Visualization of a Universal Unfolding of the Pitchfork

John Kluesner

December 13, 2013

1 Introduction

In computational science and applied mathematics, differential equations are of very high use. One often studies how the solution of the differential equation changes when a parameter is altered. When there is a drastic variation in the general behaviour of the solutions of the differential equation, we call it a bifurcation.

This report explores a certain type of bifurcation: the pitchfork. In particular, we study a universal unfolding of the pitchfork:

$$G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2$$

in the differential equation $\dot{x} + G(x, \lambda; \alpha) = 0$ where λ is the main bifurcation parameter and $\alpha \in \mathbf{R}^2$ are the auxiliary parameters. We use theory from *Groups and Singularities in Bifurcation Theory* by Martin Golubitsky and David Schaeffer throughout and often without proof.

The first section of the report will detail the mathematics that supports the visualization. Mainly, we will detail how to find the transition variety of $G(x, \lambda; \alpha)$. The mathematics will be limited as the main purpose of the report will be creating and justifying the design of the visualization.

The dataset used is generated using Python 2.7 with the NumPy library. We create a VTR file using the EVTK library created by Paulo Herrera. Then, we load the dataset into ParaView to create the visualization. In particular, we map λ to the x-axis, α_1 to the y-axis, x to the z-axis, and we use α_2 as time. The mesh used is a rectilinear grid. The main visualization is an isosurface at $G(x, \lambda; \alpha_2) = 0$, which is generated using ParaView's contour filter and colored to show stability. Then, we add slices on the isosurface to show the bifurcation diagrams.

The final result is a visualization that displays all perturbed and unperturbed bifurcation diagrams of the pitchfork. We show that it follows the guidelines of visualization set out by Helen Wright in *Introduction to Scientific Visualization* and Stephen Kosslyn's psychological principles of good graph design in *Graph Design for the Eye and Mind.*

2 Mathematical Background

This section of the report borrows heavily from *Groups and Singularities in Bifurcation Theory* by Martin Golubitsky and David Schaeffer. Because the main content of this report is the visualization, I will not give precise definitions of many of the mathematical terms; I will instead give a more geometric interpretation.

First, let's consider $G(x)$ in the differential equation $\dot{x} + G(x) = 0$. We call $x^* \in \mathbf{R}$ a fixed point if $G(x^*) = 0$. Further, we call a fixed point stable if for any solution $x(t)$ with initial condition close to x^*, then $x(t)$ tends to x^*. If a fixed point isn't stable, we call it unstable. This leads us to the first theorem will we be using for the visualization.

Theorem 1. Consider the differential equation $\dot{x} + G(x) = 0$[1]. A fixed point is stable if $G_x(x^*) > 0$, and unstable if $G_x(x^*) < 0$.

We can sketch an informal proof of this rather easily. Consider a solution of the form $x(t) = p(t) + x^*$; that is, a solution that is perturbed slightly from a fixed point solution. Then, $\dot{p} = \dot{x} = -G(x)$. So, we take the Taylor Series of $G(x)$ expanded around x^*:

$$\begin{aligned} G(x) &= G(x^*) + (x - x^*)G_x(x^*) + \cdots \\ &= G(x^*) + p(t)G_x(x^*) + \cdots \\ &= p(t)G_x(x^*) + \cdots , \end{aligned}$$

and then if we disregard everything past the ellipsis, we get $\dot{p} = -G(x) = -G_x(x^*)p(t)$. This shows us that if $G_x(x^*) < 0$, then $p(t)$ has exponential growth; and if $G_x(x^*) > 0$, then $p(t)$ has exponential decay. It should be noted that Theorem 1 tells us nothing about the stability of the fixed point if $G_x(x^*) = 0$. For a more formal proof, see chapter 1, section 4 in Golubitsky and Schaeffer [1].

Next, let's discuss bifurcations. For that we have to consider $\dot{x} + G(x, \lambda) = 0$ where λ is called our main bifurcation parameter. Let $n(\lambda)$ denote the number of real, distinct fixed points at λ. A bifurcation occurs at λ^* if $n(\lambda) \neq n(\lambda^*)$ for all λ sufficiently close to λ^*.

The most important object of the visualization will be bifurcation diagrams, so let's review those. Consider the pitchfork $G(x, \lambda) = x^3 + \lambda x$ in $\dot{x} + G(x, \lambda) = 0$. Note that the fixed points are the real solutions of $x(x^2 + \lambda) = 0$. This leads to the following table:

$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
$n(\lambda) = 3$	$n(\lambda) = 1$	$n(\lambda) = 1$
stable: $x^* = \pm\sqrt{-\lambda}$	stable: $x^* = 0$	stable: $x^* = 0$
unstable: $x^* = 0$		

Observe how at $\lambda = 0$ the number of fixed points changes from 3 to 1; i.e., it's a bifurcation. Now, we can draw the bifurcation diagram, which is a λ-x^* plot. The equations are given by the 3rd and 4th rows of the table above. The bifurcation diagram can be seen in Figure 1.

It often happens that if you're working with a differential equation, you have multiple parameters to deal with. So, let's consider adding more parameters to $G(x, \lambda)$. We denote

[1] Because our convention is $\dot{x} + G(x) = 0$ instead of $\dot{x} = G(x)$, the stability test may be the opposite of similar theorems proved in other sources. We use this convention to be consistent with Golubitsky and Schaeffer.

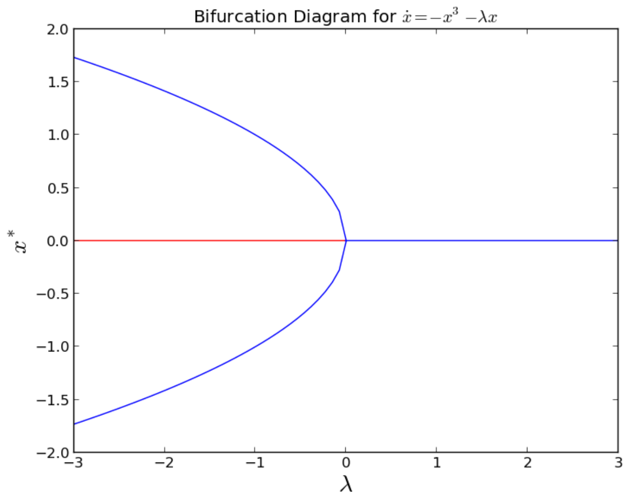

Figure 1: The bifurcation diagram for $\dot{x} + x^3 + \lambda x = 0$. The blue lines indicate stable fixed points and the red lines indicate unstable fixed points. Observe how the number of fixed points changes at $\lambda = 0$.

these parameters as an n-tuple $\alpha \in \mathbf{R}^n$ and call them auxiliary parameters. They are defined as less significant than the main bifurcation parameter λ; the effect they have on the system is as a perturbation to the bifurcation diagram. There are multiple ways to add auxiliary parameters to the pitchfork, here are a few examples:

1) $G_1(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1$,

2) $G_2(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x + \alpha_2$, or

3) $G_3(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2$.

We call G_1, G_2, and G_3 unfoldings of the pitchfork.

A good question to ask is which unfoldings are the most important; which one can we study to understand all the others as well. That question is answered by considering universal unfoldings. A universal unfolding is an unfolding that contains all possible perturbed bifurcation diagrams and doesn't have any redundant auxiliary parameters. For a more precise definition, see chapter 3, sections 1 in [1]. Proving something is a universal unfolding is a very involved process. We'll state without proof that G_3 is a universal unfolding of the pitchfork. For some believability, we'll show that the unfoldings G_1 and G_2 are also in G_3. Observe that

$$G_3(x, \lambda, ; 0, -\alpha_1) = x^3 + \lambda x + \alpha_1 = G_1(x, \lambda; \alpha).$$

We say that G_1 factors through G_3. Next, observe that G_2 factors through G_3:

$$G_3(x, \lambda + \alpha_1; 0, -\alpha_2) = x^3 + \lambda x + \alpha_1 x + \alpha_2 = G_2(x, \lambda; \alpha).$$

Additionally, we call the number of auxiliary parameters in a universal unfolding the codimension; the pitchfork has codimension 2.

Now that we have a universal unfolding of the pitchfork, we will describe persistent diagrams. We call a bifurcation diagram of a universal unfolding persistant if a small perturbation doesn't change the general structure of the diagram. We focus mostly on nonpersistant diagrams; that is, diagrams such that a small perturbation changes the bifurcation diagram in a significant way. It's sort of like the auxiliary parameters create a bifurcation of the bifurcation diagram. In [1], Golubitsky and Schaeffer prove a theorem that implies the following theorem.

Theorem 2. Let G be a universal unfolding with codimension 2. Then

$$\begin{aligned}
\mathscr{B} &= \{\alpha \in \mathbf{R}^2 \mid \exists x, \lambda \text{ such that } G = G_x = G_\lambda = 0 \text{ at } (x, \lambda, \alpha)\},\\
\mathscr{H} &= \{\alpha \in \mathbf{R}^2 \mid \exists x, \lambda \text{ such that } G = G_x = G_{xx} = 0 \text{ at } (x, \lambda, \alpha)\}, \text{ and}\\
\mathscr{D} &= \{\alpha \in \mathbf{R}^2 \mid \exists x_1, x_2, \lambda, x_1 \neq x_2 \text{ such that } G = G_x = 0 \text{ at } (x_i, \lambda, \alpha) \text{ for } i = 1, 2\}.
\end{aligned}$$

classify all nonpersistent diagrams.

Also, we call $\Sigma = \mathscr{B} \cup \mathscr{H} \cup \mathscr{D}$ the transition set as it acts as transitions between persistent diagrams. So, let's explore these sets. First, we can show that $\mathscr{D} = \emptyset$ for the universal unfolding of the pitchfork using the following lemma.

Lemma 1. Let $u \in \mathbf{R}[x]$ be a monic polynomial with degree 3. Then, u and the derivative of u do not share two distinct roots.

Proof. For the sake of contradiction assume there exists a monic polynomial $u \in \mathbf{R}[x]$ with degree 3 such that x_1, x_2 are distinct roots of u and u'. Then, without loss of generality, u can be expressed as either

$$u(x) = (x - x_1)^2(x - x_2), \quad \text{or} \quad u(x) = (x - x_1)(x - x_2)(x - x_3).$$

If $u(x) = (x - x_1)^2(x - x_2)$, then

$$\begin{aligned}
u'(x) &= 2(x - x_1)(x - x_2) + (x - x_1)^2\\
&= (x - x_1)(2(x - x_2) + (x - x_1))\\
&= (x - x_1)(2x - 2x_2 + (x - x_1))\\
&= (x - x_1)(3x - 2x_2 - x_1).
\end{aligned}$$

So, for x_2 to be a root of u', we would need $3x_2 - 2x_2 - x_1 = 0$, but that implies $x_2 = x_1$, which is a contradiction. Next, if $u(x) = (x - x_1)(x - x_2)(x - x_3)$, then

$$u'(x) = (x - x_1)(x - x_2) + (x - x_2)(x - x_3) + (x - x_1)(x - x_3).$$

For x_1 to be a root of u', we would need $u'(x_1) = (x_1 - x_2)(x_1 - x_3) = 0$. Observe that $x_1 \neq x_2$, so $x_1 = x_3$, but that leads back to the first case. Thus, u and u' do not share two distinct roots.

□

So, let's figure out what $\mathscr{B}$ and $\mathscr{H}$ are. First observe that

$$\begin{aligned} G &= x^3 + \lambda x + \alpha_1 x^2 - \alpha_2, \\ G_x &= 3x^2 + \lambda + 2\alpha_1 x, \\ G_{xx} &= 6x + 2\alpha_1, \text{ and} \\ G_\lambda &= x. \end{aligned}$$

To determine $\mathscr{B}$, note that if $G_\lambda = 0$, then $x = 0$. Further, if $G_x = 0$ then $\lambda = 0$. So, if $G = 0$, then $\alpha_2 = 0$ as well. Thus, $\mathscr{B} = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid \alpha_2 = 0\}$.

To determine $\mathscr{H}$, first, note that $G_{xx} = 0$ gives us $x = -\alpha_1/3$. Second, $G_x = 0$ gives us $\lambda = \alpha_1^2/3$. Finally, $G = 0$ gives us $\alpha_2 = -\alpha_1^3/27$. Thus, $\mathscr{H} = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid \alpha_2 = -\alpha_1^3/27\}$. Therefore, the transition set is

$$\Sigma = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid \alpha_2 = -\alpha_1^3/27 \text{ or } \alpha_2 = 0\}.$$

After obtaining Σ, we're ready to create the transition variety, which is a α_1-α_2 plot showing where the different persistent diagrams are located. This can be seen in Figure 2. The graph shows us how Σ sections off the α_1-α_2 plot into 4 parts, each corresponding to a different persistent diagram. The nonpersistent diagrams occur at points in Σ.

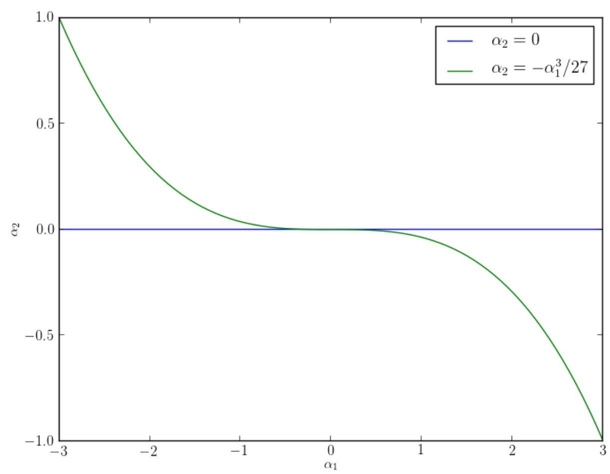

Figure 2: The transition variety for the universal unfolding of the pitchfork: $G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2$. Note that the plot is blocked off into 4 sections by the transition set, which corresponds to 4 different persistent diagrams. The nonpersistent diagrams occur at points in the transition set.

The last part of the mathematical theory we need is the next theorem.

Theorem 3. Let G be a function such that $G(x_0, \lambda_0; \alpha_0) = 0$. A necessary condition for a bifurcation to occur at $(x_0, \lambda_0; \alpha_0)$ is $G_x(x_0, \lambda_0; \alpha_0) = 0$.

Theorem 3 can be easily proven using The Implicit Function Theorem. For a proof, see Appendix I in [1].

3 Generating the Dataset

The dataset is generated using Python 2.7. The external libraries used are NumPy[2] and EVTK[3]. NumPy is used for its fast array arithmetic in generating the dataset. EVTK is a library created by Paulo Herrera and is used to generated a VTR file which will be opened in ParaView.

In generating the dataset, the first problem we run into is that we have 4 independent variables. However, all we care about are the fixed points; i.e., when

$$G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2 = 0.$$

What we want to do is use ParaView's contour filter to create an isosurface at $G(x, \lambda; \alpha) = 0$. To do that, we instead use the contour filter at

$$G(x, \lambda; \alpha_1) = x^3 + \lambda x + \alpha_1 x^2 = \alpha_2$$

and vary α_2 over time. This allows us to conveniently use time as our 4th dimension.

Convention for bifurcation diagrams is to have x^* mapped to the y-axis and λ mapped to the x-axis, as in Figure 1. So for our mesh, we map λ to the x-axis, α_1 to the y-axis, and x to the z-axis. This way we can use slices of the isosurface to show standard bifurcation diagrams.

For choosing the extents of the grid, I have found $-3 \leq x,\ \alpha_1,\ \lambda \leq 3$ to work well through experimenting. Next, we have to choose the right type of grid. Note that by Theorem 3, there is a bifurcation at $x = \lambda = \alpha_1 = \alpha_2 = 0$. So, we should focus more points around the origin. A good way to do this would be $f : [-3, 3] \to [-3, 3]$ defined as $f(t) = t^3/9$. Therefore, we are using a rectilinear grid.

Note that $G_x(x, \lambda; \alpha_1) = 3x^2 + \lambda + 2\alpha_1 x$ is free of α_2; so we can use Theorem 1 to test for stability. Specifically, we use the signum function as the only importantance is the sign of $G_x(x, \lambda; \alpha_1)$ rather than the actual value. However, by Theorem 3, the values such that $G_x(x, \lambda; \alpha_1) = 0$ do have significance. So, we create an isosurface at $G_x = 0$ on the isosurface of $G = \alpha_2$ to find the bifurcation points.

This tells us that our mesh will be a 3 dimensional rectilinear grid of (λ, α_1, x) triples and the attributes will be

1) G – evaluations of the function $G(x, \lambda, \alpha_1) = x^3 + \lambda x + \alpha_1 x^2$ used to create isosurfaces at various values of $G(x, \lambda, \alpha_1) = \alpha_2$. This creates a contour of fixed points for our universal unfolding of the pitchfork;
2) G_x – evaluations of the function $G_x(x, \lambda, \alpha_1) = 3x^2 + \lambda + 2\alpha_1$ used to locate the bifurcation values using a contour filter at $G_x(x, \lambda, \alpha_1) = 0$; and
3) stability – $\mathrm{sign}(G_x(x, \lambda, \alpha_1))$ used to test for stability by Theorem 1.

Note that the three attributes above are scalars.

The code that generates the dataset will appear at the end of this report.

---

[2] NumPy can be downloaded at `http://sourceforge.net/projects/numpy/files/`

[3] EVTK can be downloaded at `https://bitbucket.org/pauloh/pyevtk`

4 Overview of the Visualization

The point of this section is to give a high level overview of the visualization. The focus will be what the visualization shows and the thought process used when creating it.

What this visualization intends to show is all the different perturbed and unperturbed bifurcation diagrams of the pitchfork. So, what is a perturbed bifurcation diagram? We showed a bifurcation diagram earlier for the standard pitchfork bifurcation, $\dot{x}+x^3+\lambda x = 0$; that was an unperturbed bifurcation diagram. A perturbed bifucation diagram would be what would happen if we changed that equation slightly. For example, Figure 3 shows the bifurcation diagram for $\dot{x}+x^3+\lambda x-0.138=0$, and Figure 4 shows the bifurcation diagram for $\dot{x}+x^3+\lambda x-2x^2=0$. We keep the same convention with red denoting unstable fixed points, and blue denoting stable fixed points. Notice how they slightly alter the pitchfork bifurcation diagram in Figure 1.

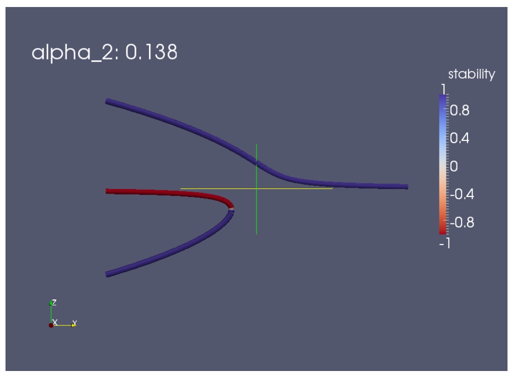

Figure 3: The bifurcation diagram for $\dot{x}+x^3+\lambda x-0.138=0$. Notice how it looks similar to the pitchfork bifurcation in Figure 1.

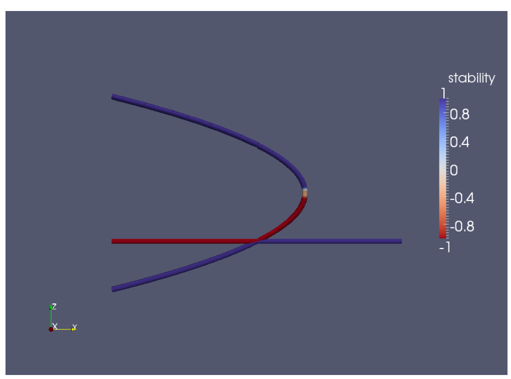

Figure 4: The bifurcation diagram for $\dot{x}+x^3+\lambda x-2x^2=0$. Notice how it looks similar to the pitchfork bifurcation in Figure 1.

The question we're trying to answer in this visualization is "How can we use ParaView to show all of these bifurcation diagrams?" We get our answer using the theory in section 1 of this report. In particular, we'll be using the universal unfolding of the pitchfork

$$G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2 \tag{1}$$

and the transition set

$$\Sigma = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid \alpha_2 = -\alpha_1^3/27 \ \text{ or } \ \alpha_2 = 0\}. \tag{2}$$

To do this, we would like to create a mesh of the variables in (1) and use ParaView's contour filter to make an isosurface showing the fixed points. From here, we can use slices to create the bifurcation diagrams. For example, Figure 5 shows this technique using the unfolding $G(x, \lambda; \alpha_1) = x^3 + \lambda x + \alpha_1 x^2$. The only problem with this approach is that (1) has 4 independent variables. So, we have to use a four dimensional plot. The fourth dimension we choose is time.

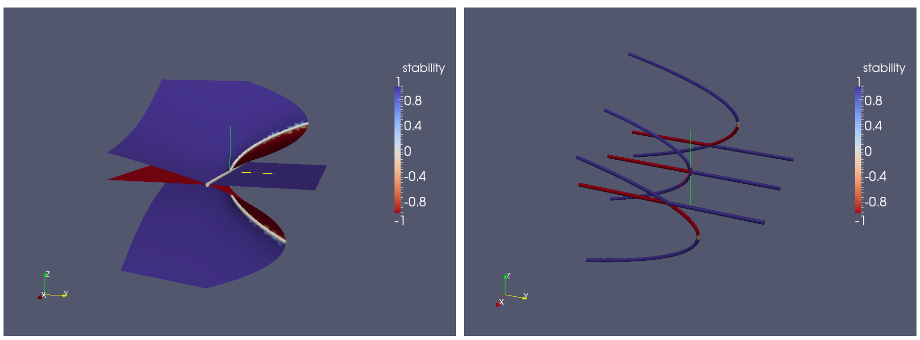

Figure 5: Visualizations of the bifurcations of the unfolding $G(x, \lambda; \alpha_1) = x^3 + \lambda x + \alpha_1 x^2$. The left panel shows the isosurface when $G(x, \lambda; \alpha_1) = 0$, and the right panel is slices of that isosurface for $\alpha_1 = -2, 0, 2$. Notice how the slices create bifurcation diagrams, perturbed and unperturbed.

Now let's make the visualization that incorporates all the variables in (1). We map λ to the x-axis, α_1 to the y-axis, x to the z-axis, and α_2 to time. To get the bifurcations diagrams, we use the slicing technique described in the last paragraph. We have to carefully choose the slices though. To find the correct slices, we need to recall the transition variety, Figure 2. We're going to choose three values of α_1: one negative, one positive, and one at zero. Good choices are $\alpha_1 = -2, 0, 2$. See Figure 6 for a geometric reasoning of this claim.

Next, if you look at the right panel of Figure 5, it's hard to tell how α_1 perturbs the bifurcation diagrams. This is because we are treating α_1 as a discrete value, but it's actually continuous. However, the isosurface visualization in the left panel of Figure 5 doesn't have this problem. So, we have to find a way to incorporate both the isosurface and the slices into 1 visualization, while having the slices more salient. A good way to do this is to have the isosurface included with a reduced opacity.

One more piece of information that deserves some salience is where the bifurcations occur. Luckily, we can use Theorem 3, to find these values. We include an isosurface on the isosurface where $G_x = 0$. A good choice of color for this isosurface is white because

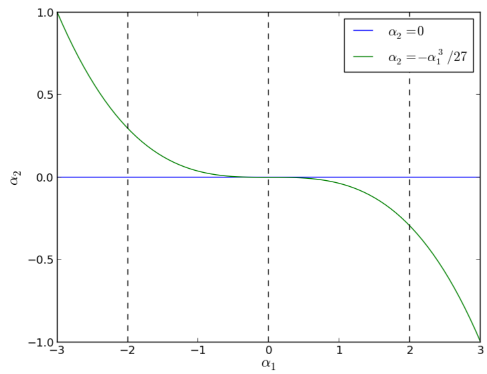

Figure 6: The transition variety shown earlier with dashed lines representing the fixed values of $\alpha_1 = -2, 0, 2$. Observe that the dashed lines cover all persistent and nonpersistent diagrams.

white has no chance of being confused with red or blue; there is no connection between white and red or blue. This all leads to the final visualization as can be seen in Figure 7.

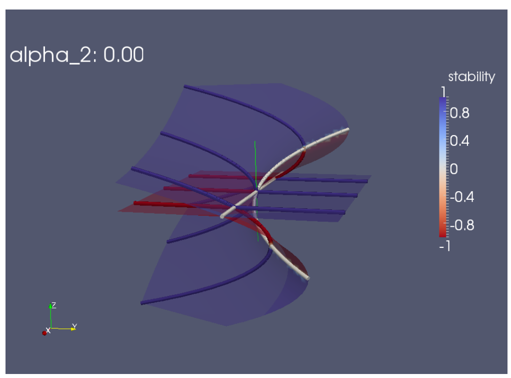

Figure 7: The final visualization of the universal unfolding of the pitchfork: $G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2$ with $\alpha_2 = 0$.

Now that we have a visualization we're happy with, we have to animate it. This is again something we have to be careful about. Because the most interesting parts of the visualization are the nonpersistent diagrams, it would make sense to pause the animation where these occur, and using (2), we can pinpoint where these are located: $\alpha_2 = \pm 8/27, 0$. So, we make our animation in a way that it slows down when we're in a small neighborhood around $\pm 8/27$, and then pauses when $(\alpha_1, \alpha_2) \in \Sigma$.

In conclusion, we have created a visualizion that indeed shows all the perturbed and unperturbed bifurcations diagrams of the pitchfork.

5 Justifying The Visualization

The main point of this section is to present and justify the visualization. The purpose of the visualization is to capture all bifurcation diagrams, persistent and nonpersistent. We will use Helen Wright's *Introduction to Scientific Visualization* [3], Stephen Kosslyn's *Graph Design for the Eye and Mind* [2], and the theorems and figures in section 2 for justification of the design. It will have some overlap with the other sections. This is intentional.

First, we will classify our dataset. Our independent variables are λ, α_1, and x. Each is continuous as they are real numbers. Our dependent variables are G, G_x and stability, which are all scalars. So, according to the table on page 65 of Wright's book, the visualization technique we should be using is a colored isosurface. Specifically, we will be making an isosurface at $G = \alpha_2$ to show the fixed points and we will color it by stability. Additionally, we will add a second isosurface at $G_x = 0$ onto the first isosurface; this second isosurface will be used to display the bifurcation values. Examples of what this isosurface looks like can be found in Figure 8.

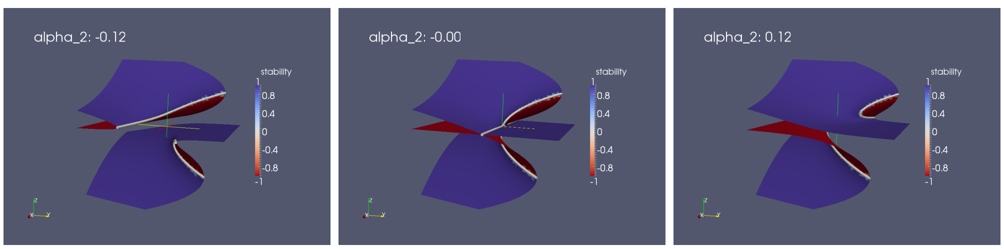

Figure 8: Three isosurfaces when $G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2 = 0$. The left panel is when $\alpha_2 < 0$, the middle panel is when $\alpha_2 = 0$, and the right panel is when $\alpha_2 > 0$. The blue areas show stable fixed points, the red areas show unstable fixed points, and the white parts show the bifurcation points. Observe that there is an obvious change in the isosurface when $\alpha_2 = 0$.

Coloring the isosurface by stability is justified by Kosslyn's Principle of Informative changes: the difference from unstable to stable fixed points is very important and thus carries sufficient information to warrant a color change. Also, we choose to use blue to representing stable and red to represent unstable. This is consistent with other authors in this field, so it follows Kosslyn's Principle of Compatibility. Additionally, we color the $G_x = 0$ isosurface as white. This is a good color choice because white has no implicit connection to either red or blue; it will not be incorrectly labeled as stable or unstable. This is consistent with Theorem 1 because the fixed points on this isosurface are notclassified by our stability test: if $G_x = 0$, the fixed point could be stable, unstable, or even half-stable.

This isosurface is a good start; however, it violates Kosslyn's Principle of Relevance because it shows too much information instead of focusing on the bifurcation diagrams. To fix this issue, we refer back to the transition variety (Figure 2). We see that if we fix three values of α_1: one less than 0, one equal to 0, and one greater than 0, we can show every bifurcation diagram. Good choices are $\alpha_1 = -2, 0, 2$; see Figure 9 for a geometric reasoning for this claim.

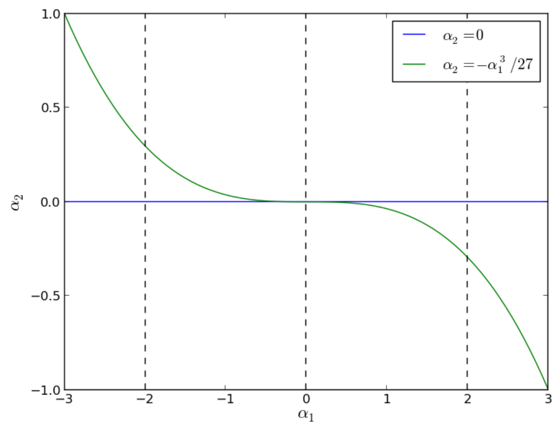

Figure 9: The transition variety shown earlier with dashed lines representing the fixed values of $\alpha_1 = -2, 0, 2$. Observe that the dashed lines cover all persistent and nonpersistent diagrams.

Fixing values of α_1 is equivalent to using slices on the isosurface in ParaView. See Figure 10 for examples of the visualization using the slices. This is good because it focuses the visualization back to the bifurcation diagrams. However, using only the slices has its drawbacks. For example, you lose a lot of information about how α_1 perturbs the bifurcation diagrams by making it seem like a discrete value. This form of the visualization therefore breaks Kosslyn's Principle of Compatibility.

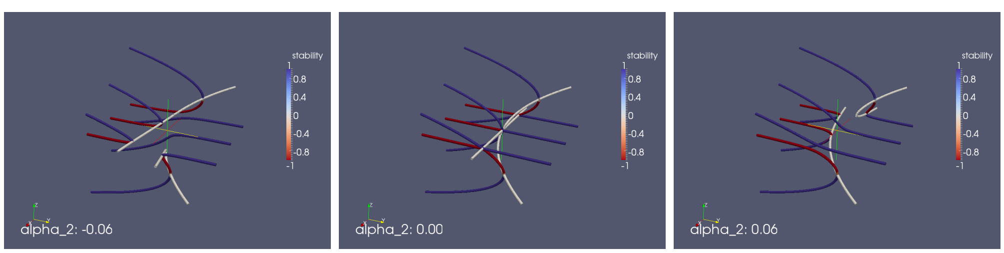

Figure 10: Three slices of the isosurfaces when $G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2 = 0$. The three slices are at the values $\alpha_1 = -2, 0, 2$. Observe that it's hard to tell how α_1 perturbs the bifurcation diagrams.

What we have to do is find a way to incorporate the isosurface in Figure 8 and the slices in Figure 10 within a single visualization. A good way to do this is to have the isosurface be at a reduced opacity as can be seen in Figure 11. This visualization correctly focuses on the bifurcation diagrams and doesn't reduce α_1 into appearing as a discrete value. By lowering the opacity, we follow Kosslyn's Principle of Discriminability because it allows us to clearly discern the slices from the isosurface.

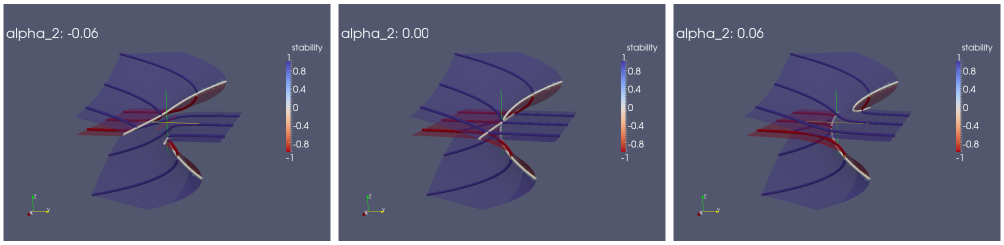

Figure 11: Examples of the final visualization of the universal unfolding of the pitchfork: $G(x, \lambda; \alpha) = x^3 + \lambda x + \alpha_1 x^2 - \alpha_2$. When the isosurface is animated, all bifurcation diagrams, persistant and nonpersistant, will appear as one of the slices.

Next, we will justify the animation. First, recall the transition set

$$\Sigma = \{(\alpha_1, \alpha_2) \in \mathbf{R}^2 \mid \alpha_2 = -\alpha_1^3/27 \text{ or } \alpha_2 = 0\}.$$

Because we are fixing values of α_1, we know what values of α_2 will have nonpersistent diagrams. In particular, we will have nonpersistence at $\alpha_2 = -8/27, 0, 8/27$. So, we will put emphasis on these points when creating our animation. This follows Kosslyn's Principle of Salience.

To emphasize the nonpersistence points, we will slow the animation down as α_2 approaches these points. Then, we will freeze the animation directly at these points. Also, when $|\alpha_2| > 8/27$, there will be no more nonpersistent diagrams. So, once α_2 is out of a small neighborhood of $\pm 8/27$, we will speed the animation up significantly. This is useful because it shows that the bifurcations diagrams will not change in any significant way. Figure 12 shows the keyframes used for the animation. Addionally, we should change the number of frames up from 10 to at least 101. Setting up the animation this way follows Kosslyn's Principle of Relevence.

	Time	Interpolation	Value
1	0	Ramp	-3
2	0.1	Ramp	-0.6
3	0.2	Ramp	-0.296
4	0.3	Ramp	-0.296
5	0.45	Ramp	0
6	0.55	Ramp	0
7	0.7	Ramp	0.296
8	0.8	Ramp	0.296
9	0.9	Ramp	0.6
10	1	Ramp	3

Figure 12: The keyframs that will be used in ParaView's animation view. Observe that the animation will pause at 0 and $\pm 0.296 \approx \pm 8/27$.

In conclusion, we have created a visualization that shows every single bifurcation diagram using visualization techniques such as slices, contours, and color. We have proven that every persistent and nonpersistent diagram appears by using the theory in Martin Golubitsky and David Schaeffer's Singularities and Groups in Bifurcation Theory. We showed the visualization is effective by citing Helen Wright's *Introduction to Scientific Visualization*, and Stephen Kosslyn's *Graph Design for the Eye and Mind.* Thus, our visualization meets the desired goal.

6 Bibliography

[1] Golubitsky, Martin, and David G. Schaeffer. *Singularities and Groups in Bifurcation Theory.* New York: Springer-Verlag, 1985.

[2] Kosslyn, Stephen Michael. *Graph Design for the Eye and Mind.* New York: Oxford UP, 2006.

[3] Wright, Helen. *Introduction to Scientific Visualization.* [London]: Springer, 2007.

7 Code

```
"""
This script creates a visualization of a universal unfolding of a pitchfork
bifurcation:

         G(x,lambda; alpha) = x^3 + lambda*x + alpha_1*x^2 - alpha_2

where dx/dt + G(x, lambda; alpha) = 0, lambda is the main parameter, and
alpha_1 and alpha_2 are auxiliary parameters.

This script will output a file pitchforkMesh.vtr. It is intended to be used in
ParaView. Steps to use the visualization in the most satisfying way are
outlined below.

To be able to run this script, you must have the EVTK package installed, which
can be found here:      https://bitbucket.org/pauloh/pyevtk/overview

The purpose of the visualization is to view all of the perturbed and
unperturbed bifurcation diagrams of the pitchfork. For the mathematical theory,
see "Singularities and Group in Bifurcation Theory" by Martin Golubitsky and
David G. Schaeffer.

Steps to create the pipeline and animation:
    1) Add a contour at G = 0. This represents the fixed points. Color it by
       stability.
    2) Create an animation with the isosurface changing from -3 to 3. This
       represents the alpha_2 pertubation parameter. Add 4 keyframes:
           time: 0     value: -3
           time: 0.1   value: -0.6
           time: 0.2   value: -0.296
           time: 0.3   value: -0.296
           time: 0.45  value: 0
           time: 0.55  value: 0
           time: 0.7   value: 0.296
           time: 0.8   value: 0.296
           time: 0.9   value: 0.6
           time: 1     value: 3
    3) Add 3 slices at (-2,0,0), (0,0,0), and (2,0,0) each with normal (1,0,0)
       to the contour.
    4) Add a tube on each slice.
    5) Add a contour to the isosurface at G_x = 0. These set of points are
       where the bifurcations occurs. Add a tube onto the contour and color it
       by a solid color; white works fine.

Steps to make an interesting visualization:
    1) Change the color map to diverging with red to blue; i.e., flip the
       default color map.
    2) Set visibility of all but the tubes and the isosurface at G = 0 to off.
    3) Set the opacity of the isosurface to 0.4.
    4) Press play.
```

```
Author:       John Kluesner
Date:         11 Dec, 2013
Email:        stringman45@gmail.com
"""

import numpy as np
from evtk.hl import gridToVTK

# Generate arrays of the values for lambda, alpha, and x. Restructure the grid
# to make more grid points around the origin as this is where a bifurcation
# occurs.
A = np.linspace(-3,3,101)
x_vals = A**3/9
lambdas = A**3/9
alphas = A**3/9

# Create the meshes that will be used to generate the attributes. The purpose
# of these meshes is to make good use of vectorization.
x_grid = x_vals.reshape(1, 1, len(x_vals))
x_grid = x_grid.repeat(len(lambdas), axis=1)
x_grid = x_grid.repeat(len(alphas), axis=0)
lambda_grid = lambdas.reshape(1, len(lambdas), 1)
lambda_grid = lambda_grid.repeat(len(x_vals), axis=2)
lambda_grid = lambda_grid.repeat(len(alphas), axis=0)
alpha_grid = alphas.reshape(len(alphas), 1, 1)
alpha_grid = alpha_grid.repeat(len(x_vals), axis=2)
alpha_grid = alpha_grid.repeat(len(lambdas), axis=1)

# Create the attribute for G = x^3 + lambda*x + alpha_1*x^2. The alpha_2
# auxiliary parameter will be included using ParaView.
G = x_grid**3 + lambda_grid*x_grid + alpha_grid*x_grid**2

# Create the attribute for G_x = 3*x^2 + lambda + 2*alpha*x
# and use the signum function as a test for stability.
Gx = 3*x_grid**2 + lambda_grid + 2*alpha_grid*x_grid
sgn_Gx = np.sign(Gx)

# Use EVTK to create the VTR file.
# The parameters for gridToVtk: (file name, x-coords of mesh, y-coords of mesh,
#                                 z-coords of mesh, pointData).
gridToVTK("./pitchforkMesh", alphas, lambdas, x_vals, \
            pointData={"G":G, "stability":sgn_Gx, "G_x":Gx})
```