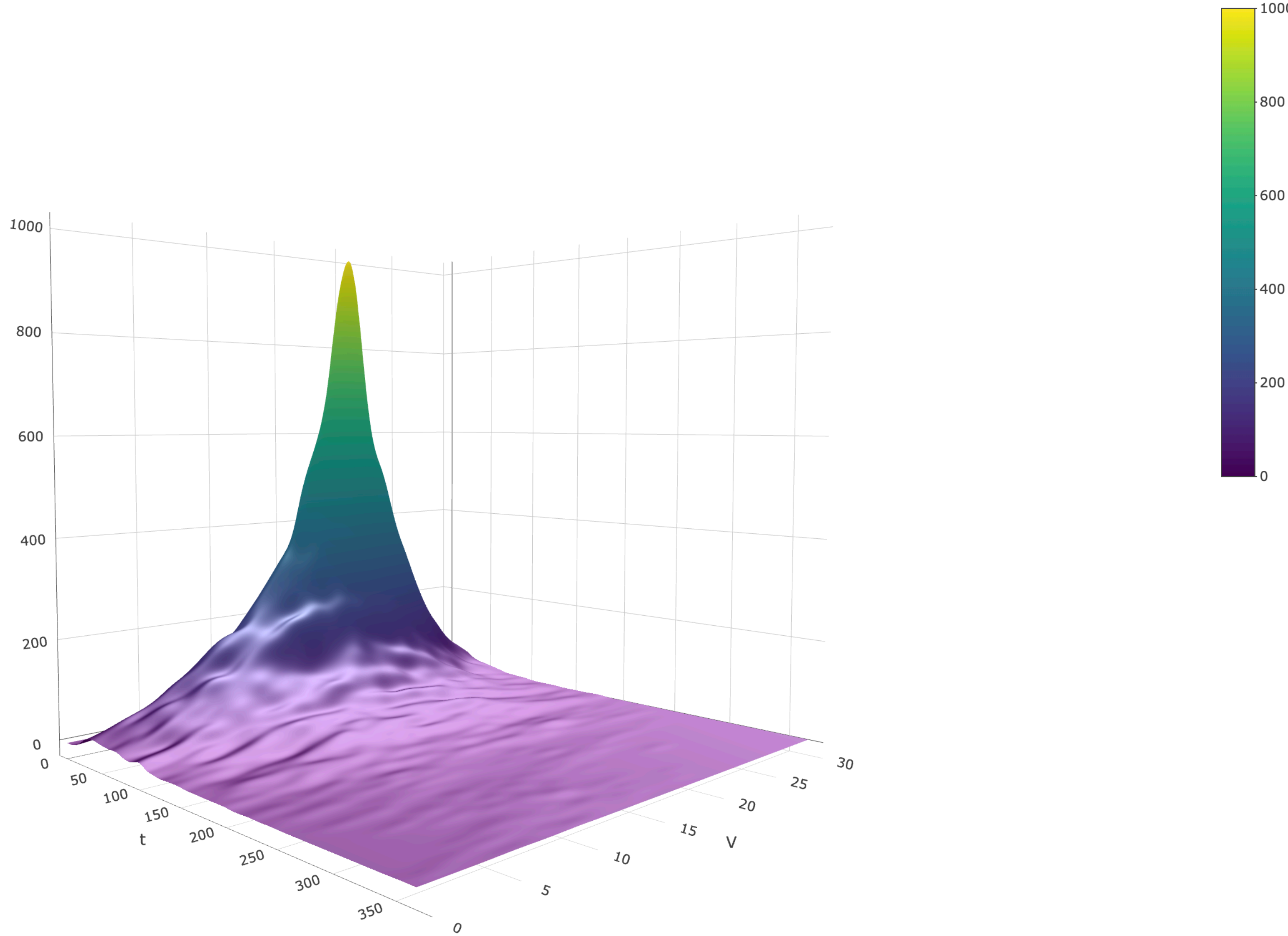


Figure 6: Fibonacci

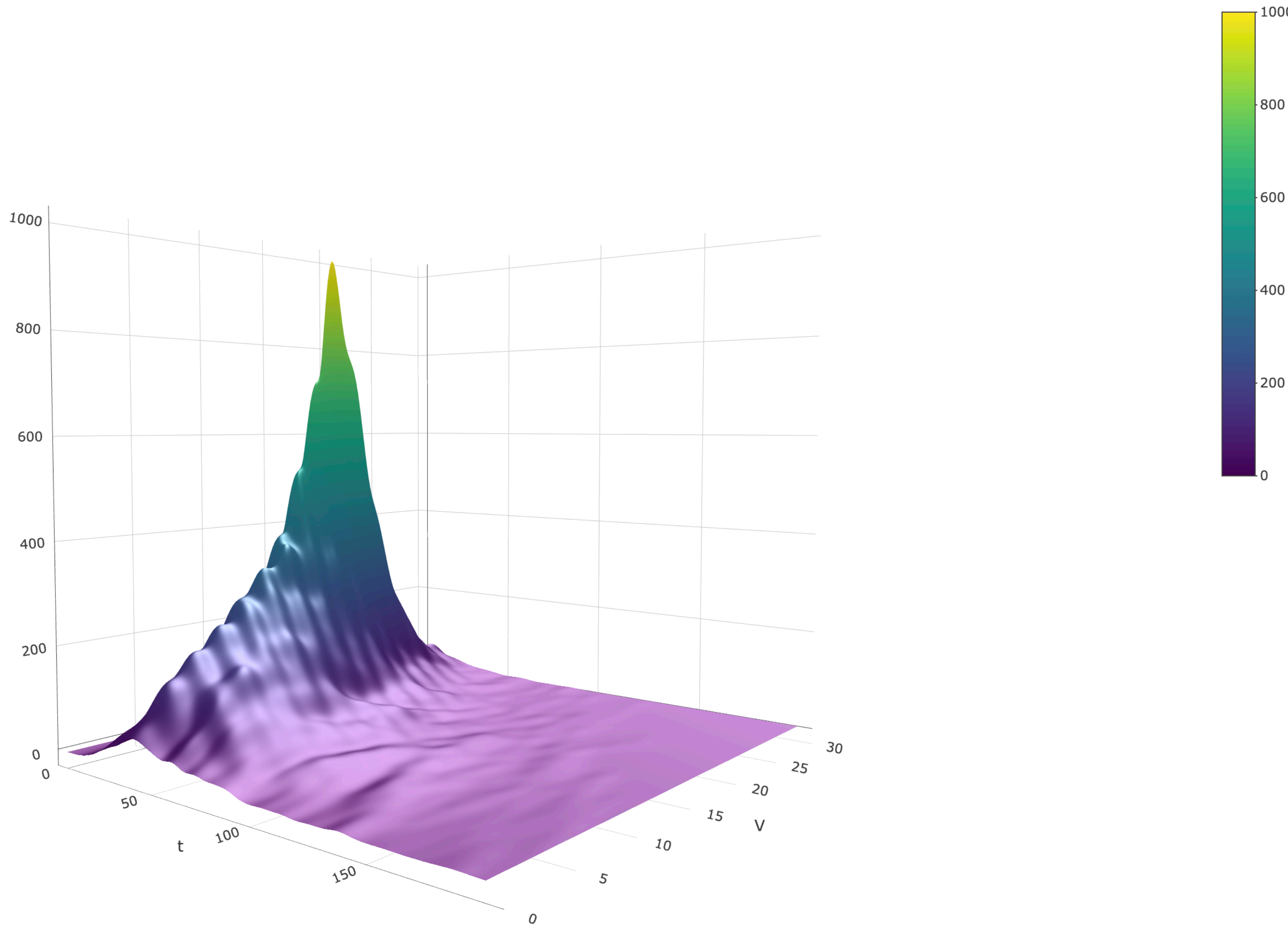


Figure 7: Martingale