Faculty of Mathematics
Part III Essays: 2019-20

Titles 1 − 69

**Department of Pure Mathematics
& Mathematical Statistics**

Titles 70 − 116

**Department of Applied Mathematics
& Theoretical Physics**

Introductory Notes

Overview. As explained in the Part III Handbook, in place of a three-hour end-of-year examination paper you may submit an essay written during the year. This booklet contains details of the approved essay titles, together with general guidelines and instructions for writing an essay. A timetable of relevant events and deadlines is included on page (iii).

Credit. The essay is equivalent to one three-hour examination paper and marks are credited accordingly.

Essay Titles. The titles of essays in this booklet have been approved by the Part III Examiners.[1] If you wish to write an essay on a topic not covered in this booklet you should approach your Part III Subject Adviser/Departmental Contact or another member of the academic staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board (email: secretary.board@maths.cam.ac.uk) **not later than 1 February**. The new essay title will require the approval of the Part III Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional essays approved by the Part III Examiners will be announced and added to this booklet not later than 1 March. All essay titles are open to all candidates. If you request an essay title you are under no obligation to write the corresponding essay. Essay titles **cannot** be approved informally: the only allowed essay titles are those which appear in the final version of this document (available on the Faculty web site).[2]

Interaction with the Essay Setter. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions. There is a range of practices across the Faculty for the other two meetings depending on the nature of the essay and whether, say, there is a need for further references and/or advice about technical questions. The setter may comment on an outline of the essay (for example in the second meeting), and may offer general feedback (for example, on mathematical style in general terms, or on whether clearer references to other sources are required) on a draft of the essay in the final meeting. The setter is not allowed to give students an expected grade for their essay.

Content of Essay and Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography. Candidates are reminded that mathematical content is more important than style.

Presentation of Essay. Your essay should be legible and may be either handwritten or produced on a word processor. There is no prescribed length for the essay in the University Ordinances, but the Faculty Board Advice to the Part III Examiners suggests that 5,000-8,000 words is a normal length, and exceptionally long essays (i.e. more than twice this maximum) are discouraged. If you are in any doubt as to the length of your essay, please consult either the essay setter or your Part III Subject Adviser/Departmental Contact.

Standard of Essays. The Faculty Board of Mathematics has approved the wording of descriptors to be used as broad guidance for Assessors (i.e. the academics who have set and mark the essays) to

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[1] The titles are also published in the University's journal of record, i.e. the Cambridge University Reporter.
[2] All additional titles will also be published in the Cambridge University Reporter.

determine the appropriate quality mark for an essay. These descriptors are reproduced in Appendix I.

Academic Misconduct and Plagiarism. Before starting your essay you must read
- both the University's statement on the *Definition of Academic Misconduct* available at the URL https://www.plagiarism.admin.cam.ac.uk/definition,
- and the *Faculty Guidelines on Plagiarism and Academic Misconduct* available at the URL https://www.maths.cam.ac.uk/internal/faculty/facultyboard/plagiarism; the latter is reproduced in Appendix II starting on page (vii) of this document.

The University takes a very serious view of academic misconduct in University examinations. The powers of the University Disciplinary Panels extend to the amendment of academic results or the temporary or permanent removal of academic awards, and the temporary or permanent exclusion from membership of the University. Fortunately, incidents of this kind are very rare.

Signed Declaration. The essay submission process includes signing the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff who set the essay. If the setter is unsure about your situation you should consult the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) as soon as possible.

***Viva Voce* Examination.** The Part III Examiners have power, at their discretion, to examine a candidate *viva voce* (i.e. to give an oral examination) on the subject of her or his essay, although this procedure is not often used.

Time Management. It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on lecture courses. You might find it helpful to draw up an essay-writing timetable with plenty of allowance for slippage; then try your hardest to keep to it.

Final Decision on Whether to Submit an Essay. You are not asked to state which essay (if any) and which written papers you have chosen for examination until the beginning of the Easter term. At that point, you will be sent the appropriate form to complete. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 30 April 2020**. **This deadline will be strictly adhered to.**

Essay Submission. You should submit your essay to the Chair of Part III Examiners (c/o Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 30 April 2020**. **This deadline will be strictly adhered to.**

- Together with your essay you should include a completed and signed *Essay Submission Form* as found on page (iv) of this document.
- The title page of your essay should bear **only** the essay title. Please **do not** include your name or any other personal details on the title page or anywhere else on your essay.
- It is important that you ensure that the pages of your essay are fastened together in an appropriate way, e.g. by stapling or binding them. However, please do not bind or staple the *Essay Submission Form* to your essay, but instead attach it loosely, e.g. with a paperclip.

Extension of Submission Deadline. If an extension is likely to be needed due to **exceptional and unexpected developments**, a letter of application and explanation demonstrating the nature of such developments is required from the candidate's Director of Studies. This application should be sent to the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) by the submission date as detailed above.[3] It is expected that such an extension would be (at most) to the following Monday at 12 noon. A student who is dissatisfied with the decision of the Director of Taught Postgraduate Education can request within seven days of the decision, or by the submission date (extended or otherwise), whichever is earlier, that the Chair of the Faculty review the decision. The provision of any such extension will be reported to the Part III Examiners.

Return of Essays. It is not possible to return essays to candidates. You are therefore advised to make your own copy before handing in your essay.

Further Guidance. Advice on writing an essay is provided in two Wednesday afternoon talks listed below. Slides from these talks will subsequently be made available on the Part III Academic Support Moodle (see https://www.vle.cam.ac.uk/course/view.php?id=144111).

Feedback. If you have suggestions as to how these notes might be improved, please write to the *Chair of Part III Examiners* (c/o Undergraduate Office, Centre for Mathematical Sciences).

Timetable of Relevant Events and Deadlines

Wednesday 13 November 4:15pm	Talk (MR2) *Planning your essay: reading, understanding, structuring.*	
Wednesday 29 January 4:15pm	Talk (MR2) *Writing your essay: from outline to final product.*	
Saturday 1 February	Deadline for Candidates to request additional essays.	
Thursday 30 April, Noon	Deadline for Candidates to return form stating choice of papers and essays.	
Thursday 30 April, Noon	Deadline for Candidates to submit essays.	
Thursday 28 May	Part III Examinations begin.	

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[3] Alternatively, the University's procedure can be invoked via the Examination Access and Mitigation Committee; see the *Guidance notes for dissertation and coursework extensions* linked from https://www.student-registry.admin.cam.ac.uk/about-us/EAMC.

MATHEMATICAL TRIPOS, PART III 2019-20
Essay Submission Form

To the Chair of Examiners for Part III of the Mathematical Tripos

I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: ....Date: ....

Title of Essay: ....

....

Essay Number: ....

Your Name: ....College: ....

Assessor comments:

Any essay comments we receive will be sent to your College immediately following the publication of results. Comments are not mandatory, and your assessor may not provide them. If you would prefer to receive your comments by email, please provide your preferred email address below:

Email: ....

Appendix I: Essay Descriptors for Part III of the Mathematical Tripos

The Part III Committee believes that the essay is a key component of Part III. It also believes that it is entirely reasonable and possible that candidates may obtain higher marks for essays than in their examination, both because of the typical amount of effort devoted to the essay, and also the different skill set which is tested compared to a time-limited written examination. In light of these beliefs, as well as the comments of both the internal examiners and the external examiners, the Part III Committee believes that it is appropriate to suggest the following descriptors for the various possible broad grade ranges for an essay. The committee trysts that these guidelines prove useful in guiding the judgement of the inevitably large numbers of assessors marking essays, and that these guidelines strengthen the mechanisms by which all essays are assessed uniformly. They are not meant to be either prescriptive or comprehensive, but rather general guidance consistent with long-standing practice within the faculty.

An Essay of Distinction Standard

Typical characteristics expected of a distinction standard essay include:
- Demonstration of a clear mastery of all the underlying mathematical content of the essay.
- Demonstration of a deep understanding and synthesis of advanced mathematical concepts.
- A well-structured and well-written essay of appropriate length (5000-8000 words) with:
  - very few grammatical or presentational issues;
  - a clear introduction demonstrating an appreciation of the context of the central topic of the essay;
  - a coherent presentation of that central topic;
  - a final section which draws the entire essay to a clear and comprehensible end, summarizing well the key points while suggesting future work.

An essay of distinction standard would be consistent with the quality expected of an introductory chapter of a PhD thesis from a leading mathematics department. A more elegant presentation and synthesis than that presented in the underlying papers, perhaps in the form of a shorter or more efficient proof of some mathematical result would be one possible characteristic of an essay of distinction standard. Furthermore, it would be expected that an essay containing publishable results would be of a high distinction standard, but, for the avoidance of doubt, publishable results are **not necessary** for an essay to be of high distinction standard. An exceptionally high mark (α+) should be justified by a specific extra statement from the assessor highlighting precisely which section of the essay was of particularly distinguished quality.

An Essay of Merit Standard

Typical characteristics expected of a merit standard essay include:
- Demonstration of a good mastery of most of the underlying mathematical content of the essay.
- Demonstration of understanding and synthesis of mathematical concepts typical of the content of a Part III course.
- A largely well-structured essay of appropriate length (5000-8000 words) with:
  - some, but essentially minor, grammatical or presentational issues;
  - an introduction demonstrating an appreciation of a least some context of the central topic of the essay;
  - a reasonable presentation of that central topic;
  - a final section which draws the entire essay to a comprehensible end, summarizing the key points.

An essay of merit standard would be consistent with the quality expected of a first class standard final year project from a leading mathematics department. Such essays would not typically exhibit extensive reading beyond the suggested material in the essay description, or original content.

An Essay of Pass Standard

Typical characteristics expected of a pass standard essay include:

- Demonstration of understanding of some of the underlying mathematical content of the essay.
- An essay exhibiting some non-trivial flaws in presentation through, for example:
    - an inappropriate length;
    - repetition or lack of clarity;
    - lack of a coherent structure;
    - the absence of either an introduction or conclusion.

An essay of pass standard would be consistent with the quality expected of an upper second class standard final year project from a leading mathematics department. For the avoidance of doubt, an excessively long essay (i.e. of the order of twice the suggested maximum length or more) would be likely to be of (at best) pass standard. A key aspect of the essay is that the important mathematical content is presented clearly in (at least close to) the suggested length.

Appendix II: Faculty of Mathematics Guidelines on Plagiarism and Academic Misconduct

For the latest version of these guidelines please see
            https://www.maths.cam.ac.uk/internal/faculty/facultyboard/plagiarism

University Resources

The University publishes information on *Plagiarism and Academic Misconduct*, including
- The *University definition of academic misconduct*;
- Information for students, covering
    - *Students' responsibilities*
    - *Why does plagiarism matter?*
    - *Collusion*
- information about *Referencing* and *Study skills*;
- information on *Resources and support*;
- the *University's statement on proofreading*;
- *Plagiarism FAQs*.

There are references to the University statement
- in the Part IB and Part II Computational Project Manuals, and
- in the Part III Essay booklet.

Please read the University statement carefully; it is your responsibility to read and abide by this statement.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.
Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to *Turnitin Plagiarism Detection Software*).

The scope of plagiarism

Plagiarism may be due to
- **copying** (this is using another person's language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition, you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline to avoid plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting

A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:
- short quotations should be in inverted commas, and a reference given to the source;
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for

bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing

Paraphrasing means putting someone else's work into your own words. Para- phrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ...").   As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness

When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources

You should use web sources as if you were using a book or journal article.
The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration

 Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Links to University Information

- Information on *Plagiarism and Academic Misconduct*, including
    - *Students' responsibilities*;
    - *Information for staff*.

Contents

Dr T. Bloom

The twin prime conjecture states that there are infinitely many primes at most 2 apart, the smallest gap possible. At the other extreme, we can ask whether the gaps between primes can get unexpectedly large infinitely often. The Prime Number Theorem tells us that a typical prime p is followed by about $\log p$ consecutive integers before the next prime. This is the average behaviour.

In 1938 Rankin showed that there are infinitely many primes p which are followed by

$$\gg \log p \frac{\log \log p \log \log \log \log p}{(\log \log \log p)^2}$$

many consecutive integers. In particular, infinitely often, there are gaps between primes which are larger than average by an arbitrarily large amount. This lower bound stood for almost 80 years, and Erdős offered a \$10,000 dollar prize for improving it even slightly. This prize was claimed recently by Ford, Green, Konyagin, Maynard, and Tao, who improved the lower bound above to

$$\gg \log p \frac{\log \log p \log \log \log \log p}{\log \log \log p}.$$

This essay would give an exposition of the proof of this recent breakthrough in the analytic understanding of gaps between primes. The proof blends sieve methods, analytic number theory, and probabilistic combinatorics.

Relevant Courses

Useful: Analytic Number Theory (while little of the syllabus of this course is directly used in the proof, and so is strictly not essential, it offers a good grounding in the general type of analytic argument used).

References

[1] K. Ford, B. Green, S. Konyagin, J. Maynard, and T. Tao. *Long gaps between primes.* J. Amer. Math. Soc. 31 (2018), no. 1, 65-105.

[2] K. Ford, B. Green, S. Konyagin, and T. Tao. *Large gaps between consecutive prime numbers.* Ann. of Math. (2) 183 (2016), no. 3, 935-974.

[3] J. Maynard. *Large gaps between primes.* Ann. of Math. (2) 183 (2016), no. 3, 915-933.

2. Metric Diophantine Approximation and the Duffin-Schaeffer Conjecture
Dr T. Bloom

Diophantine approximation studies how well irrational numbers can be approximated by rational numbers. To avoid becoming tangled in pathological behaviour, metric Diophantine approximation asks such questions about 'almost all' α.

We say that $\alpha \in [0, 1]$ is ψ-approximable if there are infinitely many p, q such that $|\alpha - p/q| < \psi(q)/q$. One of the first results in the area was Khintchine's theorem in 1926: if $\psi : \mathbb{N} \to \mathbb{R}_+$ is non-increasing then

almost all $\alpha \in [0, 1]$ are ψ-approximable if and only if $\sum_q \psi(q)$ diverges.

This theorem, while very powerful, is lacking in two respects. The first is the restriction to ψ non-increasing, which rules out being able to find rational approximations with denominators restricted to certain sets, such as the primes. The second is that, in general, the theorem does not find rationals in reduced form, since it does not guarantee that $(p, q) = 1$.

This led Duffin and Schaeffer in 1941 to their famous conjecture which remedies these inadequacies. We say that $\alpha \in [0, 1]$ is ψ-approximable by reduced forms if there are infinitely many p, q with $(p, q) = 1$ such that $|\alpha - p/q| < \psi(q)/q$. The Duffin-Schaeffer conjecture states that

almost all $\alpha \in [0, 1]$ are ψ-approximable by reduced forms if and only if $\sum_q \psi(q)\dfrac{\phi(q)}{q}$ diverges.

Despite partial progress, this conjecture remained open for over 75 years, until a proof was announced in July 2019 by Dimitris Koukoulopoulos and James Maynard. This essay would discuss the conjecture and give an exposition of the proof.

Relevant Courses

Useful: Analytic Number Theory (while little of the syllabus of this course is directly used in the proof, and so is strictly not essential, it offers a good grounding in the general type of analytic argument used).

References

[1] D. Koukoulopoulos and J. Maynard. *On the Duffin-Schaeffer conjecture.* (2019) arXiv:1907.04593.

3. Maker-Breaker Games and the Neighourhood Conjecture . . . .
Dr T. Bloom

Positional games consist of a finite set of vertices, with a specified collection of winning sets, and two players. Players alternate in claiming vertices as their own. The classical view has one player winning as soon as they claim some winning set all for themselves (the classic example is noughts and crosses, or tic-tac-toe for Americans).

These games have proved resilient to analysis, and very little non-trivial is known about them. Instead, much research has focused on the asymmetric 'Maker-Breaker' games, where the first player wins if they claim a winning set, and the second player merely has to stop the first player winning.

We say that a game is k-uniform if every winning set has k vertices. One of the fundamental results of this area is the Erdős-Selfridge theorem, which says that if the number of winning sets is less than 2^{k-1} then Breaker has a winning strategy.

This is a global criterion, in that it places a bound on the total number of winning sets. It would be much better to have a local criterion, since intuitively when a player claims a vertex they should only have to worry about the impact on winning sets containing that vertex, not the whole board.

This led Beck to formulate the Neighbourhood Conjecture: that if every winning set intersects less than $2^{k-1} - 1$ many other winning sets, then Breaker has a winning strategy (regardless of how many winning sets there are overall).

This formulation was disproved by Gebauer, but a slightly weaker version of it remains open. This essay would discuss the conjecture and Gebauer's construction. There are also many links to probabilistic combinatorics and algorithmic questions in theoretical computer science which could be explored.

References

[1] D. Hefetz, M. Krivelevich, M. Stojaković, and T. Szabó. *Positional Games.* Birkhäuser (2014).

[2] H. Gebauer. *Disproof of the Neighborhood Conjecture with implications to SAT.* Combinatorica 32(5) (2012) 573-587.

4. Deformations of Algebras ....
Dr C. J. B. Brookes

Algebraic deformation theory is primarily concerned with the interplay between homological algebra and the perturbations of algebraic structures. For example one might want to deform a commutative algebra to give a non-commutative one via 'quantisation'.

In a series of papers [2], [3], [4] and [5] Gerstenhaber developed deformation theory for associative algebras. The cohomology theory required in this context is Hochschild cohomology and one finds that 2-cocycles arise from deformations. A more modern approach is to consider additional algebraic structures, including that of a Lie algebra defined on the Hochschild cocomplex of the associative algebra. Deformations correspond to Maurer-Cartan elements. This approach has arisen in the work of Kontsevich on the deformation quantisation of Poisson manifolds.

The introductory article by Fox [1] is a good place to start. It first describes the classical theory of deformations of associative algebras and then moves on to more general algebraic structures. The papers of Gerstenhaber are also very readable.

Relevant Courses

Essential: Algebra

References

[1] T. Fox, An introduction to algebraic deformation theory, J Pure and Applied Algebra, 84 (1993), 17-41.

[2] M.Gerstenhaber, On the deformation theory of rings and algebras, Ann. Math. 78 (1963) 267-288.

[3] M.Gerstenhaber, On the deformation theory of rings and algebras II, Ann. Math. 84 (1966) 1-19.

[4] M.Gerstenhaber, On the deformation theory of rings and algebras III, Ann. Math. 88 (1968) 1-34.

[5] M.Gerstenhaber, On the deformation theory of rings and algebras IV, Ann. Math. 99 (1974) 257-256.

5. Bieri-Strebel Invariants and Nonarchimedean Amoebae ....
Dr C. J. B. Brookes

In 1980 Bieri and Strebel [3] classified metabelian groups for which there is a presentation in terms of finitely many generators and finitely many relations. (A metabelian group G is one in which there is an abelian normal subgroup K where G/K is also abelian.) For a finitely generated group the space of homomorphisms of G to the additive group of the reals forms a finite dimensional real vector space V and the method was to associate with G a subset of V. This geometric set is related to the logarithmic limit set defined previously by Bergman [1] for ideals of group algebras of free abelian groups (or Laurent polynomial rings); via conjugation K may be regarded as a module over the integral group algebra of G/K. In [2] Bieri and Groves showed that these geometric sets are all polyhedral. There were extensions of the theory to general groups where the points of the set represent certain tree actions of the group [3] and [6].

Such sets also arose as the nonarchimedean amoebae of Gelfand, Kapranov and Zevelinsky (see [8] and [10] for further developments), and are related to compactifications of subvarieties of tori (see [11]). An overview can be found in ine first chapter of the book by Maclagan and Sturmfels [9].

The essayist could concentrate on the group-theoretic aspects - see for example the essay by Strebel in [7], or consider primarily the amoebic approach.

Relevant Courses

Essential: Algebra

Useful: Toric Geometry

References

[1] GM Bergman, The logarithmic limit set of an algebraic variety, Trans. Amer. Math. Soc 157 (1971),459 - 469.

[2] R Bieri and JRJ Groves, The geometry of the set of characters induced by val uations, J. Reine Angew. Math. 347 (1984) , 168 - 195.

[3] R Bieri, W Neumann and R Strebel, A geometric invariant of discrete groups, Invent math. 90 (1987), 451 - 477.

[4] R Bieri and R Strebel, Valuations and finitely presented metabelian groups, Proc London Math Soc 41 (1980), 439 - 464.

[5] R Bieri and R Strebel A geometric invariant of modules over an abelian group, J Reine Angew. Math. 322 (1981), 170 - 189.

[6] KS Brown, Trees, valuations and the Bieri- Neumann -Strebel invariant, Invent. Math. 90 (1987), 479 - 504.

[7] Group theory, essays for Philip Hall, ed KW Gruenberg and JE Roseblade, Acad emic Press 1984.

[8] M Einsiedler, M Kapranov, D Lind, Non-archimedean amoebas and tropical varieties, J. Reine Angew. Math. 601 (2006) 139 - 157.

[9] D.Maclagan, B. Sturmfels, Introduction to tropical geometry, Graduate studies in math. 161, American Math. Soc. 2015

[10] S.Payne, Adelic amoebas disjoint from open halfspaces, J. Reine Angew. Math. 625 (2008) 115 - 123.

[11] J. Tevelev, Compactifications of subvarieties of tori, American J. Math. 129 (2007), 1087 - 1104.

6. Positive Scalar Curvature, Surgery, and Cobordism ....
Dr M. Bustamante

In geometry one is often faced with the problem of constructing smooth Riemannian manifolds satisfying a prescribed curvature bound. But curvature, being a local invariant of such manifolds, tends to be affected in an uncontrolled way by the typical constructions coming from topology, especially those of a "cut and paste" nature. For example, it is known that a connected sum of many copies of complex projective spaces does not admit a Riemannian metric of positive sectional curvature. Positive *scalar* curvature, though, excels as the curvature constraint which is rigid enough so that not every manifold admits a positively curved metric, but is sufficiently soft to survive under certain cut-and-paste modifications. This essay aims at understanding this "softness" of positive scalar curvature.

You should explain the Gromov-Lawson construction [1] in detail, which shows that any manifold obtained from one of positive scalar curvature by performing surgery in codimension ≥ 3, also admits a metric of positive scalar curvature. Thus it bridges between positive scalar curvature and cobordism.

Cobordism is one of the gems of geometric topology, which has been cut, polished and engraved by the hands of many topologists. As a result, a complete classification of oriented manifolds up to cobordism is known. You should use this knowledge to describe explicit generators of the oriented cobordism ring in order to draw the following corollary: every simply connected non-spin manifold of dimension ≥ 5, admits a Riemannian metric of positive scalar curvature.

Relevant Courses

Essential: Part III Differential Geometry, Part III Algebraic Topology.

References

[1] M. Gromov, B. Lawson, *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. 111 (1980), 423-434.

[2] J. Milnor, *A procedure for killing homotopy groups of differentiable manifolds.* 1961 Proc. Sympos. Pure Math., Vol. III pp. 39–55 American Mathematical Society, Providence, R.I

[3] C. T. C. Wall, *Determination of the Cobordism Ring*, Ann. of Math. (2) 72 (1960), 292–311.

7. Ramsey Classes and Extremely Amenable Groups ....
Dr G. Conant

Ramsey's Theorem (1928) states that for any given integers $k, m, r \geq 1$, there is an integer $n \geq 1$ such that if the r-element subsets of $\{1, \ldots, n\}$ are colored with k colors, then there is an m-element set $S \subseteq \{1, \ldots, n\}$ such that all r-element subsets of S are assigned the same color.

Now let Aut($\mathbb{Q}$) denote the group of order-preserving permutations of $\mathbb{Q}$. Then Aut($\mathbb{Q}$) is a Polish group when given the pointwise-convergence topology (relative to the discrete topology on $\mathbb{Q}$). A result of Pestov from 1998 states that any continuous action of Aut($\mathbb{Q}$) on a compact Hausdorff space X has a *fixed point*, i.e., a there is some $x \in X$ such that $g \cdot x = x$ for all $g \in \text{Aut}(\mathbb{Q})$.

Using jargon from the literature, Ramsey's Theorem can be restated by saying that the class of finite linear orders has the *Ramsey property*, while the theorem of Pestov says that Aut($\mathbb{Q}$) is *extremely amenable*. It turns out that these two theorems together form one special case of a general correspondence, which characterizes extreme amenability for closed subgroups of S_∞ (the group of permutations of a countably infinite set). Specifically, Kechris, Pestov, and Todorcevic [3] proved that a closed subgroup of S_∞ is extremely amenable if and only if it is the automorphism group of the Fraïssé limit of a Fraïssé order class with the Ramsey property. In addition to finite linear orders, other examples of Fraïssé order classes with the Ramsey property include finite linearly ordered graphs, finite linearly ordered K_n-free graphs, and finite linearly ordered metric spaces with distances in $\{0, 1, \ldots, n\}$ (see [2]).

The main goal of this essay is to present a proof of the KPT correspondence. This should include a brief survey of basic tools on Fraïssé theory (see Section 7.1 of [1]), topological dynamics, and amenable groups. A good essay will discuss at least one generalization of KPT. Examples include the correspondence from [7] between *finite Ramsey degrees* and automorphism groups with metrizable *universal minimal flows*, or similar results on automorphism groups of uncountable structures (e.g., [4, 6]).

Relevant Courses

Essential: Topological groups, and a basic familiarity with first-order languages and structures (e.g., Sections 1.1 and 1.2 of [5] are more than sufficient).

References

[1] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[2] J. Hubička and J. Nešetřil, *All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)*, Adv. Math. **356** (2019), 106791, 89 pp.

[3] A. S. Kechris, V. G. Pestov, and S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), no. 1, 106–189.

[4] K. Krupiński and A. Pillay, *On the topological dynamics of automorphism groups; a model-theoretic perspective*, arXiv preprint: 1909.13285.

[5] D. Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002.

[6] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, Int. Math. Res. Not. IMRN (2016), no. 5, 1285–1307.

[7] A. Zucker, *Topological dynamics of automorphism groups, ultrafilter combinatorics, and the generic point problem*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6715–6740.

8. Toric Kähler Geometry ....
Dr R. Dervan

The natural candidate for a "best" metric on a smooth projective variety is a Kähler metric with constant scalar curvature. In the case of toric varieties, one can translate this condition into toric terms using the moment polytope (which is the "dual" perspective on toric geometry to the fan construction). The existence problem for constant scalar curvature Kähler metrics is subtle, with the guiding principle being Donaldson's conjecture that the existence of such metrics should be equivalent to the algebro-geometric notion of K-stability. K-stability, in turn, can be translated into a more combinatorial form in the setting of toric varieties, and this special case has been an important testing ground for the general theory. The goal of this essay is to establish various fundamental results concerning constant scalar curvature Kähler metrics on toric varieties.

The literature on this topic is, by now, quite large and many different topics could be discussed. The most important is the proof that the existence of a constant scalar curvature Kähler metric implies K-polystability. Other topics which could be discussed include uniqueness of toric metrics of constant scalar curvature, Abreu's formula for the scalar curvature on a toric variety, Donaldson's integration by parts formula, and Székelyhidi's work on K-unstable toric varieties. The basic reference for most of this is the toric material contained in Székelyhidi's book [2]. More advanced topics are contained in Donaldson's article [1] and Székelyhidi's paper [3].

Relevant Courses

Essential: Toric Varieties, Complex Manifolds

References

[1] S. K. Donaldson *Scalar Curvature and Stability of Toric Varieties.* J. Differential Geom. Volume 62, Number 2 (2002), 289-349pp.

[2] G. Székelyhidi *An introduction to extremal Kähler metrics.* Graduate Studies in Mathematics, 152. American Mathematical Society, Providence, RI, 2014. xvi+192pp.

[3] G. Székelyhidi *Optimal test-configurations for toric varieties.* J. Differential Geom. Volume 80, Number 3 (2008), 501-523pp.

9. The Grunwald-Wang Theorem ....
Dr T. A. Fisher

A natural local-to-global question in number theory is whether an element of a number field k that is locally (i.e. in the completion k_v) an nth power for all but finitely many places v, must be an nth power in k. In general the answer is "no" (there are counter-examples with $k = \mathbf{Q}$ and $n = 8$) but under some extra hypotheses (which are always satisfied for example if n is odd) the answer is "yes". The Grunwald-Wang theorem itself is a closely related local-to-global question about the existence of cyclic extensions of k (i.e. Galois extensions with cyclic Galois group). The essay should include sufficient background material from class field theory and group cohomology to explain the proofs. It could also discuss how the above problem (concerning divisibility in the multiplicative group) extends to other commutative algebraic groups [3], for example elliptic curves.

Relevant Courses

Essential: Algebraic Number Theory

Useful: Elliptic Curves

References

[1] E. Artin and J. Tate, *Class field theory*, (originally published 1967), AMS Chelsea Publishing, Providence, RI, 2009.

[2] J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic number theory*, (originally published 1967), LMS, 2010.

[3] R. Dvornicich and U. Zannier, Local-global divisibility of rational points in some commutative algebraic groups, *Bull. Soc. Math. France* 129 (2001), no. 3, 317–338.

[4] J. Neukirch, *Algebraic number theory*, Springer-Verlag, Berlin, 1999.

[5] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Springer-Verlag, Berlin, 2000.

10. Variation of Selmer Groups in Families of Quadratic Twists . . . .
Dr T. A. Fisher

Computing the 2-Selmer group of an elliptic curve over a number field gives an upper bound for the rank of the group of rational points. This essay should start by giving an account of 2-descent on an elliptic curve, following [1], [2], or by specialising the results in [5]. It should then explain how the 2-Selmer group may be controlled in families of quadratic twists (see for example [4]), and (if time and space permit) also give applications to the arithmetic of surfaces (see for example [3]).

Relevant Courses

Essential: Elliptic Curves

Useful: Algebraic Number Theory

References

[1] A. Brumer and K. Kramer, The rank of elliptic curves, *Duke Math. J.* 44 (1977), no. 4, 715–743.

[2] J.W.S. Cassels, *Lectures on elliptic curves*, CUP, Cambridge, 1991.

[3] Y. Harpaz and A.N. Skorobogatov, Hasse principle for Kummer varieties, *Algebra & Number Theory* 10 (2016), no. 4, 813–841.

[4] B. Mazur and K. Rubin, Ranks of twists of elliptic curves and Hilbert's tenth problem, *Invent. Math.* 181 (2010), no. 3, 541–575.

[5] E.F. Schaefer, 2-descent on the Jacobians of hyperelliptic curves, *J. Number Theory* 51 (1995), no. 2, 219–232.

11. Countable Ordinals, Proof Theory and Fast-Growing Functions ....
Dr T. E. Forster

A good point of departure for this essay would be the Part II ST&L exercise in which the student is invited to show how, for every countable ordinal, a subset of $\Re$ can be found that is wellordered to that length in the inherited order. The obvious way to do this involves induction on countable ordinals and leads swiftly to the discovery of fundamental sequences. These can be put to work immediately in the definition of hierarchies of fast-growing functions. This leads in turn to the Schmidt conditions, which the student should explain carefully. There is a wealth of material on how proofs of totality for the faster-growing functions in this hierarchy have significant—indeed *calibratable*—consistency strength. One thinks of Goodstein's function and Con(PA), or of Paris-Harrington. There is plenty here from which the student can choose what to cover.

The Doner-Tarski hierarchy of functions (addition, multiplication, exponentiation . . .) invites a transfinite generalisation and supports a generalisation of Cantor Normal form for ordinals. Nevertheless, the endeavour to notate ordinals beyond ϵ_0 does not use those ideas, but rather the enumeration of fixed points: such is the *Veblen hierarchy*. From this one is led to the impredicative Bachmann notation, with Ω and the ϑ function.

Relevant Courses

Essential:
Part II Set Theory and Logic

References

Schwichtenberg-Wainer Proofs and Computations Cambridge University Press (go to
http://www.cambridge.org/us/academic/subjects/mathematics/logic-categories-and-sets/proofs-and-computations)

http://www.math.ucsb.edu/~doner/articles/Doner-Tarski.pdf

http://www.dpmms.cam.ac.uk/~tf/cam_only/fundamentalsequence.pdf

http://www.dpmms.cam.ac.uk/~tf/cam_only/TMStalk.pdf

12. Wellquasiorders and Betterquasiorders ....
Dr T. E. Forster

A well-quasi-order is a reflexive transitive relation with no infinite descending chains and no infinite antichains. Although this definition may not sound particularly natural there are many natural examples, at least one of which is famous: the theorem of Seymour and Robertson that finite graphs under the graph minor relation form a WQO (although this is far beyond the scope of a Part III essay!) There is Laver's theorem that the isomorphism types of scattered total orders (orders in which the rationals cannot be embedded) form a WQO. A proper treatment of this proof would almost be enough by itself for an essay. Kruskal's theorem states that finite trees with nodes labelled with elements of a WQO are also WQO-ed. This has a very striking finitisation which is associated with a very fast-growing function and a consistency proof for PA.

The class of WQO's lacks certain nice closure properties and the project to patch this up leads to a concept of *Better*-quasi-ordering. The class of BQOs is algebraically nicer.

These combinatorial ideas have wide ramifications in Graph Theory, Logic and Computer Science (lack of infinite descending chains is always liable to be connected with termination of processes) and the area has a good compact literature and some meaty theorems. Recommended for those of you who liked the Logic course and the Combinatorics course.

A Big Plus for this topic is that there is no textbook! (The setter dreams of writing one and the current draught is linked below) There is a wealth of literature. Interested students should consult the setter.

Relevant Courses

Essential: None

Useful: Combinatorics, Logic and Set Theory

References

http://www.dpmms.cam.ac.uk/~tf/BQObok.pdf

13. Quine's Set Theory NF ....
Dr T. E. Forster

The set theory revealed to the world in Quine's 1937 article *New Foundations for Mathematical Logic* was a bit of a backwater for a very long time, largely because of unanswered questions about its consistency. Recently Randall Holmes has come up with a multi-layered construction of a model using many different ideas. The best bet is that the proof is correct, but it is still working its way through the refereeing process and in any case is too complex for a Part III essay. However a good essay could be written on the background that the construction uses: there are many ideas and they are all good.

Such an essay could cover any (but not all!) of the following. Specker's results connecting NF and Simply Typed Set Theory *via* Typical Ambiguity axioms; Holmes' work on tangled types and tangled cardinals arising from Jensen's Ramsey-powered proof of Con(NFU) (= NF with atoms); Rosser's counting axiom and the refutation of AC, including a brief treatment of the relevant parts of cardinal arithmetic without choice (cardinal trees, amorphous cardinals); a study of the Rieger-Bernays permutation method (as used in the proof of independence of the axiom of foundation from ZF which you may rememnber from Part II Set Theory) would be a useful (but not essential) preparation for Holmes' proof, and in any case the method is a topic of central importance in the study of NF; a treatment of this material can be easily given and the exercise is instructive; (Fraenkel-Mostowski permutation methods are relevant but that is an essay topic on its own account, and is in any case unavailable because it was set last year); Church-Oswald models for set theories with a universal set are another topic that could be covered, useful and interesting but less central to the consistency project—tho' germane to NF. Finally another interesting topic (under investigation in Cambridge) not directly relevant to the consistency question but worthy of treatment is the failure of cartesian-closedness in the category of sets according to NF, and more generally a category-theoretic treatment of the world of NF sets.

Narrating the way in which these various and diverse ideas (all of them interesting on their own account) get tied together would be an instructive and enjoyable exercise.

The only comprehensive references at this stage are [1] and [2], but there is a wealth of other material linked from `http://math.boisestate.edu/~holmes/holmes/nf.html` and the setter will be happy to supply more detailed information on demand.

Relevant Courses

Essential: Part II Logic and Set Theory or equivalent.

References

[1] Forster, T.E., Set Theory with a Universal set. Oxford Logic Guides. OUP 1992, second edition 1995.

[2] Holmes, M. R. Elementary set theory with a universal set. volume 10 of the Cahiers du Centre de logique, Academia, Louvain-la-Neuve (Belgium), 241 pages, ISBN 2-87209-488-1. available online at `http://www.logic-center.be/cahiers/cahiersangl.html`

14. Mirror Symmetry ....

Professor M. Gross

Mirror symmetry is a geometric phenomenon first noticed by string theorists circa 1990. In string theory, one expects a 10-dimensional universe, and hence one would posit a space-time geometry $\mathbf{R}^4 \times X$ where X is a a very small compact six-dimensional manifold. Various physical considerations lead to X often in fact being a non-singular three-dimensional variety over the complex numbers of a special sort, known as a *Calabi-Yau manifold*. Initial evidence suggested that such manifolds came in pairs $X, \check{X}$, with a relationship on Hodge numbers given by $h^{p,q}(X) = h^{3-p,q}(\check{X})$. Calculations by Candelas, de la Ossa, Green and Parkes [1] then suggested that certain invariants of X, namely counts of holomorphic curves in X, could be calculated via a very different procedure on $\check{X}$, namely so-called period integrals. [1] carried this calculation out for X the quintic threefold in projective four-space, and obtained predictions for the number of rational curves in X of every degree. For algebraic geometers, this was a surprising result, and has led to what is now a huge field.

There is currently a vast literature on mirror symmetry; I include below several references providing an entrance into this literature. Reference [2] gives a good overall exposition of the state of the art in the 1990s, and my chapter in [3] covers in great detail the original calculations of [1]. Reference [4] gives a much narrower but more modern point of view leading to current developments in the field.

A successful essay should not attempt to communicate all aspects of mirror symmetry, and should stay narrowly focused on a few well-chosen topics. In addition, this essay *must* involve a mathematical rather than a string-theoretic discussion of the field, and no physics background is required for this essay. An incomplete list of possible directions would include: (1) The Candelas et al mirror symmetry calculation for genus 0 invariants. (2) An investigation of the Batyrev construction of mirror pairs. (3) An investigation of Gromov-Witten invariants and the proof of genus 0 mirror symmetry for the quintic. (4) An investigation of Homological Mirror Symmetry. (5) An investigation of approaches using tropical geometry.

(1) is covered in references [1] and [3], and (2),(3) are covered in [2], although there are more modern approaches to (3), e.g., due to Gathmann [5]. For (4), there are many possible references, but this would require a very solid grounding in symplectic geometry. [4] covers much of the material of (5).

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Toric Varietes.

Optional: Part III Complex Manifolds, Part III Symplectic Geometry.

References

[1] P. Candelas, X. de la Ossa, P. Green, L. Parkes, *An exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds,* Phys. Lett. B 258 (1991), no. 1-2, 118–126.

[2] David Cox, Sheldon Katz, *Mirror symmetry and algebraic geometry.* Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999. xxii+469 pp.

[3] Mark Gross, Daniel Huybrechts, Dominic Joyce, *Calabi-Yau manifolds and related geometries.* Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003.

[4] Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, **114**. American Mathematical Society, Providence, RI, 2011. xvi+317 pp

[5] Andreas Gathmann, *Relative Gromov-Witten invariants and the mirror formula,* Math. Ann. 325 (2003), no. 2, 393–412.

15. Donaldson-Thomas Invariants and Bridgeland Stability Conditions . . . . .
Professor M. Gross

Donaldson-Thomas invariants were introduced in [5] as a way of defining a type of invariant of a Calabi-Yau three-fold given by counting vector bundles. The idea is the moduli space of vector bundles with fixed Chern classes is "expected" to have dimension zero, and hence one can count the number of points in the moduli space. In fact the dimension of the moduli space isn't always zero, but [5] defined a *virtual fundamental class* on the moduli space which is a cycle of dimension zero, which in turn yields a number. Since the introduction of the theory, there has been a huge surge of interest and applications, and in particular the theory is now more viewed as a theory of invariants of triangulated categories with stability conditions.

Instead of counting vector bundles on a Calabi-Yau threefold X, one instead works with the *derived category of coherent sheaves* of X: this is a category whose objects are complexes of coherent sheaves on X, and where the notion of an exact sequence of objects is replaced by the notion of a distinguished triangle. They were originally developed to understand duality, and the standard exposition is [1]. It is important to have a notion of *stable* object in the category, generalizing the notions of stable vector bundles. Indeed, only stable bundles have well-behaved moduli spaces, and one similarly needs well-behaved moduli spaces of objects in the derived category. The key notion of stability in the derived category was introduced by Bridgeland in [4]; the study of Bridgeland stability is itself now a major subject. See also Chapter 5 of [3] for a gentle introduction to both the derived category and stability conditions.

After absorbing the definitions of these concepts, the essay writer can explore various aspects of the subject. This could include the wall-crossing behaviour of Donaldson-Thomas invariants under change of stability condition, as developed by Kontsevich and Soibelman [4], or the application to curve-counting, of which [7] is a good survey. There is a great deal of literature to be explored.

Relevant Courses

Essential: Part III Algebraic Geometry

References

[1] R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics 20, Springer-Verlag, 1966.

[2] R. Thomas, "Derived categories for the working mathematician," arXiv:math/0001045.

[3] P. Aspinwall et al, *Dirichlet Branes and Mirror Symmetry*, Clay Mathematics Monographs, **4**, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2009. x+681 pp.

[4] T. Bridgeland, "Stability conditions on triangulated categories," arXiv:math/0212237.

[5] R. Thomas, "A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations," arXiv:math/9806111.

[6] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," arXiv:0811.2435

[7] R. Pandharipande and R. P. Thomas, "13/2 ways of counting curves," arXiv:1111.1552.

16. Categories of Relations ....
Professor P. T. Johnstone

Categories whose morphisms behave like relations rather than functions can be studied in various ways. The objective of the study is to identify those morphisms (commonly called maps) which correspond to actual functions, and to relate properties of the whole category to properties of its subcategory of maps. One highly successful approach, originally developed by Peter Freyd, is developed in detail in [1], and more succinctly in [2]; other approaches include that of Carboni and Walters [3], which makes more explicit use of 2-categorical ideas. It is suggested that an essay might take as its goal the characterization of those allegories (or cartesian bicategories) whose categories of maps are toposes; alternatively, one might give a detailed comparison of these two (and possibly other) approaches.

Relevant Courses

Essential: Category Theory
Useful:

References

[1] P.J. Freyd and A. Scedrov, *Categories, Allegories* (North–Holland, 1990).

[2] P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium* (Oxford U.P., 2002), Chapter A3

[3] A. Carboni and R.F.C. Walters, Cartesian bicategories I, J. Pure Appl. Algebra 49 (1987), 11-32.

17. Synthetic Differential Geometry ....
<div align="right">

Professor P. T. Johnstone
</div>

In 1967, F.W. Lawvere suggested that the traditional analytic approach to differential geometry might be replaced by a 'synthetic' approach, in which one would begin by directly axiomatizing (a category containing) the category of smooth manifolds. Lawvere's axioms are incompatible with classical logic, and thus with the traditional conception of what a smooth manifold is: it was not until the development of elementary topos theory in the 1970s that it became possible to give explicit models for them. An essay on this topic could either concentrate on developing the axiomatics (for which Anders Kock's first book [1] is probably still the best introduction, although Kock's later book [2] and René Lavendhomme's [3] are also recommendable); or, more ambitiously, it could describe the construction of a 'well-adapted' model of the axioms, in which the classical category of manifolds is nicely embedded. (The latter would require the development of a good deal of topos theory; suitable references would include [4] and [5].)

Relevant Courses

Essential: Category Theory

Useful: Differential Geometry

References

[1] A. Kock, *Synthetic Differential Geometry*, Cambridge University Press (L.M.S. Lecture Notes series), 1981 (second edition 2006).

[2] A. Kock, *Synthetic Geometry of Manifolds*, Cambridge University Press (Cambridge Tracts in Mathematics), 2010.

[3] R. Lavendhomme, *Basic Concepts of Synthetic Differential Geometry*, Kluwer, 1996.

[4] I. Moerdijk and G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer–Verlag, 1991.

[5] P.T. Johnstone, 'Synthetic Differential Geometry', chapter F1 of *Sketches of an Elephant: a Topos Theory Compendium* (not published yet, but preprint copies available from the author).

18. Lagrangian Tori in $\mathbb{R}^6$ ....
<div align="right">

Dr A. M. Keating
</div>

Lagrangian submanifolds are distinguished half-dimensional submanifolds of symplectic manifolds – for instance, in the case of $\mathbb{C}^n$, the tori $\{(z_1, \ldots, z_n) : |z_i| = a_i\}$, for some positive constants a_i. The nicest condition that one can impose on a closed Lagrangian submanifold in $\mathbb{C}^n$ is for it to be monotone; in the aforementioned example, this amounts to requiring that all of the a_i be equal. Up to suitable notions of equivalence, there is a unique (automatically monotone) Lagrangian circle in $\mathbb{C}$. In $\mathbb{C}^2$, it is widely expected that there are two. The goal of this essay is to give an account of a beautiful result of Auroux, who, in contrast, produced an infinite collection of monotone Lagrangian tori in $\mathbb{C}^3$.

This essay should readily build on parts of the Symplectic Topology course. After briefly recalling relevant definitions from the course, the essay should start by explaining Auroux' construction from [1]; you may find it helpful to understand the perspective of Section 5 of [1], which draws on constructions in [2]. To tell the different tori apart, Auroux uses an invariant

which comes from counting certain pseudo-holomorphic discs; the essay should proceed to give an account of this. You may choose to treat various amounts of Floer-theoretic background as a 'black-box'.

Relevant Courses

Essential: Differential geometry; symplectic geometry; basic notions from algebraic topology

Useful: Algebraic geometry

References

[1] D. Auroux, *Infinitely many monotone Lagrangian tori in* $\mathbb{R}^6$, Invent. Math., 201(3):909–924, 2015 (arXiv:1407.3725)

[2] Y. Lekili, M. Maydanskiy, *The symplectic topology of some rational homology balls*, Comm. Math. Helv. 89 (2014), 571–596 (arXiv:1202.5625).

For the symplectic geometry background (to be covered in the Lent term course):

[4] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition, Oxford University Press, 2017

[5] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Springer, 2nd edition (2008), also available at *https://people.math.ethz.ch/∼acannas/*

19. Complements of Hyperplane Arrangements ....
Dr A. M. Keating

A hyperplane arrangement is a finite collection of affine hyperplanes in $\mathbb{C}^n$. These have been the object of considerable research, notably regarding the topological properties of their complements in $\mathbb{C}^n$. The goal of this essay is to study some of these properties. It should begin by discussing the fundamental group of the complement of a hyperplane arrangement, with starting point the Zariski–Van Kempen theorem. Several directions are then possible, for instance: Hattori's result on the topology of the complement of a generic arrangement; Deligne's proof that a simplicial arrangement gives a $K(\pi, 1)$ Eilenberg-MacLane space; the description of the cohomology ring of the complement in terms of generators and relations.

Relevant Courses

Essential: Algebraic topology

Useful: Algebraic geometry, differential geometry

References

[1] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, Berlin, 1992

[2] A. Dimca, *Hyperplane arrangements. An introduction*, Springer, Cham, 2017.

[3] R. Randell,*The fundamental group of the complement of a union of complex hyperplanes*, Invent. Math. 69 (1982), no. 1, 103–108

[4] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Invent. Math. 88 (1987), no. 3, 603–618.

[5] W. Arvola,*The fundamental group of the complement of an arrangement of complex hyperplanes*, Topology 31 (1992), no. 4, 757–765

[6] A. Hattori, *Topology of* C^n *minus a finite number of affine hyperplanes in general position*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), no. 2, 205–219.

[7] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972), 273–302.

20. Dirac Operators ....
Dr A. G. Kovalev

The Dirac operator, for smooth functions from $\mathbf{R}^n$ to $\mathbf{C}^N$, may be defined as a first order differential operator whose square is the Laplacian. (Thus the simplest example of Dirac operator would be the usual derivative of complex-valued functions on $\mathbf{R}$.) Unlike the Laplacian, which is well-defined on every oriented Riemannian manifold, the construction of Dirac operator requires the existence of a certain vector bundle, called the spinor bundle, over the base manifold. The essay could begin by explaining the significance of spinor bundles (cf. [1]), and why a Dirac operator can always be constructed when the dimension of the base manifold is 3 or 4. Operators of Dirac type arise in many geometrically natural differential equations, for example in the construction of Seiberg–Witten invariants of smooth 4-dimensional manifolds. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` and discuss further. The first two or three sections in [2] would be a good introductory reading (and a source of useful exercises!).

Relevant Courses

Essential: Differential Geometry, Algebraic Topology

Useful: Complex manifolds

References

[1] J. Roe, *Elliptic operators, topology and asymptotic methods*, Pitman Res. Notes in Math., Longman, 1988 (or second edition 1998).

[2] N. Hitchin, *The Dirac operator*. Invitations to geometry and topology, Oxford Univ. Press, Oxford, 2002, pages 208–232.

[3] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1–55.

[4] H.B. Lawson, Jr. and M.-L. Michelson, *Spin geometry*, Princeton University Press, 1989.

21. The Hodge Decomposition Theorem ....
Dr A. G. Kovalev

The concept of Laplace operator $\Delta = -(\partial/\partial x_1)^2 - \ldots - (\partial/\partial x_n)^2$ for functions on the Euclidean space $\mathbf{R}^n$ may be extended to oriented Riemannian manifolds. The construction uses a certain duality, called the Hodge star, and the resulting 2nd order differential operator is well-defined on differential forms. The celebrated Hodge decomposition theorem implies a natural isomorphism between the kernel of this Laplacian (i.e. the space of harmonic differential

forms) and the de Rham cohomology for a compact oriented manifold without boundary [1]. The theory admits a nice extensions to compact manifolds with boundary and to non-compact Riemannian manifolds with tubular ends (informally, 'with boundary at infinity'). In the latter case, there are also far-reaching relations between the topology and curvature in dimension 4 (say), including a generalization of the Gauss–Bonnet theorem for surfaces. Interested candidates are welcome to contact me (`A.G.Kovalev@dpmms`) for further details; section 3.5 of `http://www.dpmms.cam.ac.uk/~agk22/riem1.pdf` could be a good preliminary reading.

Relevant Courses

Essential: Differential geometry

Useful: Algebraic topology, Complex manifolds, Elliptic Partial Differential Equations

References

[1] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer 1983. Chapter 6.

[2] S. Cappell, D. DeTurck, H. Gluck, E. Y. Miller, *Cohomology of harmonic forms on Riemannian manifolds with boundary*, arxiv:math.DG/0508372 or Forum Math. **18** (2006), 923–931.

[3] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 97–118.

[4] R.B. Melrose, *The Atiyah–Patodi–Singer index theorem*, A K Peters Ltd., 1993.

22. Atiyah–Singer Index Theorem ....
Dr A. G. Kovalev

The main object of study in this essay is elliptic differential operators. One well-known example of elliptic operator is the Laplacian, another, and perhaps more important in this essay, is a Dirac operator. The elliptic property can be defined for operators acting on functions, and more generally sections of vector bundles, over smooth manifolds. If a manifold is compact then every elliptic operator over it has finite-dimensional kernel and cokernel. The difference between the two latter dimensions is called the (Fredholm) *index*. A remarkable theorem due to Atiyah and Singer asserts that the index of elliptic operator (over a compact manifold), an *analytic* quantity, can be computed entirely from *topological* invariants of the base manifold and vector bundle(s). Several different proofs of this theorem are now known and the essay can discuss aspects of some proof and/or applications. Interested candidates are welcome to further discuss the possibilities with me (`A.G.Kovalev@dpmms`); the presentation [1] provides a nice introduction to the topic.

Relevant Courses

Essential: Algebraic topology, Differential geometry

Useful: Analysis of Partial Differential Equations, Elliptic partial differential equations

References

[1] R. Mazzeo, *The Atiyah–Singer Index Theorem: What it is and why you should care.*
https://www2.bc.edu/patrick-orson/indextheory/asit3.pdf

[2] M. Atiyah, R. Bott, V.K. Patodi. On the heat equation and the index theorem. *Invent. Math.* **19** (1973), 279–330.

[3] B. Booss, D.D. Bleecker, *Topology and analysis. The Atiyah-Singer index formula and gauge-theoretic physics.* Springer-Verlag, 1985.

[4] L.I. Nicolaescu, *Notes on the Atiyah–Singer Index Theorem.*
http://www.nd.edu/~lnicolae/ind-thm.pdf

23. Topology of Configuration Spaces ....

Dr M. Krannich

The space $C_n(M)$ of n distinct but indistinguishable points in a manifold M has a surprisingly rich topology, even in the simplest case of Euclidian spaces, and has been object of study for algebraic topologists for several decades.

This essay focusses on different patterns in the homology of these configuration spaces. It should begin by explaining the *Fadell–Neuwirth fibration* [1] and the resulting connection to *Artin's braid groups*. The main focus should then be a result of McDuff [2] which shows that the homology of $C_n(M)$ agrees in a range of degrees with the homology of a seemingly quite different space of sections built from the tangent bundle of M. This is an instance of a principle called *scanning* which has lead to several other striking results, even in recent years.

McDuff's methods do not provide an estimate of how large this range of degrees actually is, but one can show via a different approach (see e.g. [3]) that the homology groups $H_k(C_n(M))$ are independent of n as long as $n \geq 2k$ if M has nonempty boundary. This is an example of a *homological stability* result, aspects of which could be discussed in a second part of the essay.

Relevant Courses

Essential: Part III Algebraic Topology

References

[1] E. Fadell and L. Neuwirth, *Configuration Spaces*, Math. Scand. 10 (1962): 111-118.

[2] D. McDuff, *Configuration spaces of positive and negative particles*, Topology 14 (1975), 91–107.

[3] O. Randal-Williams, *Homological stability for unordered configuration spaces*, Q. J. Math. 64 (2013), no. 1, 303–326.

24. (No) Wandering Domains ....

Dr H. Krieger

A holomorphic self-map f of the Riemann sphere can have stable regions - known as *Fatou components* - where the long-term behaviour of points under iteration is predictable. Sullivan's celebrated No Wandering Domains theorem [5] establishes that these components do not wander:

that is, if U is a Fatou component of f, then the set $\{U, f(U), f^{\circ 2}(U), f^{\circ 3}(U), \dots\}$ is a finite collection of components.

In this essay, you will apply the basic theory of complex dynamics in one variable to understand the proof of Sullivan's theorem. You can then proceed in a number of directions: (1) wandering domains in transcendental dynamics [2], (2) wandering domains in higher-dimensional complex dynamics [1], or (3) no wandering domains in p-adic dynamics [3]. In each case, you will first develop the basic dynamical theory for the relevant setting.

Relevant Courses

Essential: Part II Riemann Surfaces, Differential Geometry (Part II or Part III), Complex Dynamics (Part III).

Useful: Algebraic geometry (for direction (2)), Number Fields / Theory (for direction (3)).

References

[1] M. Astorg, X. Buff, R. Dujardin, H. Peters, J. Raissy. A two-dimensional polynomial mapping with a wandering Fatou component, *Annals of Mathematics* (2016), **184**, 263–313.

[2] I. N. Baker. Wandering domains in the iteration of entire functions, *Proc. London Math. Soc.* (1984) **49**, no. 3, 563–576.

[3] R. Benedetto. p-adic dynamics and Sullivan's no wandering domains theorem, *Compositio Mathematica* (2000), **122**, 281–298.

[4] L. Carleson, T. Gamelin. Complex dynamics. Springer Universitext Tracts in Mathematics (1993), Springer-Verlag New York, Inc.

[5] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains, *Annals of Mathematics* (1985), **122** no. 2, 401–418.

25. Effective Diophantine Approximation and Unlikely Intersections .... Dr H. Krieger

The complex plane $\mathbb{C}$ parametrizes isomorphism classes of elliptic curves via the j-invariant. The principle of unlikely intersections predicts that a curve $f(x, y) = 0$ in $\mathbb{C}^2$ with no modular component should contain only finitely many points for which both coordinates are the j-invariant of an elliptic curve which admits an additional structure (known as complex multiplication). This finiteness was established by André in 1998, but his proof was ineffective; that is, it did not provide for a given curve any way to find all points on the curve with this property. In 2012 an effective version was proved independently by Kühne [5] and Bilu-Masser-Zannier [2], using the theory of linear forms in logarithms. This is a special case of what is known as the *effective André-Oort conjecture*.

The main goal of this essay will be to understand the theory of Weil heights in arithmetic geometry (see [4]) and the technique of linear forms in logarithms (see [3]), and to explain how they are used to provide effective bounds for questions of unlikely intersections as discussed above. An interested student might then proceed to related questions of the arithmetic geometry of the complex plane as moduli space of elliptic curves such as [1], or other instances of unlikely intersections (see [6]), or further results in effective Diophantine geometry (see [3]).

Relevant Courses

Essential: Part II Number Fields and Part III Elliptic Curves.

Useful: Part III Algebraic Number Theory and Algebraic Geometry.

References

[1] Y. Bilu, P. Habegger, L. Kühne. Effective bounds for singular units. Preprint arXiv:1805.07167.

[2] Y. Bilu, D. Masser, U. Zannier. An effective Theorem of André for CM-points on a plane curve, *Math. Proc. Camb. Phil. Soc.* (2013), **154**, 145–152.

[3] Y. Bugeaud. Linear Forms in Logarithms and Applications, IRMA Lectures in Mathematics and Theoretical Physics, 28, Zürich: European Mathematical Society, 2018.

[4] M. Hindry, J. H. Silverman. Diophantine Geometry: An Introduction. Springer Graduate Texts in Mathematics (2000), Springer-Verlag New York, Inc.

[5] L. Kühne. An effective result of André-Oort type, *Annals of Mathematics* (2012), **176**, 651–671.

[6] U. Zannier. Some problems of unlikely intersections in arithmetic and geometry. Annals of mathematics studies (2012), Princeton University Press.

26. Wadge Determinacy and the Semi-Linear Ordering Principle . . . .
Professor B. Löwe

One of the surprising consequences of the Axiom of Determinacy AD is that it implies that sets of real numbers (as usual in set theory, we are using Baire space ω^ω rather than the space $\mathbb{R}$ as the "real numbers") are semi-linearly ordered by the relation $\leq_W$ defined by

$$A \leq_W B \text{ if and only if there is a continuous reduction of } A \text{ to } B.$$

Here, *semi-linearly ordered* means that for any A and B, either $A \leq_W B$ or $\omega^\omega \backslash B \leq_W A$. This fact is known as *Wadge's Lemma* and does not use the determinacy of all games, but only of a subclass of games known as *Wadge games*. In a series of papers [1,2,3], Andretta explored the relationship between the determinacy of Wadge games and the semi-linear ordering principle.

The aim of this essay is to understand and describe Andretta's result that the determinacy of Wadge games and the Semi-Linear Ordering Principle are equivalent (e.g., [3, Theorem 27]). Time permitting, the essay could also discuss other game-related reducibilities such as Borel reductions and their corresponding semi-linear ordering principles (cf. [3, §6] and [4]).

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent) and Part III *Infinite Games*.

Useful: Part Ib *Metric and Topological Spaces* (or equivalent).

References

[1] A. Andretta. Equivalence between Wadge and Lipschitz determinacy. *Ann. Pure Appl. Log.* 123 (2003):163–192.

[2] A. Andretta. More on Wadge determinacy. *Ann. Pure Appl. Log.* 144 (2006):2–32.

[3] A. Andretta. The SLO principle and the Wadge hierarchy. In: S. Bold, B. Löwe, T. Räsch, J. van Benthem (eds.), *Foundations of the Formal Sciences V. Infinite Games.* Studies in Logic Vol. 11 (College Publications 2007), 1–38.

[4] A. Andretta, D. A. Martin. Borel-Wadge degrees. *Fundam. Math.* 177 (2003): 175–192.

27. Determinacy and the Strong Compactness of $\aleph_1$ ....
Professor B. Löwe

Strongly compact cardinals and *supercompact cardinals* are two very strong large cardinal axioms that lie beyond the limits of our current set-theoretic techniques: one of the most famous open questions of the theory of large cardinals is whether these two axioms have the same strength in ZFC.

Without the Axiom of Choice, there are alternative definitions in terms of fine and normal ultrafilters on $\wp_\kappa(\lambda)$, respectively. In the ZF-context, both axioms turn out to be much weaker than in the ZFC-context; in particular, Becker proved that the Axiom of Determinacy AD implies that $\aleph_1$ is $\aleph_2$-supercompact [1]. There is a more general proof by Solovay that gives that $\aleph_1$ is $<\Theta$-supercompact under the stronger assumption of the Axiom of Real Determinacy $AD_\mathbb{R}$ [2]. This proof does not work with the weaker assumption AD, and, again, it is a famous open question whether AD alone implies that $\aleph_1$ is $<\Theta$-supercompact.

However, a modification of the proof does yield *fine* ultrafilters on $\wp_{\aleph_1}(\lambda)$ for all $\lambda < \Theta$ (instead of normal ultrafilters), and thus AD implies that $\aleph_1$ is $<\Theta$-strongly compact. The modified Solovay proof is well-known, but not very well documented in the published literature.

This essay aims at understanding the Solovay argument for supercompactness under the assumption of $AD_\mathbb{R}$ and then modify it to give the result for strong compactness under AD.

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent). Part III *Infinite Games.*

References

[1] H. S. Becker. AD and the supercompactness of $\aleph_1$. *J. Symb. Log.* 46 (1981): 833-842

[2] R. M. Solovay. The independence of DC from AD. In: A. S. Kechris, Y. N. Moschovakis (eds.), *Cabal Seminar 76–77. Proceedings. Caltech-UCLA Logic Seminar 1976-1977.* Lecture Notes in Mathematics, Vol. 689. (Springer-Verlag 1978), 171–184.

28. The Strength of the Weak Vopěnka Principle ....
Professor B. Löwe

Vopěnka's Principle says that for any proper class of structures, there is one that is elementarily embeddable into another; equivalently, no proper class of graphs is rigid. Vopěnka's Principle is very strong: its large cardinal strength is close to a so-called *almost huge* cardinal [2, §24].

In their book [1], Adámek and Rosicky re-formulated this principle category-theoretically: it is equivalent to the statement 'the category of ordinals cannot be fully embedded into the category of graphs.' Based on this formulation, Adámek and Rosicky then considered a "formally similar" statement: 'the opposite of the category of ordinals cannot be fully embedded into the category of graphs'. They proved that it follows from Vopěnka's principle and called it *weak Vopěnka's principle* WVP. It was also established that it has large cardinal strength by showing that it implies the existence of proper class of measurable cardinals. That lower bound is much lower than the upper bound given by Vopěnka's principle, and so determining the precise large cardinal strength of WVP was one of the major open problems in the area.

The problem was recently solved by Wilson in [3] where he shows that VWP is equivalent to 'the class of ordinals is Woodin,' a large cardinal statement much weaker than the strength of Vopěnka's principle.

The aim of the essay is to understand and describe the background and the proof of Wilson's result.

Relevant Courses

Essential: Part II *Logic and Set Theory* (or equivalent).

Useful: Part III *Category Theory*. Part III *Infinite Games*.

References

[1] J. Adámek, J. Rosicky. *Locally presentable and accessible categories.* London Mathematical Society Lecture Note Series, Vol. 189. (Cambridge University Press 1994)

[2] A. Kanamori. *The Higher Infinite. Large cardinals in set theory from their beginnings.* Springer Monographs in Mathematics. Second Edition (Springer-Verlag 2003)

[3] T. M. Wilson. The large cardinal strength of the weak and semi-weak Vopěnka principle. Preprint June 2019. arXiv:1907.00284.

29. Blocks with a Cyclic Defect Group ....
Dr S. Martin

This topic is perhaps the highlight of the classical theory of modular representation theory as initiated and developed by Richard Brauer, and then refined further by Sandy Green. In this theory a central role is played by the p-blocks of the modular group algebra and by certain p-subgroups of G called defect groups. As such, the more complicated the defect group, the more representation theoretically complicated is the p-block (for example, blocks of defect zero - where the defect group is trivial - are matrix algebras wherein everything is semisimple). Given a block of a finite group with a cyclic defect group, there is combinatorial gadget called a Brauer tree which describes the structure of the indecomposable projective modules completely - in the

sense that we know their 'module diagram' [4,5]. One could adopt Green's original approach [2] and [1, Ch 5] or use some methods from the representation theory of algebras as in [3, Chapter 6] to give a construction of these trees. At least one non-trivial example should be included for illustration of the theory. Generalisations of these methods [4,5] could be mentioned if you have time.

Relevant Courses

Algebra, Modular representation theory

References

[1] J.L. Alperin, Local representation theory, CUP (1986).

[2] J.A. Green, Walking around the Brauer tree, *J. Australian Math. Soc.* **17** (1974), 197–213.

[3] D.J. Benson, Representations and cohomology vol I. (Basic representation theory of finite groups and associative algebras), 2nd edn., Cambridge Studies in Advanced Mathematics **30**. CUP (1998).

[4] D.J. Benson and J.F Carlson, Diagrammatic methods for modular representations and cohomology, Commun. in Algebra **15**, 53–121 (1987).

[5] D.J. Benson and J.H. Conway, Diagrams for modular lattices, J. Pure Appl. Algebra 37 (1985), 111-116.

30. Auslander-Reiten Theory ....
Dr S. Martin

The aim of the essay is to provide an introduction to the notion of an almost split sequence, including their (far from obvious) existence and uniqueness properties and their use in the representation theory of finite-dimensional algebras. For a modern, comprehensive treatment refer to [1], but more gentle introductions appear in Chapter I of [3] and Chapter 4 of [2]. The volume of conference proceedings [4] has a nice introductory article by one of the founders of the subject, Maurice Auslander, called *A survey of existence theorems for almost split sequences* (pp 81–89).

Formulation of the existence theorem for such sequences will first necessitate a discussion of selected preliminary aspects of the general theory such as homology, categories, posets, (maybe) derived categories and the dual of the transpose, and possibly other material such as coverings. Although ostensibly concerned with finite-dimensional algebras over fields, one can work in the more general class of rings called artin algebras (these are algebras A over commutative artin rings R with A a finitely-generated R-module). This more general context widens the applicability of the theory.

Possible examples of their use include the Auslander-Reiten quiver, finiteness, Cartan matrices, coverings, biseriality, hereditary algebras and translation quivers.

Relevant Courses

Essential: Algebra, Modular epresentation Theory, Commutative Algebra

Useful: Category Theory, Homological Algebra

References

[1] M. Auslander, I. Reiten, S. O. Smalø, Representation theory of artin algebras, CUP (1994).

[2] D.J. Benson, Representations and cohomology Volume I, CUP (1991)

[3] K. Erdmann, Blocks of tame representation type and related algebras, LNM 1428, Springer-Verlag (1980)

[4] P.J. Webb (ed), Representations of Algebras, LMS LNS 116, CUP 1986.

31. Bott Periodicity and the J-Homomorphism ....
Dr O. Randal-Williams

A theorem of Freudenthal says that the groups $\pi_{n+k}(S^k)$, of homotopy classes of continuous maps from an $(n+k)$-sphere to a k-sphere, become independent of k as long as k is large enough: these are the *stable homotopy groups of spheres* π_n^s. A motivating problem in the subject of homotopy theory is to determine these groups. (Until very recently the state of the art was up to $n \sim 60$; recent breakthroughs take us up to $n \sim 90$).

This essay will focus on understanding one systematic part of these stable homotopy groups, the so-called *image of J*, using deep properties of the topology of unitary groups. In the first part of the essay you should prove Bott's periodicity theorem $\Omega^2 U \simeq U$, from which all the homotopy groups of the infinite unitary group U may be immediately determined. There are many proofs of this result, but you should follow that of [1, Section 2] (cf. [2, p. 404]). This will require you to learn about characteristic classes, classifying spaces, spectral sequences, and other basic tools from homotopy theory.

In the second part of the essay you should describe the homomorphism $J : \pi_k(U) \to \pi_k^s$, construct Adams' invariant $e : \pi_k^s \to \mathbb{Q}/\mathbb{Z}$ using the Chern character, and explain how the composition $e \circ J$ may be evaluated in terms of Bernoulli numbers. For this you should look at [3, Section 7] (cf. [4]). There are several adjacent topics which an ambitious essay might include, which can be discussed with me.

Relevant Courses

Essential: Part III Algebraic Topology

References

[1] E. Dyer and R. Lashof, *A topological proof of the Bott periodicity theorems.* Ann. Mat. Pura Appl. (4) 54 (1961), 231-254.

[2] R. M. Switzer, *Algebraic topology–homotopy and homology.* Die Grundlehren der mathematischen Wissenschaften, Band 212. Springer-Verlag, New York-Heidelberg, 1975.

[3] J. F. Adams, *On the groups $J(X)$. IV.* Topology 5 (1966), 21-71.

[4] E. Dyer, *Chern characters of certain complexes.* Math. Z. 80 (1962/63), 363-373.

32. Chromatic Polynomials of Graphs and Toric Geometry ....
Dr D. Ranganathan

Given a finite graph G, the chromatic polynomial $\chi(q)$ controls the number of proper colourings of G with q colours. A remarkable conjecture of Read, and Rota–Heron–Walsh, states that the coefficients of this polynomial satisfy an elementary quadratic inequality, – namely, they form a log-concave sequence. Despite being a statement about all graphs, no elementary proof of this fact was found and the conjecture remained open for nearly 50 years. Finally, Read's conjecture and its generalizations were proved very recently in groundbreaking work of Huh, Huh–Katz, and Adiprasito–Huh–Katz, in varying degrees of generality.

Log concavity appears in algebraic geometry for apparently unrelated reasons. The cohomology of a smooth projective variety is constrained by *Hodge theory*, which forces the dimensions of certain cohomology groups to exhibit such a log concavity property. Huh and Huh–Katz proved Read's conjecture and generalizations by relating these two worlds – graph theory and Hodge theory – using tropical geometry and the intersection theory on toric varieties.

This essay will give an introduction to intersection theory on toric varieties and a relevant part of tropical geometry that governs it, and explain how this and the *Khovanskiii–Teissier* inequalities on toric varieties produces a proof of Read's conjecture. An additional possibility would be to explain the more general work of Adiprasito–Huh–Katz on the Rota–Heron–Walsh conjecture for arbitrary matroids by developing the Hard Lefschetz theorem and Hodge–Riemann relations for matroids.

Relevant Courses

Essential: Toric Geometry, Part II Algebraic Topology.

Useful: Part III Algebraic Geometry, Part III Algebraic Topology.

References

[1] Adiprasito, K., Huh, J., and Katz, E. (2018). Hodge theory for combinatorial geometries. Annals of Mathematics, 188(2), 381-452.

[2] Baker, Matthew. "Hodge theory in combinatorics." Bulletin of the American Mathematical Society 55.1 (2018): 57-80.

[3] Huh, June, and Eric Katz. "Log-concavity of characteristic polynomials and the Bergman fan of matroids." Mathematische Annalen 354.3 (2012): 1103-1116.

[4] Huh, June. "Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs." Journal of the American Mathematical Society 25.3 (2012): 907-927.

[5] Huh, June. "Combinatorial applications of the Hodge-Riemann relations." Proceedings of the International Congress of Mathematicians 3 (2018), 3079-3098.

33. One Calculation in Gromov–Witten Theory ....
Dr D. Ranganathan

For an algebraic manifold X, Gromov–Witten theory is concerned with a space $\mathcal{M}(X)$ of algebraic curves inside X of a fixed "shape", i.e. homology class and genus. It is a string theory inspired take on the age old subject of *enumerative algebraic geometry* or *curve counting*. These

questions go back a long way. In the 1800's Zeuthen used an amazing argument to show that there were 23011191144 smooth degree 4 plane curves tangent to 14 general lines in $\mathbb{P}^2$.

Calculations in Gromov–Witten theory are notoriously hard, but invariably beautiful. Attacking them requires a slew of techniques passing through moduli theory, equivariant cohomology, degeneration methods, Hodge theory, and more.

This essay will look at any one of the more substantial calculations in this subject, internalize the techniques involved, give a detailed introduction of these techniques, and use that to give an exposition of the calculation. Different calculations have different flavours and difficulties, but are all part of the enumerative geometry canon. Here are some possible lines of exploration.

- There are exactly 12 nodal cubics through 8 general points in $\mathbb{CP}^2$. This can be done using classical techniques – blowups of algebraic surfaces and Euler characteristic calculations. This reveals various motivic generalizations of curve counting.

- The generating function for counts of covers of an elliptic curve by genus g Riemann surfaces is a quasimodular form of weight $6g - 6$. The main tool is the representation theory of the symmetric group.

- Kontsevich proved a remarkable recursion for counts of rational curves of degree d in $\mathbb{P}^2$ using equivariant cohomology, stable maps, and the WDVV equations.

Relevant Courses

Essential: Part III Algebraic Geometry, Part II Algebraic Topology.

Useful: Symplectic Topology, Part III Algebraic Topology.

References

[1] Beauville, Arnaud, and Beauville Arnaud. Complex algebraic surfaces. No. 34. Cambridge University Press, 1996.

[2] Dijkgraaf, R. (1995). Mirror symmetry and elliptic curves. In The moduli space of curves (pp. 149-163). Birkhäuser Boston.

[3] Hori, K., Thomas, R., Katz, S., Vafa, C., Pandharipande, R., Klemm, A., Vakil, R. and Zaslow, E., 2003. Mirror symmetry (Vol. 1). American Mathematical Soc..

[4] Kock, J., and Vainsencher, I. (2007). An invitation to quantum cohomology: Kontsevich's formula for rational plane curves (Vol. 249). Springer Science.

[5] Roth, Mike. "Counting covers of an elliptic curve." Unpublished notes, available at http://www.mast.queensu.ca/mikeroth (2009).

34. Spectral Estimates and Singularity Formation ....
Professor P. Raphael

Spectral estimates play a fundamental role in the study of non linear waves. Recently, new applications of abstract semi group decay estimates have been used for the study of singularity formation for non linear wave equations. The aim of this Essay is to:

1. Collect the necessary material on unbounded operator, semi group, Hille Yosida theorem, accretive operators and growth bound for compact perturbations of maximal accretive operators.

2. Analyze the applicability of the general method to the singularity formation problem in nonlinear wave equations and prove the suitable spectral gap estimate.

Relevant Courses

Introduction to non linear Analysis. Introduction to PDE.s

References

[1] Engel, K.J; Nagel, R., One parameter semi groups for linear evolution equations, Springer.

[2] Donninger, R.; Schörkhuber, B., Stable self-similar blow up for energy subcritical wave equations, *Dyn. Partial Differ. Equ.* **9** (2012), no. 1, 63–87.

35. On Self Similar Solutions to Non Linear Evolution Equations ....
Professor P. Raphael

The construction of self similar solutions to non linear evolution equations is a fundamental step for the understanding of the associated singularity formation problem. However, there exist few known tools to construct such objects. The aim of this essay is to:

1. Review some of the known results for compressible fluid/the nonlinear Schrödinger equation in various super critical regimes. The analysis involves both phase portrait methods and non linear bifurcation arguments.

2. Analyze the extension of these results to various settings.

Relevant Courses

Introduction to non linear Analysis. Introduction to PDE's.

References

[1] Collot, C.; Raphaël, P.; Szeftel, J., On the stability of self similar blow up for the energy super critical heat equation, to appear in *M*emoirs AMS.

[2] Martel, Y.; Raphael, P., Self-similar blow-up profiles for slightly supercritical nonlinear Schrödinger equations, preprint 2019.

[3] Meyer-Ter-Venh, J.; Schalk, C., Self similar compression waves in gas dynamics, Z. Naturfosch. **37** a, 955–969 (1982)

[4] Glangetas, L.; Merle, F., Existence of self similar blow up solutions for Zakharov equation in dimension two, part I, Comm. Math. Phys. **160**, 173-215 (1994)

36. Sutured Floer Homology ....
Professor J. A. Rasmussen

This essay involves a family of invariants known as Floer homology for 3-manifolds. These invariants can be defined in several ways, but they all share certain common properties. One of the most important is the fact that Floer homology detects the minimal genus of embedded surfaces. In other words, if $x \in H_2(M)$, we can ask for the minimal genus of an embedded orientable surface Σ representing x. (If Σ is disconnected, we minimize over a related quantity called the complexity.) This minimal genus can be determined from the Floer homology of M.

The best proof of this fact passes through an invariant known as sutured Floer homology, which was originally defined by Juhasz. Kronheimer and Mrowka gave an axiomatic treatment of this theory which starts from some basic properties of Floer homology, defines the sutured Floer homology, and then deduces that it detects the genus. The essay should give an account of Kronheimer and Mrowka's work, taking the axioms for granted. A good essay will go on to explain a related topic; possibilities include the proof of Gabai's decomposition theorem, Juhasz's definition of SFH, a proof that knots in S^3 are determined by their complements, or the fact that SFH detects fibredness.

Relevant Courses

Essential: Algebraic Topology

Useful: Differential Geometry

References

[1] A. Juhasz, Holomorphic discs and sutured manifolds, *Alg. Geom. Top* 6 (2006) 1429?1457, `arXiv:math/0601443`

[2] P. Kronheimer and T. Mrowka, Knots, sutures, and excision, *JDG* 84 (2010), 301-364, `arXiv:0807.4891`

[3] M. Scharlemann, Sutured manifolds and generalized Thurston norms, *JDG* 29 (1989), 557?61.

37. Webs and Foams ....
Professor J. A. Rasmussen

This essay is about polynomial invariants of knots in S^3 and their categorifications. Witten, Reshetikhin, and Turaev showed that for each finite dimensional representation V of a simple Lie algebra $\mathfrak{g}$, there is a polynomial invariant of knots in S^3. (For example, the invariant associated to the 2-dimensional representation of $\mathfrak{sl}_2$ is the Jones polynomial.) The process of categorification replaces polynomial invariants of knots by graded vector spaces whose graded Euler characteristic gives back the original polynomial. The most famous example is Khovanov homology, which categorifies the Jones polynomial.

Murakami, Ohtsuki, and Yamada gave a simple combinatorial definition of these knot polynomials in the case where $\mathfrak{g} = \mathfrak{sl}_n$ and V is an exterior power of the vector representation. They first define polynomial invariants of webs, which are oriented, weighted planar trivalent graphs with zero flux, and then use these to construct the knot polynomials. To categorify these polynomials, we use foams, which are two-dimensional analogs of webs.

The essay should first give an account of the MOY state model for the $\mathfrak{sl}_n$ polynomials and then discuss foams and how they are used to categorify the $\mathfrak{sl}_n$ polynomial. A good essay will discuss Robert and Wagner's recent work on the combinatorial evaluation of closed foams.

Relevant Courses

Essential:

Useful: Algebraic Topology, Lie Algebras and their Representations

References

[1] Murakami, H., Ohtsuki, T., and Yamada, S., Homfly polynomial via an invariant of colored plane graphs, *Enseign. Math.* 44 (1998), 325-360.

[2] L-H. Robert and E. Wagner, A closed formula for the evaluation of $\mathfrak{sl}_N$ foams, `arXiv:1702.04140`

[3] D. Bar-Natan, On Khovanov's categorification of the Jones polynomial, *Alg. Geom. Top.* 2 (2002), 337-370, arXiv:math/0201043

[4] M. Khovanov, $\mathfrak{sl}(3)$ link homology, *Alg. Geom. Top.* 4 (2004), 1045-1081 `arXiv:math/0304375`

38. Analytic Theory of Automorphic Forms ....
Professor A. J. Scholl

The aim of this essay is to understand some of the analytic theory of automorphic forms. Specifically, the essay should give an account of the spectral decomposition of the space $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}))$, including the description of the continuous spectrum using Eisenstein series.

The course on Modular Forms and L-functions is essential, and anyone tackling the essay will want to be comfortable with basic notions of functional analysis.

Related courses

Modular Forms and L-functions

References

[1] A. Borel: Automorphic forms on $SL_2(\mathbb{R})$. Cambridge University Press, 1997

[2] R. Godement: The decomposition of $L^2(G/\Gamma)$ for $\Gamma = SL(2, \mathbb{Z})$. In: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. vol.9, pp.211—224

[3] T. Kubota: Elementary theory of Eisenstein series. Wiley, 1973

39. The Field of Norms ....
Professor A J Scholl

Fontaine and Wintenberger discovered a remarkable connection between local fields of characteristic zero and those of characteristic p. They showed that the absolute Galois group of a "sufficiently ramified" infinite extension of $\mathbb{Q}_p$ is canonically isomorphic to the absolute Galois group of a local field of characteristic p. This leads to a description of p-adic representations

of the absolute Galois group of a p-adic local field in terms of so-called (ϕ, Γ)-modules, and an explicit description of their Galois cohomology. The aim of this essay will be to describe the construction of the "field of norms" and of (ϕ, Γ)-modules. Time permitting, the essay might also touch on one of the following:

- Herr's proof [3] of Tate local duality using (ϕ, Γ)-modules;

- perfectoid fields [4].

Relevant Courses

Essential: Local fields.

References

[1] J.-P. Wintenberger, Le corps des normes de certaines extensions infinies de corps locaux; applications, Ann. Scient. École Norm. Sup. (4) **16** (1983), 59–89

[2] J.-M. Fontaine: Représentations p-adiques des corps locaux. In: The Grothendieck Festschrift Vol.II, (P. Cartier et al., ed.), 249–309. Birkhäuser, 1990

[3] L. Herr: $\Phi - \Gamma$ modules and Galois cohomology. In "Invitation to higher local fields", Part II, section 6, 263–272

[4] K. S. Kedlaya: New methods for (ϕ, Γ)-modules. Mathematical Sciences **2** (2015), 20

[5] A. J. Scholl. Higher fields of norms and (ϕ, Γ)-modules. Doc. Math (2006), 685–709

40. The Goldman Bracket ....
Professor I. Smith

Take two simple closed curves on a surface Σ of positive genus which meet at a single point. There are two ways of locally resolving their intersection to produce two new (unions of) simple curves. This geometric surgery operation underlies a Lie algebra structure on the vector space generated by free homotopy classes of closed curves on Σ, known as the Goldman bracket. The Goldman Lie algebra appears naturally in symplectic geometry; it admits a natural representation on the Lie algebra of functions on (symplectic) representation varieties of surfaces; it has links to knot theory; and it admits various high-dimensional generalizations (string topology, symplectic cohomology). This essay will describe basic algebraic features of the Goldman bracket, and then develop some of its deeper aspects, including its connection to the symplectic geometry of character varieties.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry, Symplectic Topology.

Useful: Mapping class groups.

References

[1] W. Goldman. *The symplectic nature of fundamental groups of surfaces.* Adv. in Math. 54 (1984), no. 2, 200 − 225.

[2] M. Chas. *Combinatorial Lie bialgebras of curves on surfaces.* Topology 43 (2004), no. 3, 543 − 568.

[3] P. Etingof. *Casimirs of the Goldman Lie algebra of a closed surface.* Int. Math. Res. Not. (2006).

[4] H. Abbaspour and M. Zeinalian. *String bracket and flat connections.* Algebr. Geom. Topol. 7 (2007), 197 − 231.

41. Dynamics on K3 Surfaces ....

Professor I. Smith

K3 surfaces are all diffeomorphic to a smooth quartic surface in complex projective 3-space. They play a special role in the classification of complex surfaces, and have rich complex dynamics. The entropy of a holomorphic automorphism of a complex algebraic variety is given by the logarithm of its spectral radius for the action on cohomology, which means that dynamical questions can be approached lattice-theoretically. K3 surfaces admit automorphisms of positive entropy given by Salem numbers. Their construction makes extensive use of the Torelli theorem for K3 surfaces, Coxeter groups, and more. This essay will explain the Torelli theorem, discuss the Gromov-Yomdin theorem on topological entropy, construct some explicit positive entropy automorphisms, and discuss open questions.

Relevant Courses

Essential: Algebraic Topology, Complex Dynamics

Useful: Differential Geometry, Complex Manifolds, Algebraic Geometry

References

[1] Curt McMullen. *Dynamics on K3 surfaces: Salem numbers and Siegel discs.* J. Reine Angew. Math. 545 (2002).

[2] Curt McMullen. *Automorphisms of projective K3 surfaces with minimum entropy.* Invent. Math. 203 (2016).

[3] S. Cantat. *Dynamique des automorphismes des surfaces K3.* Acta Math. 187 (2001), 1– 57.

[4] Mikhail Gromov. *On the entropy of holomorphic maps.* Enseign. Math. 49 (2003).

42. Gentle Algebras and Fukaya Categories of Surfaces ....

Dr J. Smith

The Fukaya category $\mathcal{F}(X)$ of a symplectic manifold X encodes information about Lagrangian submanifolds in X and their intersections. The morphism groups are defined using Floer theory, which involves counting holomorphic curves in X, but we shall restrict to the case where X is a surface Σ (of real dimension 2), where a purely combinatorial description is possible.

For each generating object L in $\mathcal{F}(\Sigma)$ one obtains an equivalence between the $\mathcal{F}(\Sigma)$ and the derived category of dg-modules over the algebra $A = \hom(L, L)$. Choosing different generators L may give different algebras A, but by construction all of them have equivalent derived categories. These derived equivalences may be non-obvious from a purely algebraic perspective.

It turns out that A is a *gentle algebra*—of significant recent interest in representation theory— and Lekili–Polishchuk [3] showed how to reverse the above construction, associating a surface to each gentle algebra. From this they gave a new sufficient condition for derived equivalence of gentle algebras, and a geometric description of the 'AAG' derived invariants [1].

In this essay you will understand and sketch the description of the Fukaya category in [2, Sections 3.1–3.5], explain in detail the key results of [3], and compute illustrative examples of derived equivalences and inequivalences. Technical results from [2] may be taken as black boxes. If time permits, you could then cover related topics that interest you, e.g. from [4].

Relevant Courses

Essential: Algebraic Topology

Useful: Algebra

References

[1] Diana Avella-Alaminos and Christof Geiss. *Combinatorial derived invariants for gentle algebras.* J. Pure Appl. Algebra 212 (2008), no. 1, 228–243. arXiv:math/0607348

[2] Fabian Haiden, Ludmil Katzarkov, and Maxim Kontsevich. *Flat surfaces and stability structures.* Publ. Math. Inst. Hautes Études Sci. 126 (2017), 247–318. arXiv:1409.8611

[3] Yankı Lekili and Alexander Polishchuk. *Derived equivalences of gentle algebras via Fukaya categories.* Math. Ann. (2019). arXiv:1801.06370

[4] Sebastian Opper. *On auto-equivalences and complete derived invariants of gentle algebras.* arXiv:1904.04859

43. Colouring Triangle-Free Graphs ....
Professor A. G. Thomason

A graph G of maximum degree Δ has an independent set of size $|G|/\Delta$, which can be found quickly by the greedy algorithm. But, unless the graph is locally very dense, a larger set can be found. In fact, if the graph is triangle-free then the graph behaves more like a random graph; there is an independent set of size $\Omega(|G| \log \Delta / \Delta)$. This was first proved by Ajtai, Komlós and Szemerédi by the *semi-random method*, which they introduced to analyse the random greedy algorithm. It has since been shown that the chromatic number is $O(\Delta / \log \Delta)$, and there have been corresponding extensions to hypergraphs. An essay on this topic would discuss the semi-random method, explaining its effect in some simple cases and describing some more involved applications.

Relevant Courses

Combinatorics and Extremal Graph Theory are relevant: nothing is essential.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers. *J. Combin. Theory Ser(A)* **29** (1980), 354–360.

[2] M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, Extremal uncrowded hypergraphs, *J. Combinatorial Theory, Ser. A* **32** (1982), 321–335.

[3] J.H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms* **7** (1995), 173–207.

[4] A. Frieze and D. Mubayi, On the Chromatic Number of Simple Triangle-Free Triple Systems, *Electronic J. Combinatorics* **15** (2008), R121.

44. Flag Algebras in Graph Theory ....
Professor A. G. Thomason

A substantial number of questions in extremal graph theory have been attacked successfully in recent years by the method of flag algebras. The method's most spectacular early success was the determination of the minimum number of triangles in a graph subject to a given number of edges. The effectiveness of the method comes to a great extent from the fact that it permits a computer to find, in effect, an optimal proof by solving a practically feasible semi-definite program. In some cases the method can be pushed even further, to the point of establishing a calculus for finding a minimum. There are other limitations to the method too which are now understood better. An essay would describe the basic flag algebra method, together with some examples, and then explore further aspects.

Relevant Courses

Extremal Graph Theory is relevant.

References

[1] A. Razborov, On the minimal density of triangles in graphs, *Combinatorics, Probability and Computing* **17** (2008), 603–618.

[2] R. Baber and J. Talbot, Hypergraphs do jump, *Combinatorics, Probability and Computing* **20** (2011), 161–171.

[3] H. Hatami, J. Hladky, D. Král', S. Norine and A. Razborov, On the number of pentagons in triangle-free graphs, *J. Combinatorial Theory (Ser A)* **120** (2013), 723–732.

45. Which Vertex-Transitive Graphs are Quasi-Isometric to Cayley Graphs?
Dr M. C. H. Tointon

A graph Γ is called *vertex transitive* if for every two vertices x, y of Γ there exists an automorphism of Γ mapping x to y. A particularly tractable type of vertex-transitive graph is a *Cayley graph*: given a group G with a finite symmetric generating set S, the Cayley graph $\Gamma(G, S)$ has the elements of G as its vertices, with an edge between vertices x and y if and only if there exists $s \in S$ such that $x = ys$. The Petersen graph is a famous example of a vertex-transitive graph on 10 vertices that is not a Cayley graph, so the two notions are not equivalent. However, for some time it was an open question whether all vertex-transitive graphs

had the weaker property of being *quasi-isometric* to Cayley graphs – roughly, this would mean that every vertex-transitive graph 'looked like a Cayley graph from far enough away'.

In an effort to answer this question, Diestel and Leader [1] constructed a certain vertex-transitive graph that they conjectured not to be quasi-isometric to any Cayley graph. Eskin, Fisher and Whyte [2] subsequently confirmed this conjecture. On the other hand, Trofimov had previously shown that every vertex-transitive graph with a property called *polynomial growth* was quasi-isometric to a Cayley graph [3]. The purpose of this essay is to present proofs of these results.

Relevant Courses

The notions of quasi-isometry and Cayley graphs will be discussed in the Graduate course on Geometric Group Theory. Attendance at that course, while not strictly necessary (as the papers are fairly self-contained), is therefore strongly recommended. It will also be useful to have some basic knowledge of graph theory.

References

[1] R. Diestel and I. Leader. A conjecture concerning a limit of non-Cayley graphs, *J. Algebraic Combin.* **14** (2001), 17–25.

[2] A. Eskin, D. Fisher and K. Whyte. Quasi-isometries and rigidity of solvable groups, *Pure Appl. Math. Q.* **3** (2007), Special Issue: In honor of Grigory Margulis. Part 1, 927–947.

[3] W. Woess. Topological groups and infinite graphs, *Discrete Math.* **95** (1991), 373–384.

46. Equidistribution of Roots of Polynomials ....

Dr P. P. Varjú

The essay will discuss the phenomenon that polynomials with small integer coefficients tend to have their roots accumulated near the unit circle and they are approximately evenly distributed there. Precise statements of this kind have been proved by Erdős and Turán [3] and Bilu [2]. These results have been revisited by many authors because of their importance to number theory.

Relevant Courses

No courses are required but basic knowledge of Galois theory and Fourier analysis is very useful for this essay.

References

[1] Amoroso, F. ; Mignotte, M. *On the distribution of the roots of polynomials.* Ann. Inst. Fourier (Grenoble) 46 (1996), no. 5, 1275–1291.

[2] Bilu, Y. *Limit distribution of small points on algebraic tori.* Duke Math. J. 89 (1997), no. 3, 465–476.

[3] Erdős, P. ; Turán, P. On the distribution of roots of polynomials. Ann. of Math. (2) 51, (1950). 105–119.

[4] Granville, A. *The distribution of roots of a polynomial.* Equidistribution in number theory, an introduction, 93–102, NATO Sci. Ser. II Math. Phys. Chem., 237, Springer, Dordrecht, 2007.

[5] Petsche, C. *A quantitative version of Bilu's equidistribution theorem.* Int. J. Number Theory 1 (2005), no. 2, 281–291.

[6] Rumely, R. *On Bilu's equidistribution theorem.* Spectral problems in geometry and arithmetic (Iowa City, IA, 1997), 159–166, Contemp. Math., 237, Amer. Math. Soc., Providence, RI, 1999.

[7] Soundararajan, K. *Equidistribution of zeros of polynomials.* 2018. https://arxiv.org/abs/1802.06506

47. Weak Convergence and Nonlinear PDE ....
Dr C. M. Warnick

When solving PDE, one often proceeds by constructing an approximating sequence, and argues by weak compactness that a (weakly) convergent subsequence exists. Where the PDE in question is nonlinear, demonstrating that the limiting object solves a suitable equation can be a delicate matter, and often requires one to understand in detail how a weakly convergent sequence can fail to be strongly convergent. Broadly speaking, this can happen for two reasons, either concentration or rapid oscillation, and different methods are required to deal with both problems. In this essay, weak convergence methods for nonlinear PDE will be studied. The essay should cover issues associated to weak convergence: possible topics include the defect measure; Young measure; concentrated compactness; compensated compactness. Possible applications include the calculus of variations and homogenization theory.

Relevant Courses

Essential: Analysis of PDE

Useful: Introduction to nonlinear analysis

References

[1] "Weak convergence methods for nonlinear partial differential equations", L. C. Evans, AMS 1990

[2] DiPerna, R. J. (1985). Measure-valued solutions to conservation laws. Archive for Rational Mechanics and Analysis, 88(3), 223-270.

[3] Lions, P. L. (1984, March). The concentration-compactness principle in the Calculus of Variations. The locally compact case, part 1. In Annales de l'Institut Henri Poincare (C) Non Linear Analysis (Vol. 1, No. 2, pp. 109-145).

48. Rigidity Theorems for Hyperbolic Groups ....
Dr H. Wilton

A finitely generated group is called *word-hyperbolic* if triangles in its Cayley graph are uniformly thin. This condition defines a vast class of groups, first introduced by Gromov [5], and enables the geometric techniques developed by Thurston when studying hyperbolic 3-manifolds to be

applied in a much wider setting.

The idea of the essay is to explore rigidity theorems for hyperbolic groups. A typical result is Paulin's theorem, which says that the outer automorphism group of a torsion-free hyperbolic group Γ is infinite if and only if Γ splits as an amalgamated free product or HNN extension over a cyclic subgroup [3, 6].

A successful essay, after describing some of the basic theory of hyperbolic groups [4, Chapters III.H and III.Γ], will explain the basic strategy used to prove theorems like Paulin's theorem (sometimes called the *Bestvina–Paulin method*): if rigidity fails (in this case, if the outer automorphism group is infinite), then one can apply a limiting argument to extract an action on an $\mathbb{R}$-tree [1]; one then applies Rips' classification of actions on $\mathbb{R}$-trees [2] to deduce a contradiction. More advanced essays will go into other results in the same vein, such as Rips–Sela's proof that rigid hyperbolic groups are co-Hopfian [7].

Relevant Courses

Essential: Part II Algebraic topology
Useful: Part III Geometric group theory

References

[1] Mladen Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002. http://www.math.utah.edu/~bestvina/eprints/handbook.ps

[2] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Inventiones Mathematicae*, 121(2):287—321, 1995.

[3] M. R. Bridson and G. A. Swarup. On Hausdorff–Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)*, 40(3-4):267–289, 1994.

[4] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[5] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[6] Frédéric Paulin. Outer automorphisms of hyperbolic groups and small actions on **R**-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.

[7] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geometric and Functional Analysis*, 4(3):337–371, 1994.

49. Outer Space ....
Dr H. Wilton

The action of a group Γ on itself by conjugation defines a natural map $\Gamma \to \mathrm{Aut}(\Gamma)$. Its image is the (normal) subgroup of *inner automorphisms*, and the quotient $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is called the

outer automorphism group Out(Γ). When Γ = F_n, the non-abelian free group of rank n, the group Out(F_n) is especially complicated and interesting.

An important idea in geometric group theory is that one can study interesting groups by constructing nice spaces on which they act. For Out(F_n), Culler and Vogtmann [4] constructed a certain space of graphs, now known as 'Culler–Vogtmann Outer Space' and denoted by $\mathcal{CV}_n$. Topological properties of $\mathcal{CV}_n$ translate into group-theoretic properties of Out(F_n). For instance, Culler and Vogtmann showed that $\mathcal{CV}_n$ is contractible, from which it follows that Out(F_n) has finite cohomological dimension.

The idea of this essay is to describe the construction of $\mathcal{CV}_n$, to give a proof that it is contractible, and to deduce the corresponding results about Out(F_n). The original Culler–Vogtmann proof of contractibility is quite combinatorial, but a more transparent geometric proof was given by Skora — see [3] or [5]. The required results about cohomological dimension can be found in [2]. An excellent essay might go on to describe more general deformation spaces (along the lines of [3] or [5]), or to prove the existence of train tracks [1].

Relevant Courses

Essential: Part II Algebraic topology *Useful:* Part III Algebraic topology

References

[1] Mladen Bestvina. A Bers-like proof of the existence of train tracks for free group automorphisms. `arXiv:1001.0325`, 2010.

[2] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[3] Matt Clay. Contractibility of deformation spaces of *G*-trees. *Algebr. Geom. Topol.*, 5:1481–1503 (electronic), 2005.

[4] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[5] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups, Geometry, and Dynamics*. Volume 1, Issue 2, 2007, pp. 135-181.

50. The Statistics of Manifold Data ....
Professor J. A. D. Aston

Many observed data are constrained by their intrinsic features or geometry. This is especially true when the objects under analysis are shapes or images, as in many cases the angle of view and the magnification of the shape is unimportant [1]. This is also linked to the study of data which arises in particularly spaces, such as data observed on a (hyper-)sphere [2,3], or where the observations are types of matrices, such as those which are positive definite [4]. All these settings yield data that are inherently non-Euclidean, but most statistical analysis is predicated on the data coming from a Euclidean space.

The idea of this essay will be to review some of the recent advances in shape and related statistics, many of which are based on concepts from differential geometry. There is then considerable scope in the essay. Theoretical investigation could be undertaken into some of the underlying metrics that are used in statistical shape analysis. Methodology for certain special cases of shape data could be compared, such as different methods for the statistical analysis of samples of positive definite matrices. Alternatively, data analysis could be undertaken for shape observations derived from images, for example.

Relevant Courses

Useful: Differential Geometry, Modern Statistical Methods, Topics in Statistical Theory

References

[1] IL Dryden and KV Mardia (1998). *Statistical Shape Analysis.* Wiley

[2] S Jung, IL Dryden and JS Marron. (2012) Analysis of principal nested spheres, *Biometrika* **99**:551–568.

[3] KV Mardia and P Jupp (2000). *Directional Statistics.* Wiley.

[4] D Pigoli, JAD Aston, IL Dryden and P Secchi. (2014) Distances and Inference for Covariance Functions, *Biometrika* **101**:409–422.

51. Measuring the Unmeasurable ....
Professor J. A. D. Aston

We are often only able to measure quantities with a lack of complete precision, either because we can only take a few samples, or because the measurement device has an inherent error associated with it. Statistics is often the required to help quantify the uncertainty in the data. However, in some cases, there is some inherent reason why people involved in the system do not want it to be measured, for example those engaged in illegal activity. A prominent example of this is modern slavery, where it might be useful to estimate the population of people subjected to modern slavery but for obvious reasons the population itself is hidden. Despite this, statistical techniques can be used to try to estimate the size of the population [1].

This essay would aim to look at some of the statistical ways that hidden populations might be measured. This could take the form of an methodological essay looking at methods for estimating hidden populations and suggesting new approaches, to a more applied essay which explores a particular area where hidden populations are of interest. This could be anywhere from the modern slavery example to understanding ecological populations, and the essay could look methods for analysing this population through traditional capture-recapture methods or other techniques.

Relevant Courses

Useful: Modern Statistical Methods

References

[1] Multiple-systems analysis for the quantification of modern slavery: classical and Bayesian approaches' Bernard W. Silverman, Journal of the Royal Statistical Society, Series A, to appear (available online)

52. The Edge Reinforced Random Walk ....
Dr S. A. Bacallado

The edge-reinforced random walk (ERRW) [1] on an undirected graph $G = (V, E)$ is a stochastic process with $\mathbb{P}(X_{n+1} = j \mid X_1, \ldots, X_{n-1}, X_n = i) = \beta + C_{i,j}(n)$ where $C_{i,j}(n)$ is the number of transitions across (i, j) in either direction in $X_1, \ldots, X_n$. This process has a representation as a random walk in a random environment, namely if $(X_n)_{n \in \mathbb{N}}$ is recurrent, there is some probability measure μ on stochastic matrices such that,

$$\mathbb{P}(X_2 = i_2, \ldots, X_n = i_n \mid X_1 = i_1) = \int \prod_{t=1}^{n-1} K_{i_t, i_{t+1}} \mu(dK).$$

This representation makes the ERRW a useful prior for Bayesian analysis of reversible Markov chains [2]. One possibility for the essay is to review applications of the ERRW and non-reversible generalisations in Bayesian statistics [2-5]. Alternatively, the essay could focus on the relationship between the ERRW, vertex-reinforced jump process, and supersymmetric hyperbolic sigma models revealed in [6], which is used in that paper to derive a formula for the density of μ and to solve longstanding problems on the recurrence of $(X_n)_{n \in \mathbb{N}}$ on the graph $\mathbb{Z}^3$.

Relevant Courses

Essential: Advanced Probability.

References

[1] D. Coppersmith and P. Diaconis. Random walk with reinforcement. Unpublished manuscript, (1987).

[2] P. Diaconis, and S.W.W. Rolles. Bayesian analysis for reversible Markov chains. *The Annals of Statistics*, 34.3, pp. 1270–1292, (2006).

[3] S. Bacallado. Bayesian analysis of variable-order, reversible Markov chains. *The Annals of Statistics*, 39.2, pp. 838–864, (2011).

[4] S. Bacallado. Bayesian nonparametric analysis of reversible Markov chains. *The Annals of Statistics*, 41.2, pp. 870–896, (2013).

[5] S. Bacallado, et al. Bayesian regularization of the length of memory in reversible sequences. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 78.4, pp. 933–946, (2016).

[6] C. Sabot, and P. Tarres. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. arXiv:1111.3991 (2011).

53. Approximate Bayesian Computation and Optimal Transport ....
Dr S. A. Bacallado

In many scientific problems it is possible to formulate a probability model $\{P_\theta : \theta \in \Theta\}$ and to simulate data Y from any distribution P_θ in the family. However, the likelihood function $\theta \mapsto P_\theta(y)$ is intractable, which makes maximum likelihood estimation and Bayesian inference of θ prohibitive. Approximate Bayesian Computation (ABC), reviewed in [1], is a family of likelihood-free procedures for estimation and uncertainty quantification which may be applied in this setting. Bernton et al. [2] propose using the Wasserstein distance between the observed data and data generated from P_θ within ABC procedures. The goal of this essay is to review the results in this paper, putting them in the larger context of ABC techniques, and providing background on the Wasserstein distance and optimal transport. Students are encouraged to carry out a simulation study to evaluate the method.

Relevant Courses

Useful: Modern Statistical Methods.

References

[1] M.A. Beaumont. Approximate Bayesian computation. *Annual Review of Statistics and Its Application*, vol 6., pp. 379–403, (2019).

[2] E. Bernton et al. Approximate Bayesian computation with the Wasserstein distance. *Journal of the Royal Statistical Society: Series B (Methodology)*, 81, pp. 235–269, (2019).

54. Hydrodynamic Limit of the Simple Exclusion Process ....
Dr R. Bauerschmidt

The simple exclusion process is a simple example of a stochastic particle system. Particles occupy the vertices of $\mathbb{Z}^d$, at most one per vertex, and jump to a neighbouring vertex independently provided that the target vertex is not occupied. Hydrodynamic limits of interacting particle systems describe their evolution of large space and time scales. For the simple exclusion process, assuming that the particles jump symmetrically to their neighbours, this hydrodynamic evolution is described by the heat equation. The goal of this essay is to prove this, first introducing all of the relevant concepts.

Relevant Courses

Essential:
Advanced Probability
Stochastic Calculus and Applications

References

[1] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[2] T. Seppäläinen. Translation invariant exclusion processes. Available at `https://www.math.wisc.edu/~seppalai/excl-book/etusivu.html`.

55. Nonparametric Bayesian Estimation of Discretely Observed Compound Poisson Processes ....

Dr A. J. Coca

Statistical inverse problems (SIPs) are central in fields such as engineering, imaging, physics and finance. An important source of (generally nonlinear) SIPs is the area of estimation of discretely observed stochastic processes. A particularly relevant class of stochastic processes is jump processes and, within it, compound Poisson processes (CPPs) enjoy a privileged position: they are one of the "off-the-shelf" processes first used in practice to model jumps; and, from a more theoretical perspective, they sit at the intersection of fundamental classes of jump processes such as Lévy processes, renewal processes, Poisson point processes, etc., whilst they still retain much of the mathematical structure and challenges of each class constituting a tractable but representative model to work with.

Consequently, the problem of nonparametric estimation of discretely observed CPPs has received much attention in the last two decades, starting with [1]. Most of the existing literature studies non-Bayesian methodologies, and we are only now beginning to understand the Bayesian solution to this problem. (Note that the Bayesian method is particularly attractive for SIPs, as it provides a unified way to construct estimators unlike many traditional methods that require very problem-specific procedures; on the flip side, this complicates the study of the method and partly explains the delay in understanding it.) Indeed, convergence rates of posterior distributions for discretely observed CPPs are only partially understood: very recently, [4] have solved the problem when the jumps take values in the natural numbers; and, [5] and [6] partly addressed (but did not solve) the problem in the more interesting case of the jumps having an absolutely continuous distribution. These three results are based on exploiting the seminal result for posterior convergence rates in 'direct problems' in [3] in the spirit summarised in [7].

The objective of this project is two-fold: to understand, summarise and explain the general result in [3] and its adaptation to SIPs using the general-purpose idea in [7]; and, to understand, summarise and explain the application of this idea to the problem at hand by considering the works [4], [5] and [6], possibly together with some unpublished results that I will share with the students between the first and second meetings, and which solve the problem in full generality. A more ambitious project can additionally look into empirical Bayes procedures for the problem at hand by applying the results in [2].

Relevant Courses

Partly useful: Topics in Statistical Theory (Part III); Advanced Probability (Part III); Principles of Statistics (Part II); Mathematics of Machine Learning (Part II).

References

[1] Buchmann, B. and Grübel, R.: Decompounding: an estimation problem for Poisson random sums. Annals of Statistics, **31**(4), 1054–1074 (2003)

[2] Donnet, S., Rivoirard, V., Rousseau, J. and Scricciolo, C.: Posterior concentration rates for empirical Bayes procedures with applications to Dirichlet process mixtures. Bernoulli, **24**(1), 231–256 (2018)

[3] Ghosal, S., Ghosh, J. K., and van der Vaart, A. W.: Convergence rates of posterior distributions. Annals of Statistics, **28**(2), 500–531 (2000)

[4] Gugushvili, S., Mariucci, E., and van der Meulen, F.: Decompounding discrete distributions: A non-parametric Bayesian approach. arXiv:1903.11142

[5] Gugushvili, S., van der Meulen, F. and Spreij, P.: Nonparametric Bayesian inference for multidimensional compound Poisson processes. Mod. Stoch. Theory Appl., **2**(1), 1–15 (2015)

[6] Gugushvili, S., van der Meulen, F. and Spreij, P.: A non-parametric Bayesian approach to decompounding from high frequency data. Stat. Inference Stoch. Process., **21**(1), 53–79 (2018)

[7] Knapik, B. and Salomond, J.-B.: A general approach to posterior contraction in nonparametric inverse problems. Bernoulli, **24**(3), 2091–2121 (2018)

56. Convergence Rates of Variational Posterior Distributions and The Gaussian Sequence Model ....

Dr A. J. Coca

Given a statistical model, the Bayesian method is a universal and elegant way of constructing a probability distribution over the possible parameters that may have given rise to our data. During much of the 20th century, these methods were not considered as highly as others due to philosophical reasons or simply because they required too much computational power to be implemented. With the advent of computer power, they gained popularity in both practice and theory. In particular, in the last two decades the frequentist study of nonparametric and high-dimensional Bayesian procedures has grown dramatically.

However, modern statistical applications have created many new challenges for the use of the methodology, due to increased model complexity and data dimensionality. A prominent example is the impracticability of drawing directly from the posterior distribution using, e.g., Markov Chain Monte Carlo methods. A simple, yet very successful, way around it is variational Bayes, which consists in approximating the posterior by a more tractable distribution by minimising an objective function—generally the Kullback–Leibler divergence—of the former over a so-called variational family. A recent review of the technique can be found in [1]. More recently, the work in [4] has established general conditions to derive convergence rates of variational posterior distributions in nonparametric and high-dimensional models, paralleling the seminal results for posterior convergence rates in [2] and [3].

The objective of this project is two-fold: to understand, summarise and explain the general results of [2], [3] and [4], including the mathematical techniques to prove them; and to specialise their conclusions to the Gaussian sequence model, a fundamental model within nonparametric statistics. A more ambitious project can additionally look into an intriguing connection noted in [4] between variational Bayes and a seemingly unrelated methodology called empirical Bayes.

Relevant Courses

Partly useful: Topics in Statistical Theory (Part III); Principles of Statistics (Part II); Mathematics of Machine Learning (Part II).

References

[1] Blei, D., Kucukelbir, A., and McAuliffe, J.: Variational inference: A review for statisticians. Journal of the American Statistical Association, **112**(518), 859–877 (2017)

[2] Ghosal, S., Ghosh, J. K., and van der Vaart, A. W.: Convergence rates of posterior distributions. Annals of Statistics, **28**(2), 500–531 (2000)

[3] Ghosal, S., and van der Vaart, A. W.: Convergence rates of posterior distributions for noniid observations. Annals of Statistics, **35**(1), 192–223 (2007)

[4] Zhang, F. and Gao, C.: Convergence Rates of Variational Posterior Distributions. Annals of Statistics, to appear. arXiv:1712.02519v4

57. The Abelian Sandpile Model ....
Dr T. M. Hutchcroft

The *Abelian Sandpile Model* was introduced by Bak, Tang, and Wiesenfeld [1] as a toy model exhibiting *self-organized criticality*, a notion that they introduced. This is the phenomenon by which certain randomized dynamical systems tend to exhibit critical-like behaviour at equilibrium despite being defined without reference to any parameters that can be varied to produce a phase transition in the traditional sense.

Let $d \geq 1$, let $\mathbb{Z}^d$ denote the d-dimensional hypercubic lattice, and let $\Lambda = [-n, n]^d$ be a large box. A **sandpile configuration** on Λ is a function $\eta : \Lambda \to \{0, 1, \ldots\}$, which we think of as a collection of indistinguishable particles (grains of sand) located on the vertices of Λ. We say that η is **stable** at a vertex x if $\eta(x) < \deg(x) = 2d$, and otherwise that η is **unstable** at x. We say that η is stable if it is stable at every x, and that it is unstable otherwise. If η is unstable at x, we can **topple** η at x, redistributing $2d$ of the grains of sand at x equally among its neighbours. Grains of sand redistributed to neighbours of x outside of Λ are lost. Dhar [2] proved that carrying out successive topplings will eventually result in a stable configuration and, moreover, that the stable configuration obtained in this manner does not depend on the order in which the topplings are carried out: This property justifies the model's description as *Abelian*.

We can define a Markov chain on the set of stable sandpile configurations on Λ as follows: At each time step, a vertex of Λ is chosen uniformly at random, an additional grain of sand is placed at that vertex, and the resulting configuration is stabilized. Although this Markov chain is not irreducible, it has a unique closed communicating class, consisting of the *recurrent* configurations, and the stationary measure of the Markov chain is simply the uniform measure on the set of recurrent configurations.

We are particularly interested in the statistical properties of *avalanches*, that is, the random function describing the number of times each vertex topples when a single grain of sand is added to an equilibrium sandpile and the resulting sandpile is stabilized. Avalanches are expected to have interesting critical-type behaviour: The size of an avalanche should have a power law tail, and a typical large avalanche should have a fractal like structure, the exact nature of which should depend strongly on the dimension d but not on the specific choice of lattice $\mathbb{Z}^d$. This is now fairly well understood in high ($d \geq 5$) dimensions [6], but remains very poorly understood in low dimensions.

A successful essay will begin by discussing the connections between the sandpile model, random walks, and uniform spanning trees, before moving on to discuss further, more advanced topics; the survey [3] gives a good selection of possibilities.

Relevant Courses

Essential: Random Walks and Uniform Spanning Trees

Useful: Advanced Probability, Percolation and Related Topics

References

[1] P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality. *Phys. Rev. A (3)*, 38(1):364–374, 1988.

[2] D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64(14):1613–1616, 1990.

[3] A. A. Járai. Sandpile models. *Probability Surveys*, Volume 15 (2018), 243-306.

[4] S. N. Majumdar and D. Dhar. Equivalence between the abelian sandpile model and the $q \to 0$ limit of the potts model. *Physica A*, (185):129–145, 1992.

[5] A. A. Járai and F. Redig. Infinite volume limit of the abelian sandpile model in dimensions $d \geq 3$. *Probab. Theory Related Fields*, 141(1-2):181–212, 2008.

[6] T. Hutchcroft. Universality of high-dimensional spanning forests and sandpiles. *Probability Theory and Related Fields*, pages 1–65, 2018.

58. Bayesian Additive Regression Trees ....
Professor R. Nickl

Regression tree structures are used widely in modern statistics and machine learning, a notable example being 'random forest' type algorithms. A Bayesian version of such algorithms has been proposed in [2, 3], known as Bayesian CART or BART. It has provided successful adaptive methodology for nonparametric regression, classification, variable selection and causal inference, among others, see [1] for various references. In particular it provides candidates for 'confidence sets' via the posterior distribution, thus going beyond mere 'point estimators' arising from algorithmic outputs.

There has been very little in terms of theoretical guarantees for these algorithms, but in a recent preprint [1] substantial progress has been made towards our understanding of Bayesian tree regression. Specifically a connection to certain wavelet shrinkage methods (when the tree is based on Haar-type wavelet functions) is revealed.

This essay should review the basic ideas of BART and Bayesian CART, following the original references [2, 3], discuss some applications, and describe the theoretical results from [1], supplying also a reasonable summary of the necessary mathematical background (most of which can be found in [4]).

Relevant Courses

Background in analysis, statistics and probability is helpful.

References

[1] Castillo, I., and Rockova, V. (2019). Multiscale Analysis of Bayesian CART. https://arxiv.org/abs/1910.07635

[2] Chipman, H., E. I. George, and R. E. McCulloch (1997). Bayesian CART model search. Journal of the American Statistical Association 93, 935–960.

[3] Chipman, H., E. I. George, and R. McCulloch (2010). BART: Bayesian additive regression trees. Annals of Applied Statistics 4, 266–298.

[4] Giné, E., and Nickl, R. *Mathematical foundations of infinite-dimensional statistical models.* Cambridge University Press, Cambridge (2016).

59. Random Walks on Height Functions ....

Professor J. R. Norris

According to Donsker's invariance principle, any zero-mean, finite-variance random walk on the integers converges weakly under diffusive scaling to a Brownian motion. The diffusivity of the limit Brownian motion is simply the one-step variance of the random walk. The essay will examine the phenomenon of convergence to Brownian motion in a more general setting.

Suppose we are given a finite bipartite graph G with edge set E. Let us say that a function $f : G \to \mathbb{Z}$ is a height function if $|f(x) - f(y)| = 1$ whenever $(x, y) \in E$, and say that two height functions f and g are neighbours if $|f(x) - g(x)| = 1$ for all $x \in G$. Consider the random walk $(F_n)_{n \geq 0}$ on the set of height functions, that is, the random process which moves in each time step from its present state to a randomly chosen neighbour.

The aim of the essay is to show that the average height process $\bar{F}_n = \frac{1}{|G|} \sum_{x \in G} F_n(x)$ converges under diffusive scaling to a Brownian motion and to determine, at least in some special cases, the diffusivity of the limit. See Chapter 7 in [2] for an introduction to diffusion approximation. Ideas from [3] on correctors may also be useful. Some aspects of this essay may be open problems. Original work will receive special credit but is not necessary for an essay of Distinction standard.

Relevant Courses

Essential: None

Useful: Advanced Probability

References

[1] E. Boissard, S. Cohen, Th. Espinasse & J. Norris. Diffusivity of a random walk on random walks. Random Structures Algorithms 47 (2015), no. 2, 267–283.

[2] S.N. Ethier & T.G. Kurtz. *Markov processes: characterization and convergence.* Wiley (2005).

[3] M.J. Luczak & J.R. Norris. Averaging over fast variables in the fluid limit for Markov chains: application to the supermarket model with memory. Ann. Appl. Probab. 23 (2013), no. 3, 957–986.

60. Brownian Motion on a Riemannian Manifold ....

Professor J. R. Norris

Brownian motion on a Riemannian manifold is the unique random process which satifies the two conditions that it is a martingale and that its quadratic variation is given by the metric tensor. Properties of this process are then closely related to both local and global properties of the manifold.

The essay will present an account of one or more constructions of Brownian motion on a Riemannian manifold and will discuss ways to characterize Brownian motion in terms of discrete approximations, as a Markov process, using stochastic differential equations, and via the heat equation. Then some further topics can be chosen in which the behaviour of Brownian motion is analysed. Examples of such topics are: recurrence and transience, behaviour under projections, Brownian bridge and geodesics, long-time behaviour, the case of Lie groups.

The nature of this essay is a synthesis of material in a well developed field. Given the availability of many relevant sources, special credit will be given for an attractive and coherent account.

Relevant Courses

Essential: None

Useful: Advanced Probability, Stochastic Calculus and Applications, Differential Geometry

References

[1] Elworthy, K. D. Stochastic differential equations on manifolds. London Mathematical Society Lecture Note Series, 70. Cambridge University Press, Cambridge-New York, 1982

[2] Emery, Michel. Stochastic calculus in manifolds. Universitext. Springer-Verlag, Berlin, 1989.

[3] Grigor'yan, Alexander. Heat kernel and analysis on manifolds. AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009

[4] Hsu, Elton P. Stochastic analysis on manifolds. Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

61. High-Dimensional Statistical Inference with Heterogeneous Missingness
Professor R. J. Samworth

One of the ironies of working with Big Data is that missing data play an ever more significant role. For instance, a common approach to handling missing data is to perform a so-called *complete-case analysis* ([1]), where we restrict attention to individuals in our study with no missing attributes. To see how this approach is no longer viable in high-dimensional regimes, consider an $n \times d$ data matrix in which each entry is missing independently with probability 0.01. When $d = 5$, a complete-case analysis would result in around 95% of the individuals (rows) being retained, but even when we reach $d = 300$, only around 5% of rows will have no missing entries.

Although missing data is a classical topic in Statistics, recent years have also witnessed increasing emphasis on understanding the performance of methods for dealing with missing data in high-dimensional problems, including sparse regression ([2], [3]), classification ([4]), sparse principal component analysis ([5]) and covariance and precision matrix estimation ([6], [7]). However, these works all assume either a homogeneous missingness probability, or present theoretical guarantees that depend on a lower bound on the missingness probability in each column, which essentially amounts to the same thing. It is only very recently ([8], [9]) that tools have been developed to handle the heterogeneous missingness mechanisms that are ubiquitous in real data sets, from health data to film ratings.

This essay would begin by reviewing traditional approaches to handling missing data, but the main focus should be on recent high-dimensional developments. There is scope for an ambitious candidate to attempt to study the effect of heterogeneous missingness in a new high-dimensional context, such as those mentioned in the previous paragraph or others.

Relevant Courses

Useful: Modern Statistical Methods, Topics in Statistical Theory

References

[1] Little, R. J. and Rubin, D. B. (2014) *Statistical Analysis with Missing Data*, John Wiley & Sons, Hoboken.

[2] Loh, P.-L. and Wainwright, M. J. (2012) High-dimensional regression with noisy and missing data: provable guarantees with nonconvexity. *Ann. Statist.*, **40**, 1637–1664.

[3] Belloni, A., Rosenbaum, M. and Tsybakov, A. B. (2017) Linear and conic programming estimators in high dimensional errors-in-variables models. *J. Roy. Statist. Soc., Ser. B*, **79**, 939–956.

[4] Cai, T. T. and Zhang, L. (2018) High-dimensional linear discriminant analysis: optimality, adaptive algorithm, and missing data. Available at `arXiv:1804.03018`.

[5] Elsener, A and van de Geer, S. (2018) Sparse spectral estimation with missing and corrupted measurements. `arXiv:1811.10443`.

[6] Loh, P.-L. and Tan, X. L. (2018) High-dimensional robust precision matrix estimation: Cellwise corruption under ϵ-contamination. *Electron. J. Statist.*, **12**, 1429–1467.

[7] Lounici, K. (2014) High-dimensional covariance matrix estimation with missing observations. *Bernoulli*, **20**, 1029–1058.

[8] Zhu, Z., Wang, T. and Samworth, R. J. (2019) High-dimensional principal component analysis with heterogeneous missingness. Available at `https://arxiv.org/abs/1906.12125`.

[9] Yang, Y., Wang, T., Zhu, Z. and Samworth, R. J. (2019) Sparse linear regression with heterogeneous missingness. *In preparation.*

62. Estimation of Heterogeneous Treatment Effects ....
Dr Q. Zhao and Professor R. J. Samworth

In the potential outcomes model for causal inference [1], we consider independent copies of quadruples (X, A, Y^0, Y^1), where X is a covariate taking values in $\mathbb{R}^d$, A is a treatment indicator taking values in $\{0, 1\}$, and Y^a is the observed response when $A = a$ (we do not observe Y^{1-a}). There is rapidly growing interest in understanding how the treatment effect varies in subgroups define by X, where the quantity of interest is the so-called *conditional average treatment effect* [2], given by $\tau(x) := \mathbb{E}(Y^1|X = x) - \mathbb{E}(Y^0|X = x)$. Under the no unmeasured confounders assumption ([3]), that is, (Y^0, Y^1) and A are conditionally independent given X, the function $\tau(\cdot)$ is identified by the difference between two regression functions, $\tau(x) = \mathbb{E}(Y^1|A = 1, X = x) - \mathbb{E}(Y^0|A = 0, X = x)$.

A number of statistical and machine learning methods have been proposed to estimate $\tau(\cdot)$ in this setting. In a recent data challenge, researchers were invited to apply methods they

developed to analyze a common education dataset ([4]). The high-level ideas, analyses of results, and references to the methodological papers can be found in the reports in a 2019 special issue of the open-access journal *Observational Studies*. This essay could review some of these recent innovations.

Alternatively, one may also consider estimation of integral functionals of $\tau(\cdot)$, for instance using higher-order influence functions ([5]). Most existing approaches rely on restrictive assumptions on the support of the covariate being compact, however, and appropriate functions being bounded away from their extremes. Such restrictions could potentially be alleviated using recent developments in the theory of functional estimation [6,7], and this may provide an interesting new direction for an ambitious candidate.

Relevant Courses

Useful: Causal Inference, Modern Statistical Methods, Statistical Learning in Practice, Topics in Statistical Theory.

References

[1] Holland, P. W. (1986). Statistics and causal inference. *J. Amer. Statist. Assoc.*, **81**: 945--960.

[2] Robins, J., Li, L., Tchetgen, E., van der Vaart, A. (2008) Higher order influence functions and minimax estimation of nonlinear functionals. *IMS Collections Probability and Statistics: Essays in Honor of David A. Freedman*, **2**, 335–421.

[3] Rosenbaum, P. R., Rubin, D. B. (1983). The central role of the propensity score in observational studies for causal effects. *Biometrika*, **70**, 41–55.

[4] Carvalho, C., Feller, A., Murray, J., Woody, S., Yeager, D. (2019). Assessing treatment effect variation in observational studies: Results from a data challenge. *Observational Studies*, **5** 21–35.

[5] Robins, J. M., Li, L., Mukherjee, R., Tchetgen Tchetgen, E. and van der Vaart, A. (2017) Minimax estimation of a functional on a structured high-dimensional model. *Ann. Statist.*, **45**, 1951–1987.

[6] Berrett, T. B., Samworth, R. J. and Yuan, M. (2019) Efficient multivariate entropy estimation via k-nearest neighbour distances. *Ann. Statist.*, **47**, 288–318.

[7] Berrett, T. B. and Samworth, R. J. (2019) Efficient functional estimation and the super-oracle phenomenon. arXiv:1904.09347.

63. Optimal Transport Methods in Statistics ....
Professor R. J. Samworth and Dr Y. Zemel

The problem of optimal transport was introduced in 1781 by Monge [8] and concerns the most efficient way to transport a distribution of objects to a distribution of locations. Its convex relaxation ([6]) gives rise to probability metrics known as Wasserstein, or optimal transport, distances. Within mathematics, these have found applications in, e.g., partial differential equations ([13]) and probability theory ([11]).

In Statistics, Wasserstein distances have been used in a myriad of setups. Classical examples are deviations from Gaussianity ([2]), goodness-of-fit testing ([9]), and concentration of measure

([7]). More recent applications include multivariate quantiles and related notions ([5]), generative adversarial networks ([1]), approximate Bayesian computation ([3]), and convergence of stochastic gradient descent ([4]). Optimal transport distances have also proven successful in data analysis of objects with complex geometric structures such as images ([12]) or point processes ([10]).

This essay could survey a small number of these topics in some depth, with a focus on the statistical aspects of the problem at hand. However, it is also a highly active current research area, and an ambitious candidate may also propose their own research question to address.

Relevant Courses

Essential: No specific course is necessary, but students should have mathematical and statistical maturity.

Useful: Topics in Statistical Theory, Probability and Measure, Advanced Probability.

References

[1] Arjovsky M, Chintalah S, Bottou L. 2017. Wasserstein Generative Adversarial Networks. *Proceedings of Machine Learning Research* 70:214-223.

[2] Bickel PJ, Freedman DA. 1981. Some asymptotic theory for the bootstrap. *Ann. Stat.* 9:1196–1217.

[3] Bernton E, Jacob PE, Gerber M, Robert CP. 2017. Inference in generative models using the Wasserstein distance. *J. Roy. Stat. Soc. Ser. B 81* 2:235-269.

[4] Chizat L, Bach F. 2018. On the Global Convergence of Gradient Descent for Over-parameterized Models using Optimal Transport In *Advances in Neural Information Processing Systems 31*, eds. Bengio, S, Wallach, H, Larochelle, H, Grauman, K, Cesa-Bianchi, N, Garnett, R, pages 3036–3046.

[5] Chernozhukov V, Galichon A, Hallin M, Henry M. 2017. Monge–Kantorovich depth, quantiles, ranks and signs. *Ann. Stat.* 45:223–256.

[6] Kantorovich LV. 1942. On the translocation of masses. *(Dokl.) Acad. Sci. URSS 37* 3:199–201.

[7] Ledoux M. 2005. The Concentration of Measure Phenomenon. Providence, RI: Am. Math. Soc.

[8] Monge G. 1781. Mémoire sur la théorie des déblais et des remblais. In *Histoire de l'Acad. R. Sci de Paris*. Paris: L'Imprimerie Royale, 666–704.

[9] Munk A, Czado C. 1998. Nonparametric validation of similar distributions and assessment of goodness of fit. *J. R. Stat. Soc. B* 60:223–241.

[10] Panaretos VM, Zemel Y. 2016. Amplitude and phase variation of point processes. *Ann. Stat.* 44:771–812.

[11] Rachev ST, Rüschendorf L. 1998. *Mass Transportation Problems*. New York: Springer.

[12] Rubner Y, Tomasi C, Guibas LJ. 2000. The earth mover's distance as a metric for image retrieval. *Int. J. Comput. Vis.* 40:99–121.

[13] Villani C. 2008. *Optimal Transport: Old and New.* Berlin: Springer.

64. Scaling Limits of Critical Random Graphs ....
<div align="right">Dr P. Sousi</div>

The simplest model of a random network is the Erdös Rényi random graph. It has vertex set $\{1, \ldots, n\}$ and each pair of vertices is independently joined by an edge with probability p. When $p = c/n$, the size of the giant component (largest connected subgraph) undergoes a phase transition when c changes from $c < 1$ (subcritical) to $c > 1$ (supercritical). In the subcritical regime, the size of the largest component is of order $\log n$, while in the supercritical one it is of order n. In the critical regime ($c = 1$), the size of the giant is of order $n^{2/3}$. Understanding finer properties and scaling of the critical random graph has attracted a lot of interest in the last two decades.

Consider now $p = 1/n + \lambda/n^{4/3}$ with $\lambda \in \mathbb{R}$. Aldous [2] in 1997 obtained a scaling limit result for the sizes of all components as well as their surplus (minimal number of edges that should be removed to get a tree). The limit random variable is described in terms of a Brownian motion with parabolic drift that depends on λ. More recently, Addario-Berry, Broutin and Goldschmidt [3] established a scaling limit for the connected components viewed as metric spaces endowed with the graph distance suitably rescaled. The fundamental building block of the limiting object is the continuum random tree.

A successful essay will give an overview of the different results, explain a construction of the continuum random tree and include proofs (or overviews of proofs) of the important results.

Relevant Courses

Useful: Advanced Probability

References

[1] D. Aldous (1991). The continuum random tree. I. *Ann. Probab.* 19, no. 1, 1-28.

[2] D. Aldous (1997). Brownian excursions, critical random graphs and the multiplicative coalescent. *Ann. Probab.* 25, no. 2, 812-854.

[3] L. Addario-Berry, N. Broutin and C. Goldschmidt (2012). The continuum limit of critical random graphs. *Probab. Theory Relat. Fields.* 152, no. 3-4, 367-406.

65. Fundamental Theorems of Robust Finance ....
<div align="right">Dr M. R. Tehranchi</div>

The first fundamental theorem of asset pricing says that, in an arbitrage-free market with no dividends, the time-0 price of each asset is a weighted average of the possible time-T prices of that asset, where $T > 0$. Despite the success of this theorem in modelling financial markets, it has come under criticism recently. In particular, the definition of arbitrage and the above notion of weighted average depend on the assumption that there is a unique probability measure $\mathbb{P}$ governing the future evolution of asset prices. However, in reality, this measure $\mathbb{P}$ is not known with certainty and can only be estimated from data. Robust finance attempts to extend the fundamental theorems to this uncertain case.

A successful essay will survey the recent literature on robust finance, explaining the various new notions of no-arbitrage that have been proposed as well as their dual formulations in terms of martingale measures.

Relevant Courses

Essential: Advanced Financial Models, Advanced Probability

Useful: Stochastic Calculus & Applications

References

[1] P. Beissner. Coherent price systems and uncertainty-neutral valuation. *Risks* 7(3):98 (2019)

[2] B. Bouchard and M. Nutz. Arbitrage and duality in nondominated discrete-time models. *Annals of Applied Probability* **25**(2): 823–859 (2015)

[3] M. Burzoni, M. Frittelli and M. Maggis. Universal arbitrage aggregator in discrete-time markets under uncertainty. *Finance & Stochastics* **20**(1): 1–50. (2016)

66. Rough Volatility ....

Dr M. R. Tehranchi

A typical stochastic volatility model for an asset price is of the form

$$dS_t = S_t(r \ dt + \sigma_t \ dW_t^S)$$

where W^S is a Brownian motion, the interest rate (assumed here to be constant) is r and the spot volatility process is $(\sigma_t)_{t \geq 0}$. In many of the models used in practice, the spot volatility is also assumed to solve a stochastic differential equation of the form

$$d\sigma_t = b(\sigma_t)dt + c(\sigma_t)dW_t^\sigma \tag{1}$$

where W^σ is another Brownian motion. However, recently evidence from various directions point to the idea that the sample paths of the spot volatility process for many assets are rougher than the Hölder $\frac{1}{2} - \varepsilon$ smoothness of a typical diffusion process.

There has been an explosion of papers proposing and analysing so-called rough volatility models. One way to construct a rough volatility model is to let σ solve an equation such as (1) but now let W^σ be a fractional Brownian motion.

A successful essay will pick a thread of research and survey the recent literature in that area. Possible directions are the notions of existence and uniqueness of the solutions of the rough versions of popular stochastic volatility model (the mathematical challenges involve the lack of Markov and semi-martingale property) or the micro-structural foundations of such models.

Relevant Courses

Essential: Advanced Financial Models, Stochastic Calculus & Applications

Useful: Advanced Probability

References

[1] J. Gatheral, Th. Jaisson, M. Rosenbaum. Volatility is rough. *Quantitative Finance* **18**(6) 933-949 (2018).

[2] E. Abi Jaber, M. Larsson, and S. Pulido. Affine Volterra processes. *Annals of Applied Probability* **29**(5): 3155-3200 (2019)

[3] O. El Euch, M. Fukasawa, M. Rosenbaum. The microstructural foundations of leverage effect and rough volatility. *Finance & Stochastics* **22**:241-280 (2018)

67. Estimating Causal Treatment Effects: A Theoretical and Computational Framework for Model Evaluation, Selection and Development ....

Professor M. van der Schaar

We propose to investigate a theoretical framework for causal inference model evaluation and selection using observational data. In analogy to cross-validation in the standard supervised learning setup, the framework will enable (i) accurately estimating and evaluating the performance of different causal inference models and (ii) selecting the best performing model for a given observational dataset.

The project has broad theoretical and practical impact.

Tasks are:

1. To derive an analytic form for the first-order influence function. Using such analytic expressions, we will devise an algorithm to evaluate the higher-order influence functions by recursively evaluating the first-order influence functions of lower order influence functions. We will also analyze the conditions under which our validation procedures are consistent and efficient.

2. To investigate a practical automated causal inference framework to select the best causal inference algorithm for the observational study at hand.

References

[1] A. M. Alaa, M. van der Schaar, *Validating Causal Inference Models via Influence Functions*, International Conference on Machine Learning (ICML), 2019.

[2] A. M. Alaa, M. van der Schaar, *Limits of Estimating Heterogeneous Treatment Effects: Guidelines for Practical Algorithm Design*, ICML, 2018.

[3] A. M. Alaa, M. van der Schaar, *Bayesian Nonparametric Causal Inference: Information Rates and Learning Algorithms*, IEEE Journal of Selected Topics in Signal Processing (JSTSP), 2018.

68. Causal Inference with Interference ....

Dr Q. Zhao

A key assumption in causal inference is the so-called stable unit treatment value assumption (SUTVA) [1], which asserts that the potential outcome on one unit should be unaffected by the assignment of treatments to other units ("no interference" of the treatment effect). Due to the increasing availability of social network data and experiments, there is growing interest in relaxing SUTVA and understanding interference [2–5]. However, asymptotic theory for causal effect estimators with interference is still incomplete and often relies on strong or nontransparent assumptions [4,6–8].

This essay could give a critical review of the recent literature on causal inference with interference. Another option is to derive some asymptotic theory for commonly used causal estimators in simple interference models. For example, suppose the treatments W_i iid $\sim$ Bernoulli(π), $i = 1, \ldots, n$ for some known $0 < \pi < 1$ and the outcome is generated by the following model

$$Y_i = \alpha + \beta W_i + \frac{\gamma}{d_i} \sum_{j \in \mathcal{N}_i} W_j + \epsilon_i, \ i = 1, \ldots, n,$$

where $\mathcal{N}_i$ contains the neighbours of i in a given graph, $d_i = |\mathcal{N}_i|$, and the noise ϵ_i is iid with mean zero and finite fourth moment and is independent of $W_1, \ldots, W_n$. This essay could derive some asymptotic results for the following difference-in-means estimator

$$\hat{\beta}_n = \frac{1}{n\pi} \sum_{i=1}^{n} W_i Y_i - \frac{1}{n(1-\pi)} \sum_{i=1}^{n} (1 - W_i) Y_i, \text{ as } n \to \infty,$$

based on central limit theorems derived from Stein's method [9, 10].

Relevant Courses

Useful: Causal Inference, Advanced Probability.

References

[1] Rubin, D. B. (1980). Comment on "Randomization analysis of experimental data: The Fisher randomization test". *J. Amer. Statist. Assoc.*, **75**, 591–593.

[2] Hudgens, M. G., Halloran, M. E. (2008). Toward causal inference with interference. *J. Amer. Statist. Assoc.*, **103**, 832–842.

[3] Tchetgen, E. J. T., VanderWeele, T. J. (2012). On causal inference in the presence of interference. *Stat. Methods Med. Res.*, **21**, 55–75.

[4] Aronow, P. M., Samii, C. (2017). Estimating average causal effects under general interference, with application to a social network experiment. *Ann. Appl. Stat.*, **11**, 1912–1947.

[5] Athey, S., Eckles, D., Imbens, G. W. (2018). Exact p-values for network interference. *J. Amer. Statist. Assoc.*, **113**, 230–240.

[6] Sävje, F., Aronow, P. M., Hudgens, M. G. (2017). Average treatment effects in the presence of unknown interference. arXiv:1711.06399.

[7] Ogburn, E. L., Sofrygin, O., Diaz, I., van der Laan, M. J. (2017). Causal inference for social network data. arXiv:1705.08527.

[8] Chin, A. (2018). Central limit theorems via Stein's method for randomized experiments under interference. arXiv:1804.03105.

[9] Chen, L. H., Shao, Q. M. (2004). Normal approximation under local dependence. *Ann. Probab.*, **32**, 1985–2028.

[10] Chatterjee, S. (2008). A new method of normal approximation. *Ann. Probab.*, **36**, 1584–1610.

69. Unmeasured Confounding in High-Dimensional Data ....
Dr R. D. Shah and Dr Q. Zhao

Unmeasured confounding is the dragon in causal inference and, unfortunately, is difficult to avoid in any empirical scientific work using observational data. Many methods have been proposed to remove or reduce the bias due to unmeasured confounding, such as instrumental variables [1, Chap. 23–24] and difference in differences [2].

Modern high-dimensional data presents a new opportunity to overcome unmeasured confounding. In several problems with unmeasured confounding, it is shown that one may leverage

sparsity of the causal effects to obtain unbiased estimates [3–6]. However, the practice of estimating unmeasured confounders and then adjust for them is highly controversial. This is best represented in a series of exchanges [7–10] on an article that highlights the importance of measuring multiple proxies of the unmeasured confounder [11].

This essay could review and critique the recent developments that attempt to use aspects of high-dimensional data to overcome unmeasured confounding. Another option would be to focus more closely on some particular identification strategies and compare associated methods or slightly modify existing procedures with the aim of obtaining improved performance in certain settings.

Relevant Courses

Useful: Causal Inference, Modern Statistical Methods.

References

[1] Imbens, G. W., Rubin, D. B. (2015). *Causal Inference in Statistics, Social, and Biomedical Sciences.* Cambridge University Press.

[2] Abadie, A. (2005). Semiparametric difference-in-differences estimators. *Rev. Econ. Stud.*, **72**, 1–19.

[3] Wang, J., Zhao, Q., Hastie, T., Owen, A. B. (2017). Confounder adjustment in multiple hypothesis testing. *Ann. Statist.*, **45**, 1863-1894.

[4] Guo, Z., Kang, H., Tony Cai, T., Small, D. S. (2018). Confidence intervals for causal effects with invalid instruments by using two-stage hard thresholding with voting. *J. Roy. Stat. Soc. B*, **80**, 793–815.

[5] Ćevid, D., Bühlmann, P., Meinshausen, N. (2018). Spectral deconfounding and perturbed sparse linear models. arXiv:1811.05352.

[6] Shah, R. D., Frot, B., Thanei, G. A., Meinshausen, N. (2018). RSVP-graphs: Fast high-dimensional covariance matrix estimation under latent confounding. arXiv:1811.01076.

[7] D'Amour, A. (2019). Comment: Reflections on the deconfounder.
arXiv:1910.08042.

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[10] Wang, Y., Blei, D. M. (2019). The blessings of multiple causes: A reply to Ogburn et al. (2019). arXiv:1910.07320.

[11] Wang, Y., Blei, D. M. (2019). The blessings of multiple causes. *J. Amer. Statist. Assoc.*, to appear.

70.   Precise Minimal Supersymmetric Standard Model Predictions of the Higgs Boson Mass ....
Professor B. C. Allanach

The Minimal Supersymmetric Standard Model (MSSM) is still regarded by many to be an attractive TeV-scale extension to the Standard Model (however, in this essay, it is not necessary

to review supersymmetry or the MSSM at all). The prediction of a Higgs boson whose properties match those of the experimentally discovered particle is an obvious priority. The calculation of its mass in particular has rather large radiative corrections, and is calculated to a relatively high order in perturbation theory, in various different schemes and approximations. It is a subject of active research as to which scheme or approximation links the Higgs boson mass prediction most precisely to the rest of the model.

The purpose of this essay is to find out and present the issues in the precision Higgs mass prediction in the MSSM, while providing an overall context.

The first half of the essay will set the scene in terms of experimental data for the Higgs boson discovery, and some analytical predictions for the lightest CP even Higgs boson mass prediction in terms of the other parameters of the model. The second half should address the important issues coming from approximations and different schemes, along with current attempts to address them, and their short-comings.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

References

[1] D. de Florian *et al.* [LHC Higgs Cross Section Working Group], arXiv:1610.07922 [hep-ph].

[2] P. Kant, R. V. Harlander, L. Mihaila and M. Steinhauser, JHEP **1008** (2010) 104 [arXiv:1005.5709 [hep-ph]].

[3] R. V. Harlander, J. Klappert and A. Voigt, arXiv:1708.05720 [hep-ph].

[4] S. P. Martin, Adv. Ser. Direct. High Energy Phys. **21** (2010) 1 [Adv. Ser. Direct. High Energy Phys. **18** (1998) 1] [hep-ph/9709356].

(and references therein)

71. Walking Deeper on Dynamic Graphs: Learning Latent Representations with Random Walks for Image Classification ....
Dr A. I. Aviles-Rivero and Professor C.-B. Schönlieb

In the era of big data, graph representation is a natural and powerful tool for representing big data in real-world problems [1],[2],[4]; some examples include data coming from medical records, social networks, recommendation systems and transport systems. A challenging question when using graph representation is how to learn *latent representations* on multi-label networks for several classification tasks, and a seminal algorithm for this is the *DeepWalk* technique using random walks [1].

We propose two questions for investigation in this essay. Firstly, we hope that students will develop a rigorous mathematical underpinning for the DeepWalk algorithm, in the spirit of convergence guarantees.

Secondly, we seek to investigate the connection of DeepWalk to dynamic graphs. Many real-world events are dynamic - for example, in a social network new users are constantly added- while most of the body of literature is based on the unrealistic assumption that the graph is static. From the learning point of view, this assumption has a negative impact in the computations, as

the graph has to be re-learned each time that an instance changes. We also hope that students will also discuss some open questions that they find interesting.

Relevant Courses

Essential: None

Useful: Background knowledge in Machine Learning and Statistics is helpful, as is probability to the level of Part II Applied Probability. Some content from Part III Mixing Times of Markov Chains, on the long-time behaviour of random walks on graphs, may also be useful.

References

[1] B. Perozzi, R. Al-Rfou and S. Skiena. Deepwalk: Online learning of social representations. ACM SIGKDD International Conference on Knowledge Discovery and Data Mining pp. 701-710, 2014.

[2] A.I Aviles-Rivero, N. Papadakis, R. Li, SM. Alsaleh, R. T Tan and C-B Schönlieb. Beyond Supervised Classification: Extreme Minimal Supervision with the Graph 1-Laplacian. arXiv preprint:1906.08635

[3] HP Sajjad, A Docherty and Y. Tyshetskiy, Y. Efficient representation learning using random walks for dynamic graphs. arXiv preprint arXiv:1901.01346.

[4] M Valko. Lecture Notes on Graphs in Machine Learning, `http://researchers.lille.inria.fr/~valko/hp/mva-ml-graphs.php`, 2019.

72. Massive Neutrinos in Cosmology ....
Dr T. Baldauf

Neutrino oscillation experiments have revealed that neutrinos have a non-zero mass. While they behave like radiation initially, they become non-relativistic at late times, which makes them a part of the dark matter mass budget of the late time universe. This "hot" dark matter component erases structures on small scales and allows us to constrain their mass from the clustering statistics of Large-Scale Structure (LSS). These constraints are much tighter than those of terrestrial experiments like KATRIN which are trying to infer the electron neutrino mass from the endpoint energy of beta decay.

In this essay you will review the role of neutrinos in cosmology, starting from big bang nucleosynthesis and the formation of the Cosmic Microwave Background (CMB). You will then study their impact on structure formation and review the constraints that can be gleaned from upcoming LSS surveys in combination with CMB lensing and the Ly-α forest. The core of the essay will focus on how neutrinos can be included in structure formation simulations or how their effects can be treated perturbatively.

Relevant Courses

Essential: Particle Physics, Cosmology

Useful: Advanced Cosmology

References

[1] J. Lesgourgues, G. Mangano, G. Miele and S. Pastor, "Neutrino Cosmology,"

[2] J. Lesgourgues and S. Pastor, "Massive neutrinos and cosmology," Phys. Rept. **429** (2006) 307 [astro-ph/0603494].

[3] K. N. Abazajian *et al.*, "Neutrino Physics from the Cosmic Microwave Background and Large Scale Structure," Astropart. Phys. **63** (2015) 66 [arXiv:1309.5383].

[4] M. Archidiacono, T. Brinckmann, J. Lesgourgues and V. Poulin, "Neutrino properties from cosmology," [arXiv:1705.00496].

[5] A. Font-Ribera, P. McDonald, N. Mostek, B. A. Reid, H. J. Seo and A. Slosar, "DESI and other dark energy experiments in the era of neutrino mass measurements," JCAP **1405** (2014) 023 [arXiv:1308.4164].

[6] D. Blas, M. Garny, T. Konstandin and J. Lesgourgues, "Structure formation with massive neutrinos: going beyond linear theory," JCAP **1411** (2014) 039 [arXiv:1408.2995].

[7] M. Shoji and E. Komatsu, "Massive Neutrinos in Cosmology: Analytic Solutions and Fluid Approximation," Phys. Rev. D **81** (2010) 123516 [arXiv:1003.0942].

[8] Y. Ali-Haimoud and S. Bird, "An efficient implementation of massive neutrinos in non-linear structure formation simulations," Mon. Not. Roy. Astron. Soc. **428** (2012) 3375 [arXiv:1209.0461].

[9] E. Castorina, C. Carbone, J. Bel, E. Sefusatti and K. Dolag, "DEMNUni: The clustering of large-scale structures in the presence of massive neutrinos," JCAP **1507** (2015) 043 [arXiv:1505.07148].

[10] M. Levi and Z. Vlah, "Massive neutrinos in nonlinear large scale structure: A consistent perturbation theory," [arXiv:1605.09417].

73. Classical Approaches to Simulating Quantum Dynamics ....
Dr B. Béri

Understanding the time evolution of complex quantum systems is required in a number of research areas including thermalisation, nonequilibrium phases of matter, or many-body quantum chaos, just to name a few. Gaining insights into the quantum dynamics is sometimes possible through the classical simulation of the system. The purpose of this essay is to describe various approaches to this and the complementary nature of the corresponding challenges, including (but not necessarily limited to): (i) matrix-product-state methods and their limitations in terms of the systems and time scales accessible due to the entanglement content of the states of interest, and (ii) methods based on the stabiliser formalism and the corresponding restrictions on the type of efficiently simulatable dynamics.

Relevant Courses

Essential: None

Useful: Part II Quantum Information and Computation, Part III Quantum Computation

References

[1] M. A. Nielsen and I. L. Chuang, *"Quantum Computation and Quantum Information"*, Cambridge University Press (2010).

[2] S. Paeckel, *et al.*, *"Time-evolution methods for matrix-product states"*, arXiv:1901.05824.

[3] Y. Li, X. Chen, and M. P. A. Fisher, *"Measurement-driven entanglement transition in hybrid quantum circuits"*, arXiv:1901.08092.

[4] J. Iaconis, S. Vijay, and R. Nandkishore, *"Anomalous Subdiffusion from Subsystem Symmetries"*, arXiv:1907.10629.

74. Effective Transport in Heterogeneous Media ....
Dr M. Bruna

The transport of a solute through heterogeneous media such as porous media depends on microscopic features such as the structure of the porous matrix, the fluid flow and the nature of the interactions between the solid and liquid phases. However, the complexity of the microscopic problem means that in practice it is often desirable to obtain an effective-medium equation from which the macroscopic transport can be obtained directly. This concept of upscaling is relevant in many biological, environmental and industrial applications.

The method of multiple scales is a common homogenisation technique to analyse periodic media. It provides a good approximation near the centre of mass of the solute distribution, but fails to capture the tails of the distribution. However in some applications even minute concentrations far from the centre of mass are of critical importance, such as in the case of the spread of a hazardous chemical. The theory of large deviations offers a way to accurately resolve the entire distribution.

This essay will start by reviewing homogenisation techniques for periodic media. Then it will explore the connections between multiple scales with the theory of large deviations. A possible direction is also to compare the two approximations for a particular problem through computer simulations.

Relevant Courses

Essential: Perturbation methods

Useful: Numerical Solution of Differential Equations

References

[1] Haynes, P. H., and Vanneste, J. (2014). Dispersion in the large-deviation regime. Part 1: shear flows and periodic flows. Journal of Fluid Mechanics, 745, 321-350.

[2] Luckins, E., Breward, C. J. W., Griffiths, I. M., & Wilmott, Z. (2019). Homogenisation problems in reactive decontamination. European Journal of Applied Mathematics, 1-24.

[3] Pavliotis, G., and Stuart, A. (2008). Multiscale methods: averaging and homogenization. Springer Science & Business Media.

[4] Tzella, A., and Vanneste, J. (2016). Dispersion in rectangular networks: effective diffusivity and large-deviation rate function. Physical review letters, 117(11), 114501.

75. Stochastic Models of Diffusion ....
<div align="right">**Dr M. Bruna**</div>

Many biological processes can be described in terms of diffusing and reacting species (e.g. enzymes). Such reaction-diffusion processes can be mathematically modelled using either stochastic models or deterministic partial differential equations (PDEs). For example, the dispersal of an enzyme can be modelled as a random walk on a uniform lattice or a Brownian motion. In both cases, the probability that the enzyme is at a given position in space and time satisfies a diffusion PDE in the appropriate continuum limit. The presence of reactive boundaries (for example a cell membrane) complicates the connection between a given stochastic model and its deterministic counterpart. In particular, given a stochastic model, what is the corresponding effective boundary condition for the PDE? This is particularly important in the case of hybrid models (where stochastic discrete models are used in parts of the domain where finer details are necessary, and a PDE model is used in the rest [3]). The question above is addressed in [2] for four possible stochastic models of diffusion using asymptotic methods.

The purpose of this essay is to present and compare different models to describe the dispersal of biological organisms, with an emphasis on the connection between a given stochastic model and its deterministic counterpart. Depending on the student's interests, the next step could be to work through the asymptotic results in [3] and present them in a complete way, or implement various stochastic simulations algorithms and compare the different models through some numerical examples.

Relevant Courses

Essential: None

Useful: Perturbation methods, Theoretical Physics of Soft Condensed Matter, Stochastic Simulation Algorithms.

References

[1] Chandrasekhar, S. (1943). Stochastic problems in physics and astronomy. Reviews of modern physics, 15(1), 1.

[2] Erban, R., and Chapman, S. J. (2007). Reactive boundary conditions for stochastic simulations of reaction-diffusion processes. Physical Biology, 4(1), 16.

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[4] Othmer, H. G., Dunbar, S. R., and Alt, W. (1988). Models of dispersal in biological systems. Journal of mathematical biology, 26(3), 263-298.

76. The Unruh Effect ....
<div align="right">**Dr J. N. Butterfield**</div>

The Unruh effect is, very roughly speaking, that an observer accelerating through the vacuum state of a free quantum field on Minkowski spacetime sees—not *no* particles—but a thermal bath of particles (at a temperature that depends on the observer's acceleration). Since 1976,

when the effect was first discovered by Unruh [1] (and in similar independent work by Davies and Fulling), one can discern three main areas of development.

First, one seeks to better understand the effect as just stated, i.e. as a feature of the free quantum field in flat spacetime. Second, one generalizes the effect to curved spacetimes. Third, one relates it to the Hawking effect: the basic idea being that hovering outside the event horizon of a black hole is like accelerating through a vacuum.

This essay will concentrate on the first area of development, emphasizing the algebraic approach to quantum theory, i.e. (3) below (matching the emphasis in a Lent Term 2020 course). Cf. [2] for references to both: (a) the second and third areas, and (b) avenues (1) and (2) of the first area.

Investigations in the first area, i.e. of the 'original' Unruh effect, have taken three main avenues.
(1): To understand it in terms of how a particle detector can fire when accelerating through the vacuum. This was Unruh's (and Davies') original approach.
(2): To understand it by: (i) writing the Minkowski vacuum as a superposition of tensor products of Fulling quanta, i.e. quanta obtained by quantizing in the left and right Rindler wedges, defining positive frequencies in terms of the Rindler time coordinate ρ instead of the inertial time coordinate t; and then (ii) tracing out the left Rindler wedge, getting a thermal state on the right Rindler wedge.
(3): To understand it in terms of algebraic quantum theory, specifically modular theory. The basic idea is that the restriction of the Minkowski vacuum state to the right Rindler wedge is a thermal state. More precisely: it is, thanks to the Tomita-Takesaki theorem, a KMS state: which is a generalized notion of thermal state, with an associated time-evolution with respect to which it is an equilibrium state. And for this state, the associated time-evolution is that of the congruence of constant acceleration worldlines in the wedge. [3] gives some references..

In the philosophical literature [4], Earman (2011) is a survey of all three avenues, giving reason to favour (and giving most details about) the third avenue. (He also discusses generalizing the effect to curved spacetimes, i.e. the second area above.) Clifton and Halvorson (2001, 2001a) give a detailed account of the definitions, and behaviours, of relevant number operators; and how the disjointness of the relevant representations undermines the second avenue, (2) above.

The aim of the essay is to survey one or more of these main avenues, (1) to (3): preferably at least two, but not necessarily all three.

Relevant Courses

Essential: None

Useful: Philosophical aspects of quantum field theory

References

[1]: W. Unruh (1976), Notes on black hole evaporation, *Physical Review D* **14**, 870-892.

[2]: N. Birrell and P. Davies *Quantum Fields in Curved Space*, Cambridge University Press 1984, Chapters 1 to 4; and Fulling, S. *Aspects of quantum field theory in curved spacetime*, LMS Student Texts 17, Cambridge University Press 1989, up to Chapter 6. These books include references to the original papers by Davies, Fulling and Unruh (and of course others) about the flat spacetime case, as well as treatments of curved spacetimes and the connection to black holes. Cf. also Earman's article in [4].
More recent surveys by one of the creators of the field, R. Wald, exhibit the importance of

algebraic methods. For example, his *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago University Press 1994; and his 'Black hole thermodynamics', *Living Reviews in Relativity* 2001. A review emphasizing applications is: L. Crispino et al, 'The Unruh effect and its applications', *Reviews of Modern Physics* 2008; arXiv:0710.5373

[3]: The idea as stated here is in: J. Bisognano and E. Wichmann (1976), 'On the duality condition for quantum fields', *Journal of Mathematical Physics* **17** 303-321; and G. Sewell (1982), 'Quantum fields on manifolds: PCT and gravitationally induced thermal states', *Annals of Physics* **141** 201-224.
A brief pedagogic introduction to algebraic methods for quantum theory on curved spacetime, is: B. Kay, ' Quantum field theory in curved spacetime', in *Encyclopedia of Mathematical Physics*, ed. J-P. Francoise et al. Academic Press 2006, volume 4; gr-qc/0601008.

[4]: Earman (2011), The Unruh Effect for Philosophers, *Studies in History and Philosophy of Physics* **42** 81-97; R. Clifton and H. Halvorson (2001), 'Are Rindler Quanta real? Inequivalent particle concepts in QFT', *British Journal Philosophy of Science* **52**, 417-470; R. Clifton and H. Halvorson (2001a), 'Entanglement and opens systems in algebraic QFT', *Studies in History and Philosophy of Physics* **32**, 1-31.

77. Symmetry and Symplectic Reduction ....
Dr J. N. Butterfield

Symplectic reduction is a large subject in both classical and quantum mechanics. One starts from Noether's theorem in a classical Hamiltonian framework, and thereby the ideas of: Lie group actions; the co-adjoint representation of a Lie group G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$; Poisson manifolds (a mild generalization of symplectic manifolds that arise when one quotients under a symmetry); conserved quantities as momentum maps. With these ideas one can state the main theorems about symplectic reduction. Main texts for this material include [1].

The flavour of these theorems is well illustrated by the *Lie-Poisson* reduction theorem. It concerns the case where the natural configuration space for a system is itself a Lie group G. This occurs both for the pivoted rigid body and for ideal fluids. For example, take the rigid body to be pivoted, so as to set aside translational motion. This will mean that the group G of symmetries defining the quotienting procedure will be the rotation group SO(3). But it will also mean that the body's configuration space is given by $G = $ SO(3), since any configuration can be labelled by the rotation that obtains it from some reference-configuration. So in this example of symplectic reduction, the symmetry group acts on itself as the configuration space. Then the theorem says: the quotient of the natural phase space (the cotangent bundle on G) is a Poisson manifold isomorphic to the dual $\mathfrak{g}^*$ of G's Lie algebra. That is: $T^*G/G \cong \mathfrak{g}^*$. There are several 'cousin' theorems, such as the *Marsden-Weinstein-Meyer* theorem. For a philosopher's exposition of the Lie-Poisson reduction theorem, cf. [2].

The essay should, starting from this basis, expound one or other of the following two topics. (Taking on both would be too much.)
(A): The first topic is technical and concerns the application of these classical ideas to quantum theory: more specifically, the interplay between reduction and canonical quantization. Physically, this is a large and important subject, since it applies directly to some of our fundamental theories, such as electromagnetism and Yang-Mills theories. The essay can confine itself to the more general aspects: which are well introduced and discussed by Landsman and Belot; [3].
(B): The second topic is more philosophical. It concerns the general question under what circumstances should we take a state and its symmetry-transform to represent the same state of

affairs—so that quotienting under the action of the symmetry group gives a non-redundant representation of physical possibilities? This question can be (and has been) discussed in a wholly classical setting. Indeed, the prototype example is undoubtedly the question debated between Newton (through his *ammanuensis* Clarke) and Leibniz: namely—in modern parlance—whether one should take a solution of, say, Newtonian gravitation for N point-particles and its transform under a Galilean transformation to represent the same state of affairs. This topic is introduced by the papers in [4]. In particular, Dewar discusses how, even when we are sure that a state and its symmetry-transform represent the same state of affairs, quotienting can have various disadvantages.

Relevant Courses

Essential: None

Useful: Symmetries, Fields and Particles; Philosophical Aspects of Classical and Quantum Mechanics.

References

[1]: R. Abraham and J. Marsden (1978), *Foundations of Mechanics*, second edition: Addison-Wesley; V. Arnold (1989), *Mathematical Methods of Classical Mechanics*, Springer, (second edition); J. Marsden and T. Ratiu (1999), *Introduction to Mechanics and Symmetry*, second edition: Springer-Verlag.

[2]: J. Butterfield (2006) On Symmetries and Conserved Quantities in Classical Mechanics, in W. Demopoulos and I. Pitowsky (eds.), *Physical Theory and its Interpretation*, Springer; 43 - 99; Available at: http://arxiv.org/abs/physics/0507192; J. Butterfield (2006). On Symplectic Reduction in Classical Mechanics, in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, North Holland; 1 - 131. Available at: http://arxiv.org/abs/physics/0507194.

[3]: G. Belot (1998), 'Understanding electromagnetism', *British Journal for the Philosophy of Science* **49**, p. 531-555; G. Belot (2003), 'Symmetry and gauge freedom', *Studies in History and Philosophy of Modern Physics* **34** 189-225;
N. Landsman (2006), 'Between Classical and Quantum', Section 4. in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, North Holland; 1 - 131. Available at: http://arxiv.org/abs/physics/0507194.
N. Landsman (2017). *Foundations of Quantum Theory*: Sections 5.6-5.12. Springer. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3

[4]: Four papers by G. Belot: (2000), 'Geometry and motion', *British Journal for the Philosophy of Science* **51**, p. 561-596; (2001), 'The principle of sufficient reason', *Journal of Philosophy* **98**, p. 55-74; (2003), 'Notes on symmetries', in Brading and Castellani (ed.s) (2003), pp. 393-412. (2013), 'Symmetry and equivalence', in R. Batterman (ed.), *Oxford Handbook of Philosophy of Physics* Oxford University Press, 2013. All Belot papers are available at:
https://sites.google.com/site/gordonbelot/home/papers-etc
Caulton, A. (2015). 'The Role of Symmetry in the Interpretation of Physical Theories'. *Studies in History and Philosophy of Modern Physics*, **52**, pp. 153-162. N. Dewar (2017) 'Sophistication about symmetries', *British Journal for the Philosophy of Science*: available at: https://academic.oup.com/bjps/advance-article-abstract/doi/10.1093/bjps/axx021/4111183

<div align="right">**Dr J. N. Butterfield**</div>

The treatment of time in quantum theory, even quantum mechanics, has various subtle aspects. Some overviews and research articles are in [1]. But this essay concentrates on the variety of ways in which one can define the time it takes a quantum particle to tunnel through a barrier. This variety reflects two central features of how time is treated in quantum mechanics, namely: (i) the non-existence, in general, of a time operator, i.e. an operator that suitably represents time as a physical quantity; (ii) the time-energy uncertainty principle. These features (and others) are discussed, irrespective of tunnelling in [1].

Broadly speaking, what we call a measurement of time is really the measurement of a dynamical variable whose evolution is known (and is usually uniform like a pointer on a clock dial). So the problems of finding a unique or best definition of the time a particle takes to tunnel reflect (i) the variety of dynamical variables, of the wave-packet or of some system appropriately coupled to it, that we might choose to measure, and (ii) these variables' features (one might say: merits and demerits). [2] is a selection of the large literature about these problems.

These problems, i.e. the assessment and comparison of the various definitions, leads into matters of interpretation; on which even adherents to the overall orthodoxy, the Copenhagen interpretation, can and do differ. But the issues of interpretation become especially vivid when one considers the pilot-wave interpretation—or if one prefers: the pilot-wave alternative theory—pioneered by de Broglie and Bohm. For in the pilot-wave theory, there is a uniquely natural definition of the tunnelling time. Indeed, on first inspection this definition suggests there should surely be some *experimentum crucis*, whereby we can break the surprising and recalcitrant empirical equivalence of the pilot-wave theory and the orthodox theory. But in fact, matters are not simple: as the papers in [3] show.

The aim of the essay is to survey these developments. It would be enough to survey the orthodox literature, of which [2] is a selection; or to specialize to the pilot-wave theory, as in [3]. But it should be possible to report on both topics, and to relate them.

Relevant Courses

Essential: None

Useful: Philosophical aspects of classical and quantum mechanics

References

[1]: General introductions include: Hilgevoord, J. (2002), Time in quantum mechanics, *American Journal of Physics* **70**, 301-306; Butterfield, J. (2012), Time in quantum physics, `http://philsci-archive.pitt.edu/9287/`. A large collection of research articles is Muga, G., Sala Mayato, R. and Egusquiza I., ed.s (2008) *Time in Quantum Mechanics*, Springer: two volumes. A recent article is Roberts, B. (2014), A general perspective on time observables. *Studies in History and Philosophy of Physics* **47**, 50-54, `http://philsci-archive.pitt.edu/10600/`. These examples cover the topics mentioned, i.e.: (i) problems about defining time as a physical quantity, and (ii) subtleties about the time-energy uncertainty principle.

[2]: A standard review is: Hauge, E. and Stovneng J. (1989), Tunneling times: a critical review, *Reviews of Modern Physics* **61**, 917–936. The main previous articles include: Salecker H.

and Wigner E. (1958), Quantum limitations of measurements of space-time distances, *Physical Review* **109**, 571-577; Peres, A. (1980), "Measurement of time by quantum clocks". *American Journal of Physics* **48**, 552–557; Buettiker, M. (1983). "Larmor precession and the traversal time for tunneling", *Physical Review B* **27**, 6178-6188.

[3]: Leavens C. R. (1990), Transmission, reflection and dwell times within Bohm's causal interpretation of quantum mechanics. *Solid State Communications* **74**; Leavens, C.R. and Aers, G.C. (1996), Bohm trajectories and the tunnelling time problem', in *Springer Series in Surface Sciences: vol. 29*, ed. Robert Gomer, Springer, pp. 105–140; Cushing J.T. (1995), Quantum tunnelling times: a crucial test for the causal program?, *Foundations of Physics* **25**, 269-280.

79. Gravitational Lensing in Cosmology ....
Professor A. Challinor

A century ago, observations of bending of light by the gravitational field of the Sun provided an early confirmation of a prediction of the theory of General Relativity. One hundred years on, such *gravitational lensing* has become an essential tool in modern cosmology. Radiation from extragalactic sources, such as distant galaxies [1] and the cosmic microwave background (CMB) [3], is deflected by the gravitational field of all the intervening large-scale structure in the Universe. The power of lensing is that it is sensitive to the distribution of all matter, not just the luminous, baryonic matter seen in galaxy surveys. Lensing has many applications in cosmology including: (i) constraining the growth of large-scale structure and the geometry of the Universe from correlations in the shapes of galaxies [2] and anisotropies in the CMB [4]; (ii) determination of the masses of individual dark matter haloes and hence, e.g., galaxy clusters (whose abundance as a function of mass and redshift is a very sensitive cosmological probe) [5,6]; (iii) measurements of the expansion rate of the Universe from strong gravitational lensing time delays [7,8]; and even (iv) probing a stochastic background of gravitational waves [9]. However, lensing is also a hindrance, blurring our view of the polarization properties of the CMB when searching for the B-mode imprint of primordial gravitational waves [10] and adding stochasticity to the luminosity distance measured from distant standard sirens [11].

Your essay should begin by reviewing the leading-order calculation of gravitational lensing in a cosmological context using perturbation theory (e.g., [1,3]). You should also discuss the observational manifestations of lensing, for example, in the shape, size and flux of galaxies and the anisotropies and polarization of the CMB. One or two cosmological applications of lensing could then be studied in detail, presenting the underlying physics, reviewing the current state of the art observationally, and the difficulties faced and potential solutions. These difficulties might include theoretical corrections to the leading-order perturbative results, e.g., corrections beyond the Born approximation [12].

Relevant Courses

Essential: Cosmology; Field Theory in Cosmology

Useful: General Relativity

References

[1] Bartelmann M. & Schneider P., Physics Reports, 340, 291 (astro-ph/9912508)

[2] Hildebrandt H., et al., 2019, arXiv:1812.06076

[3] Lewis A. & Challinor A., 2006, Physics Reports, 429, 1 (arXiv:astro-ph/0601594)

[4] Planck Collaboration, 2018, arXiv:1807.06210

[5] Hoekstra H., et al., 2013, Space Science Reviews, 177, 75 (arXiv:1303.3274)

[6] Baxter E., et al., 2018, MNRAS, 476, 2674 (arXiv:1708.01360)

[7] Refsdal S., 1964, MNRAS, 128, 307

[8] Wong K.C., et al., 2019, arXiv:1907.04869

[9] Dodelson S., Rozo E., Stebbins A., 2003, Phys. Rev. Lett., 91, 021301 (arXiv:astro-ph/0301177)

[10] Knox L. & Song Y.-S., 2002, Phys. Rev. Lett., 89, 011303 (arXiv:astro-ph/0202286)

[11] Holz D.E., & Hughes, S.A., 2005, ApJ, 629, 15 (arXiv:astro-ph/0504616)

[12] Krause E. & Hirata C.M., 2010, Astronomy and Astrophysics, 523, A28 (arXiv:0910.3786)

80. The Thermal Instability of Subglacial Lakes ....
Dr L.-A. Couston and Dr J. R. Taylor

A liquid heated from below is unstable to thermal convection when it is subject to a heat flux or an imposed temperature difference greater than a critical value. The thermal instability of a liquid confined between two horizontal isothermal plates and with constant thermodynamical properties has been extensively studied and is known as Rayleigh-Bénard convection (RBC) [1]. The thermal instability of a liquid with variable thermodynamical properties has been comparatively less studied, despite the fact that water–arguably the most important liquid we know of–has variable thermodynamical properties [2,3,4].

This essay will explore the thermal instability of subglacial lakes on Earth, most of which lie between the Antarctic bedrock and ice sheets, which can be up to 10km thick. Subglacial lakes are subject to geothermal heating from below and are in a high-pressure, low-temperature environment. The thermodynamical properties of water in such conditions are strongly dependent on pressure and temperature, such that the value of the critical heat flux for convection to set in, as a function of lake depth and ice cover thickness, remains uncertain.

This essay will examine the stability of an idealised subglacial lake. Specifically, consider a subglacial lake bounded by two flat, horizontal surfaces. Assume that the temperature at the top surface is held fixed at the local freezing point, while a constant heat flux can be applied at the bottom surface to represent geothermal heating. No-slip boundary conditions can be applied to the velocity, while the density of the fluid is a nonlinear function of temperature. The objective is to find the critical heat flux for the onset of convection as a function of various parameters in the system.

The essay should include (i) a discussion of the variability of the thermodynamical properties of water in subglacial high-pressure low-temperature environments, (ii) a discussion of the model problem, including assumptions made and the solution method, i.e. how to set-up the eigenvalue problem and solve it to get the critical heat flux, and (iii) an analysis of the stability of the problem for a chosen set of parameters. Candidates may also want to discuss the effect of planetary rotation and salt on the instability.

Relevant Courses

Essential: None

Useful: Hydrodynamics Stability, Fluid Dynamics of the Solid Earth

References

[1] Chandrasekhar, S. (1961). Hydrodynamic and hydromagnetic stability. Courier Corporation

[2] Veronis, G. (1963). Penetrative Convection. The Astrophysical Journal, 137, 641

[3] Thoma, M., Grosfeld, K., Smith, A. M., & Mayer, C. (2010). A comment on the Equation of State and the freezing point equation with respect to subglacial lake modelling. Earth and Planetary Science Letters, 294(1-2), 80-84

[4] McDougall, T. J., & Barker, P. M. (2011). Getting started with TEOS-10 and the Gibbs Seawater (GSW) oceanographic toolbox. SCOR/IAPSO WG, 127, 1-28

81. Contextuality and Nonlocality in Quantum Information ....

Dr N. de Silva

Contextuality [1] is a concept from the foundations of quantum mechanics denying the classical assumption of a physical reality in which systems possess properties independently of how they are observed. It generalises *nonlocality* [2]: a concept that crystallises the challenge posed to a reality obeying strict classical causality by the 'spooky action at a distance' of quantum entanglement lamented by Einstein. The contradictions between classically realist assumptions and quantum theory are witnessed by the violation of probabilistic inequalities [3] or, in extreme cases, logical paradoxes [4].

Contextuality and nonlocality have recently emerged as promising hypotheses as origins for the advantages of quantum technologies. As signatures of genuinely quantum behaviour, they are naturally suited to probing the classical-quantum computational boundary. This hypothesis suggest a unified source of quantum advantages in computation and communication and deep connections between the foundations of computer science and the foundations of physics.

This aim of this essay is to trace the evolution of notions of contextuality and nonlocality from thought experiments in early debates on the foundations of quantum mechanics through to the modern mathematical frameworks [5,6] that formalise and widen the scope of these ideas. It should also touch upon their application in contemporary efforts to understand the nature of quantum advantages in computation and communication, e.g. nonlocal games [7], contextuality as a resource in quantum computation [8]. The essay, depending on the particular interests of the student, could focus on physical, mathematical, or computational aspects.

Relevant Courses

Essential: Part III Quantum Computation

Useful: Part III Quantum Information Theory

References

[1] S. Kochen and E. Specker. The problem of hidden variables in quantum mechanics (1967). *J. Math. Mech.*

[2] J. Bell. On the Einstein-Podolsky-Rosen paradox (1964). *Physics.*

[3] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt. Proposed experiment to test local hidden-variable theories (1969). *Phys. Rev. Lett.*

[4] D. Greenberger, M. Horne, A. Shimony, and A. Zeilinger. Bell's theorem without inequalities (1990). *Am. J. Phys.*

[5] S. Abramsky and A. Brandenburger. The sheaf-theoretic structure of non-locality and contextuality (2011). *New J. Phys.*

[6] A. Acín, T. Fritz, A. Leverrier, and A. B. Sainz. A combinatorial approach to nonlocality and contextuality (2015). *Comm. Math. Phys.*

[7] R. Cleve, P. Høyer, B. Toner, and J. Watrous. Consequences and limits of nonlocal strategies (2004). *Proc. 19th IEEE CCC.*

[8] M. Howard et al., Contextuality supplies the 'magic' for quantum computation (2014). *Nature.*

82. Classical Simulation of Quantum Computation ....
Dr N. de Silva

While it is widely expected that quantum computers can fundamentally outperform classical computers in accomplishing certain tasks, our understanding of the precise nature and scope of these advantages is at an early stage. The complexity-theoretic study of classical algorithms for the simulation of quantum circuits is a key tool in answering the fundamental open questions of quantum computation such as: for which problems do quantum computers offer an advantage? and which features of quantum theory are essential for achieving these advantages?.

Classes of quantum circuits whose behaviour can be efficiently simulated by a classical algorithm cannot, by definition, offer any computational advantage over classical computation. The canonical example of such a simulation algorithm is that of Gottesman-Knill [1] which asserts that any quantum circuit whose components are restricted to stabiliser operations can be efficiently classically simulated. This can be surprising at first blush as such circuits can generate high degrees of entanglement and superposition: two features of quantum theory that are prominently deployed in quantum algorithms.

Instead, the Gottesman-Knill theorem and subsequently developed simulation techniques paint a complex portrait of how quantum advantage can be achieved. By supplementing simulable classes of circuits with the minimal resources needed to promote their power, we identify the sources and signatures of quantum computational advantage [2,3,4,5,6]. The result is a rich variety of methods for bridging the classical-quantum computational gap through subtle interplays of quantum behaviours with classical computational and dynamical resources. The question of which features of quantum theory enable quantum computational advantage is thus shown to defy naive answers and in need of being framed and considered more precisely.

The aim of this essay is to review some of these simulation algorithms and to consider their implications for questions of quantum computational advantage.

Relevant Courses

Essential: Part III Quantum Computation

Useful: Part III Quantum Information Theory

References

[1] S. Aaronson and D. Gottesman. Improved simulation of stabilizer circuits (2004). *Phys. Rev. A.*

[2] R. Jozsa and A. Miyake. Matchgates and classical simulation of quantum circuits (2008). *Proc. Roy. Soc.*

[3] R. Jozsa and M. Van den Nest. Classical simulation complexity of extended Clifford circuits (2014). *Quant. Inf. Comp.*

[4] M. Howard et al., Contextuality supplies the 'magic' for quantum computation (2014). *Nature.*

[5] S. Bravyi, G. Smith, and J. Smolin. Trading classical and quantum computational resources (2016). *Phys. Rev. X.*

[6] S. Bravyi, D. Gosset, and R. König. Quantum advantage with shallow circuits (2018). *Science.*

83. Twistor Transform ....
Dr M. Dunajski

One of the most remarkable achievements of Penrose's twistor program is the link it provides between solutions to certain linear and non-linear differential equations of mathematical physics and unconstrained holomorphic geometry of the twistor space. The essay would review the subject concentrating on linear massless fields four space-time dimensions. You should also explore one (or more!) of the following:

(a) Mathematical aspects of the construction such as isomorphisms between sheaf cohomology classes and massless fields.

(b) Connections with integrability.

(c) Generalisations to non-abelian gauge fields.

Relevant Courses

Applications of Differential Geometry to Physics or Differential Geometry. A firm knowledge of basic complex analysis (e.g. IB Complex Methods, or Complex Analysis) is essential.

References

[1] Penrose, R. (1969), Solutions of the zero-rest-mass equations. J. Math. Phys. **10**, 38.

[2] Dunajski, M. (2009) Solitons, Instantons and Twistors. Oxford Graduate Texts in Mathematics **19**, Oxford University Press.

[3] Huggett, S. A. and Tod, K. P. (1994) An Introduction to Twistor Theory, LMS Student Texts 4 (second edition).

[4] Witten, E. (2003) Perturbative Gauge Theory As A String Theory In Twistor Space, hep-th/0312171.

[5] Atiyah, M., Dunajski, M. and Mason, L. (2017) Twistor theory at fifty: from contour integrals to twistor strings. `arXiv:1704.07464`. Proceedings of the Royal Society, **473**. 1

[6] Penrose, R. Rindler, W. (1986) *Spinors and Space–time* Vol I, II (Chapter 9). CUP.

84. Bundle Methods for Nonsmooth Optimization ....
Dr H. Fawzi

The topic of this essay is optimization of *nonsmooth* convex functions. One of the simplest algorithms for nonsmooth optimization is the subgradient method, which is well understood theoretically but is slow to converge in practice. Another more sophisticated algorithm which works very well in practice is the so-called *bundle method* [1]. The convergence analysis of this algorithm is more intricate and is still not well understood [2,3]. The goal of this essay is to survey the different variants of bundle methods, their convergence rates, and perform numerical comparison on some test problems, e.g., from semidefinite programming [4].

Relevant Courses

Essential: Topics in Convex Optimisation

References

[1] Lemaréchal, Claude, Arkadii Nemirovskii, and Yurii Nesterov. "New variants of bundle methods." *Mathematical Programming* 69, no. 1-3 (1995): 111-147.

[2] Kiwiel, Krzysztof C. "Efficiency of proximal bundle methods." *Journal of Optimization Theory and Applications* 104, no. 3 (2000): 589-603.

[3] Du, Yu, and Andrzej Ruszczyński. "Rate of convergence of the bundle method." *Journal of Optimization Theory and Applications* 173, no. 3 (2017): 908-922.

[4] Helmberg, Christoph, and Franz Rendl. "A spectral bundle method for semidefinite programming." *SIAM Journal on Optimization* 10, no. 3 (2000): 673-696.

85. Mathematical Phyllotaxis ....
Professor J. R. Gog

The appearance of Fibonacci numbers in plant structures, such as sunflower spiral counts, has fascinated mathematicians for centuries. Almost all recent work has been variation on a standard model in which organs are treated as point nodes successively placed on a cylinder according to a given model. The model hypotheses represent long-known botanical principles that new nodes cannot be placed too close or too far away from the existing nodes, and locally lead to lattice-like structures. As parameters of the model, such as the diameter of the cylinder, are changed, the lattice can transition to another lattice with a different spiral count. It can often be proved that these transitions move lattice counts to higher Fibonacci numbers. While mathematically compelling, empirical validation of the standard model is as yet weak. There is also no complete mathematical characterisation of the classes of models which will generate Fibonacci structure.

This essay will discuss the mathematics of lattices on cylinders [1] and show the similarity in bifurcation tree structure of spiral counts from two or more different models (e.g. [2] and [3]). Students with a leaning towards pure mathematics could discuss number theoretic links, such as between the bifurcations of lattice count pairs and an elegant conformal mapping of the space of lattice space due to Levitov which allows a compact description of the possible bifurcation structure [4]. More applied students could discuss the ways in which spiral counts 'obvious' to the eye have been mathematically modelled, or compare the mathematical assumptions of node

placement models to classic 19th century biological hypotheses such as Hofmeister's Rule, and more modern molecular understandings.

Relevant Courses

Essential: Mathematical Phyllotaxis

References

[1] Roger V. Jean. *Phyllotaxis: A Systemic Study in Plant Morphogenesis*, Cambridge University Press (1994)

[2] Stephane Douady. 'The Selection Of Phyllotactic Patterns'. In: book *Symmetry in plants* ed. by Roger V. Jean and D. Barabé. World Scientic (1998) pp. 335–358

[3] Ralph O. Erickson. 'The Geometry of Phyllotaxis'. In: book *The growth and functioning of leaves* ed. by J. E. Dale and F. L. Milthorpe. Cambridge University Press (1983)

[4] Pau Atela, Christophe Gole, and Scott Hotton. 'A Dynamical System for Plant Pattern Formation: A Rigorous Analysis'. *Journal of Nonlinear Science* 12.6 (2002) pp. 641–676

86. β-Plane Turbulence and Jets ....
Professor P. H. Haynes

The β-plane is a mathematical construction that includes the important dynamical effect of spherical geometry on a fluid on the surface of a rotating planet, which is that the vertical component of rotation varies with latitude, but excludes less important purely geometric effects. The simplest form of β-plane dynamics is plane two-dimensional flow with the standard property that relative vorticity ζ is materially conserved in the absence of forcing/dissipation applied instead to the absolute vorticity $\zeta + \beta y$, where β is a constant and y is the north-south coordinate. Turbulent flow on the β-plane, where the 'turbulence' must be forced externally if the flow has no vertical structure, but may be forced by dynamical instabilities when there is vertical variation, has the striking property of self-organisation into alternating westward and eastward jets. There are many naturally occurring examples of this type of behaviour, in the Earth's atmosphere and ocean and in the atmospheres of giant planets.

An essay on this topic should review the dynamics of jet formation in two-dimensional flows where the turbulence is generated by external forcing. A good starting point for a student who chooses this essay would be [1] , a relatively recent research paper which cites several earlier papers such as [2] and [3] (which suggested a 'new paradigm' for jet formation). The later part of an essay might focus on a particular sub-topic, such as the application of quasilinear models, e.g. [4], or the dynamics of ocean jets as revealed by numerical models, e.g. [5], or recent observations e.g. [6].

The equations for 2-D flow on a β-plane are relatively easy to code and an essay might include some research based on numerical simulations. It would be important to choose problems for which the computational resources required are not too large and any student who is potentially interested in this approach is advised to discuss their ideas with the essay setter.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Hydrodynamic Stability, Fluid Dynamics of Climate (Neither is essential.)

References

[1] Scott, R.K, Dritschel, D.G., 2012: The structure of zonal jets in geostrophic turbulence. J. Fluid Mech., 711, 576–598.

[2] Maltrud, M. E. , Vallis, G. K., 1991: Energy spectra and coherent structures in forced two-dimensional and beta-plane turbulence. J. Fluid Mech. 228, 321–342.

[3] Baldwin, M. P., Rhines, P. B., Huang, H.-P., McIntyre, M. E., 2007: The jet-stream conundrum. Science 315, 467–468.

[4] Srinivasan K., Young, W.R., 2012: Zonostrophic Instability. J. Atmos. Sci., 69, 1633–1656.

[5] Berloff, P., Karabasov, S., Farrar, J. T., Kamenkovich, I., 2011: On latency of multiple zonal jets in the oceans. J. Fluid Mech. 686, 534–567.

[6] Cravatte, S., Kestenare, E., Marin, F., Dutrieux, P., Firing, E., 2017: Subthermocline and intermediate zonal currents in the tropical Pacific Ocean: Paths and vertical structure. Journal of Physical Oceanography, 47(9), 2305–2324.

87. Effective Field Theories for Ultracold Quantum Gases ....
Dr J. B. Hofmann

As an atomic gas is cooled to extremely low temperatures, it is no longer described by classical statistical physics but must be described using quantum mechanics. Such ultracold quantum gases are now routinely created in experiment and probed with great precision (indeed, they are the coldest objects in the universe). In particular, the interaction strength between the atoms can be very strong, and it is a considerable theoretical challenge to describe a strongly interacting quantum gas.

It turns out that the low-energy physics of a quantum gas is universal, i.e., it is not sensitive to every microscopic detail of the atoms. This makes quantum gases of very general appeal since we are not just describing the complicated microscopic details of one particular setup but can make general statements about strongly-interacting non-relativistic quantum systems. Effective field theory methods originally introduced in high-energy and nuclear physics have been extremely successful in describing universal aspects of strongly-interacting gases, and the essay is supposed to explore the quantum field theory of ultracold gases.

The essay should begin in a first part with a discussion of universality and effective field theories, and introduce one particular effective field theory for cold gases, the zero-range model. The second part should present one advanced topic from the field theory of quantum gases. Such an advanced topic could be (but is not limited to) three-body scattering and the Efimov effect; non-relativistic conformal symmetries and quantum anomalies; or universal relations and the operator-product expansion.

Relevant Courses

*Essential:* Quantum Field Theory

Useful: Statistical Field Theory, Advanced Quantum Field Theory

References

[1] W. Zwerger, *The BCS-BEC Crossover and the Unitary Fermi Gas*, Springer-Verlag (Heidelberg, 2012)

[2] E. Braaten, M. Kusunoki, D. Zhang, *Scattering Models for Ultracold Atoms*, Annals of Physics **323**, 1770 (2008) [arXiv:0709.0499]

[3] E. Braaten and H.-W. Hammer, *Universality in few-body systems with large scattering length*, Physics Reports **428**, 259 (2006) [arXiv:0410417]

88. Tidal Instabilities in Planetary Cores ....
Professor R. R. Kerswell

The study of rotating fluids is profoundly important for our understanding of geophysical and astrophysical phenomena. One particular branch of the subject looks at the effect of disturbing a uniformly-rotating body of contained fluid to model the likely dynamics within the molten-iron-filled cores of terrestial planets and satellites. The gravitational forces of neighbouring celestial objects can cause the rotation vector of the planet or satellite to vary periodically in direction (precession) and amplitude (libration) or simply distort the flow so that it is no longer circular (tides). In each case, the normal modes of the system - inertial waves - can get excited with the core flow becoming 3-dimensional and unsteady and possibly even turbulent [1]. The theory for how the inertial waves get excited is well established with good agreement between theory and experiment although there are still some surprises [2]. What is not so clear is what happens to the initial instability as it grows typically surrounded by others. Recent experimental and numerical work [3,4] has started to collect data on what can happen indicating that either a vortex-dominated or wave-dominated regime exists. This essay would review the instability mechanisms (e.g the 'elliptical instability' mechanism for tidal instability [5]) and then discuss the possible nonlinear regimes of the instability. This essay would provide good preparation for a student interested in doing a PhD in this research area.
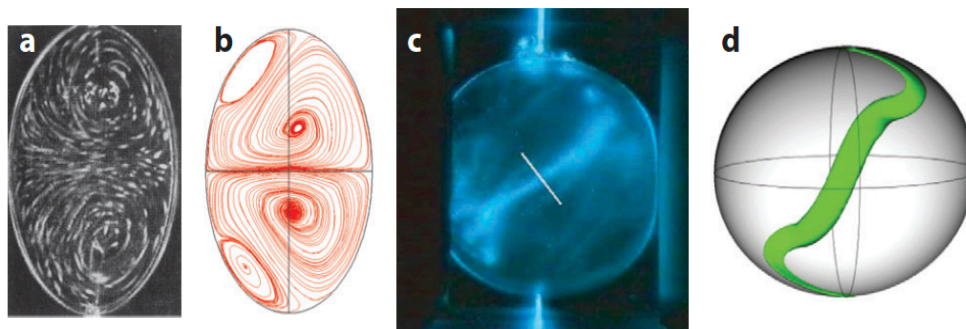
Relevant Courses

Essential: Fluids II, Hydrodynamic Stability, Methods
Useful: Dynamical Systems

References

[1] Le Bars, M., Cebron, D. and Le Gal, P. "Flows driven by libration, precession and tides" *Ann. Rev. Fluid Mech.* **47**, 163-193, 2015.

[2] Lin, Y, Marti, P. and Noir, J. "Shear-driven parametric instability in a precessing sphere" *Phys. Fluids* **27**, 046601, 2015.

[3] Le Reun, T., Favier, B. and Le Bars, M. "Experimental study of the nonlinear saturation of the elliptical instability: inertial wave turbulence versus geostrophic turbulence" *J. Fluid Mech.* **879**, 296-326, 2019.

[4] Le Reun, T., Favier, B., Barker, A.J. and Le Bars, M. "Inertial wave turbulence driven by elliptical instability" *Phys. Rev. Lett.* **034502**, 2017.

[5] Kerswell, R.R. "Elliptical Instability" *Ann. Rev. Fluid Mech.* **34**, 83-113, 2002.

89. Neural Networks to Linearise Nonlinear Dynamics ....
Professor R. R. Kerswell

Neural networks are being increasingly used to study dynamical systems with a view to extract some feature or essence hitherto hidden. This essay is about a certain strand of recent work (e.g. see [1] and refs herein) which seeks to use machine learning to uncover a coordinate transformation which best *linearizes* the dynamics of a nonlinear system. That this is even being contemplated is due to Koopman operator theory (Koopman 1931) which discusses how a finite-dimensional nonlinear system can be replaced by an infinite-dimensional linear system and the seemingly-related and coincident idea of Carleman linearization (Carleman 1932). The approach used in [1] (see also [2]) is to use a neural network to force the dynamics onto a representation that evolves linearly in time just as Koopman eigenfunctions should. A number of low-dimensional ODEs are treated to illustrate the approach although the ultimate area of application could be a time-dependent and 3-dimensional fluid flow. Depending on the interests of the student, this essay could concentrate on machine learning techniques (e.g [3]) or Koopman analysis (e.g [4]) or a combination of both. This essay would be good preparation for a machine-learning thesis project in fluid mechanics.
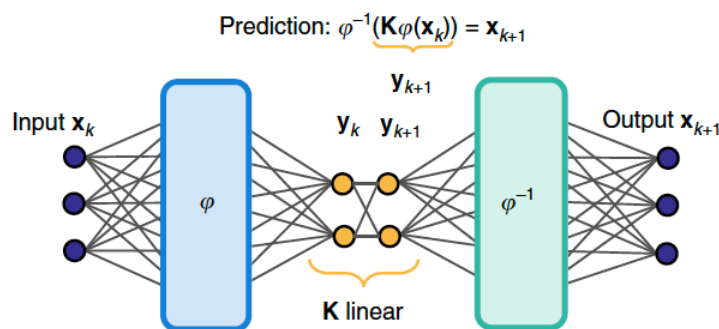

Figure 1: from [1]

Relevant Courses

Essential: Dynamical systems, Fluids II, Methods

Useful: Hydrodynamic Stability

References

[1] Lusch, B., Kutz, J.N. & Brunton, S.L. "Deep learning for universal linear embeddings of nonlinear dynamics" *Nature Commun.* **9**, 4950, 2018.

[2] Otto, S.E. & Rowley, C.W. "Linearly recurrent autoencoder networks for learning dynamics" *SIAM J. App. Dyn. Sys.* **18**, 558-593, 2019.

[3] Brunton, S.L, Noack, B.R. & Koumoutsakos, P. "Machine learning for fluid mechanics" *Ann. Rev. Fluid Mech.* **52**, 477-508, 2020.

[4] Mezic, I "Analysis of fluid flows via spectral properties of the Koopman operator" *Ann. Rev. Fluid Mech.* **45**, 357-378, 2013.

90. Viscous Fingering Instabilities ....
Dr K. Kowal

The interface between two fluids can be made morphologically unstable, resulting in complex pattern formation frequently encountered in porous media and biological systems. Such phenomena are widespread in nature and industry, ranging from crude oil recovery, hydrology, and filtration, to the self-organisation of collective biological systems and medical applications. In most cases, these instabilities occur when a less viscous fluid displaces a more viscous one, for example water displacing syrup, either by injection or by gravity when the interface separates two fluids of different densities. In rectangular geometries, such a system becomes unstable to the formation of a single finger known as the Saffman-Taylor finger. In radial geometries, multiple fingers form by successive tip-splitting. The onset of such instabilities can be examined through linear stability analyses.

The essay should review the mechanism of the instability using linear stability theory. The candidate may choose to investigate various stabilising mechanisms, including the effects of surface tension or mixing at the interface, anisotropy, or the effects of non-Newtonian fluids (e.g. power-law fluids, viscoplastic fluids). A simplified Hele-Shaw geometry may be used. The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate fluid mechanics

Useful: Slow Viscous Flow

References

[1] Saffman, P. G., Taylor, G. I. (1958) The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London Ser. A*, 245, 312-329

[2] Homsy, G. M. (1987) Viscous Fingering in Porous Media. *Ann. Rev. Fluid Mech.* 19, 271-311.

[3] Lindner, A., Bonn, D., Meunier, J. (2000) Viscous fingering in a shear-thinning fluid. *Phys. Fluids*, 12, 256-261

[4] Kowal, K. N. and Worster, M. G. (2019) Stability of lubricated viscous gravity currents. Part 1. Internal and frontal analyses and stabilisation by horizontal shear. *J. Fluid Mech.*, 871, 970–1006

91. Basal Sliding of Glacial Ice Sheets ....
Dr K. Kowal

Ice sheets are large bodies of ice, such those of Greenland and Antarctica, that slowly deform, or spread, under their own weight. Glacial ice appears to behave as a solid on small length and time scales; however, over large scales and under substantial pressure due to their own weight, ice sheets begin to flow as a viscous fluid, much like the viscous fluids we regularly see and eat, like honey and syrup. As such, understanding the flow of thin films of viscous fluids helps us to understand large-scale ice-sheet dynamics. These dynamics are also strongly affected by what is going on beneath ice sheets. The presence of meltwater and glacial till greatly accelerates the flow and results in rapid ice discharge towards the oceans. Such effects are frequently modelled using appropriate basal boundary conditions, also known as sliding laws, or examined by coupling the flow to an underlying layer of material, such as a less viscous fluid. However, limited access to the underside of ice sheets makes it difficult to predict the exact sliding mechanisms.

The essay should review the fluid mechanics of ice sheets and the influence of different basal conditions on their large-scale dynamics. The candidate may choose to illustrate their discussion by considering simple models of steady, uniform viscous shear flow down an incline using lubrication theory, subject to various basal boundary conditions, for which exact solutions are obtainable. Newtonian rheology may be used. The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate fluid mechanics

Useful: Fluid Dynamics of the Solid Earth, Slow Viscous Flow

References

[1] Weertman, J. (1957) On the sliding of glaciers. *J. Glaciol.* 3, 33–38.

[2] Boulton, G. S., Hindmarsh, R. C. A. (1987) Sediment deformation beneath glaciers: rheology and geological consequences. *J. Geophys. Res.* 92, 9059-9082.

[3] Schoof, C. , Hewitt, I. (2013) Ice-sheet dynamics. *Annu. Rev. Fluid Mech.* 45, 217-239.

[4] Kowal, K. N., Worster, M. G. (2015) Lubricated viscous gravity currents. *J. Fluid Mech.* 766, 626-655.

[5] Kowal, K. N., Pegler, S.S., Worster, M. G. (2016) Dynamics of laterally confined marine ice sheets. J. Fluid Mech. 790.

92. Thermocapillary Instabilities ....
Dr K. Kowal

An interface between two immiscible fluids is subject to interfacial, or surface, tension, which may depend on various scalar fields, such as the temperature and solute concentration, as well as on the concentration of surfactants, or compounds that decrease surface tension. When surface tension depends on the temperature, gradients along the interface induce shear stresses that result in thermocapillary fluid flow. Thermocapillary flows are ubiquitous in nature and industry, such as in crystal growth, welding, the manufacture of silicon wafers, electron beam

melting of metals and the rupture of thin films. In many of these, the transport of heat can increase significantly as a result of additional mixing processes triggered by thermocapillary instabilities. The onset of these instabilities can be examined using linear stability theory.

The essay should review the mechanisms of the instability using linear stability theory as well as characterise the different modes of instability that occur in *static* and *dynamic* thermocapillary liquid layers. The candidate may choose to examine the long-wave evolution of thermocapillary waves, their nonlinear dynamics, or the effects of surfactants and thermophoresis (the Soret effect). The exact direction of the essay depends on the interests of the candidate.

Relevant Courses

Essential: Undergraduate fluid mechanics

Useful: Slow Viscous Flow, Hydrodynamic Stability

References

[1] Pearson, J. R. A. (1958) On convection cells induced by surface tension. *J. Fluid Mech.* **4** (5), 489–500.

[2] Smith, M. K. and Davis, S. H. (1983) Instabilities of dynamic thermocapillary liquid layers. Part 1. Convective instabilities. *J. Fluid Mech.* **132** 119–144.

[3] Davis, S. H. (1987) Thermocapillary instabilities. *Ann. Rev. Fluid Mech.* **19**, 403–435

[4] Oron, A., Davis, S. H. and Bankoff, S. G. (1997) Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69** (3)

[5] Oron, A. (2000) Nonlinear dynamics of three-dimensional long-wave Marangoni instability in thin liquid films. *Phys. Fluids* **12**, 1633

[6] Saenz, P. J., Valluri, P., Sefiane, K., Karapetsas, G., and Matar, O. K. (2013) Linear and nonlinear stability of hydrothermal waves in planar liquid layers driven by thermocapillarity. *Phys. Fluids* **25**, 094101

93. Fluid Dynamics of Cell Locomotion ....
Professor E. Lauga

Fluid motion impacts the biological world on a large range of length scales, from the swimming of whales and birds to the dynamics of red blood cells in the circulation and gas exchange in our lungs. At the level of individual cells, fluid dynamics occurs typically at low Reynolds number and often plays a critical role in cellular fluid transport and force generation. One well-studied example is the locomotion of microorganisms such as bacteria, spermatozoa and algae. These organisms possess slender organelles called flagella whose time-periodic motion in a fluid environment gives rise to the motility of the cells.

In this essay, candidates will review the hydrodynamic principles behind the locomotion of microorganisms. A solid essay will present a survey of the research literature, including both the qualitative physical mechanisms at play in cell hydrodynamics and the quantitative mathematical tools derived to capture it. An excellent essay will apply these tools to a detailed mathematical description of at least one model organism.

Relevant Courses

Essential: None

Useful: Slow Viscous Flow

References

[1] *Mathematical Biofluiddynamics*, J. Lighthill (1975) SIAM.

[2] Life at low Reynolds number, E. M. Purcell (1977) Am. J. Phys., 45, 3.

[3] The hydrodynamics of swimming microorganisms, E. Lauga and T. R. Powers (2009) Rep. Prog. Phys., 72, 096601.

[4] Green algae as model organisms for biological fluid dynamics, R. E. Goldstein (2015) Annu. Rev. Fluid Mech., 47, 343.

94. Unfolding Plug-and-Play Priors ....
Dr J. Liang and Professor C.-B. Schönlieb

Regularisation methods Signal/image processing and inverse problems are often highly ill-posed. In order to find meaningful solutions, a common strategy is to impose certain prior information to the solution in the form of appropriate regularisation terms in the recovery model. The most well-known prior in inverse problems and imaging is the total variation [1]. To solve the corresponding recovery problem, optimisation methods are needed.

Imaging denoising methods As suggested by the name, image denoising aims to remove the noise in a degraded image. For example taking a photo in a very low light scene, the image will be very noisy due to high ISO values. Over the past decades, by exploiting the structure of images, many advanced image denoising methods are proposed in the literature, such as non-local means [2] and BM3D [3].

The above two concepts have mostly been developed in parallel, until recently, people discovered that the combination of the two can provide very surprising results. In [4], for image recovery problems, the authors proposed using denoising methods as implicit priors and plugging them into the optimisation methods, state-of-the-art outputs are obtained, see also [5].

The basic goal of this essay is to review the literature to date, implement the method and apply it to some medical image problems. An optional task of this essay is building a (practical) connection to iterative regularisation.

Relevant Courses

Essential: Inverse Problems, MATLAB programming

Useful: Topics in Convex Optimisation

References

[1] Rudin, L. I., Osher, S., & Fatemi, E. (1992). Nonlinear total variation based noise removal algorithms. Physica D: nonlinear phenomena, 60(1-4), 259-268.

[2] Buades, A., Coll, B., & Morel, J. M. (2005, June). A non-local algorithm for image denoising. In 2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition (CVPR'05) (Vol. 2, pp. 60-65). IEEE.

[3] Dabov, K., Foi, A., Katkovnik, V., & Egiazarian, K. (2007). Image denoising by sparse 3-D transform-domain collaborative filtering. IEEE Transactions on image processing, 16(8), 2080-2095.

[4] Venkatakrishnan, S. V., Bouman, C. A., & Wohlberg, B. (2013). Plug-and-play priors for model based reconstruction. In 2013 IEEE Global Conference on Signal and Information Processing (pp. 945-948). IEEE.

[5] Chan, S. H., Wang, X., & Elgendy, O. A. (2016). Plug-and-play ADMM for image restoration: Fixed-point convergence and applications. IEEE Transactions on Computational Imaging, 3(1), 84-98.

[6] Osher, S., Burger, M., Goldfarb, D., Xu, J., & Yin, W. (2005). An iterative regularization method for total variation-based image restoration. Multiscale Modeling & Simulation, 4(2), 460-489.

95. The Curse of Dimensionality: Safe Screening Rules and Geometric Adaptation ....
Dr J. Liang and Professor C.-B. Schönlieb

In data science, high dimensional statistics and many other fields, the curse of dimensionality poses various challenges in both modelling and numerical methods. Because of this and as a consequence of the "big data" era, dimension reduction is becoming increasingly important.

In this essay, we focus on a particular class of dimension reduction techniques, namely LASSO related problems [1] and safe screenings rules [2]. Screening rules allow to discard irrelevant variables from the early state of the optimisation in LASSO problems, such that the dimension is progressively reducing [3]. Recently, the advances in geometric properties of non-smooth optimisation [4] make it possible to build a geometric interpretation of safe screening rules and propose further adaptive refinements.

The goal of this essay is to review the literature to date, implement the methods, and possibly extend the safe screening technique to non-LASSO type problems.

Relevant Courses

Essential: Topics in Convex Optimisation, MATLAB programming

Useful: Inverse Problems, Modern Statistical Methods, Statistical Learning in Practice

References

[1] Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society: Series B (Methodological), 58(1), 267-288.

[2] Fan, J., & Lv, J. (2008). Sure independence screening for ultrahigh dimensional feature space. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 70(5), 849-911.

[3] Fercoq, O., Gramfort, A., & Salmon, J. (2015). Mind the duality gap: safer rules for the lasso. arXiv preprint arXiv:1505.03410.

[4] Liang, J. (2016). Convergence rates of first-order operator splitting methods (Doctoral dissertation).

96. The Axion, Lattice QFT and Cosmology ....
Dr M. McCullough

The axion is a popular solution to the QCD strong-CP problem and also an attractive dark matter candidate which, theoretically, kills two puzzles with one stone. The finite-temperature behaviour of the QCD-generated axion potential is an important input for predicting the cosmological abundance of axions and is also interesting, from a theoretical perspective, due to its relation to the physics of anomalies, instantons, and topological properties of QCD.

The first part of this essay will be to review the Strong-CP problem and its proposed solutions, including the axion. The second will be to calculate the zero-temperature axion potential from chiral perturbation theory and sketch the cosmological behaviour of this potential. The final task is to review the state of the art in calculations of the temperature-dependence of the axion potential, including for temperatures close to the QCD scale, and then discuss the impact of this temperature-dependence for the predicted axion abundance now. This last part will require some appreciation for the techniques of lattice QFT.

Relevant Courses

Essential: Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory, Cosmology

References

[1] J. E. Kim and G. Carosi, "Axions and the Strong CP Problem," Rev. Mod. Phys. **82** (2010) 557

[2] G. Grilli di Cortona, E. Hardy, J. Pardo Vega and G. Villadoro, "The QCD axion, precisely," JHEP **1601** (2016) 034

[3] C. Bonati, M. D'Elia, M. Mariti, G. Martinelli, M. Mesiti, F. Negro, F. Sanfilippo and G. Villadoro, "Axion phenomenology and lattice QCD," PoS LATTICE **2016** (2016) 088.

97. Lessons from the S Matrix Program ....
Dr S. Melville

Before the development of QCD, there was the S Matrix Program [1].
The goal was to find enough symmetry principles and physical axioms so that scattering amplitudes could be fixed uniquely [2], rather than computed from an explicit Lagrangian.

Although eventually superseded by the QCD Lagrangian, there are many important lessons to be learned from the S-Matrix Program. For instance, studies of the S-Matrix formed the basis of early string theory, went on to underpin modern bootstrap techniques, and today is combined with Effective Field Theory ideas to produce useful "positivity bounds" [3].

This essay will review how physical scattering amplitudes are constrained by,

   (i) Lorentz invariance,

(ii) unitarity,

(iii) cauality,

(iv) locality,

with the aim of using these tools to constrain a simple low-energy Effective Field Theory (such as the $(\partial\phi)^4$ interaction for a single scalar field [3], or the scattering lengths for pions [4]).

Relevant Courses

Essential: Quantum Field Theory

Useful: Symmetries, Fields and Particles; Advanced Quantum Field Theory; Physics Beyond the Standard Model

References

[1] "*S*-matrix theory of strong interactions",
Chew, G. F. 1961.
Benjamin, New York (Available from: https://archive.org/details/smatrixtheoryofs00chew)

[2] "The Analytic *S* Matrix",
R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne 1966
Cambridge University Press (Google-Books-ID: VWTnlTyjwjMC)

[3] "Causality, analyticity and an IR obstruction to UV completion",
A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi 2006
JHEP, 0610:014 (arXiv:hep-th/0602178)

[4] "Dispersion Relation Bounds for $\pi\pi$ Scattering",
Manohar, A. V. and V. Mateu 2008
Phys. Rev., D77:094019 (arXiv:0801.3222)

98. The Brewer-Dobson Circulation ....
Dr A. Ming

The Brewer-Dobson Circulation (BDC) is a large scale meridional circulation in the stratosphere (the part of the atmosphere between about 10-50km altitude) that slowly transports air upwards in the tropics, poleward and then downwards over mid and high latitudes.

The strength and variability of this circulation is important in setting the composition of the stratosphere. For example, ozone depleting substances such as chlorofluorocarbons are inert in the troposphere but in the middle and upper stratosphere are broken down by radiation into simpler molecules. Their atmospheric lifetimes, set by the average time taken for a molecule starting in the troposphere to reach the middle stratosphere, are determined by the strength of the BDC. The BDC also has an important effect on water vapour in the stratosphere. Air entering the stratosphere in the tropics, in the upward part of the BDC, encounters a cold region, the temperature of which is itself dependent on the strength of the BDC, where the dehydration of the air parcels is important in setting the concentrations of stratospheric water vapour. Although these concentrations are very low, stratospheric water vapour makes an important contribution to the 'greenhouse effect' and is also a source of hydrogen oxide radicals which control many key chemical reactions important for ozone. The water vapour signal shows

multi-timescale variations from daily to decadal which dominated by temperature variations and the tropical upwelling strength, with the BDC playing an important role in both.

The BDC cannot be explained without careful consideration of dynamics since, for example, the poleward transport requires a force to change the angular momentum, otherwise, according to the 'ballerina effect' the rotation of air relative to the Earth would increase, which is not observed.

This essay should begin by discussing the dynamical frameworks used to understand the BDC, including the Transformed Eulerian Mean framework [1] and the 'downward control principle' [2]. The essay should then review the current status of understanding the BDC and challenges in quantifying it. Possible topics to cover in the second half of the essay include discussing in more detail of one or two of the topics reviewed in [1] such as discussing the main drivers of variability in the BDC (e.g. [3]), the predicted changes to the BDC under climate change or the methods of estimating the current BDC from dynamical and chemical observations (e.g. [4]). These papers offer starting points for further reading.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Fluid Dynamics of the Climate

References

[1] Butchart, N. (2014), The Brewer-Dobson circulation, Rev. Geophys., 52, 157–184, doi:10.1002/2013RG000448.

[2] Haynes, P. H., C. J. Marks, M. E. McIntyre, T. G. Shepherd, and K. P. Shine (1991), On the "downward control" of extratropical diabatic circulations by eddy-induced mean zonal forces, J. Atmos. Sci., 48, 651–678, doi:10.1175/1520-0469(1991)048¡0651:OTCOED¿2.0.CO;2.

[3] Fueglistaler, S., M. Abalos, T. J. Flannaghan, P. Lin, and W. J. Randel (2014), Variability and trends in dynamical forcing of tropical lower stratospheric temperatures, Atmos. Chem. Phys., 14, 13,381–13,412, doi:10.5194/acpd-14-13381-2014.

[4] Abalos, M., B. Legras, F. Ploeger, and W. J. Randel (2015), Evaluating the advective Brewer-Dobson circulation in three reanalyses for the period 1979–2012, J. Geophys. Res. Atmos., 120, 7534–7554, doi:10.1002/2015JD023182.

99. Subglacial Drainage: Formation and Stability of Roëthlisberger Channels
Dr J. A. Neufeld

Ice sheets have exhibited significant sensitivity to warming conditions through their subglacial environment, where basal lubrication plays a key role in determining the volume of ice exported to the world's oceans. The water routed to the base of many glaciers may either flow in a distributed network, or be focused in a series of channels, often called Roëthlisberger channels. The efficiency with which these channels export melt water is determined by their cross section, which is set by a balance between turbulence melting back the channel walls and viscous creep closure. Roëthlisberger channels have been observed in many glaciers, but mainly near the glacial terminus, where they often widen as they exit beneath the glacier. The ability of these channels to transport large volumes of water is clear, but the stability of these channels to longitudinal perturbations has yet to be assessed.

An analogous system is that of the partially melting, upwelling mantle where melting, creates a porous matrix in which the buoyant rise of melt rises is balanced by viscous compaction of the solid matrix. This system has been modelled in the laboratory by a viscously deforming pipe, the results of which suggest the system is unstable to solitary waves.

This essay will review the formation and fluid mechanics of glacial Roëthlisberger channels starting with the theory of their cross-section. The essay will then explore the mathematical links between turbulent meltwater channels and viscous compaction of porous materials. In particular, the essay should address, using rigorous mathematical analysis or analogy to the partial melting example, to what extent Roëthlisberger channels may suffer longitudinal variations in their cross-section and hence to what extent natural variability of the melt water export may be anticipated.

Relevant Courses

Useful: Slow Viscous Flow, Fluid dynamics of the Solid Earth

References

[1] H. Roëthlisberger. Water pressure in intra- and subglacial channels. (1972) J. Glaciol., **11**(62):177–203.

[2] J. F. Nye. The flow law of ice from measurements in glacier tunnels, laboratory experiments and the Jungfraufpoirn borehole experiment. Proc. R. Soc. Lond. Ser. A 219, pages 477?489, 1953.

[3] D.R. Scott, D.J. Stevenson, J.A. Whitehead Jr. (1986). Observations of solitary waves in a viscously deformable pipe. Nature, **319**, 759–761.

[4] Schoof, C., Hewitt, I. (2013). Ice-Sheet Dynamics. Ann. Rev. Fluid Mech., **45**(1), 217–239.

100. Warped Astrophysical Discs ....
Professor G. I. Ogilvie

In a spherically symmetric gravitational potential, orbital motion is possible in any plane containing the centre of the potential. A warped astrophysical disc is a fluid flow dominated by orbital motion, in which the orbital plane varies continuously with radius and possibly with time. Both the shape of the disc and its mass distribution evolve as a result of internal torques that transport angular momentum within the disc, as well as any external torques applied to the disc.

Astrophysical discs are expected to be warped whenever a misalignment occurs in the system, as in the classic problem in which a black hole is fed with gas having an orbital angular momentum that is not parallel to the spin of the black hole; indeed, the shape adopted by the disc in this situation is a problem of considerable interest. It is also possible for a warp to arise through instability of an aligned system, as in the case of accretion on to a magnetized star.

Linear and nonlinear theories of the dynamics of warped discs have been discussed since the 1970s [1], and increasing use is being made of global numerical simulations (e.g. [2]). The aim of this essay would be to review the subject concisely, with an emphasis on recent developments of a theoretical, computational or observational nature.

An introduction to the subject and a selection of useful references can be found in Sections 1 and 2 of reference [3]. Use of the ADS archive `adsabs.harvard.edu` is highly recommended. Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

Essential: Dynamics of Astrophysical Discs

Useful: Astrophysical Fluid Dynamics

References

[1] Bardeen, J. M. & Petterson, J. A. (1975). The Lense–Thirring effect and accretion disks around Kerr black holes. *Astrophys. J. Lett.* **195**, L65–L67

[2] Lodato, G. & Price, D. J. (2010). On the diffusive propagation of warps in thin accretion discs. *Mon. Not. R. Astron. Soc.* **405**, 1212–1226 (`arxiv.org/abs/1002.2973`)

[3] Ogilvie, G. I. & Latter, H. N. (2013). Local and global dynamics of warped astrophysical discs. *Mon. Not. R. Astron. Soc.* **433**, 2403–2419 (`arxiv.org/abs/1303.0263`)

101. Wave Attractors in Rotating and Stratified Fluids ....
Professor G. I. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics which might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.

2. The relation, if any, between the propagation of rays within a container and the existence of invisicid normal modes.

3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.

4. The roles of nonlinearity and instability in wave attractors.

5. The relevance of wave attractors to tidal dissipation in astrophysical systems.

Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

Useful: Astrophysical Fluid Dynamics

References

[1] Maas, L. R. M. & Lam, F.-P. A. (1995). *J. Fluid Mech.* **300**, 1

[2] Maas, L. R. M., Benielli, D., Sommeria, J. & Lam, F.-P. A. (1997). *Nature* **388**, 557

[3] Rieutord, M., Georgeot, B. & Valdettaro, L. (2001). *J. Fluid Mech.* **435**, 103

[4] Ogilvie, G. I. (2005). *J. Fluid Mech.* **543**, 19

[5] Jouve, L. & Ogilvie, G. I. (2014). *J. Fluid Mech.* **745**, 223

102. The Cosmological Bootstrap ....
Dr E. Pajer

In most cases of phenomenological interest, QFT cannot be solved exactly and perturbative methods are needed to make predictions for observables. The path from fields and Lagrangians to amplitudes and correlators passes by the well-developed machinery of Feynman diagrams. It has long been realised that this prescription for calculating observables obscures much of the structure of the final result. The poster child of this phenomenon are amplitudes for massless fields with spin, which can be computed very effectively referring only to on-shell quantities such as helicities and momenta, as opposed to needing degrees of freedom that are removed by constraints or gauge redundancy. Many "on-shell methods" for amplitudes are known for Lorentz invariant theories and new structures are being discovered in this very active research field. Conversely, much less is known about correlations functions in cosmology (i.e. in-in expectations values of operators in curved spacetime as opposed to in-out amplitudes in flat spacetime), where the expansion of the universe spontaneously break time translations and boosts. New ideas on how to address this shortcoming have been proposed recently, specifically in applications to de Sitter spacetime and the closely related inflationary cosmology. The main idea is to use symmetries and general principles such as unitarity, locality and causality to constrain the form that correlators can take in any theory.

In this essay, one starts by reviewing the most important on-shell methods to compute amplitude in flat spacetime, focussing on massless particles and the spinor-helicity formalism. Nice and pedagogical reviews on this are available in the literature [1-3]. Then one moves on to cosmology where similar ideas have been developed for correlators in de Sitter spacetime or its cousin the inflating universe [4-7].

Relevant Courses

Essential: Quantum Field Theory, General Relativity.

Useful: Cosmology

References

[1] H. Elvang and Y. t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

[2] P. Benincasa, "New structures in scattering amplitudes: a review," Int. J. Mod. Phys. A **29**, no. 5, 1430005 (2014) doi:10.1142/S0217751X14300051 [arXiv:1312.5583 [hep-th]].

[3] C. Cheung, "TASI Lectures on Scattering Amplitudes," doi:10.1142 9789813233348 0008 arXiv:1708.03872 [hep-ph].

[4] J. M. Maldacena and G. L. Pimentel, "On graviton non-Gaussianities during inflation," JHEP **1109**, 045 (2011) doi:10.1007/JHEP09(2011)045 [arXiv:1104.2846 [hep-th]].

[5] N. Arkani-Hamed and J. Maldacena, "Cosmological Collider Physics," arXiv:1503.08043 [hep-th].

[6] N. Arkani-Hamed, D. Baumann, H. Lee and G. L. Pimentel, "The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities," arXiv:1811.00024 [hep-th].

103. Relativistic Quantum Mistrustful Cryptography ....
Dr D. Pitalúa-García

In *mistrustful cryptography*, two or more parties perform a given task without trusting each other. For example, in *coin tossing* Alice and Bob are at distant locations and want to agree on a random bit a, with the security guarantee that a is not biased by the other party. *Bit commitment* is another example, where Alice *commits* to b by giving a proof to Bob that she has chosen a bit b at some time t without telling him the value of b, until a later time $t' > t$ when Alice *unveils* the value of b to Bob. After some quantum protocols for tasks in mistrustful cryptography were proposed in the 90s as unconditionally secure, i.e. with unbreakable security based only on the laws of quantum physics, some no-go theorems claimed that various tasks in mistrustful cryptography could not achieve unconditional security even using quantum protocols [1,2,3,4]. However, it was later shown by Kent [5,6] that protocols where parties have two or more collaborating agents at distant locations can exploit the relativistic physical principle stating that information cannot travel faster than the speed of light to evade these no-go theorems and achieve unconditional security. Various protocols and tasks in relativistic quantum mistrustful cryptography have been investigated since then [7–9].

An ideal essay will review the literature on this research area, showing a clear understanding of the main physical ideas and mathematical concepts. You are not expected to reproduce the mathematical security proofs nor are expected to cover all papers. For example, you might choose to focus on bit commitment by covering Refs. [1,2,5,8,9], or to focus on secure computations by covering Refs. [3,4,6,7].

Relevant Courses

Essential: None

Useful: Part II Principles of Quantum Mechanics, Part II Quantum Information and Computation

References

[1] D. Mayers, "Unconditionally Secure Quantum Bit Commitment is Impossible," Physical Review Letters **78**, 3414 (1997).

[2] H.-K. Lo and H. F. Chau, "Is Quantum Bit Commitment Really Possible?," Physical Review Letters **78**, 3410 (1997).

[3] H.-K. Lo and H. Chau, "Why quantum bit commitment and ideal quantum coin tossing are impossible," Physica D: Nonlinear Phenomena **120**, 177 (1998), proceedings of the Fourth Workshop on Physics and Consumption.

[4] H.-K. Lo, "Insecurity of quantum secure computations," Physical Review A **56**, 1154 (1997).

[5] A. Kent, "Unconditionally secure bit commitment," Physical Review Letters **83**, 1447 (1999).

[6] A. Kent, "Coin Tossing is Strictly Weaker than Bit Commitment," Physical Review Letters **83**, 5382 (1999).

[7] R. Colbeck, "Impossibility of secure two-party classical computation," Physical Review A **76**, 062308 (2007).

[8] A. Kent, "Unconditionally Secure Bit Commitment by Transmitting Measurement Outcomes," Physical Review Letters **109**, 130501 (2012).

[9] T. Lunghi, J. Kaniewski, F. Bussieres, R. Houlmann, M. Tomamichel, S. Wehner, and H. Zbinden, "Practical Relativistic Bit Commitment," Physical Review Letters **115**, 030502 (2015).

104. Does the Quantum State Directly Represent Physical Reality? ....
Dr D. Pitalúa-García

The quantum state is the main mathematical object in quantum theory. However, its relation to physical reality is not completely understood. On the one hand, assuming that there cannot be instantaneous disturbance on a distant physical system, it was argued by Einstein, Podolsky and Rosen [1] that the quantum state does not provide a complete description of physical reality. The EPR paper led to Bell's theorem [2], stating that there are not locally causal hidden variable theories reproducing all the predictions of quantum theory. On the other hand, under some physical assumptions, for example that physical systems that are prepared independently have independent physical states, Pusey, Barrett and Rudolph [3] showed that the quantum state represents a state of physical reality. It is debatable whether the assumptions made in the PBR theorem are physically sensible. Different versions of the PBR theorem have been found, for example in Refs. [4,5], making different physical assumptions. By dropping some of the assumptions made in these theorems, physical models can be constructed in which the quantum state does not represent physical reality directly [6].

An ideal essay will review the literature on this research area, showing a clear understanding of the PBR theorem [3], and of its interconnection with the EPR argument [1] and with Bell's theorem [2]. The discussion of Refs. [4–6] is not expected to be in great detail, but should give a broader perspective on the PBR theorem.

Relevant Courses

Essential: None

Useful: Part II Principles of Quantum Mechanics, Part II Quantum Information and Computation, Philosophical Aspects of Classical and Quantum Mechanics (graduate non-examinable course)

References

[1] A. Einstein, B. Podolsky and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?," Physical Review **47**, 777–780 (1935).

[2] J. S. Bell, "On the Einstein-Podolsky-Rosen Paradox," Physics **1**, 195–200 (1964).

[3] M. F. Pusey, J. Barrett and T. Rudolph, "On the reality of the quantum state," Nature Physics **8**, 475–478 (2012).

[4] J. Barrett, E. G. Cavalcanti, R. Lal and O. J. E. Maroney, "No ψ-epistemic model can fully explain the indistinguishability of quantum states," Physical Review Letters **112**, 250403 (2014).

[5] O. J. E. Maroney, "How statistical are quantum states?," arXiv:1207.6906v2 (2013).

[6] P. G. Lewis, D. Jennings, J. Barrett and T. Rudolph, "Distinct quantum states can be compatible with a single state of reality," Physical Review Letters **109**, 150404 (2012).

105. Advantages, Limitations and Challenges in Photoacoustic Imaging . . . .
<div align="right">Dr O. Rath Spivack</div>

Hybrid imaging techniques have attracted considerable attention in the past few years as a promising emerging medical imaging tool [1]. These techniques combine a high contrast modality with a high resolution modality, and have the potential for delivering the advantages of both techniques. Examples include ultrasound elastography, magnetic resonance elastography, photoacoustic tomography, magnetic resonance electric impedance tomography, and others. They are mostly not yet established as standard imaging techniques in a clinical setting, and are still the object of much theoretical investigation.

This essay should concentrate on photoacoustic tomography, describe the physical effect on which it is based, and explain the equations used to model the wave propagation and the inverse problem [2,3].

An overview should be given of the type of inverse problems that need solving, and of the range of validity of the models used [4,5]. Then, the essay could focus on the theoretical results for stability and uniqueness of solutions and the conditions needed [6], or on the validity of the approximations in applications, and perhaps the merits of different numerical methods for the solution [7,8]. The choice should depend on your interests and background.

Finally, the essay should give a view of current challenges and possible avenues for addressing them, especially in relation to the focus chosen in the main part of the essay.

Relevant Courses

Essential: Basic knowledge of PDEs, and particularly the wave equation, from any undergraduate course.

Useful: The Part III courses 'Inverse problems', or 'Numerical Solutions of Differential Equations', further background in PDEs, or in wave propagation in random media, or in numerical techniques such as Finite Element Methods may be useful, depending on the choice for the focus of this essay.

References

[1] Ammari H An Introduction to Mathematics of Emerging Biomedical Imaging, Springer (2008)

[2] Cox B, Laufer JG, Arridge SR, Beard PC Quantitative spectroscopic photoacoustic imaging: a review *Journal of Biomedical Optics* 17(6), 061202 (2012)

[3] Xu M and Wang LV Photoacoustic imaging in biomedicine *Rev. Sci. Instruments*, 77, 041101 (2006)

[4] Arridge SR & Schotland, JC Optical tomography: forward and inverse problems *Inverse Problems* 25, 123010 (2009)

[5] Ammari H, Bossy E, Jugnon V, Kang H Mathematical Modeling in Photoacoustic Imaging of Small Absorbers *SIAM Review* 52(4) 677–695 (2010)

[6] Bal G, Jollivet A, Jugnon V Inverse transport theory of photoacoustics *Inverse Problems* 26, 025011 (2010)

[7] Martí-López L, Bouza-Domínguez J, Hebden JC, Arridge SR, Martínez-Celorio RA Validity conditions for the radiative transfer equation *J. Opt. Soc. Am.* A 20(11), 2046-2056 (2003)

[8] Kim AD and Ishimaru A Optical diffusion of continuous-wave, pulsed, and density waves in scattering media and comparisons with radiative transfer *Applied Optics* 37(22) 5313-5319 (1998)

106. Quasinormal Modes of Black Holes ....
Professor H. S. Reall

Quasinormal modes are linear perturbations of a black hole that describe damped oscillations of the hole. A quasinormal mode has a definite, complex, frequency. Any black hole has a characteristic spectrum of quasinormal frequencies. If a hole is perturbed then its quasinormal modes describe how it relaxes back to equilibrium. This is observable in the gravitational waves produced by a black hole merger.

The essay should explain what quasinormal modes are and why they are relevant for the late time behaviour of a perturbed black hole. It should then discuss some applications. These might include: quasinormal modes of the Kerr black hole (via the Teukolsky equation); methods for calculating quasinormal modes e.g. WKB or continued fractions; the role of quasinormal modes in the AdS/CFT correspondence [3]; a rigorous mathematical approach to quasinormal modes [4]; or the connection between quasinormal modes and strong cosmic censorship [5].

Relevant Courses

Essential: General Relativity, Black Holes.

References

[1] K. Kokkotas and B. Schmidt "Quasinormal modes of stars and black holes", Living Reviews in Relativity 2 (1999), 2.

[2] E. Berti, V. Cardoso and A. Starinets, "Quasinormal modes of black holes and black branes", Class. Quant. Grav. 26, 163001 (2009)

[3] G. Horowitz and V. Hubeny, "quasinormal modes of AdS black holes and the approach to thermal equilibrium", Phys. Rev. D62, 024027 (2000)

[4] C. Warnick, "on quasinormal modes of asymptotically anti-de Sitter black holes", Commun. Math. Phys. 333, 959 (2015)

[5] O. Dias, H. Reall and J. Santos, "strong cosmic censorship: taking the rough with the smooth", JHEP 1810,001 (2018)

107. String Theory on Orbifolds ....
Dr R. A. Reid-Edwards

The basic principles underlying string theory as a quantum theory of gravity have still not yet been identified and, as such, much of our current understanding of the theory comes from the study of string theory on particular backgrounds. String theory is still only well-understood on a relatively small number of backgrounds. Orbifolds are an important class of backgrounds which allow for some of the more novel aspects of string theory to be explored.

Starting with a circle of radius R with coordinate $X \sim X + 2\pi R$, we could impose the $\mathbb{Z}_2$ identification $X = -X$. The resulting space is the interval $S^1/\mathbb{Z}_2$ with end points (or *fixed points*) at $X = 0, \pi R$. This is a simple example of an orbifold. More generally, an orbifold may be constructed by taking the quotient of a manifold $\mathcal{M}$ by discrete group Γ. If the group is not freely acting then the orbifold will have fixed points and the space will be singular. It is remarkable that string theory makes sense on such spaces, even at the singular points. The basic idea is to quantise the string on the space $\mathcal{M}$, a torus for example, and obtain the Hilbert space $\mathcal{H}$. One then projects out all states in $\mathcal{H}$ that are not invariant under Γ. For a field theory, such as general relativity, the story would end here and we could only hope to make sense of the theory away from the singular points. In string theory, we must go further. There will be states in the orbifold Hilbert space that were not in $\mathcal{H}$, that wind around the fixed points, and are only physical due the action of Γ. These *twisted sectors* add an important, purely string theoretic, contribution to the physics not to be found in field theory. It is in this way that string theory can make sense on such singular spaces.

It is envisaged that this essay would comprise two parts. The first part would discuss some general aspects orbifold constructions, illustrating the key ideas explicitly in simple examples. The physical interpretation of the twisted sectors and questions of modular invariance could be discussed. In the second part of the essay a particular topic could be explored in more depth. Such topics include, but are not restricted to; the moduli space of $c = 1$ conformal field theories, asymmetric orbifolds and non-geometric backgrounds, supersymmetry breaking using orbifolds, mirror symmetry and the relationship between orbifolds and smooth Calabi-Yau manifolds.

Relevant Courses

Essential: Quantum Field Theory, String Theory

Useful: Advanced Quantum Field Theory, Supersymmetry

References

[1] "String theory. Vol. 1: An introduction to the bosonic string,", J. Polchinski, CUP.

[2] "String theory and particle physics: An introduction to string phenomenology" L. Ibanez & A. Uranga, 2012, CUP.

[3] "Strings on Orbifolds", L. Dixon, J. Harvey, C. Vafa, E. Witten. Jul 1985. 9 pp. Nucl.Phys. B261 (1985) 678-686
"Strings on Orbifolds 2", L. Dixon, J. Harvey, C. Vafa, E. Witten. Feb 1986. 30 pp. Nucl.Phys. B274 (1986) 285-314

108. Wess-Zumino-Witten Models ....
Dr D. B. Skinner

This essay will study WZW models in two dimensions. These are a particular type of Quantum Field Theory where the basic field describes a map from a Riemann surface to a Lie group. Unlike most QFTs, which can only be studied perturbatively, WZW models are exactly solvable despite being inherently strongly coupled. This comes from a rich mathematical structure that interweaves conformal invariance of the Riemann surface, current algebra and the representation theory of loop groups (certain infinite dimensional Lie groups), and a deep relation to Chern-Simons theory in 2+1 dimensions. Their physical applications are as diverse as Euclidean black holes, a three-dimensional version of AdS/CFT and a description of plateau transitions in the quantum Hall effect.

Relevant Courses

Essential: Quantum Field Theory and Symmetries, Particles & Fields are both essential

Useful: Advanced Quantum Field Theory, String Theory.

References

[1] P. di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer (1996).

[2] K. Gadewzki, *Conformal Field Theory: A Case Study*, `hep-th/9904145`.

[3] E. Witten, *On Holomorphic Factorization of WZW and Coset Models*, Comm. Math. Phys. **144** (1992).

[4] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Remarks on the Canonical Quantization of Chern-Simons-Witten Theory*, Nucl. Phys. **B326** (1989).

109. Spontaneous Scalarization of Neutron Stars in Scalar-Tensor Gravity .
Dr U. Sperhake

General relativity (GR) has passed a plethora of experimental and observational tests, including the most recent strong-field tests made available by the detection of gravitational waves by LIGO. In spite of its tremendous success as the theory of gravity, there remain important open questions that motivate searches for possible modifications of GR [1]. These include the measurements of galactic rotation curves, the cosmic microwave background and the difficulties in reconciling GR with quantum physics.

Observational tests of GR face a considerable challenge. An alternative candidate theory must simultaneously meet two almost contradictory requirements: (i) It must satisfy all the observational tests that have already been performed for GR and (ii) it must lead to *new* effects that would allow for experimental or observational discrimination. The most concrete example of such a smoking gun signature of a viable alternative theory of gravity is the *spontaneous scalarization* of neutron stars in scalar-tensor gravity [2]. The strong nature of this effect has lead to significant constraints of scalar-tensor gravity for the case of a massless scalar obtained from observations of binary pulsars (see the discussions in [1] or [3]). These constraints, however, do not apply for most massive scalar-tensor theories of gravity [4] and may lead to smoking gun signatures in the gravitational wave emission from stellar core collapse [5].

This essay should provide a description of the formulation of this class of scalar-tensor gravity (sometimes referred to as "two-derivative" or "Bergmann-Wagoner" scalar-tensor theory) and of the effect of spontaneous scalarization including an intuitive picture of the effect (the essay may follow here either of the pictures provided in Ref. [2] or Ref. [4]). The essay should further discuss why the observational constraints do not apply for a sufficiently large mass of the scalar field and how this may lead to the smoking gun signatures found in [5]. A possible extension of the essay beyond reviewing the literature consists in the derivation of the expressions for the phase, frequency and amplitude of the dispersed gravitational-wave signal, namely Eqs. (10,11) in Ref. [5].

Relevant Courses

Essential: General Relativity

Useful: Black Holes, Cosmology

References

[1] E. Berti *et al*, Class. Quant. Grav. **32** (2015) 243001, arXiv:1501.07274 [gr-qc].

[2] T. Damour, G. Esposito-Farèse, Phys. Rev. Lett. **70** (1993) 2220.

[3] D. Gerosa, U. Sperhake, C. D. Ott, Class. Quant. Grav. **33** (2016) 135002, arXiv:1602.06952 [gr-qc].

[4] F. M.Ramazanoglu, F. Pretorius, Phys. Ref. D **93** (2016) 064005, arXiv:1601.07475 [gr-qc].

[5] U. Sperhake *et al*, Phys. Rev. Lett. **119** (2017) 201103, arXiv:1708.03651 [gr-qc].

110. Variational Hybrid Quantum-Classical Algorithms ....
Dr S. Strelchuk

Modern quantum algorithms require computational resources which are currently beyond the reach of state of the art implementations. But even minimal quantum resources can be made useful if we use them in conjunction with powerful classical optimization methods. This approach has been exploited in Variational Quantum Eigensolver (VQE) [1]. It makes use of Ritz's variational principle to prepare approximations to the ground state and its energy. In this algorithm, the quantum computer is used to prepare a class of variational 'trial' states which are characterized by a set of parameters. Then, the expectation value of the energy is estimated and used by a classical optimizer to generate a new set of improved parameters which are then used to prepare the next iteration of trial states. The advantage of VQE over purely classical simulation techniques is that it is able to prepare trial states that cannot be generated by efficient classical algorithms.

This essay should discuss the algorithm and its applications [2-3].

Relevant Courses

Part III Quantum Computing (M16) is recommended.

References

[1] McClean, Jarrod R., et al. "The theory of variational hybrid quantum-classical algorithms." New Journal of Physics 18.2 (2016): 023023.

[2] Kandala, Abhinav, et al. "Hardware-efficient variational quantum eigensolver for small molecules and quantum magnets." Nature 549.7671 (2017): 242.

[3] O'Malley, P. J. J., et al. "Scalable quantum simulation of molecular energies." Physical Review X 6.3 (2016): 031007.

111. A Stronger Subadditivity Relation ....
Dr S. Strelchuk

Strong subadditivity of the von Neumann entropy is one of the most fundamental results with numerous applications in Quantum Information Processing [1]. It states that for a tripartite quantum state ρ_{CRB}:

$$S(CB)_\rho + S(RB)_\rho \geq S(CRB)_\rho + S(B)_\rho,$$

where $S(A)_\rho = -\text{Tr}\rho\log\rho$ denotes the von Neumann entropy of the designated system. Alternatively, one may re-write it as $I(C:R|B)_\rho \geq 0$, where $I(C:R|B)_\rho$ is the quantum conditional mutual information evaluated on ρ_{CRB}. In recent years, there were several improvements to this inequality which presented a non-trivial function $g(\rho_{BCR})$, so that $I(B:C|R)_\rho \geq g(\rho_{BCR})$ [2]. One such function is based on the trace distance to the set of separable states [3]. Another refinement connects it with the ability to reconstruct the state from its bipartite reductions: the conditional mutual information is an upper bound on the regularized relative entropy distance between the quantum state and its reconstructed version [4-7].

This essay should discuss these refinements and their applications.

Relevant Courses

Part III Quantum Information Theory (M24) and Quantum Computing (M16) are recommended.

Useful:

The following textbooks may be used for guidance:

a) Nielsen, M. A., Chuang, I. L. (2000). Quantum Computation and Quantum Information. CUP.

b) Sutter, D., (2018). Approximate quantum Markov chains. In Approximate Quantum Markov Chains. Springer, Cham.

References

[1] Lieb, E. H., Ruskai, M. B. (1973). Proof of the strong subadditivity of quantum-mechanical entropy. Journal of Mathematical Physics, 14(12), 1938–1941.

[2] Fawzi, O., Renner, R. (2015). Quantum Conditional Mutual Information and Approximate Markov Chains. Communications in Mathematical Physics, 1-37.

[3] Brandão F. G. S. L., Christandl M., Yard J. (2011). Faithful Squashed Entanglement. Communications in Mathematical Physics 306, 805.

[4] Brandão, F. G. S. L., Harrow, A. W., Oppenheim, J., Strelchuk, S. (2015). Quantum Conditional Mutual Information, Reconstructed States, and State Redistribution. Physical Review Letters, 115(5), 050501.

[5] Sutter, D., Tomamichel, M., Harrow, A. W. (2015). Monotonicity of Relative Entropy via Pinched Petz Recovery Map.

[6] Li, K., Winter, A. (2014). Squashed entanglement, k-extendibility, quantum Markov chains, and recovery maps.

112.   Analysis of Static Monopoles in the (Einstein)-Yang-Mills-Higgs Systems ....
Dr D. M. A. Stuart

The $SU(2)$ Yang-Mills-Higgs equations have monopole solutions whose basic properties are described in the book [2]. There is a solution with generalized radial symmetry for general positive values of the Higgs self-coupling constant λ, and also much larger families of solutions for the case of zero λ (the Bogomolny case), whose existence can be proved analytically by the glueing construction described in Chapter four of [2].

Monopole solutions also exist in the Einstein-Yang-Mills-Higgs system and describe self-gravitating monopoles, and it is possible to analyze their Newtonian (non-relativistic) limit through rigorous analytical means, see [4].

Relevant Courses

Useful: Quantum field theory, advanced quantum field theory, General relativity, Symmetries particles and fields, Distribution theory and applications, Analysis of PDEs, Elliptic PDEs.

References

[1] G. Androulakis and S. Dostoglou, On the stability of monopole solutions. *Nonlinearity* **11** (1998), no. 3, 377-408.

[2] A. Jaffe and C. taubes, *Vortices and Monopoles* Birhauser, Basel (1980).

[3] P. Breitenlohner, P. Forgács and D. Maison, Classification of static, spherically symmetric solutions of the Einstein-Yang-Mills theory with positive cosmological constant, *Comm. Math. Phys.* **261**, no. 3, 569–611 (2006).

[4] T.A. Oliynyk, An existence proof for the gravitating BPS monopole *Ann. Inst. Henri Poincaré* , **7** 199-232 (2006).

113. Higher Form Symmetries in Quantum Field Theories ....
Professor D. Tong

Symmetries are one of the most important concepts in quantum field theory, with results of Noether, Goldstone and 't Hooft ensuring that they have deep implications for the dynamics of the theory. Recently a class of more subtle symmetries in quantum gauge theories has been uncovered. These *higher form symmetries*, also known as *generalised symmetries*, are characterised by having transformation parameters that are p-forms, as opposed to functions.

The purpose of this essay is to describe these symmetries, the way they act on non-local operators such as electric Wilson lines and magnetic 't Hooft lines, their relation to discrete gauge symmetries, and their implication for the quantum dynamics of certain theories.

Relevant Courses

Essential: General Relativity (for differential geometry), Advanced Quantum Field Theory (for the path integral) and the Standard Model (for Goldstone's theorem)

References

[1] Generalised symmetries are discussed in Section 3.6 of my lecture notes on Gauge Theory. These can be downloaded at http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html

[2] The original paper is by D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, "'*Generalized Global Symmetries*," JHEP **1502**, 172 (2015) [arXiv:1412.5148 [hep-th]].

[3] A nice application of these ideas was described in D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, "*Theta, Time Reversal, and Temperature,*" JHEP **1705**, 091 (2017) [arXiv:1703.00501 [hep-th]].

114. Traversable Wormholes ....
Dr A. C. Wall

A wormhole is a topologically nontrivial bridge connecting two different regions of space. The wormhole is called traversable if it is possible for a causal signal to be transmitted from one end to the other. (For example, the maximally extended Schwarzschild solution, a.k.a Einstein-Rosen bridge, is an example of a *non*traversable wormhole, since anyone who jumps in will hit the singularity, instead of making it through to the other side.)

An important theorem of general relativity says that traversable wormholes cannot exist if the stress-energy tensor satisfies certain "energy conditions". However, there are loopholes that permit these energy conditions to be violated. One such loophole is if the two sides of the wormhole are coupled to each other with an interaction; in this case, the wormhole can become traversable due to the quantum effects of ordinary matter fields.

The essay should describe a proposed method for constructing traversable wormholes, explain why it works conceptually, and whether or not it can be used to violate causality. The essay may also explore the relationship between traversable wormholes and other quantum effects. The suggested references [2-5] exploit the particular loophole described above, but you are also permitted to critically examine other articles which propose methods for making traversable wormholes.

Relevant Courses

Essential: Black Holes, Quantum Field Theory

Useful: Gauge/Gravity Duality, Quantum Information Theory, Standard Model

References

[1] J.L. Friedman, K. Schleich, D.M. Witt, "Topological Censorship", Phys.Rev.Lett. 71 (1993), arXiv:gr-qc/9305017.

[2] P. Gao, D. Jafferis, A.C. Wall, "Traversable Wormholes via a Double Trace Deformation", JHEP 1712 (2017) 151, arXiv:1608.05687

[3] J. Maldacena, D. Stanford, Z. Yang "Diving into traversable wormholes", Fortsch. Phys. 65 (2017), 1700034, arXiv:1704.05333

[4] J. Maldacena, A. Milekhin, F. Popov, "Traversable wormholes in four dimensions", arXiv:1807.04726

[5] G.T. Horowitz, D. Marolf, J.E. Santos, D. Wang, "Creating a Traversable Wormhole", Class.Quant.Grav. 36 (2019), 205011, arXiv:1904.02187.

115. The Positive Mass Theorem ....
Dr C. M. Warnick

The positive mass theorem is an important milestone in mathematical relativity. For an asymptotically flat spacetime, a definition of the total mass (or equivalently energy) of the spacetime was given by Arnowitt, Deser and Misner in the '60s. Their definition reads off the mass from the asymptotic behaviour of the Riemannian metric, h, induced on a Cauchy surface, Σ, by the spacetime metric (see the discussion in §11.2 of [1]). A priori, there is no reason to believe that the quantity they define should be positive, and indeed one may find Riemannian $3-$manifolds for which this is not the case. The fact that (Σ, h) embeds into a solution of Einstein's equations implies certain constraints on the metric h (in particular non-negativity of the scalar curvature). In '79 Schoen and Yau showed [2] that these constraints are sufficient to establish positivity of the mass through an ingenious argument involving minimal surfaces. Subsequently, in '81 Witten gave a simpler proof involving spinors. This essay should give an overview of the problem of mass in General Relativity, and discuss in detail at least one of the proofs of the positive mass theorem.

Relevant Courses

Essential: General Relativity

Useful: Analysis of PDE; Differential Geometry; Black Holes

References

[1] General Relativity; Robert Wald, Chicago 1984.

[2] On the proof of the positive mass conjecture in general relativity; Richard Schoen; Shing Tung Yau, Comm. Math. Phys. 65 (1979), no. 1, 45–76

[3] A new proof of the positive energy theorem; Edward Witten; Comm. Math. Phys. (1981) 80: 381.

116. The Running Coupling in Lattice Field Theory ....
Dr M. B. Wingate

In quantum field theories, couplings depend on regularization scheme. The most familiar examples are couplings defined using perturbation theory, wherein divergent loop-integrals are

regulated by a momentum cutoff or dimensional regularization, and some subtraction scheme is imposed.

In some situations, it is advantageous to instead define a coupling based on a "physical" scheme. For example, a coupling constant can be defined in terms of a potential between static charges [1], Wilson loops [2], or a mass gap in a finite volume system [2]. These quantities can be computed nonperturbatively using numerical methods for lattice field theories, thereby giving physical, nonperturbative definitions of the coupling.

A successful essay will touch upon the following points:

1. In asymptotically free theories, where the continuum limit of the lattice theory coincides with the weak coupling limit, couplings defined differently can be related in perturbative expansions. Using one or more examples, discuss how lattice calculations can be used to determine the strong coupling constant of QCD, e.g. the $\overline{\text{MS}}$ coupling at the scale $\mu = M_Z$.

2. The β-function governing the running of the coupling may be computed nonperturbatively, numerically, using a step-scaling procedure [3,4]. Outline the procedure in general and with a well-chosen example.

The essay need not offer a full introduction to lattice methods. It is sufficient to introduce only those notions used later in the essay. It is also not intended that the essay should focus too much on real-world details such as the running of the coupling as the scale is varied through quark masses. To first approximation you may assume QCD is SU(N) Yang-Mills theory without matter.

Relevant Courses

Essential: Quantum Field Theory, Advanced Quantum Field Theory

Useful: Statistical Field Theory

References

[1] G.P. Lepage and P. Mackenzie, Phys. Rev. D **48**, 2250 (1993).

[2] C.T.H. Davies *et al.*, Phys. Rev. D **56**, 2755 (1997).

[3] M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B **359**, 221 (1991).

[4] M. Lüscher, R. Sommer, U. Wolff, and P. Weisz, Nucl. Phys. B **389**, 247 (1993).