Verification of a Mechanical-Load Clamp

Joshua D. Salvi

April 25, 2014

We wish to verify and calibrate each of the independent virtual impedance elements in a mechanical-load clamp. This note derives simplifications of the clamp's command for well-controlled conditions. We begin with the formal introduction of the mechanical-load clamp and simplify the equations after this introduction. The equation of motion for a hair bundle coupled to a flexible stimulus fiber is

$$m_{HB}\ddot{X} + \xi_{HB}\dot{X} + K_{HB}X - F_A = -\xi_{XX}\dot{X} - \xi_{\Delta X}\dot{\Delta} + K_{SF}(\Delta - X), \tag{1}$$

in which X is the position of the hair bundle and m_{HB}, ξ_{HB}, and K_{HB} are the bundle's mass, drag coefficient, and stiffness, respectively. Δ is the position of the base of the stimulus fiber, K_{SF} is the fiber's stiffness, ξ_{XX} is the drag coefficient owing to motion at the fiber's base, and $\xi_{\Delta X}$ is the drag coefficient owing to motion at the fiber's tip. F_A is any active force produced by the hair bundle. We can define motion at the fiber's base as the amplified error signal between a commanded position and the bundle's current position:

$$\Delta = \beta V_O = \beta G V_E = \alpha\beta G(X_C - X). \tag{2}$$

We can then introduce virtual impedance elements to load a hair bundle

$$m_{HB}\ddot{X} + \xi_{HB}\dot{X} + K_{HB}X - F_A = -m_V\ddot{X} - \xi_V\dot{X} - K_V X + F_E, \tag{3}$$

where m_V, ξ_V, K_V and F_E are the bundle's virtual mass, drag coefficient, stiffness, and external load. Combining Equations 1 and 3 allows us to apply this load with the clamp:

$$0 = \xi_{XX}\dot{X} + \xi_{\Delta X}\dot{\Delta} - K_{SF}(\Delta - X) - m_V\ddot{X} - \xi_V\dot{X} - K_V X + F_E. \tag{4}$$

Note that all hair-bundle forces have been eliminated from Equation 4. Using Equation 2, we can solve for the commanded position of the hair bundle:

$$K_{SF}X_C - \xi_{\Delta X}\dot{X}_C = \frac{m_V - (\xi_V - \xi_{XX} + \alpha\beta G\xi_{\Delta X})\dot{X} - [K_V - K_{SF}(1+\alpha\beta G)]X + F_E}{\alpha\beta G}. \tag{5}$$

It is Equation 5 that provides the mechanical-load clamp with control of all four virtual elements through adjustment of a commanded position and, by extension, the command voltage.

Isolation of Variables

We may now use Equation 5 and simplify it to plan calibration experiments that test the load clamp's virtual stiffness, virtual drag, and virtual mass. Initially, we will set all but one of the three terms to zero and solve for an equation that is experimentally tractable. Furthermore, a number of assumptions will be made to ensure that the experiments and subequent fits are meaningful. These assumptions can in principle be controlled during the experiment.

Virtual Stiffness

Setting the virtual mass and virtual drag coefficient in Equation 5 to zero results in a new solution for the commanded position

$$K_{SF}X_C - \xi_{\Delta X}\dot{X}_C = \frac{-(-\xi_{XX}+\alpha\beta G\xi_{\Delta X})\dot{X}-[K_V-K_{SF}(1+\alpha\beta G)]X+F_E}{\alpha\beta G}. \tag{6}$$

If we assume that $K_{SF}X_C \gg \xi_{\Delta X}\dot{X}_C$ and $\xi_{XX}\dot{X} \gg \xi_{\Delta X}\alpha\beta G\dot{X}$, Equation 6 becomes

$$\alpha\beta G K_{SF}X_C = \xi_{XX}\dot{X} - [K_V - K_{SF}(1+\alpha\beta G)]X + F_E. \tag{7}$$

The above assumptions are true if the stimulus fiber is stiff and gain G is small. If the system is at rest, such that the bundle's velocity is small,

$$\alpha\beta G K_{SF}X_C = [K_V - K_{SF}(1+\alpha\beta G)]X + F_E, \tag{8}$$

$$X = \frac{F_E - \alpha\beta G K_{SF}X_C}{K_V - K_{SF}(1+\alpha\beta G)}. \tag{9}$$

Equation 9 provides a solution to calibration of the virtual stiffness, with all other impedances zero. One can set values for an external force and a virtual stiffness, and the commanded position can increase in discrete steps. Plotting the bundle's mean position versus its total stiffness yields a relationship consistent with Hooke's law of elasticity. In the special case where the external force is large and gain G is small,

$$X = \frac{F_E}{K_V - K_{SF}}. \tag{10}$$

The calibration protocol for Equation 10 follows that of Equation 9, but a fit to the plot of position versus stiffness should yield a simpler solution.

Virtual Drag

Setting the virtual mass and virtual stiffness in Equation 5 to zero results in another solution for the commanded position

$$\alpha\beta G K_{SF}X_C - \alpha\beta G\xi_{\Delta X}\dot{X}_C = -(\xi_V - \xi_{XX} + \alpha\beta G\xi_{\Delta X})\dot{X} - [-K_{SF}(1+\alpha\beta G)]X + F_E. \tag{11}$$

If we assume that $K_{SF}X_C \gg \xi_{\Delta X}\dot{X}_C$, $\xi_V\dot{X} \gg \xi_{\Delta X}\alpha\beta G\dot{X}$, $\xi_{XX}\dot{X} \gg \xi_{\Delta X}\alpha\beta G\dot{X}$, $K_{SF}X \gg K_{SF}\alpha\beta G X$, Equation 11 becomes

$$\alpha\beta G K_{SF} X_C = -(\xi_V - \xi_{XX})\dot{X} + K_{SF} X + F_E. \tag{12}$$

The first assumption is true for a stiff stimulus fiber, and the last three assumptions are true for small values of G. We can then solve for the bundle's velocity as a function of force and drag

$$\dot{X} = \frac{F_E - \alpha\beta G K_{SF} X_C + K_{SF} X}{\xi_V - \xi_{XX}}. \tag{13}$$

If the external force is very large and the gain G is small, Equation 13 simplifies to

$$\dot{X} = \frac{F_E + K_{SF} X}{\xi_V - \xi_{XX}}. \tag{14}$$

Equation 14 provides a system in which the virtual mass and external force can be adjusted, and the bundle's velocity can be calculated. In doing so, a plot of the bundle's velocity versus total drag should provide an inverse relationship analogous to that in Equations 9 and 10.

Virtual Mass

Setting the virtual drag coefficient and virtual stiffness in Equation 5 to zero results in a third solution for the commanded position

$$\alpha\beta G K_{SF} X_C - \alpha\beta G \xi_{\Delta X} \dot{X}_C = -m_V \ddot{X} - (-\xi_{XX} + \alpha\beta G \xi_{\Delta X})\dot{X} + K_{SF}(1 + \alpha\beta G)]X + F_E. \tag{15}$$

If we assume that $K_{SF} X_C \gg \xi_{\Delta X} \dot{X}_C$ and $\xi_{XX} \dot{X} \gg \xi_{\Delta X} \alpha\beta G \dot{X}$, Equation 15 simplifies to

$$\alpha\beta G K_{SF} X_C = -m_V \ddot{X} + \xi_{XX} \dot{X} + K_{SF}(1 + \alpha\beta G)]X + F_E. \tag{16}$$

These assumptions are true if the fiber's stiffness is large and the gain G is small. We can then solve for the hair bundle's acceleration

$$\ddot{X} = \frac{F_E + \xi_{XX} \dot{X} - \alpha\beta G K_{SF} X_C + K_{SF}(1 + \alpha\beta G)]X}{m_V}. \tag{17}$$

Additionally, if the fiber's drag coefficient is small relative to the external force and if G is small, Equation 17 becomes

$$\ddot{X} = \frac{F_E + K_{SF} X}{m_V}. \tag{18}$$

Equation 18 provides a verification of the virtual mass in this clamp system. With inputs of an external force and a virtual mass, one may calculate the bundle's acceleration upon initiation of the load clamp. Plotting the bundle's acceleration versus its virtual mass provides a fit similar to that in Equations 9 and 14. The force in Equations 14 and 18 is dependent both upon the external force applied and the fiber's stiffness.

Summary

By isolating each of the impedance elements and setting all others to zero, we may track the bundle's motion and subsequently extract the forces applied to that bundle. While these protocols do not explicitly demonstrate independence of each of the four impedance elements, they do verify each of their individual contributions upon isolation. Subsequent calibration then requires that combinations of these parameters be applied and the bundle motion tracked. The relative contributions of each impedance may then be extracted from this information.