CHAPTER 1

Elementary Techniques

Review of asymptotic notation. We write $f(x) = O(g(x))$ if there exists some constant $C > 0$ such that $|f(x)| \le C\,|g(x)|$ for all sufficiently large x. We will also use the Vinogradov notation $f \ll g$ to denote the same thing (so that $f = O(g)$ and $f \ll g$ are equivalent).

We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We write $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Observe that

$$f \sim g \text{ if and only if } f = (1 + o(1))g.$$

1. Arithmetic functions

An arithmetic function is simply a function on the natural numbers[1], $f : \mathbb{N} \to \mathbb{R}$. An arithmetic function is multiplicative if

$$f(nm) = f(n)f(m) \text{ whenever } (n, m) = 1,$$

and is completely multiplicative if $f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{N}$.

An important operation on the space of arithmetic functions is that of multiplicative convolution:

$$f \star g(n) = \sum_{ab=n} f(a)g(b).$$

This is since we are looking primarily at multiplicative functions.

If f and g are both multiplicative functions, then so too is $f \star g$. The most obvious arithmetic function is the constant function:

$$\mathbf{1}(n) = 1.$$

We recall the definition of the Möbius function,

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ where } p_i \text{ are distinct primes and} \\ 0 & \text{otherwise (i.e. if } n \text{ is divisible by a square).} \end{cases}$$

$\delta(n) = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$

A fundamental relationship is that of Möbius inversion, which says that the Möbius function acts as an inverse to multiplicative convolution:

$$\mathbf{1} \star f = g \text{ if and only if } \mu \star g = f.$$

<u>Properties</u>
- $*$ is commutative
- $*$ is associative
- $*$ has an inverse
- f, g multiplicative $\Rightarrow f*g$ multiplicative
- δ is the identity since $\delta * f = f$

A great deal of analytic number theory is concerned with a deep study of the distribution of the prime numbers. For this the 'correct' way to count primes is not, as one might expect, the indicator function

$$1_{\mathbb{P}}(n) = \begin{cases} 1 & \text{if } n \text{ is prime, and} \\ 0 & \text{otherwise,} \end{cases}$$

[1]For the purposes of this course, 0 is not a natural number.

$1 * \mu = \delta$

<u>Proof:</u> <u>$n=1$</u> clear.
- $\mu, \mathbb{1}$ multiplicative $\Rightarrow \mathbb{1} * \mu$ multiplicative.
- Show that $\mathbb{1} * \mu(p^k) = 0$. Done.

but instead the von Mangoldt function, which firstly also counts prime powers p^k, but also counts them not with weight 1, but with weight $\log p$ instead:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This is a clever idea. Since prime powers are relatively scarce, including them does not affect the analysis much.

The main reason that this function is much easier to work with than $1_{\mathbb{P}}$ directly, is the following identity.

Lemma 1.

$$\mathbf{1} \star \Lambda(n) = \log n \text{ and } \log \star \mu(n) = \Lambda(n).$$

Proof. The second identity follows from the first by Möbius inversion. To establish the first, if we let $n = p_1^{k_1} \cdots p_r^{k_r}$, then

Note that log is not multiplicative, so have to do all the work & check for $n = p_1^{k_1} \cdots p_r^{k_r}$ instead of just for prime powers.

$$\begin{aligned} \mathbf{1} \star \Lambda(n) &= \sum_{i=1}^{r} \sum_{j=1}^{k_i} \log p_i \\ &= \sum_{i=1}^{r} \log p_i^{k_i} \\ &= \log n. \end{aligned}$$

□

2. Summation

A major theme of analytic number theory is understanding the basic arithmetic functions, particularly how large they are on average, which means understanding $\sum_{n \le x} f(n)$. For example, if f is the indicator function of primes, then this summatory function is precisely the prime counting function $\pi(n)$.

We say that f has average order g if

$$\sum_{n \le x} f(n) \sim x g(x).$$

One of the most useful tools in dealing with summations is partial summation, which is a discrete analogue of integrating by parts.

Theorem 1 (Partial summation)**.** *If a_n is any sequence of complex numbers and $f : \mathbb{R}_+ \to \mathbb{R}$ is such that f' is continuous then*

$$\sum_{1 \le n \le x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, \mathrm{d}t,$$

where $A(x) = \sum_{1 \le n \le x} a_n$.

Idea: If we understand $A(x) = \sum_{1 \le n \le x} a_n$ then we understand the weighted sum for a suitably behaved x.

Proof. Let $N = \lfloor x \rfloor$. Using $a_n = A(n) - A(n-1)$

$$\begin{aligned} \sum_{1 \le n \le N} a_n f(n) &= \sum_{n=1}^{N} f(n)(A(n) - A(n-1)) \\ &= f(N)A(N) - \sum_{n=1}^{N-1} A(n)(f(n+1) - f(n)). \end{aligned}$$

We now observe that

$$\int_n^{n+1} f'(x)\,\mathrm{d}x = f(n+1) - f(n),$$

This lets us take the jump from the continuous world to the discrete world.

and so, since $A(x)$ is constant for $x \in [n, n+1)$,

$$\sum_{1\le n\le N} a_n f(n) = f(N)A(N) - \sum_{n=1}^{N-1} \int_n^{n+1} A(x)f'(x)\,\mathrm{d}x,$$

and the result follows since if $N \le x < N+1$ then

$$A(x)f(x) = A(N)f(x) = A(N)\left(f(N) + \int_N^x f'(x)\,\mathrm{d}x\right).$$

□

This is extremely useful even when the coefficients a_n are identically 1, when $A(x) = \lfloor x \rfloor = x + O(1)$.

Lemma 2.

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + O(1/x),$$

where $\gamma = 0.577\cdots$ is a constant, known as Euler's constant.

Proof. By partial summation

$\{t\} = t - \lfloor t \rfloor$

$$\begin{aligned}\sum_{n\le x} \frac{1}{n} &= \underbrace{\frac{\lfloor x \rfloor}{x}}_{=1+O(\frac{1}{x})} + \int_1^x \frac{\lfloor t \rfloor}{t^2}\,\mathrm{d}t \\ &= 1 + \underbrace{\int_1^x \frac{1}{t}\,\mathrm{d}t}_{=\log x} + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t - \int_x^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t + O(1/x) \\ &= \log x + \left(1 + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t\right) + O(1/x).\end{aligned}$$

It remains to note that the second term is a constant, since the integral converges. As $\{t\} \le 1$ $\forall t$

□

It is remarkable how little we understand about Euler's constant – it is not even known whether it is irrational or not.

γ measures the difference between continuous and discrete.

Lemma 3.

$$\sum_{1\le n\le x} \log n = x\log x - x + O(\log x).$$

Proof. By partial summation

$$\begin{aligned}\sum_{n\le x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor t \rfloor}{t}\,\mathrm{d}t \\ &= x\log x - x + O(\log x).\end{aligned}$$

□

How to do (most) of ANT:
1. Write as a multiple sum/integral
2. Change the order of summation

Note: $f(1) = 1$ for any multiplicative function f

3. Divisor function

We now turn our attention to number theory proper, and examine one of those most important arithmetic functions: the divisor function[2]

$$\tau(n) = \mathbf{1} \star \mathbf{1}(n) = \sum_{ab=n} 1 = \sum_{d|n} 1.$$

We first find its average order.

Theorem 2.

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

In particular, the average order of $\tau(n)$ is $\log n$.

Proof. A first attempt might go as follows:

$$\sum_{n \le x} \tau(n) = \sum_{ab \le x} 1 \qquad = \sum_{n\le x}\sum_{ab=n} 1$$

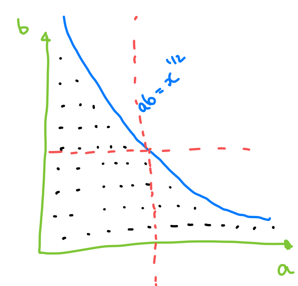
$$= \sum_{a \le x} \left\lfloor \frac{x}{a} \right\rfloor \qquad = \sum_{a\le x}\sum_{b\le x/a} 1$$

$$= x \sum_{a \le x} \frac{1}{a} + O(x)$$

$$= x \log x + \gamma x + O(x).$$

The problem is that the second term γx is lost in the error term $O(x)$. To improve the error term we use what is known as the hyperbola method, which is the observation that when summing over pairs (a, b) such that $ab \le x$ we can express this as the sum over pairs where $a \le x^{1/2}$ and where $b \le x^{1/2}$, and then subtract the contribution where $\max(a, b) \le x^{1/2}$.

since $ab = x \Rightarrow a \le x^{1/2}$ or $b \le x^{1/2}$ but $\max(a,b) \ge x^{1/2}$



Here we are applying the same proof but the Hyperbola method is used to pin down the error term

$$\sum_{ab \le x} 1 = \underbrace{\sum_{a \le x^{1/2}} \left\lfloor \frac{x}{a} \right\rfloor}_{} + \underbrace{\sum_{b \le x^{1/2}} \left\lfloor \frac{x}{b} \right\rfloor}_{} - \sum_{a,b \le x^{1/2}} 1$$

These sums are equal.

$$= 2x \sum_{a \le x^{1/2}} \frac{1}{a} - \lfloor x^{1/2} \rfloor^2 + O(x^{1/2})$$

$$= x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

□

It is a deep and difficult problem to improve the error term here – the truth is probably $O(x^{1/4+\epsilon})$, but this is an open problem, and the best known is $O(x^{0.3149\cdots})$.

We have just shown that the 'average' number of divisors of n is $\log n$. The worst case behaviour can differ dramatically from this average behaviour, however.

Theorem 3. *For any $n \ge 1$,*

$$\tau(n) \le n^{O(1/\log\log n)}.$$

In particular, for any $\epsilon > 0$, $\tau(n) = O_\epsilon(n^\epsilon)$.

[2]Alternative notation used in some places is $d(n)$ or $\sigma_0(n)$.

CHAPTER 2

Dirichlet series and the Riemann zeta function

We will now begin to harness the power of complex analysis for number theory. The main object of study will the Riemann zeta function. Before we explore the applications to number theory, we will spend some time proving various essential facts about this function.

In the rest of the course, we will use (as is traditional for this topic) the letter s to denote a complex variable, and σ and t to denote its real and imaginary parts respectively, so that $s = \sigma + it$. Before we begin, it's worth pausing to explicitly point out what we mean by n^s, where n is a natural number and $s \in \mathbb{C}$. By definition this is

$$n^s = e^{s \log n} = n^\sigma e^{it \log n}.$$

It is easy to check the multiplicative property, that $(nm)^s = n^s m^s$.

A Dirichlet series is an infinite series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for some coefficients $a_n \in \mathbb{C}$. If we denote the coefficients a_n by an arithmetic function $f(n)$ then we may write $F_f(s)$ to denote this dependence.

Lemma 9. *For any sequence a_n there is an abscissa of convergence σ_c such that $F(s)$ converges for all s with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$. If $\sigma > \sigma_c$ then there is a neighbourhood of s in which $F(s)$ converges uniformly. In particular, $\alpha(s)$ is holomorphic at s.*

Proof. It suffices to show that if $F(s)$ converges at $s = s_0$ and we take some s with $\sigma > \sigma_0$ then F converges uniformly in some neighbourhood of s. The lemma then follows by taking $\sigma_c = \inf\{\sigma : F(s) \text{ converges}\}$.

Suppose that $F(s)$ converges at $s = s_0$. If we let $R(u) = \sum_{n>u} a_n n^{-s_0}$ then by partial summation, for any s,

$$\sum_{M<n\leq N} a_n n^{-s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s)\int_M^N R(u)u^{s_0-s-1}\, \mathrm{d}u.$$

If $|R(u)| \leq \epsilon$ for all $u \geq M$, and if $\sigma > \sigma_0$, then it follows that

$$\left| \sum_{M<n\leq N} a_n n^{-s} \right| \leq 2\epsilon + \epsilon\, |s - s_0| \int_M^\infty t^{\sigma_0-\sigma-1}\, \mathrm{d}t \leq \left(2 + \frac{|s-s_0|}{\sigma - \sigma_0}\right)\epsilon.$$

There is some neighbourhood of s in which $|s - s_0| \ll \sigma - \sigma_0$, and hence by Cauchy's principle the series converges uniformly in this neighbourhood of s. $\square$

Lemma 10. *If $\sum a_n n^{-s} = \sum b_n n^{-s}$ for all s in some half-plane $\sigma > \sigma_0$ (where both series converge) then $a_n = b_n$ for all n.*

Proof. It suffices to show that if $\sum c_n n^{-s} = 0$ for all s with $\sigma > \sigma_0$ then $c_n = 0$ for all n. Suppose that $c_n = 0$ for all $n < N$. We can write

$$c_N = -\sum_{n>N} c_n (n/N)^{-\sigma}.$$

Since the sum here is convergent, the summands tend to 0, and hence $c_n \ll n^{\sigma_0}$. It follows that that this sum is absolutely convergent for $\sigma > \sigma_0 + 1$. Since each term tends to 0 as $\sigma \to \infty$, and the series is absolutely convergent, the right-hand side tends to 0, and hence $c_N = 0$. □

Lemma 11. *If $F_f(s)$ and $F_g(s)$ are two Dirichlet series, both absolutely convergent at s, then*

$$\sum_{n=1}^{\infty} f \star g(n) n^{-s}$$

is absolutely convergent and equals $\alpha(s)\beta(s)$.

Proof. We simply multiply out the product of two series,

$$\left(\sum_n a\frac{a_n}{n^s}\right)\left(\sum_m \frac{b_m}{m^s}\right) = \sum_{n,m} \frac{a_n b_m}{(nm)^s} = \sum_k \left(\sum_{nm=k} a_n b_m\right) k^{-s},$$

which is justified since both series are absolutely convergent. □

We now define the Riemann zeta function in the half-plane $\sigma > 1$ by

$$\zeta(s) = \sum_n \frac{1}{n^s}.$$

Observe that this series diverges at $s = 1$, and the series actually converges absolutely for $\sigma > 1$. By the above, $\zeta(s)$ defines a holomorphic function in this half-plane. For our applications, we need to extend this definition to be able to talk about $\zeta(s)$ for $\sigma > 0$.

Lemma 12. *For $\sigma > 1$,*

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Proof. By partial summation, for any x,

$$\sum_{1\le n\le x} n^{-s} = \frac{\lfloor x\rfloor}{x^s} + s\int_1^x \frac{\lfloor t\rfloor}{t^{s+1}}\,\mathrm{d}t.$$

The integral here is

$$s\int_1^x t^{-s}\,\mathrm{d}t - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t = \frac{s}{s-1} - \frac{s}{s-1}x^{1-s} - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Since $\sigma > 1$, if we take the limit as $x \to \infty$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t,$$

noting that the integral converges. □

The integral here is convergent for any $\sigma > 0$, and therefore the right hand side defines an analytic function for $\sigma > 0$, aside from a simple pole at $s = 1$ with residue 1. We have therefore given an analytic continuation for $\zeta(s)$ up to $\sigma = 0$.

A crash course on infinite products

3.1. **Euler products.** Since it is a topic not often covered in analysis courses, we first take a brief digression to discuss infinite products. If $a_n \in \mathbb{C}\backslash\{0\}$ then tshe infinite product

$$\prod_{n=1}^{\infty} a_n$$

is defined to be the limit $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$ if this exists and is not zero.

Lemma 13 (Cauchy criterion)**.** *If $a_n \neq 0$ then the infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if for any $\epsilon > 0$ there exists N such that*

$$\left| \prod_{n<k\leq m} a_k - 1 \right| < \epsilon$$

for all $m > n \geq N$.

In particular, $\lim_{n\to\infty} a_n = 1$. For this reason it is often convenient to change variables so that we consider the product $\prod(1 + a_n)$ instead. We say that

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges absolutely if and only if $\prod(1 + |a_n|)$ converges. The following is a simple consequence of the Cauchy criterion.

Lemma 14. *If $a_n \neq -1$ and $\prod(1 + a_n)$ converges absolutely then it converges.*

The final fundamental fact we will require is the following.

Lemma 15. *If $a_n > 0$ for all $n \geq 1$ then $\prod(1 + a_n)$ converges if and only if $\sum a_n$ converges.*

Proof. By the monotone convergence theorem, it suffices to show that the partial sums are bounded above if and only the partial products are. This follows from the inequalities

$$a_1 + \cdots + a_n < (1 + a_1)\cdots(1 + a_n) \leq e^{a_1+\cdots+a_n}.$$

□

All of the infinite products we will encounter in this course will converge absolutely. The previous lemmas have the following useful consequence: if $\sum |a_n|$ converges (and $a_n \neq -1$) then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. In particular, it is not zero!

Lemma 16. *If f is multiplicative and $\sum |f(n)| n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right).$$

Proof. Note that this product is absolutely convergent. By comparison each sum in the product is absolutely convergent. Since a product of finitely many absolutely convergent series can be arbitrarily rearranged,

$$\prod_{p\leq y} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) = \sum_{\substack{n \\ p|n \implies p\leq y}} f(n)n^{-s}.$$

Therefore the difference between the product here and the Dirichlet series is at most

$$\sum_{n>y} |f(n)|\, n^{-\sigma} \to 0 \text{ as } y \to \infty.$$

□

Corollary 1 (Euler product). *If f is completely multiplicative and $\sum |f(n)|\, n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

In particular, we note the Euler product for $\zeta(s)$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

which is valid for $\sigma > 1$. From this it follows that $\zeta(s) \neq 0$ for $\sigma > 1$. The Euler product leads to the identity

We can play these games to get almost any multiplicative function in terms of $\zeta(s)$.

Q: Express the Von Mangoldt function in terms of $\zeta(s)$

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_n \frac{\mu(n)}{n^s}.$$

Furthermore, when $\sigma > 1$, the series is absolutely convergent, and so the derivative can be computed summand by summand, leading to

e.g. since $F_{f*g} = F_f \cdot F_g$

$$\sum \frac{\tau(n)}{n^s} = \sum \frac{1*1(n)}{n^s} = \zeta(s)^2$$

$$\zeta'(s) = -\sum_n \frac{\log n}{n^s}.$$

From the Euler product we have

$$\sum \frac{|\mu(n)|}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right) = \prod_p \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - 1/p^s\right)} = \frac{\zeta(s)}{\zeta(2s)}$$

$$\log \zeta(s) = -\sum_p \log\left(1 - p^{-s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} = \sum_n \frac{\Lambda(n)}{\log n} n^{-s}.$$

Finally, taking the derivative of this, we obtain the Dirichlet series with $\Lambda(n)$ as coefficients:

"Any interesting dirichlet series can be expressed in terms of the zeta function"

$$\frac{\zeta'}{\zeta}(s) = -\sum_n \frac{\Lambda(n)}{n^s}.$$

Note: Möbius inversion looks trivial in terms of ζ

$$1 * g = f \iff \mu * f = g$$

$$\zeta \cdot F_f = F_g \iff \frac{1}{\zeta} \cdot F_g = F_f$$

4. Gamma function

4.1. The Weierstrass definition. Let

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N\right) = 0.5772157\cdots$$

and define the Gamma function $\Gamma(s) : \mathbb{C} \to \mathbb{C}$ by

Weierstrass canonical product

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} e^{-s/n} \left(1 + \frac{s}{n}\right).$$

* The following proofs about $\Gamma(s)$ are NON-EXAMINABLE however the definition of $\Gamma(s)$, and the statements are. *

This product is analytic for all $s \in \mathbb{C}$, because when $|s| \leq N/2$ the series

i.e. it is entire

$$\sum_{n=N+1}^{\infty} \left(\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right)$$

is absolutely and uniformly convergent, and so its exponential is also an analytic function. This shows that the product is an analytic function for $|s| \leq N/2$, and we then take N arbitrarily large.

Tail converges $\Leftrightarrow$ function converges

It is clear from this expression that $\Gamma(s)$ itself is analytic at all $s \in \mathbb{C}$ apart from simple poles at $s = 0, -1, -2, \ldots$. The residue at $s = -n$ is $(-1)^n/n!$.

4.2. **The Euler definition.** Inserting the definition of γ gives

$$\frac{1}{\Gamma(s)} = s \lim_{N\to\infty} e^{\left(\sum_{m=1}^N \frac{1}{m} - \log N\right)s} \prod_{n=1}^N e^{-s/n}\left(1+\frac{s}{n}\right)$$

these cancel with the respective part of γ.

$$= s \lim_{N\to\infty} N^{-s} \prod_{n=1}^N \left(1+\frac{s}{n}\right)$$

Convince myself of this $\rightarrow$?

$$= s \lim_{N\to\infty} \left(1+\frac{1}{N}\right)^s \prod_{n=1}^N \left(1+\frac{s}{n}\right)\left(1+\frac{1}{n}\right)^{-s},.$$

whence we have the following formula of Euler,

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^\infty \left(1+\frac{1}{n}\right)^s \left(1+\frac{s}{n}\right)^{-1},$$

valid for all $s \in \mathbb{C}$ except $s = 0, -1, -2, \ldots$. It follows that $\Gamma(1) = 1$. Rewriting this, we also get

$$\Gamma(s) = \lim_{N\to\infty} N^s \frac{(N-1)!}{s(s+1)\cdots(s+N-1)}.$$

4.3. **The difference equation.** By Euler's formula, if s is not a negative integer,

$$\frac{\Gamma(s+1)}{\Gamma(s)} = \frac{s}{s+1}\lim_{N\to\infty}\prod_{n=1}^N \frac{\left(1+\frac{1}{n}\right)(s+n)}{s+n+1}$$

$$= s\lim_{N\to\infty}\frac{N+1}{s+N+1} = s,$$

whence

$$\Gamma(s+1) = s\Gamma(s).$$

"It is not Γ who is one off here, it is the factorial. We should be attached to Γ & not factorials."

In particular, since $\Gamma(1) = 1$, if s is a positive integer then $\Gamma(s) = (s-1)!$.

4.4. **The reflection formula.**

$$\Gamma(s)\Gamma(1-s) = \frac{1}{s(1-s)}\prod_{n=1}^\infty \frac{1+1/n}{\left(1+\frac{s}{n}\right)\left(1+\frac{1-s}{n}\right)}$$

$$= \frac{1}{s(s-1)}\prod_{n=1}^\infty \frac{1}{\left(1+\frac{s}{n}\right)\left(1-\frac{s}{n+1}\right)}$$

$$= \frac{1}{s}\prod_{n=1}^\infty\left(1-\frac{s^2}{n^2}\right)^{-1}$$

Idea: $\sin(\pi s)$ is entire and should equal the product over it's zeros. Exercise: Prove this rigorously.

$$= \frac{\pi}{\sin(\pi s)}.$$

It follows, for example, that $\Gamma(1/2) = \sqrt{\pi}$.

4.5. **The duplication formula.** Consider the expression

$$\frac{2^{2s}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)}{2\Gamma(2s)}.$$

We claim that this is independent of s. By Euler's formula it is

$$2^{2s-1}\frac{\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^s}{(s)\cdots(s+N-1)}\cdot\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^{s+1/n}}{\left(s+\frac{1}{n}\right)\cdots\left(s+\frac{1}{n}+N-1\right)}}{\lim_{N\to\infty}\frac{1\cdot 2\cdots(2N-1)(2N)^{2s}}{2s(2s+1)\cdots(2s+2N-1)}}$$

which is

$$\lim_{N\to\infty}\frac{((N-1)!)^2N^{1/2}2^{2N-1}}{(2N-1)!},$$

and in particular independent of s. To evaluate it we set $s=1/2$, yielding

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

We have proved the duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=2^{1-2s}\pi^{1/2}\Gamma(2s).$$

4.6. **Euler's integral expression.** By integration by parts

$$\begin{aligned}\int_0^N\left(1-\frac{t}{N}\right)^N t^{s-1}\,\mathrm{d}t &= N^s\int_0^1(1-t)^N t^{s-1}\,\mathrm{d}t\\ &= N^s\frac{N!}{s(s+1)\cdots(s+N)}\\ &\to\Gamma(s)\end{aligned}$$

as $N\to\infty$ using Euler's formula. This is valid if $\sigma>0$, whence we have the formula for this region

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t.$$

5. Functional equation

Theorem 4 (Functional equation). *The zeta function $\zeta(s)$ can be extended a function meromorphic on the whole complex plane, and for all s satisfies the identity*

$$\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Equivalently,

$$\zeta(s)=\chi(s)\zeta(1-s)$$

This is the way to think about this formula. $\zeta(s) = \zeta(1-s) \times$ some analytic noise

where

$$\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)=\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Many interesting facts can be deduced from this identity. We will first use it to study the possible poles of $\zeta(s)$. We know that $\zeta(s)$ has a simple pole at $s=1$, and nowhere else for $\sigma>-1$. Suppose that ζ has a pole at s for $\sigma<0$. Then so too does $\Gamma(1-s)\zeta(1-s)$, but both $\Gamma(s)$ and $\zeta(s)$ are holomorphic for all s with $\Re s>1$, which is a contradiction. It follows that $\zeta(s)$ only has one pole in $\mathbb{C}$, which is a simple pole at $s=1$.