Mich. 2019 **RAMSEY THEORY – EXAMPLES 2** IBL

1. For which $a, b \in \mathbb{Q}$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & a & 1 & -1 & b \end{pmatrix}$$

partition regular?

Used PR ?

2. Deduce from Ramsey's theorem that whenever $\mathbb{N}$ is finitely coloured there exist x, y, z with $\{x, y, z, x+y, y+z, x+y+z\}$ monochromatic.

3. Verify directly that the matrix corresponding to the Finite Sums theorem has the columns property.

4. A rational matrix A is called *partition regular over* $\mathbb{Z}$ (resp. $\mathbb{Q}$) if whenever $\mathbb{Z} - \{0\}$ (resp. $\mathbb{Q} - \{0\}$) is finitely coloured there is a monochromatic vector x with $Ax = 0$. Show that A is partition regular over $\mathbb{Z}$ if and only if it is partition regular over $\mathbb{N}$. If A is partition regular over $\mathbb{Q}$, must it be partition regular over $\mathbb{N}$?

5. For each $k \in \mathbb{N}$, construct a rational matrix A such that A is not partition regular but, whenever $\mathbb{N}$ is k-coloured, there is a monochromatic vector x with $Ax = 0$.

6. For each $m \in \mathbb{N}$, prove that whenever the collection of finite non-empty subsets of $\mathbb{N}$ is finitely coloured there exist disjoint $F_1, \ldots, F_m$ with $\left\{\bigcup_{i \in I} F_i : \emptyset \neq I \subset [m]\right\}$ monochromatic.

7. Do the partition regular subsets of $\mathbb{N}$ form an ultrafilter?

8. Show that the sequence of principal ultrafilters $1, 2, \ldots$ in $\beta\mathbb{N}$ has no convergent subsequence. Is the topology on $\beta\mathbb{N}$ induced by a metric?

9. Prove that $\beta\mathbb{N} - \mathbb{N}$ is not separable (meaning: it has no countable dense subset).

10. Show that if $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ are distinct ultrafilters on $\mathbb{N}$ then we can find $A \in \mathcal{U}$ such that $A \notin \mathcal{U}_i$ for all i. Show also that if $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \ldots$ are distinct ultrafilters on $\mathbb{N}$ then there need not exist $A \in \mathcal{U}$ such that $A \notin \mathcal{U}_i$ for all i. What happens if we insist that each $\mathcal{U}_i$ is non-principal?

+11. How many ultrafilters are there on $\mathbb{N}$?

~~1.~~ For which $a, b \in \mathbb{Q}$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & a & 1 & -1 & b \end{pmatrix}$$

partition regular?

PR $\Leftrightarrow$ CP by Rado. To find B_1 for which $\sum_{i \in B_1} c^{(i)} = 0$ must have $B_1 = \{2,4,5\}$ or $B_1 = \{2,3,4,5\}$ by looking at the first two rows.

$B_1 = \{2,4,5\}$ For $\sum_{i \in B_1} c^{(i)} = 0$ must have $a+b=1 \Rightarrow b = 1-a$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & a & 1 & -1 & 1-a \end{pmatrix}$$

$B_2 = \{3\}$ and $B_3 = \{1\}$ works

$B_1 = \{2,3,4,5\}$ Bottom row is $a+b=0 \Rightarrow b=-a$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & a & 1 & -1 & -a \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{span} B_1 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_4 - \lambda_3 \\ \lambda_3 - \lambda_1 \\ a\lambda_1 + \lambda_2 - \lambda_3 - a\lambda_4 \end{pmatrix}$$

$$\Rightarrow \lambda_4 = \lambda_3 + 1 = \lambda_1 + 2$$

$$\cancel{a\lambda_1} + \lambda_2 - \lambda_1 - 1 - \cancel{a\lambda_1} - 2a = 0 \Rightarrow \lambda_2 = 1 + \lambda_1 + 2a$$

i.e. there is a solution. Wrong

Thus the matrix is PR iff $a+b \in \{0,1\}$.

Q1

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & a & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & a \\ 1 & a & 1 & -1 & b \end{array}$$

1. Show have to use column 2 in B_1
2. Show have to have $B_1 = \{2,3,4,5\}$ or $\{2,4,5\}$
3. $B_1 = \{2,3,4,5\}$ we have $\mathbb{R}^3$ in span B_1 so done so long as $a+b=0$ (to ensure $\sum_{i \in B_1} c^{(i)} = 0$)

$B_1 = \{2,4,5\}$ Just need to show $c_3 + c_1 \in$ span B_1 (other cases are implied by this).

Since $\sum_{B_1} c^{(i)} = 0$ can consider only columns 2,5.

Should get $\boxed{a+b=1,\ b-a=2}$

Deduce from Ramsey's theorem that whenever $\mathbb{N}$ is finitely coloured there exist x, y, z with $\{x, y, z, x+y, y+z, x+y+z\}$ monochromatic.

Using Rado:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns property e.g. with $B_1 = \{2, 4, 5, 6\}$, $B_2 = \{1, 3\}$ so by Rado WINFC $\exists$ monochromatic solution.

Without Rado: Let $c : \mathbb{N} \to [k]$ be an arbitrary finite colouring of $\mathbb{N}$ and induce a colouring $c' : \mathbb{N}^{(2)} \to [k]$ with $c'(ab) = c(b-a)$.

By Ramsey $\exists$ mono 4-set $\{a, b, c, d\}$.

Let $x = b-a$, $y = c-b$, $z = d-c$ then $x+y = c-a$, $y+z = d-b$ and $x+y+z = c-a$ which are all monochromatic.

Note: This works for higher sums, but is always falling short of the finite sums theorem.

3. Verify directly that the matrix corresponding to the Finite Sums theorem has the columns property.

At time 1, take x and the corresponding -1's in the places where x has a 1.

At time 2, take any new entry's in y and the -1's corresponding to these new entries.

$$\begin{array}{ccc|cccc} x & y & z & & & & \\ 1 & 1 & & -1 & & & \\ 1 & & 1 & & -1 & & \\ & \overset{\text{New entry}}{1} & 1 & & & -1 & \\ 1 & 1 & 1 & & & & -1 \\ (1) & (2) & & & & & \end{array}$$

<u>Q4</u> PR/$\mathbb{N}$ $\Leftrightarrow$ PR/$\mathbb{Z}$ $\Leftrightarrow$ PR/$\mathbb{Q}$ $\Leftrightarrow$ PR/$\mathbb{R}$

! this holds

What about ∞ families?

PR/$\mathbb{N}$ $\Leftrightarrow$ PR/$\mathbb{Z}$ $\underset{\not\Leftarrow}{\Rightarrow}$ PR/$\mathbb{Q}$ $\underset{\not\Leftarrow}{\Rightarrow}$ PR/$\mathbb{R}$

i.e. one such that $\nexists x$ mono : $Ax = 0$

Take a bad colouring of $\mathbb{N}$: Want to manufacture out of a colouring of $\mathbb{N}$ a colouring of $\mathbb{Z}$ that is bad.

Given colouring of $\mathbb{N}$

Reflect $\mathbb{N}$ to $-\mathbb{N}$ and create duplicate colours for each colour to give a bad colouring of $\mathbb{Z}$

Order $\mathbb{Q}$ as $\{q_1, q_2, \ldots\}$

Whenever have k-colouring of $\mathbb{Q}$.

<u>CLAIM</u>: $\exists$ finite part of $\mathbb{Q}$ $q_1, \ldots, q_n$ s.t. whenever $q_1, \ldots, q_n$ k-coloured $\exists$ mono solution to $Ax = 0$

Compactness argument. This is the usual argument. Once you have said k-coloured, solving in set $\Leftrightarrow$ solving in some finite part of the set.

4. A rational matrix A is called *partition regular over* $\mathbb{Z}$ (resp. $\mathbb{Q}$) if whenever $\mathbb{Z} - \{0\}$ (resp. $\mathbb{Q} - \{0\}$) is finitely coloured there is a monochromatic vector x with $Ax = 0$. Show that A is partition regular over $\mathbb{Z}$ if and only if it is partition regular over $\mathbb{N}$. If A is partition regular over $\mathbb{Q}$, must it be partition regular over $\mathbb{N}$?

Clearly A PR over $\mathbb{N} \Rightarrow A$ PR over $\mathbb{Z} \Rightarrow A$ PR over $\mathbb{Q}$ since $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$

A PR over $\mathbb{Z} \Rightarrow A$ PR over $\mathbb{N}$ A is $m \times n$

Fix an arbitrary c and let $x \in \mathbb{Z} - \{0\}$ be such that $c|_x$ is constant. let $y_i = |x_i|$ and $I = \{i \in [m] : x_i < 0\}$.

Then $AJy = 0$ where $y \in \mathbb{N}$ and $J_{ii} = 1$ if $i \in I$ and $J_{ij} = 0$ otherwise.

Thus AJ is partition regular over $\mathbb{N}$ and so AJ satisfies the columns property by Rado.

AIM: AJ satisfies columns property $\Rightarrow$ A does

Note: Permuting rows does not affect satisfying columns property so WLOG $J = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$ for $0 \leq r \leq s \leq n$ and $r + s = n$.

However permuting rows is unnecessary. let $A'_{(1)} \dots A'_{(n)}$ be the columns of AJ and $A_{(1)}, \dots, A_{(n)}$ be the columns of A then

$$\underset{2\times 3}{\begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 0 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & -1 & 0 \end{pmatrix}$$

$B_1 = \{1, 2, 3\} \Rightarrow CP$ $\qquad \sum_{\{123\}} A'_{(i)} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$Ax \Rightarrow$

5. For each $k \in \mathbb{N}$, construct a rational matrix A such that A is not partition regular but, whenever $\mathbb{N}$ is k-coloured, there is a monochromatic vector x with $Ax = 0$.

A not PR $\Rightarrow$ A doesn't satisfy the column's property

Want $Ax = 0$ mono for every k-colouring but $\exists$ a $k+1$ colouring for which there is no x with $Ax = 0$.

$k=1$: $A = ($

This was hard

10 colours look at $1, 2, 4, \dots, 2^{11}$ then some two have the same colour

Try $n \quad \underbrace{-1 \quad -1 \quad \dots \quad -1}_{d}$ with $d < n$ so nothing sums to 0

As long as n = sum of d 1's and 2's i.e. $d < n < 2d$ then can solve this in $\{1, 2\}$.

n = sum of d 1s and 4s $\quad n = d +$ multiple of 3

n = sum of d 1s and 8s $\quad n = d +$ multiple of 7

either 10 or 11

keep going to n = sum of d 1s and 2^{11}s

Way to make it work with 10 colours is in set X of 11 points there is always a solution.

6. For each $m \in \mathbb{N}$, prove that whenever the collection of finite non-empty subsets of $\mathbb{N}$ is finitely coloured there exist disjoint $F_1, \ldots, F_m$ with $\{\bigcup_{i \in I} F_i : \emptyset \neq I \subset [m]\}$ monochromatic.

By VdW can find ∞ subsets $M_1 \supset \ldots \supset M_N$ for N suff. large such that $M = M_N$ satisfies $M^{(a)}, M^{(a+d)}, \ldots, M^{(a+(m-1)d)}$ are all monochromatic and WLOG RED.

where ind. hyp is that $\exists\ F_1, \ldots, F_{m-1}$ all of same size.

Now by inductive argument, since M is ∞, whenever M is finitely coloured there exists disjoint $F_1, \ldots, F_{m-1}$ disjoint subsets of $M^{(d)}$ all of the same size such that $\{\bigcup_{i \in I} F_i : \emptyset \neq I \subset M\}$ is monochromatic.

Could use Finite Sums theorem instead of Van der Waerden and this would have worked!

By applying Ramsey many times can insist have M such that

$M^{(r)}$ is monochromatic for each $1 \leq r \leq k$

Then by Finite Sums can find $x_1, x_2, \ldots, x_m$ s.t. $FS(x_1, \ldots, x_m)$ is monochromatic.

Note: Could also do this proof in reverse.

7. Do the partition regular subsets of $\mathbb{N}$ form an ultrafilter?

Note: If this were true it would be an explicit ultrafilter which does not exist (was said in notes)

Build two large sets

Take first PR matrix A_1 s.t. $Ax = 0$. Take s_1 as all elements in $\underline{x}$

$\underline{A}$	$\underline{B}$
s_1	λs_1 ← large
$\lambda' s_2$	$\lambda'' s_2$
$\vdots$	

Continue enumerating all PR sets then get

$\underline{A}, \underline{B}$ disjoint and done.

Alternative: Take colouring • • • • • • • • • • • • • • • • • •

Take A as the red and B as the blue.

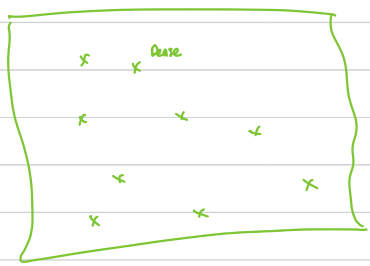
Then $2A \subseteq B$ and $2B \subseteq A$

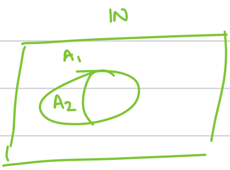
Q8 $\forall A \in \mathcal{U}$. All primes eventually belong to A where C_A is open neighbourhood of $\mathcal{U}$.

Not induced by a metric since the principal ultrafilters are dense.

Q9 $\beta\mathbb{N} - \mathbb{N}$

Given sequence of ultrafilters $\mathcal{U}_1, \mathcal{U}_2, \dots$ want to find an open set A they do not intersect.





Pick ∞ $A_1 : A_1 \notin \mathcal{U}_1$

∞ $A_2 \subset A_1 : A_2 \notin \mathcal{U}_2$

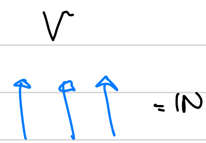
$\vdots$

Then pick $x_1 \in A_1, x_2 \in A_2, \dots$ Then $\{x_1, x_2, \dots\} \notin \mathcal{U}_1$ since it is a subset of A_1 then not in $\mathcal{U}_k$ for any k since only $\{x_1, \dots, x_{k-1}\}$ could be in it and any finite amount of the x_i doesnt affect it.

Q10 Find sets $A_1 \in \mathcal{U}$ $\quad A_2 \in \mathcal{U}$ $\quad \dots \quad A_n \in \mathcal{U}$ $\quad$ then take $A_1 \cap \dots \cap A_n$

$\qquad\qquad A_1 \notin \mathcal{U}_1$ $\quad A_2 \notin \mathcal{U}_1$ $\qquad\qquad A_n \notin \mathcal{U}$

For infinite example take $\mathcal{U}_1 = \tilde{1}$, $\mathcal{U}_2 = \tilde{2}, \dots$ then $\forall n: n \notin A$ so $A \in \mathcal{U}$



What about without principal ultrafilters? Fix some ultrafilter $\mathcal{V}$

Let $\mathcal{U}_1$ = All A s.t. $A \cap C_1$ is BIG $\leftarrow \{y : (1,y) \in A\} \in \mathcal{V}$

$\qquad \mathcal{U}_2$ = All A s.t. $A \cap C_2$ is BIG

$\qquad\vdots$

Let $\mathcal{U}$ = All A s.t. $\{n : A \cap C_n \text{ is BIG}\}$ is BIG.