Mich. 2019 **RAMSEY THEORY – EXAMPLES 1** IBL

1. How many combinatorial lines are there in $[m]^n$?

2. Show that $\mathrm{HJ}(2,k) = k$ for all k.

3. Let A be an infinite subset of the plane, with no three points of A collinear. Prove that A contains a set of 2018 points forming a convex 2018-gon.

? 4. Let c be a 2-colouring of the finite subsets of $\mathbb{N}$. Must there exist an infinite $M \subset \mathbb{N}$ such that, for each r, the colouring c is constant on $M^{(r)}$?

2 5. Prove that $\{0,1\}^{\mathbb{N}}$ (with the product topology) is compact.

6. By mirroring the proof of van der Waerden's theorem for arithmetic progressions of length 3, show that whenever $\mathbb{N}^2$ is finitely coloured there exist a, b, r such that the set $\{(a,b), (a+r,b), (a,b+r)\}$ is monochromatic. Deduce by a product argument that whenever $\mathbb{N}^2$ is 2-coloured there exist a, b, r such that the square $\{(a,b), (a+r,b), (a,b+r), (a+r,b+r)\}$ is monochromatic. Give an explicit n such that whenever $[n]^2$ is 2-coloured there exists a monochromatic square.

7. Show that for every m there is an n with the following property: whenever $[n]^{(2)}$ is 2-coloured there exists a monochromatic set M of size at least m satisfying $|M| > \min M$.

?? 8. Let A be a subset of $\mathbb{N}$ such that, whenever A is finitely coloured, there is a monochromatic arithmetic progression of length m. Must A contain an arithmetic progression of length $m+1$?

9. Let $\mathbb{N}$ be finitely coloured. Must there exist arbitrarily long monochromatic arithmetic progressions having the same common difference?

+10. Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

+11. Let c be a colouring of $\mathbb{N}$ using (possibly) infinitely many colours. Prove that, for every m, there is an arithmetic progression of length m on which c is either constant or injective.

1. How many combinatorial lines are there in $[m]^n$?

$[1]^{(1)} = 1 \qquad [m]^{(1)} = 1$

Let $f(m,n)$ be the # of combinatorial lines in $[m]^{(n)}$.

$$f(m,n) = \sum_{\#\text{active coords}} 1^{\#\text{active coords}} \cdot m^{\#\text{stat. coords}} = \sum_{i=1}^{n} \binom{n}{i} m^{n-i} 1^{i} = (m+1)^n - m^n$$

However can get $(m+1)^n$ directly by listing possibilities as $\begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \vdots & \vdots & \vdots \\ m & m & m \\ * & * & * \end{matrix} \quad \cdots$ n copies

active coord

2. Show that $\mathrm{HJ}(2,k) = k$ for all k.

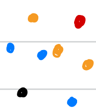
Aim: Color $\{0,1\}^{k-1}$ with k-colours such that there is no mono combinatorial line

Let $N(A) = \{x : xy \text{ is an edge with } x \notin A, y \in A\}$

Color 0 with color 1, $N(0)$ colour 2, $N(N(0))$ colour 3, ..., $N^k(0) = \underline{1}$ colour k.

Then there are no mono combinatorial lines (Note: Do not colour previously coloured neighbours in this colouring)

Example w/ $\{0,1\}^3$ with 4-colours



since a neighbour of a vertex is always coloured differently to itself.

Aim: Show any k colouring of $\{0,1\}^k$ contains a mono combinatorial line

Go for contradiction.

0 must be coloured some colour, say 1. Then each element of $N(0)$ must be coloured distinctly from 0 and from eachother, however $|N(0)| = k > k-1$ #.

Thus $k \le HJ(2,k) \le k \Rightarrow HJ(2,k) = k$.

$HJ(2,k) \leq k$: Idea: Want to say in any k colouring I see a line.

Consider $0000\ldots0$, $00\ldots01$, $00\ldots011$, $00\ldots0111$, $11\ldots11$ is $k+1$ things each of which form a line so done.

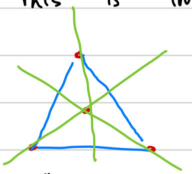
$HJ(2,k) > k-1$: Colour x by # of 1s (or $\|x\|_1$, same thing)

Reviewed

3. Let A be an infinite subset of the plane, with no three points of A collinear. Prove that A contains a set of 2018 points forming a convex 2018-gon.

WLOG A is countably infinite, else reduce it's size.

2-colour $A^{(4)}$ by colouring a 4-set CONVEX if the points form a convex 4-gon. Else colour them NON CONVEX.
Ramsey $\Rightarrow \exists$ an ∞ mono subset of $A^{(4)}$. Suppose this is NONCONVEX, this is impossible since any non-convex 4-gon is of the following form:



Adding a 5-th point anywhere leads to a convex 4-gon.
Can argue this properly by splitting triangle into 6 equal regions.

Thus $\exists\, \infty$ CONVEX subset of A.
Then done since we can take any 2018 points and they must form a convex 2018 gon (Suppose not, then $\exists$ a non-convex 4-gon by triangulating any 2018 points in our set).

Proof 2 Rotate points in A such that no two are in a vertical line, then color the three sets as HAPPY $\times{}_{\times}\times$ or SAD $\times{}^{\times}\times$, then by Ramsey can find 2018 mono points HAPPY or SAD then done.

Note: If we wanted to find n: n PTS $\Rightarrow$ convex 2018 gon, then Proof 2 would give a better bound for n.

4. Let c be a 2-colouring of the finite subsets of $\mathbb{N}$. Must there exist an infinite $M \subset \mathbb{N}$ such that, for each r, the colouring c is constant on $M^{(r)}$?

Yes. By Ramsey $\exists$ ∞ set $M_1 \subset \mathbb{N}$ such that $c|_{M_1}$ is constant.

Further, since $M_1 \cong \mathbb{N}$, by Ramsey for 2-sets $\exists$ ∞ M_2 such that $M_2^{(2)} \subset M_1^{(2)}$ and $c|_{M_2^{(2)}}$ is constant. Continuing recursively we get ∞ sets $M_1 \supset M_2 \supset M_3 \supset \dots$ such that $c|_{M_k^{(k)}}$ is constant.

Then let $M = \bigcap_{k=1}^{\infty} M_k$ then $c|_{M^{(r)}}$ is constant for each $r \geq 1$

Note: $c|_{X^{(j)}}$ is not necessarily the same as $c|_{X^{(k)}}$ for $j \neq k$.

This is not true since our intersection of infinitely many nested sets can be $\emptyset$. Find a counterexample colour.

Proof Lets ensure $1 \notin M$ by colouring ij RED $i=1$

BLUE $i \neq 1$

Now lets make sure $2 \notin M$ by colouring ijk RED if $2 \in (i,j,k)$

BLUE if not

Colour the r-sets RED if $r \in$ it

BLUE if not

3. Prove that $\{0,1\}^{\mathbb{N}}$ (with the product topology) is compact.

$\{0,1\}^{\mathbb{N}}$ with $d(f,g) = \frac{1}{\min\{n : f(n) \neq g(n)\}}$

WANT: Every sequence has a convergent subsequence

$f_1 : 0110111\ldots$ Let $f(1)$ = some value s.t. ∞ many $f_n(1)$ take this value

$f_2 : 1101111\ldots$ $\forall n \in A_1 : f_n(1) = f_1$

$f_3 : 001000\ldots$ Let $f(2)$ = " s.t. ∞ many $f_n(2)$, $n \in A_1$ take this value

$\forall n \in A_2 : f_n(2) = f(2)$

Now take an $f_{i_1} \in A_1$, $f_{i_2} \in A_2$, $f_{i_3} \in A_3$, $f_{i_4} \in A_4, \ldots$ Then this sequence is convergent

??

6. By mirroring the proof of van der Waerden's theorem for arithmetic progressions of length 3, show that whenever $\mathbb{N}^2$ is finitely coloured there exist a, b, r such that the set $\{(a,b),(a+r,b),(a,b+r)\}$ is monochromatic. Deduce by a product argument that whenever $\mathbb{N}^2$ is 2-coloured there exist a, b, r such that the square $\{(a,b),(a+r,b),(a,b+r),(a+r,b+r)\}$ is monochromatic. Give an explicit n such that whenever $[n]^2$ is 2-coloured there exists a monochromatic square.

call this $\cancel{\Delta}$ set and Δ together with its focus

Def^n (CF) Say two sets of the form $\{(a_1,b_1),(a_1+r_1,b_1)\}$, $\{(a_2,b_2),(a_2+r_2,b_2)\}$ where $a_i, b_i, r_i \in \mathbb{N}$ are colour focussed if $b_1+r_1 = b_2+r_2 = f \in \mathbb{N}$ and both sets are monochromatic with different colours

Proposition (Van-der-Waerden for length 3) $\forall k$ $\exists n$ s.t. whenever $[n]^{(2)}$ is k-coloured $\exists$ mono Δ set of length 3.

Proof: Claim: $\forall p \leq k$ $\exists n$ s.t. whenever $[n]^{(2)}$ k-coloured $\exists$ either:

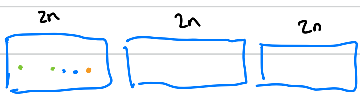
- Mono Δ set
- p colour-focused CF $\cancel{\Delta}$ sets

Then done, put $p=k$ and look at the colour of the focus.

Proof of claim: Induction on . $p=1$ ✓ (Take $n=k+1$, then two points of same colour in each direction)

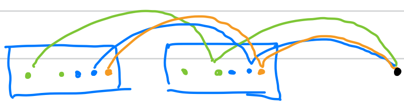
Given n suitable for $p-1$, we'll show that $(k^{2n}+1)2n$ is suitable for p. comes out of proof

So given a k-colouring of $[(k^{2n}+1)2n]^{(2)}$ with no mono Δ of length 3:



length of each block is $2n$ to ensure focus is contained in it. There are k^{2n} ways to colour a block. We have $k^{2n}+1$ blocks, so there are two blocks coloured the same.

In writing:



Focus, not necessarily within $[(k^{2n}+1)2n]$

Let $B_1, \dots, B_{k^{2n}+1}$ where $B_i := [2n(i-1)+1, 2ni]^{(2)}$

patterns for a block is k^{2n}.

Have two blocks B_s, B_{s+t} coloured the same. Now B_s contains $p-1$ colour focussed $\cancel{\Delta}$'s together with their focus, say Since blocks have length $2n$

$\{(a_1,b_1),(a_1+r_1,b_1)\}, \{(a_2,b_2),(a_2+r_2,b_2)\} \dots$ are focussed at $f=(f_1,f_2)$

However then the $p-1$ $\cancel{\Delta}$'s $\{(a_1,b_1),(a_1+r_1+2nt)\}, \dots, \{(a_{p-1},b_{p-1}),(a_{p-1}+r_{p-1},b_{p-1})\}$ are colour focussed at (f_1+4nt, f_2+4nt) Note: Focus doesn't need to be $<$. then our value

Also $\{(f_1,f_2),(f_1+2nt,f_2)\}$ is mono of a diff. colour to those.

Thus have p colour focussed $\cancel{\Delta}$s

Proof of claim ■

Prop^n follows from claim.

Proof of prop^n ■

Q6 [Continued]

Let $C(k)$ be such that whenever $[C(k)]$ is k-coloured there exists a mono Δ set, which exists from above. Let $n' = C(2)$ and let $n = C(2^{2n'}) \cdot 2n'$.

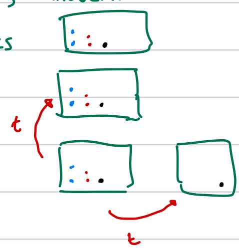
2-colour $[n]$. Splitting it into $C(2^{2n'})$ blocks of length $2n'$ gives that there are ... Hmm how to continue this?

Q6 (Proof from Examples Class) For k colours

claim: $\exists n$ s.t. $[n]^2$ k-coloured $\Rightarrow$ mono L or r colour focussed

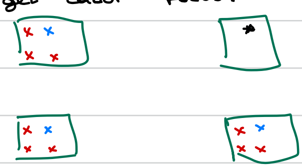
Proof of Claim START: $n = k+1$, induction: k^{n^2} possible colourings of $n \times n$ grid.

Take $k^{n^2}+1$ consecutive blocks



Product argument 9 is enough if you think about it, 3×2^9 is if you don't.

Look by product argument at the blocks of size 9×9 as points, 2^{9^2} ways of Colouring each block so 2^{81} colours get colour focus.



7. Show that for every m there is an n with the following property: whenever $[n]^{(2)}$ is 2-coloured there exists a monochromatic set M of size at least m satisfying $|M| > \min M$.

Suppose not. Then by compactness we get a 2-colouring of the naturals with no M s.t. $|M| > \min M$.

However by ∞ Ramsey there is an infinite mono set. Pick a point in the mono colouring say m, then pick the next $m+1$ pts and add them to the set.

Same argument proves $\forall m, k, r$ $\exists n$ s.t. $[m]^{(r)}$ k-coloured $\exists$ mono M with $|M| > \min M$

Intresting note: This is true in PA however PA $\nvdash$ this statement. This was first example of a nice sentence like this. End up showing statements of the form $\forall x \, \exists y \, P(x,y)$ where y grows very fast is not provable in PA.

8. Let A be a subset of $\mathbb{N}$ such that, whenever A is finitely coloured, there is a monochromatic arithmetic progression of length m. Must A contain an arithmetic progression of length $m+1$?

Yes. Suppose $\exists\, k$ such that cannot embed a copy of $[W(m,k)]$ into A, then can finitely colour A with k colours in such a way that there is no AP of length m #

Thus $\exists$ a copy of $[W(m,k)]$ (translated and/or multiplied by an integer) in A so since $W(m,k) > m+1\ \forall k$, $\exists$ an AP of length $m+1$.

Not sure about this

Proof: Want a rich enough A k-coloured $\Rightarrow \exists$ mono AP of length 3 yet A has no AP of length 4.

Use Hales - Jewitt and like we applied HJ to VdW we can take linear map here from $(x_1, \dots, x_n) \to 10^{10} x_1 + 10^{10^{10^{10^{10}}}} x_2 + \dots$

9. Let $\mathbb{N}$ be finitely coloured. Must there exist arbitrarily long monochromatic arithmetic progressions having the same common difference?

Doesn't matter about the colour since if we have arbitrarily long ones must have arbitrary long red or blue ones etc.

Intuition: No, too much to ask for?

No. Given a colouring of $\mathbb{N}$ as the Thue-Morse sequence this is not true, but it is very hard to prove!

Might try something periodic like mod 10, however this fails for d a multiple of 10. So try colour mod $\sqrt{2}$ (or π) instead.

Colour $\frac{x}{\sqrt{2}}$ mod 1 if $> 1/2$ RED, $< 1/2$ BLUE.

$^{+}$10. Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

No. Let $A = \mathbb{R}$ and color ab for $a, b \in \mathbb{R}$ RED if the usual ordering of $\{a,b\}$ agrees with the well-ordering of $\{a,b\}$, BLUE otherwise. Monochromatic uncountable set would give a subset of order type ω_1? (or it's reverse) in the usual ordering which is impossible.