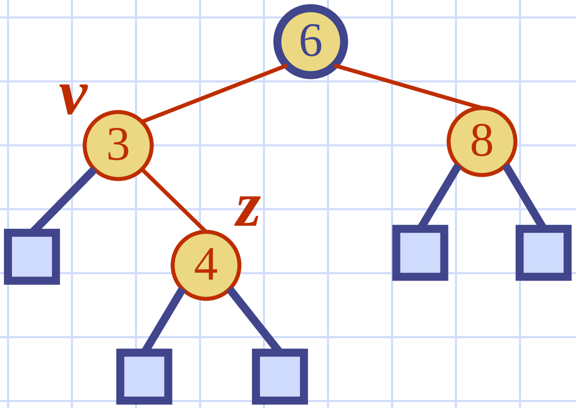
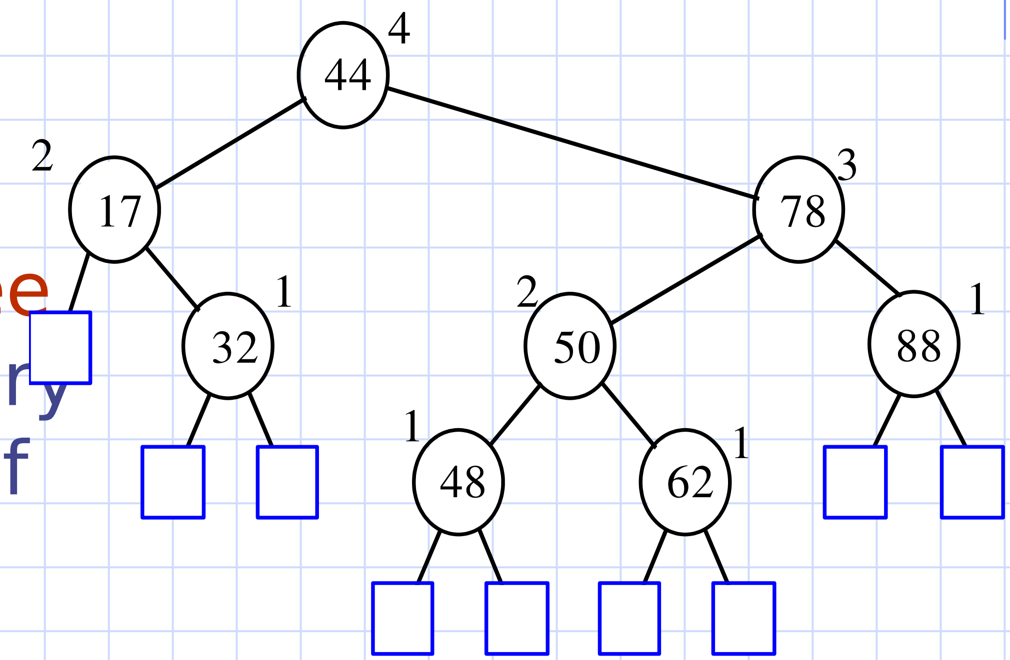
AVL Trees



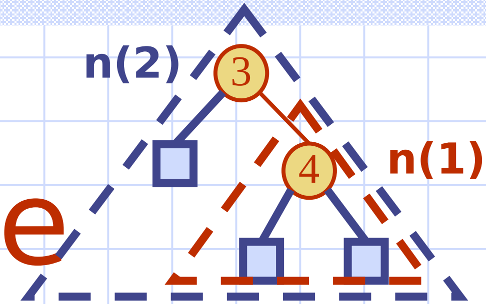
AVL Tree Definition

- AVL trees are balanced
- An AVL Tree is a binary search tree such that for every internal node v of T, the heights of the children of v can differ by at most 1



An example of an AVL tree where the heights are shown next to the nodes:

Height of an AVL Tree

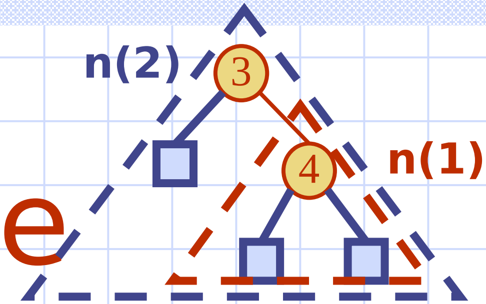


- Fact: The height of an AVL tree storing n keys is O(log n).
- Proof: Let us bound n(h): the minimum number of internal nodes of an AVL tree of height h.
- We easily see that n(1) = 1 and n(2) = 2
- For n > 2, an AVL tree of height h contains the root node, one AVL subtree of height n-1 and another of height n-2.
- That is, n(h) = 1 + n(h-1) + n(h-2)
- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2). So

  n(h) > 2n(h-2), n(h) > 4n(h-4), n(h) > 8n(n-6), … (by induction),

  n(h) > 2^in(h-2i)

Height of an AVL Tree



$$i = \left\lceil \frac{h}{2} \right\rceil - 1.$$

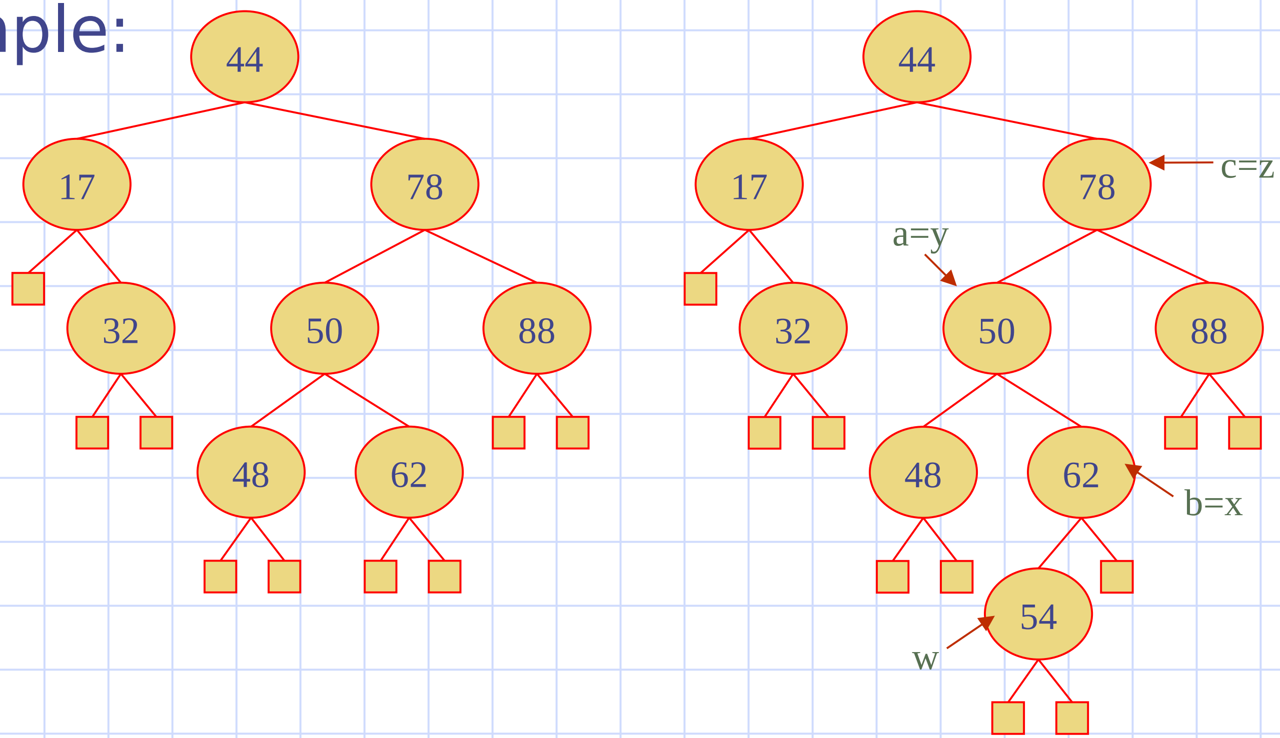
$$\begin{aligned} n(h) &> 2^{\lceil \frac{h}{2} \rceil - 1} \cdot n\left(h - 2\left\lceil \frac{h}{2} \right\rceil + 2\right) \\ &\geq 2^{\lceil \frac{h}{2} \rceil - 1} n(1) \\ &\geq 2^{\frac{h}{2} - 1}. \end{aligned}$$

Taking logarithms: h < 2log n(h) +2

Thus the height of an AVL tree is O(log n)

Insertion

- Insertion is as in a binary search tree
- Always done by expanding an external node.
- Example:



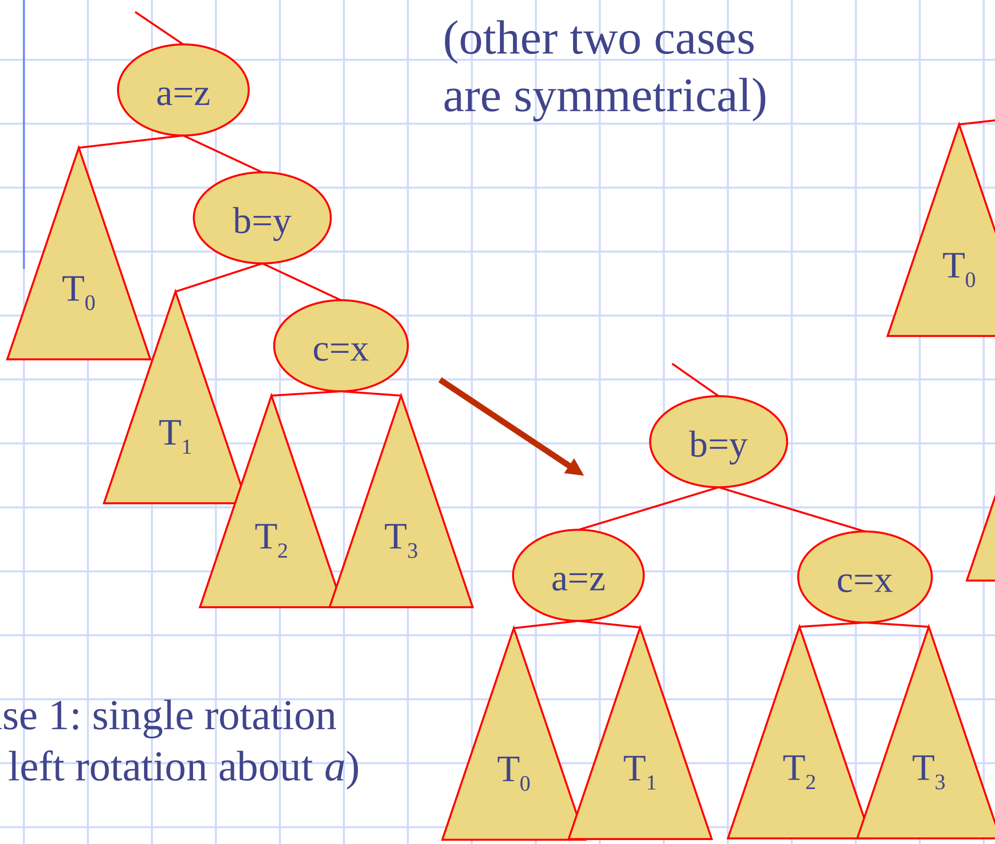
before insertion

after insertion

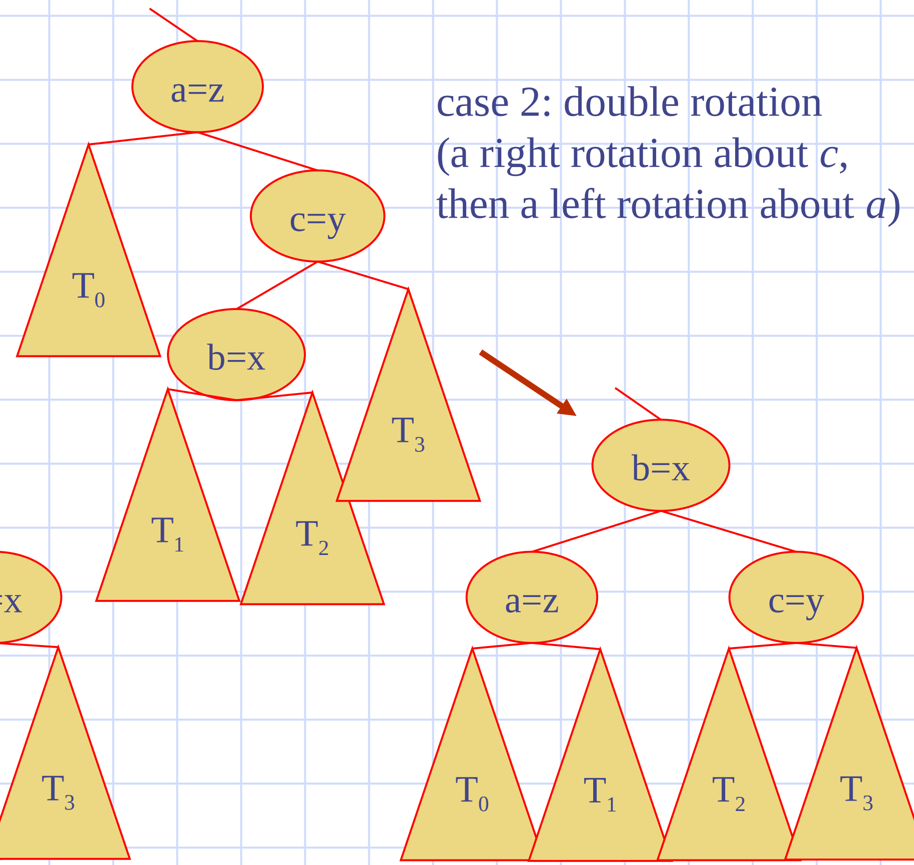
Trinode Restructuring

- let (a,b,c) be an inorder listing of x, y, z
- perform the rotations needed to make b the topmost node of the three

(other two cases are symmetrical)

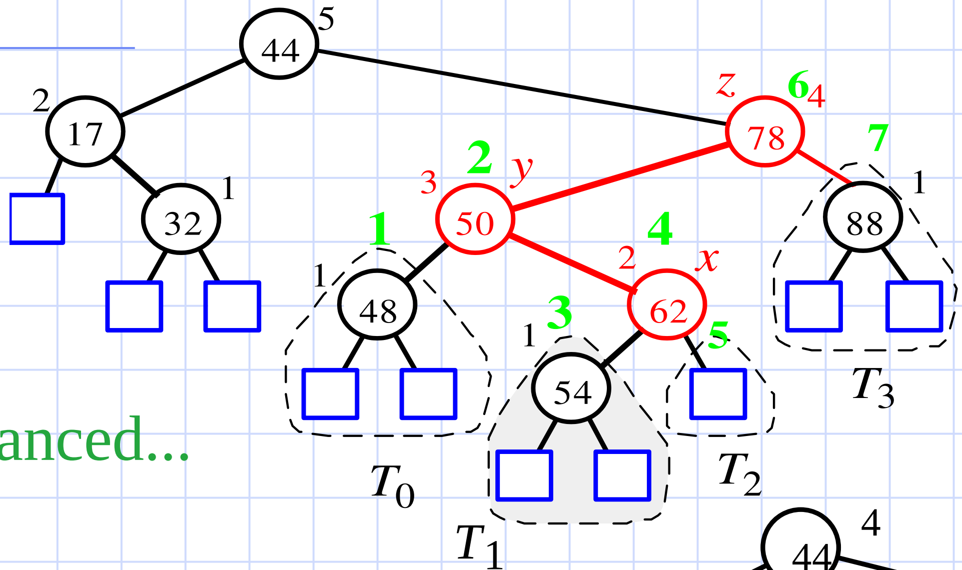


case 1: single rotation
(a left rotation about *a*)

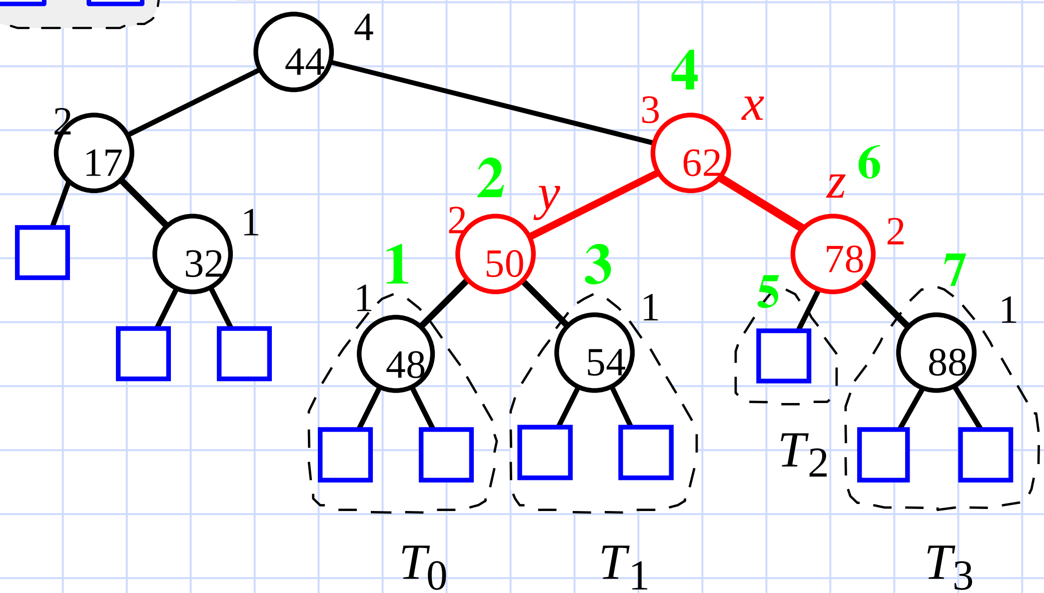


case 2: double rotation
(a right rotation about *c*,
then a left rotation about *a*)

Insertion Example, continued



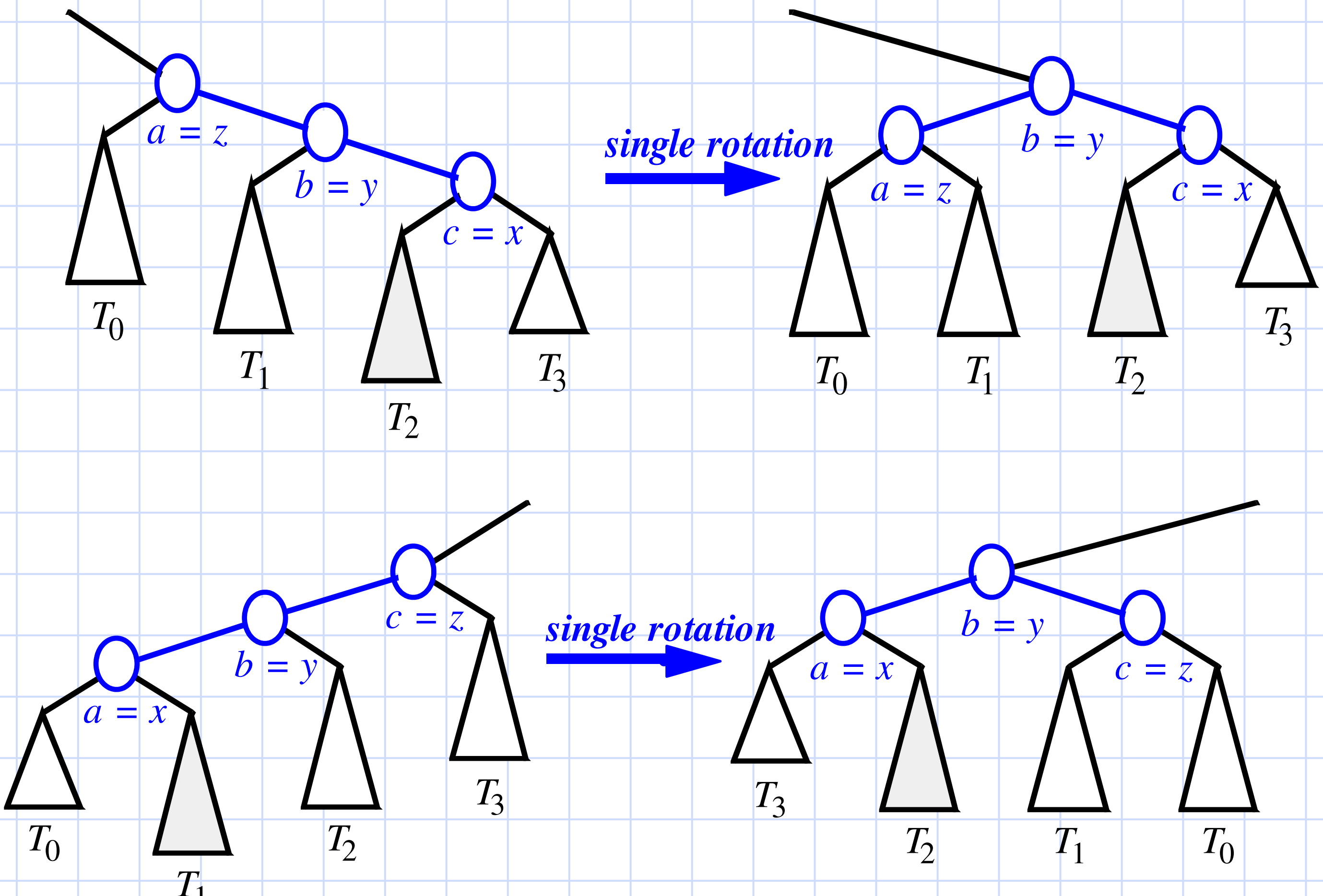
unbalanced...



...balanced

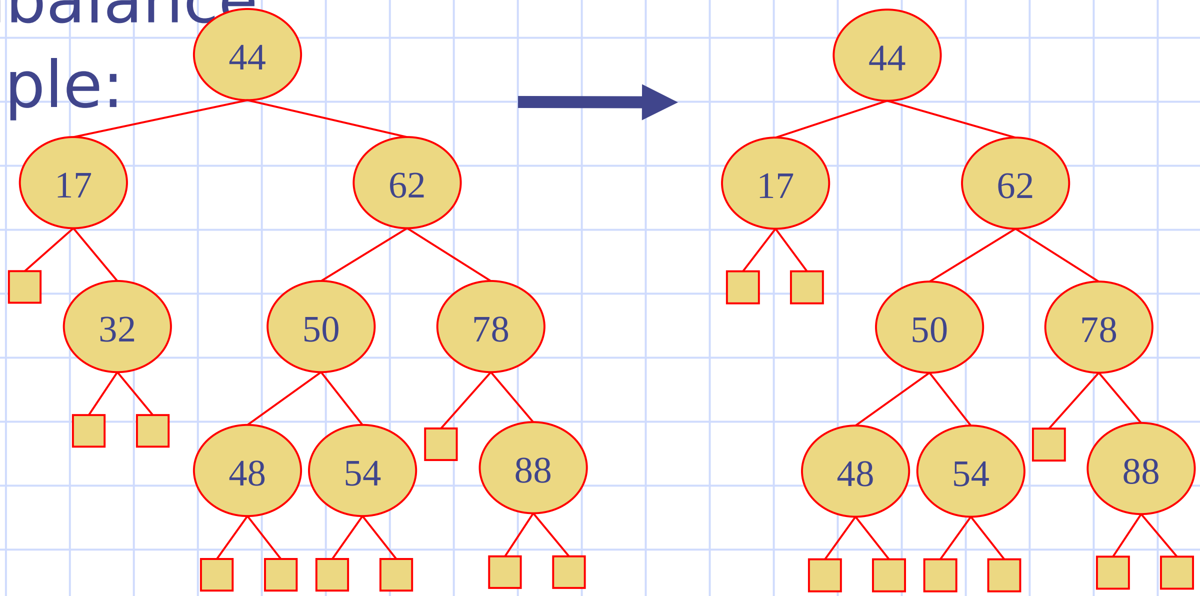
Restructuring (as Single Rotations)

- Single Rotations:



Removal

- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent, w, may cause an imbalance
- Example:

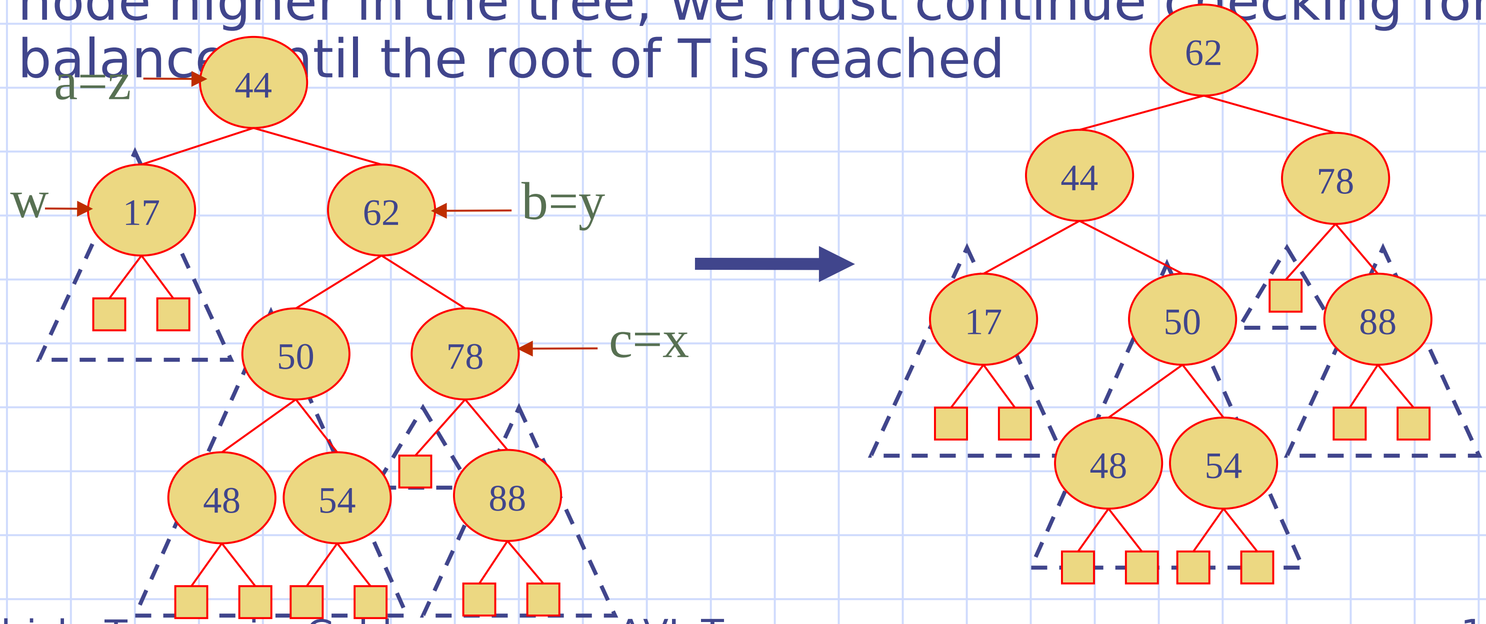


before deletion of 32

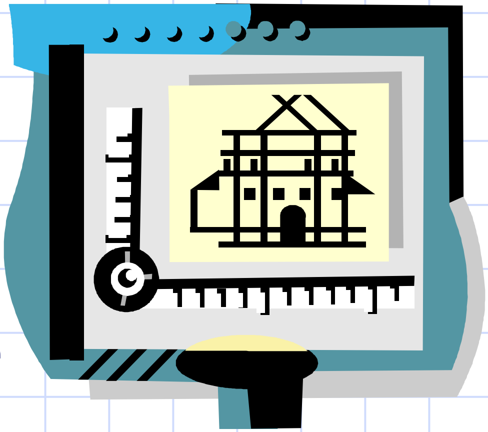
after deletion

Rebalancing after a Removal

- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height
- We perform restructure(x) to restore balance at z
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached



AVL Tree Performance



- a single restructure takes O(1) time
  - using a linked-structure binary tree
- Searching takes O(log n) time
  - height of tree is O(log n), no restructures needed
- Insertion takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)
- Removal takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)