University of
Cambridge

Mathematics Tripos

Part III

Combinatorics

Example Sheet I

December 12, 2019

Solutions by
Joshua Snyder

Introduction

These are written solutions to Combinatorics Example Sheet I. Solutions are written based on those seen in examples classes and may contain errors, likely due to the author. Solutions may be incomplete and do not usually include starred questions. These are to be used as a reference for revision **after** examples classes and should never be used beforehand. Doing so will severely impair your ability to learn and study mathematics.

Questions

> **Question** (Question 1)**.** Let $P = (V, <)$ be a finite poset. Recall that a subset $U \subset V$ is a chain if any two elements of U are comparable, and it is an antichain if no two elements of U are comparable. Show that the maximal size of an antichain in P is equal to the minimal number of chains in P that cover V.

Solution. Let $N_1 =$ maximum size of antichain, $N_2 =$minimum number of chains that cover V.

$\underline{N_2 \geq N_1}$ Given $A_1, A_2, ..., A_{N_2}$ minimal number of chains covering V. Any antichain B can contain at most one element from each A_i so $N_1 \geq |B| \geq N_2$.

$\underline{N_1 \geq N_2}$ We prove this by induction on n, the size of the partially ordered set. If P is empty the theorem is vacuously true. Thus, assume P has atleast one element and let a be a maximal element in P which exists since P is finite. By induction, assume $\exists k \; : \; P^{'} := P \; a$ can be covered by k disjoint chains $C_1, .., C_k$ and there is an antichain A_0 of size atleast k. Have $A_0 \cap C_i \neq \emptyset$. Let x_i be the maximal element of C_i belonging to an antichain of length atleast k.

Remark (Claim)**.** Let $A_0 = \{x_1, x_2, ..., x_k\}$, then A is an antichain

Proof of Claim. Let A_i be an antichain of size k that contains x_i, fix $i \neq j$ arbitrarily. Then $A_i \cup C_j \neq \emptyset$. Suppose $y \in A_i \cup C_j$. Then $y \leq x_j$ since x_j is maximal in C_j. Thus $x_i \ngeq x_j$ since $x_i \ngeq y$. Exchanging i, j gives $x_j \ngeq x_i$. ...

Now suppose $a \geq x_i$ for some $1 \leq i \leq k$. Then set

$$K = \{a\} \cup \{z \in C_i : z \leq x_i\}$$

Then by choice of x_i, $P \; K$ does not have an antichain of size k and so by induction $P \setminus K$ can be covered by $k-1$ disjoint chains as $A \; x_i$ is an antichain of size $k-1$ in $P \setminus K$. Thus P can be covered by k disjoint chains.
Else, suppose instead that $a \ngeq x_i$ for all $1 \leq i \leq k$. The $A \cup \{a\}$ is an antichain of size $k+1$ in P and P can be covered by $k+1$ chains $\{a\}, C_1, C_2, ..., C_k$.

$\square$

Remark. This proof is tedious and a very difficult Question 1. The ideas are, however, important and should be understood.

> **Question** (Question 2)**.** Let $(V, <)$ be a finite ranked poset with non-empty level sets $V_0 V_1, ..., V_n$. Suppose for $0 < i \leq n$ every $v \in V_i$ dominates exactly $d_i \geq 1$ elements of V_{i-1}, for $0 \leq i < n$ every $v \in V_i$ is dominated by exactly $e_i \geq 1$ elements of V_{i+1}, and the partial order on $V = \cup_0^n V_i$ is induced by these relations.
> Show that if $U \subset V$ is an antichain then
> $$\sum_0^n \frac{|U \cap V_i|}{|V_i|} \leq 1$$

Idea. Count number of chains of maximal length in two ways

Solution. Must have $|V_i|e_i = |V_{i+1}|d_{i+1}$ for all $0 \leq i < n$. Thus there are $|V_0|e_0 e_1 ... e_{n-1} = d_1 ... d_k |V_k| e_k ... e_{n-1}$ chains of maximal length in V.
For each maximal chain C we have $|C \cap U| \leq 1$ as U is an antichain. Every element in V_k is contained in exactly $(d_k d_{k-1} ... d_1)(e_k ... e_{n-1})$ maximal chains. Putting both of these together gives:

$$\sum_0^n |U \cap V_k|(d_k ... d_1)(e_k ... e_{n-1}) = \#\text{maximal chains} = |V_0|e_0 ... e_{n-1}$$

which upon dividing the LHS by the RHS yields the required result.

$\square$

Remark. Counting arguments like these are popular. The counting itself is not difficult, but knowing what to count often is.

> **Question** (Question 3)**.** Let $\mathcal{F} \subset \mathbb{P}(n)$ be a Sperner family i.e. let $\mathcal{F}$ be such that $A \not\subset B$ whenever $A, B \in \mathcal{F}$, $A \neq B$. Show that
> $$\sum_{k=0}^n \frac{f_k}{\binom{n}{k}} \leq 1$$
> where f_k is the number of k-sets in $\mathcal{F}$.

Solution 1.

Idea. Use the Local LYM inequality repeatedly.

Let $\mathcal{A} \subset X^{(r)}$, by the Local LYM inequality we have that

$$\frac{|\partial A|}{\binom{n}{r-1}} \geq \frac{|A|}{\binom{n}{r}}$$

i.e that the shadow of a set has higher density then the set itself.
Let $\mathcal{F}_r = \mathcal{F} \cap X^{(r)}$ so that $|\mathcal{F}_r| = f_r$. Since $|\mathcal{F}_n|/\binom{n}{n} \leq 1$ we have that

$$1 \geq \frac{|\partial \mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{F}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} \geq \frac{|\partial \mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}}$$

Where the second equality holds since the two sets are disjoint, else the family would not be Sperner. Repeating this gives the desired result. □

Solution 2.

Idea. Use the result from Question 2

Let $V = \mathbb{P}(n)$ as in Question 2, ranked by inclusion. Then every set of size k contains k sets of size $k-1$ and is contained in $k+1$ sets of size $k+1$. Thus from Question 2, $|V_k| = \binom{n}{k}$ and $|U \cap V_k| = f_k$ and so the result follows. □

Solution 3.

Idea. Pick a chain uniformly at random and use probability

Pick a chain uniformly at random in Q_n. Take $A \in X^{(r)}$. Then the probability that C coincides with A is

$$\mathbb{P}(C \text{ meets } A) = \frac{1}{\binom{n}{r}} \tag{1}$$

$$\implies \mathbb{P}(C \text{ meets } \mathcal{F}_k) = \frac{f_k}{\binom{n}{k}} \tag{2}$$

$$\implies \mathbb{P}(C \text{ meets } \mathcal{F}) = \sum_{k=o}^{n} \frac{f_k}{\binom{n}{k}} \tag{3}$$

from which the result follows since all probabilities are bounded above by 1. □

Remark. The third proof exhibits a useful idea. Picking at random and using probability to prove a result is a popular method in combinatorics and leads to the field known as Probabalistic Combinatorics.

> **Question** (Question 4)**.** Let $2 \leq 2r < n$ and let $\mathcal{F} = \mathcal{F}_r \cup \mathcal{F}_{\backslash - \nabla} \subset \mathbb{P}(n)$ be a Sperner family where $\mathcal{F}_r \subset X^{(r)}$, $\mathcal{F}_{n-r} \subset X^{(n-r)}$ and $|\mathcal{F}_r| = |\mathcal{F}_{n-r}| = m$. At most how large is m?

Solution.

Idea. Use the fact that $\partial^{n-2r}\mathcal{F}_{n-r}$ and $\mathcal{F}_r$ are disjoint.

We get that $|\partial^{n-2r}\mathcal{F}_{n-r}| + |\mathcal{F}_r| \leq \binom{n}{r}$

$$1 \geq$$

□

Remark. I cannot go from here to get more then $m \leq \frac{1}{2}\binom{n}{r}$ which one can achieve directly from the LYM inequality. The reason for this is the only proof I can think of is essentially a proof of LYM for this particular case. See a bubble down Kruskal-Katona proof for help. The answer should be $\binom{n-1}{r-1}$

> **Question** (Question 5)**.** For $2 \leq r \leq \frac{n}{2}$, let $A \subset X(r)$ be an intersecting family. (Thus $A \cap B \neq \emptyset$, whenever $A, B \in \mathcal{A}$.) Deduce from the Kruskal-Katona Theorem that $\mathcal{A} \leq \binom{n-1}{r-1}$
> What is the maximal size of an intersecting family $\mathcal{A} \subset \mathbb{P}(X)$? What about

in the case $A \subset X\check{(}\leq r)$

Solution. For $A, B \in \mathcal{A}$, have $A \cap B \neq \emptyset$, i.e. $A \not\subseteq B^c$. Writing

$$\overline{\mathcal{A}} \coloneqq \{A^c \mid A \in \mathcal{A}\} \subset X^{(n-r)},$$

this says that $\partial^{n-2r}\overline{\mathcal{A}}$ is disjoint from $\mathcal{A}$. Now suppose that $|\mathcal{A}| > \binom{n-1}{r-1}$. Then $|\overline{\mathcal{A}}| > \binom{n-1}{r-1} = \binom{n-1}{n-r}$, so by repeated application of the LYM inequality we have $|\partial^{n-2r}\overline{\mathcal{A}}| \geq \binom{n-1}{r}$. But

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$$

i.e.

$$|\partial^{n-2r}\overline{\mathcal{A}}| + |\mathcal{A}| > |X^{(r)}|.$$

a contradiction.

For $\mathcal{A} \subset \mathbb{P}(X)$ an upper bound is $2\hat{}n - 1$ since a set and it's complemement cannot both be in a Sperner family. To achieve this maximal bound, we can extend any Sperner family, but a trivial example is $\mathcal{A} = \{A : 1 \in A\}$.

For $A \subset X\check{(}\leq r)$, applying $|A \cap X\check{(}s)| \leq \binom{n-1}{s-1}$ for all $1 \leq s \leq r$ gives an upper bound of $\sum_{s=1}\hat{r}\binom{n-1}{s-1}$. This can be achieved with $\mathcal{A} = \{A : |A| \leq r$ and $1 \in A\}$ $\square$

Remark. For the first part, the numbers *had* to work as we get equality for $\mathcal{A} = \{A \in X^{(r)} \mid 1 \in A\}$.