Derivation of the Hopf Normal Form

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Normal Form From a Planar System

First, consider the following planar system:

$$\dot{x}_1 = \mu x_1 - x_2 + Ax_1(x_1^2 - x_2^2),$$
$$\dot{x}_2 = x_1 + \mu x_2 + Ax_2(x_1^2 - x_2^2).$$

Let $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. The previous equations then become:

$$\dot{z} = \dot{x}_1 - i\dot{x}_2 = \mu x_1 - x_2 + Ax_1(x_1^2 - x_2^2) - i\left[x_1 + \mu x_2 + Ax_2(x_1^2 - x_2^2)\right],$$

which simplifies to

$$\dot{z} = \mu z + iz + A(x_1^3 + ix_2^3 + x_1 x_2^2 + ix_1^2 x_2).$$

Note that the term iz may be multiplied by a frequency term ω by using the planar system

$$\dot{x}_1 = \mu x_1 - \omega x_2 + Ax_1(x_1^2 - x_2^2),$$
$$\dot{x}_2 = \omega x_1 + \mu x_2 + Ax_2(x_1^2 - x_2^2).$$

By combining the above equations, this results in the normal form of the Hopf bifurcation up to the third-order component:

$$\dot{z} = \mu z + i\omega z + Az^2\bar{z}.$$

For a supercritical Hopf bifurcation A = 1 in which the origin is globally asymptotically stable, and for a subcritical Hopf bifurcation A = -1 in which the origin is locally asymptotically stable but is surrounded by an unstable circular cycle. The Hopf frequency is then defined by ω. We can alternatively define the normal form in polar coordinates using $z = \rho e^{i\varphi}$:

$$\dot{\rho} = \rho(\mu + i\omega + A\rho^2),$$
$$\dot{\varphi} = 1.$$

Here it becomes apparent that the limit cycle can be approximated by a circle with radius $\sqrt{\mu}$ for a supercritical Hopf bifurcation or an unstable cycle with radius $\sqrt{-\mu}$ for a subcritical Hopf bifurcation.