Part III Michaelmas, 2019

COMBINATORICS

B. B.

Examples Sheet II.

If an exercise seems to make no sense, correct it and then solve it.

18. (i) Given n, determine the maximal integer k such that there is an antichain $\mathcal{A} \subset \mathcal{P}(n)$ with $\mathcal{A} \cap [n]^{(r)} \neq \emptyset$ for $r = 1, \dots, k$.

(ii) And what is the maximum for $n = 10$ if we demand that for $r = 1, \dots, k$ the number of r-subsets in $\mathcal{A}$ is at least r?

19. Let $x_1, \dots, x_n \in \mathbb{R}$ with $|x_i| > 1$ and let $r \in \mathbb{N}$. Show that the number of sums $\sum_{i \in I} x_i$ falling into an interval of length r is at most the sum of the $2r$ largest binomial coefficients $\binom{n}{k}$.

20. For $a = (a_1, \dots, a_n)$, with nonnegative integers a_i, write $D(a)$ for the constant term of the Laurent polynomial

$$F(X; a) = \prod_{\substack{1 \le i \le n \\ i \neq j}} \left(1 - \frac{X_i}{X_j}\right)^{a_i}$$

in $X = (X_1, \dots, X_n)$. Show that

$$D(a) = \binom{m}{a_1, \dots, a_n},$$

where $m = \sum_{i=1}^n a_i$.

21. Let $\mathbb{F}$ be a field and V_n the space of all $\mathbb{F}$-valued functions on $Q_n = \{0, 1\}^n$, viewed as a vector space over $\mathbb{F}$. For $i = 1, \dots, n$, let $h_i \in V_n$ be given by $h_i(a) = a_i$, where $a = (a_1, \dots, a_n) \in Q_n$, and for $I \in \mathcal{P}(n)$ set $h_I = \prod_{i \in I} h_i$. (In particular, $h_\emptyset(a) = 1$ for all $a \in Q_n$.) Check that $B = \{h_I : I \in \mathcal{P}(n)\}$ is a basis of V_n.

22. Let $p \geq 2$ be a prime, and let $v_1, v_2, \dots, v_{3p}$ be $3p$ vectors in $V = \mathbb{Z}_p \oplus \mathbb{Z}_p$ with 0 sum: $\sum_{i=1}^{3p} v_i = 0 \in V$. Give two proofs of the assertion that some p of these vectors also sum to 0, i.e. there is a set $I \subset [3p]$ with $|I| = p$ such that $\sum_{i \in I} v_i = 0 \in V$. Base one of your proofs on the previous exercise.

[Hint. For the proof based on Exercise 21, supposing that the assertion fails, set $m = 3p$ and consider the polynomial $f(X) = f(X_1, \dots, X_m)$

defined as follows

$$\left(\left(\sum_{i=1}^{m} a_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} b_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} X_i\right)^{p-1} - 1\right).$$

What are the values of f on $\{0,1\}^m$?]

23. (i) Let A and B be finite sets of integers with $2 \leq |A| \leq |B|$ and $|A+B| = a+b-1$. Show that A and B are arithmetic progressions with the same difference.

(ii) What can you say about A and B in (i) if $|A+B| = a+b$?

24. Imitate the proof of Theorem 6 in Chapter II to prove Theorem 7, the Chevalley–Warning theorem for several polynomials.

25. In Ch. II we have proved that if there are $2n+1$ seats around a circular table and $2n+1$ is a prime then every seating pattern is realizable. For what other values of n is every pattern realizable?

26. Let m be a natural number and A a non-empty subset of a field $\mathbb{F}$. Show that there is a one-variable polynomial $r(X) \in \mathbb{F}[X]$ of degree less than $|A|$ such that $r(a) = a^m$ for every $a \in A$.

27. Let $A = A_1 \times \cdots \times A_n$, where $A_1, \ldots, A_n$ are non-empty subsets of a field $\mathbb{F}$. Is it true that for every function $\varphi : A \to \mathbb{F}$ there is a polynomial $f(X) = f(X_1, \ldots, X_n) \in \mathbb{F}[X]$ such that $f(a) = \varphi(a)$ for every $a \in A$?

28. Let $\mathbb{F}$ be a finite field of order q, and let $f \in \mathbb{F}[X_1, \ldots, X_n]$ have degree $d \geq 1$. Show that f has at most dq^{n-1} zeros in $\mathbb{F}^n$.

29. (i) Let p be a prime, and let A and B be subsets of the field $\mathbb{Z}_p$ with $1 \leq a = |A| \leq b = |B|$ and $a+b \leq p+3$. Show that the *product-restricted sum*

$$A \oplus_{\text{prod}} B = \{x+y : \ x \in A, \ y \in B, \ xy \neq 1\}$$

has at least $a+b-3$ elements.

(ii) Given a prime $p \geq 3$ and integers $a \geq 2$ and $b \geq 2$ with $a+b \leq p+3$, is the bound in (i) best possible? Thus, are there sets $A, B \subset \mathbb{Z}_p$ such that $|A| = a$, $|B| = b$ and $|A \oplus_{\text{prod}} B| = a+b-3$?

18. (i) Given n, determine the maximal integer k such that there is an antichain $\mathcal{A} \subset \mathcal{P}(n)$ with $\mathcal{A} \cap [n]^{(r)} \neq \emptyset$ for $r = 1, \ldots, k$.

(ii) And what is the maximum for $n = 10$ if we demand that for $r = 1, \ldots, k$ the number of r-subsets in $\mathcal{A}$ is at least r?

(i) Construct A inductively and see when we fail, First element is WLOG 1, second element must not include 1, third element can't include 1 or atleast 1 element of the second. Assuming at each stage the antichain is constructed so as to take 1 element away from the previous one and add two new ones, can achieve

$k = \left\lfloor \frac{n-1}{2} \right\rfloor$

This is wrong!

(ii) $[n] = [10]$ so from (i) must have $k \leq \left\lfloor \frac{10-1}{2} \right\rfloor = 4$

$A = \{\{1\}, \{2,3\}, \{3,4,5\}, \{2,4,5,6,7\},$

Ans is $n-2$ ind. $n \to n+2$

Show

(ii) $k = 6$

Part (i) Take $C_1, C_2, \ldots, C_{n-2} \subset [n]$

- $\{n+1\}$
- $\{n+2\} \cup C_1$
- $\vdots$
- $\{n+2\} \cup C_{n-2}$
- $\{1, 2, \ldots, n\}$

for the inductive step.

19. Let $x_1, \ldots, x_n \in \mathbb{R}$ with $|x_i| > 1$ and let $r \in \mathbb{N}$. Show that the number of sums $\sum_{i \in I} x_i$ falling into an interval of length r is at most the sum of the $2r$ largest binomial coefficients $\binom{n}{k}$.

Sum of $2r$ largest binomial coefficients is: $\binom{n}{\lfloor n/2 \rfloor}, \binom{n}{\lfloor \frac{n}{2} \rfloor - 1}, \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}, \ldots$

$x_i = 1 \ \forall i \Rightarrow r$ largest binomial coefficients work.

$x_i = 1.5$

$x_i \to -x_i$ does not affect problem, kind of like translation.

20. For $a = (a_1, \ldots, a_n)$, with nonnegative integers a_i, write $D(a)$ for the constant term of the Laurent polynomial

$$F(X;a) = \prod_{\substack{1 \le i \le n \\ i \ne j}} \left(1 - \frac{X_i}{X_j}\right)^{a_i}$$

in $X = (X_1, \ldots, X_n)$. Show that

$$D(a) = \binom{m}{a_1, \ldots, a_n},$$

where $m = \sum_{i=1}^n a_i$.

First try, led nowhere.

WLOG $j=1$, $F(X;a) = \prod_{2 \le i \le n} \left(1 - \frac{X_i}{X_1}\right)^{a_i}$

$$\left(1 - \frac{X_2}{X_1}\right)^{a_2} \left(1 - \frac{X_3}{X_1}\right)^{a_3} \left(1 - \frac{X_1}{X_2}\right)^{a_1} \left(1 - \frac{X_1}{X_3}\right)^{a_1} \left(1 - \frac{X_2}{X_3}\right)^{a_2} \left(1 - \frac{X_3}{X_2}\right)^{a_3}$$

$$= 1 + (-1)^2 \left(\binom{\Sigma a_i}{2}\right) + (-1)^3 \left(\binom{\Sigma a_i}{3}\right) \ldots$$

For each term in the product, 1 makes no difference and if use $-\frac{X_i}{X_j}$ then add (i,j) to a set S.

Then a set S of ordered pairs contributes $(-1)^{|S|}$ to the constant term if $(\# i = k) = (\# j = k)$ for all $k = 1, \ldots, n$ and contributes 0 otherwise.

How many such sets are there? There are (Σa_i) possible pairs to choose from.

three $(1,2)$s, four $(2,3)$s, five $(1,3)$s, two $(3,2)$s, one $(2,1)$, two $(3,1)$s

$\binom{\Sigma a_i}{a_1, a_2, \ldots, a_n}$ is the number of ways of arranging $\sum_{i=1}^{n} a_i$ things with multiplicities $a_1, a_2, \ldots, a_n$

20. For $a = (a_1, \ldots, a_n)$, with nonnegative integers a_i, write $D(a)$ for the constant term of the Laurent polynomial

$$F(X;a) = \prod_{\substack{1 \le i \le n \\ i \ne j}} \left(1 - \frac{X_i}{X_j}\right)^{a_i}$$

in $X = (X_1, \ldots, X_n)$. Show that

$$D(a) = \binom{m}{a_1, \ldots, a_n},$$

where $m = \sum_{i=1}^n a_i$.

Second try

Claim: $F(a_1, \ldots, a_n) = \sum_{k=1}^{n} F(a_1, \ldots, a_k - 1, \ldots, a_n)$

Show $1 = \sum_i \prod_{j \ne i} \frac{x_j - x}{x_j - x_i}$ Lagrange Interpolation

$$\prod_{1 \le i \ne j \le n} \left(1 - \frac{X_i}{X_j}\right)^{a_i} \overset{?}{=} \sum_{k=1}^{n} \prod_{1 \le i \ne j \le n} \left(1 - \frac{X_i}{X_j}\right)^{a_i} \prod_{\substack{1 \le j \le n \\ j \ne k}} \left(1 - \frac{X_k}{X_j}\right)^{-1}$$

$$= F(a_1, \ldots, a_n)\left[\sum_{k=1}^{n} \prod_{1 \le j \le n} \left(1 - \frac{X_k}{X_j}\right)^{-1}\right]$$

$$\left(1 - \frac{X_1}{X_2}\right)^{-1} + \left(1 - \frac{X_2}{X_1}\right)^{-1} = \frac{X_2}{X_2 - X_1} - \frac{X_1}{X_2 - X_1} = 1$$

$$\left(1 - \frac{X_1}{X_2}\right)^{-1}\left(1 - \frac{X_1}{X_3}\right)^{-1} + \left(1 - \frac{X_2}{X_1}\right)^{-1}\left(1 - \frac{X_2}{X_3}\right)^{-1} + \left(1 - \frac{X_3}{X_1}\right)^{-1}\left(1 - \frac{X_3}{X_2}\right)^{-1} = 1$$

Induction on n: $$\sum_{k=1}^{n} \prod_{\substack{1 \le j \le n \\ j \ne k}} \left(1 - \frac{X_k}{X_j}\right)^{-1} = \sum_{k=1}^{n-1} \prod_{1 \le j \le n-1} \left(1 - \frac{X_k}{X_j}\right)^{-1}\left(1 - \frac{X_k}{X_n}\right)^{-1} + \prod_{1 \le j \le n-1} \left(1 - \frac{X_n}{X_j}\right)^{-1}$$

Some kind of induction gets the result, couldn't get at the details.

Suppose $F(a_1, \ldots, a_n) = \sum_{i=1}^{n} F(a_1, \ldots, a_i - 1, \ldots a_n)$ so prove by induction on $\sum a_i = m$. Case $m=1$ is trivially true. Else let $CT(f)$ denote the constant term of f then from the above $CT(F(a_1, \ldots, a_n)) = \sum_{i=1}^{n} CT(F(a_1, \ldots, a_i - 1, \ldots, a_n))$

$$\overset{\text{ind. hyp.}}{=} \sum_{i=1}^{n} \binom{m-1}{a_1, \ldots, a_i - 1, \ldots, a_n} = \binom{m}{a_1, \ldots, a_n}$$

21. Let $\mathbb{F}$ be a field and V_n the space of all $\mathbb{F}$-valued functions on $Q_n = \{0,1\}^n$, viewed as a vector space over $\mathbb{F}$. For $i = 1, \dots, n$, let $h_i \in V_n$ be given by $h_i(a) = a_i$, where $a = (a_1, \dots, a_n) \in Q_n$, and for $I \in \mathcal{P}(n)$ set $h_I = \prod_{i \in I} h_i$. (In particular, $h_\emptyset(a) = 1$ for all $a \in Q_n$.) Check that $B = \{h_I : I \in \mathcal{P}(n)\}$ is a basis of V_n.

Linearly independent: Consider $\underset{\neq \emptyset}{I_1, I_2 \in \mathcal{P}(n)}$, $\lambda, \mu \in \mathbb{F}$. $\lambda h_{I_1} + \mu h_{I_2} \equiv 0$

iff $h_{I_1} = -\frac{\lambda}{\mu} h_{I_2} = c h_{I_2}$ for some c.

Suppose $\exists\, i \in I_1 \Delta I_2$ with WLOG $i \in I_1$ and $i \notin I_2$.

Then let $v = \sum_{i \in I_1} e_i$ then $1 = h_{I_1}(v) = c\, h_{I_2}(v) = 0$ #

Thus $I_1 \Delta I_2 = \emptyset \Rightarrow I_1 = I_2$

Spanning: Function is uniquely determined by its value on points in Q^n.

$$\delta_{\underline{x}}(a) = \underbrace{\prod_{i : x_i = 1} h_i(a)}_{\in \text{ linear span of the } h_I} + \underbrace{\prod_{i : x_i = 0} (1 - h_i(a))}_{\in \text{ linear span of } h_I}$$

22. Let $p \geq 2$ be a prime, and let $v_1, v_2, \ldots, v_{3p}$ be $3p$ vectors in $V = \mathbb{Z}_p \oplus \mathbb{Z}_p$ with 0 sum: $\sum_{i=1}^{3p} v_i = 0 \in V$. Give two proofs of the assertion that some p of these vectors also sum to 0, i.e. there is a set $I \subset [3p]$ with $|I| = p$ such that $\sum_{i \in I} v_i = 0 \in V$. Base one of your proofs on the previous exercise.

[Hint. For the proof based on Exercise 21, supposing that the assertion fails, set $m = 3p$ and consider the polynomial $f(X) = f(X_1, \ldots, X_m)$ defined as follows

$$\left(\left(\sum_{i=1}^{m} a_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} b_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} X_i\right)^{p-1} - 1\right).$$

What are the values of f on $\{0,1\}^m$?]

Proof based on previous exercise:

Suppose not.

Consider $f(X_1, \ldots, X_m) = \underbrace{\left(\left(\sum_{i=1}^{m} a_i X_i\right)^{p-1} - 1\right)}_{\neq 0 \text{ iff } \sum a_i X_i = 0}\underbrace{\left(\left(\sum_{i=1}^{m} b_i X_i\right)^{p-1} - 1\right)}_{\neq 0 \text{ iff } \sum_{i=1}^m b_i X_i = 0}\underbrace{\left(\left(\sum_{i=1}^{m} X_i\right)^{p-1} - 1\right)}_{\neq 0 \text{ iff } \sum_{i=1}^m X_i = 0}$

$f(x) \neq 0$ for $x \neq 0 \Rightarrow \sum_{i=1}^m a_i x_i = 0$, $\sum_{i=1}^m b_i x_i = 0$, $\sum_{i=1}^m x_i = 0$ modulo p.

For $I = \{x_i : x_i = 1\}$, cannot have $|I| = p$ by assumption, if $|I| = 2p$ then take $I^c = [m] \setminus I$ then $|I^c| = p$ and $\sum_{i \in I^c} x_i v_i = 0$ #

Thus have $I = 0$ or $[m]$, $I = 0$ corresponds to $x = 0$, $I = [m]$ gives $f(x_1, \ldots, x_m) = -1$.

Thus $f(\underline{x}) = \begin{cases} -1 & \underline{x} = \underline{0} \text{ or } \underline{1} \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ $f(x)$ has even degree so can't use ACNS since $m = 3p$ odd?

How to fix this, multiply f by something

Let $h(\underline{x}) = (x_1 - 1)\ldots(x_m - 1) - x_1 \ldots x_m$ then $f(\underline{x}) = h(\underline{x})$ $\forall \underline{x}$

and $\deg f \leq 3p - 3 < 3p - 1 = \deg h$ so $\deg(h - f) = 3p - 1$

↑ Wanted $3p$! doesn't work.

Can't see how to fix this, don't have much "room" to multiply f by a polynomial of large degree.

Fine.

22. Let $p \geq 2$ be a prime, and let $v_1, v_2, \ldots, v_{3p}$ be $3p$ vectors in $V = \mathbb{Z}_p \oplus \mathbb{Z}_p$ with 0 sum: $\sum_{i=1}^{3p} v_i = 0 \in V$. Give two proofs of the assertion that some p of these vectors also sum to 0, i.e. there is a set $I \subset [3p]$ with $|I| = p$ such that $\sum_{i \in I} v_i = 0 \in V$. Base one of your proofs on the previous exercise.

[Hint. For the proof based on Exercise 21, supposing that the assertion fails, set $m = 3p$ and consider the polynomial $f(X) = f(X_1, \ldots, X_m)$ defined as follows

$$\left(\left(\sum_{i=1}^{m} a_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} b_i X_i\right)^{p-1} - 1\right)\left(\left(\sum_{i=1}^{m} X_i\right)^{p-1} - 1\right).$$

What are the values of f on $\{0,1\}^m$?]

Let $v_i = (a_i, b_i)$ for $1 \leq i \leq 3p$. Consider the following polynomials:

$$f_1(x) = \sum_{i=1}^{3p-1} x_i^{p-1}$$

$$f_2(x) = \sum_{i=1}^{3p-1} a_i x_i^{p-1}$$

$$f_3(x) = \sum_{i=1}^{3p-1} a_i x_i^{p-1}$$

f_1, f_2, f_3 have common zero at $\underline{0}$ and $\sum_{i=1}^{3} \deg f_i = 3p-3 < 3p-1$

So by Chevellay-... (Thm 7), f_1, f_2, f_3 have another common $\underline{0}$. Let J be a set of coordinates such that the f_i are all zero at this other common zero.

Then $|J| = p$ or $2p$ from f_1, and $\sum_{i \in J} v_i = 0$ by f_2, f_3. If $|J| = p$ then done. If $|J| = 2p$ take $\tilde{J} = [3p] \setminus J$ then $|\tilde{J}| = p$ and $\sum_{i \in \tilde{J}} v_i = 0$.

learn part 1

23. (i) Let A and B be finite sets of integers with $2 \le |A| \le |B|$ and $|A+B| = a+b-1$. Show that A and B are arithmetic progressions with the same difference.

(ii) What can you say about A and B in (i) if $|A+B| = a+b$?

(i) Order $A = \{a_1, \dots, a_n\}$ $B = \{b_1, \dots, b_m\}$, $|A| = n$, $|B| = m$ wrt normal order

Then $a_1 + b_1, a_1 + b_2, \dots, a_1 + b_m, a_2 + b_m, \dots, a_n + b_m$ are in ascending order

WLOG $a_1 = 0$, $b_l = 0$. Now if $|A+B| = a+b-1$, then

$A+B \supset \{0, a_2, \dots, a_l, b_1, \dots, b_{l-1}\}$ however RHS already has $a+b-1$ elements.

Thus $a_2 + b_{l-1}$ is either $< a_2$ or $> b_{l-1}$ $\Rightarrow$ $a_2 + b_{l-1} = 0$ so

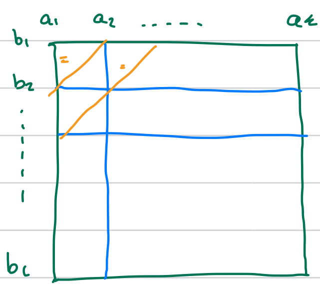
$a_2 = -b_{l-1}$.

Can continue to argue $a_3 + b_{l-1} = a_2$, $a_4 + b_{l-1} = a_3$ etc.

Thus A is AP with difference a_2. But also analogously, must have $a_2 + b_{l-2} = b_{l-1}$,

$a_2 + b_{l-3} = b_{l-2}$ ... etc.

$a_1 + b_1, a_1 + b_2, \dots, a_1 + b_l, a_2 + b_l, \dots, a_k + b_l$



any path downward from LH corner to RH corner get that each diagonal are equal and result follows

(ii) Try get box of APs in lower right corner, if can expand the box to whole thing then done. What happens if we fail?

Part (ii) is too difficult.

24. Imitate the proof of Theorem 6 in Chapter II to prove Theorem 7, the Chevalley–Warning theorem for several polynomials.

Easy

25. In Ch. II we have proved that if there are $2n+1$ seats around a circular table and $2n+1$ is a prime then every seating pattern is realizable. For what other values of n is every pattern realizable?

Let $2n+1 = kl$, $k, l > 1$. Set $d_1 = \dots = d_n = k$ and consider sets

$\{1, 1+k, \dots, 1+(l-1)k\}$

$\{2, 2+k, \dots, 2+(l-1)k\}$ $\rightarrow$ in each set there is an odd # of seats

$\vdots$ and any couple has to lie in one set

$\{k, 2k, \dots, lk\}$ so have $1+$ untaken in each set. #

26. Let m be a natural number and A a non-empty subset of a field $\mathbb{F}$. Show that there is a one-variable polynomial $r(X) \in \mathbb{F}[X]$ of degree less than $|A|$ such that $r(a) = a^m$ for every $a \in A$.

Let $A = \{a_1, \dots, a_k\}$. We want to find coefficients $b_0, b_1, b_2, \dots, b_{k-1}$ s.t.

$$b_0 + b_1 a_1 + \dots + b_{k-1} a_1^{k-1} = a_1^m$$

$$\vdots$$

$$b_0 + b_1 a_k + \dots \; b_{k-1} a_k^{k-1} = a_k^m$$

We know such $b_0, b_1, \dots, b_{k-1}$ exist if $\det M \neq 0$ where

$$M = \begin{pmatrix} 1 & a_1 & \cdots & a_1^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_k & \cdots & a_k^{k-1} \end{pmatrix}$$

But by Vandermonde $\det M = \prod_{1 \leq i < j \leq k} (a_j - a_i) \neq 0$ ■

27. Let $A = A_1 \times \cdots \times A_n$, where $A_1, \ldots, A_n$ are non-empty subsets of a field $\mathbb{F}$. Is it true that for every function $\varphi : A \to \mathbb{F}$ there is a polynomial $f(X) = f(X_1, \ldots, X_n) \in \mathbb{F}[X]$ such that $f(a) = \varphi(a)$ for every $a \in A$?

WLOG $A_1 = A_2 = \cdots = A_n = \mathbb{F}$ (worst case scenario)

Set $$f(x) = \sum_{(a_1, \ldots, a_n) \in \mathbb{F}^n} \left[\varphi(a) \prod_{\substack{a \in \mathbb{F} \\ a \neq a_1}} \frac{X_1 - a}{a_1 - a} \prod_{\substack{a \in \mathbb{F} \\ a \neq a_2}} \frac{X_2 - a}{a_1 - a} \cdots \prod_{\substack{a \in \mathbb{F} \\ a \neq a_n}} \frac{X_n - a}{a_n - a} \right]$$

Hard

28. Let $\mathbb{F}$ be a finite field of order q, and let $f \in \mathbb{F}[X_1, \dots, X_n]$ have degree $d \geq 1$. Show that f has at most dq^{n-1} zeros in $\mathbb{F}^n$.

$n=1$: Then we are happy. Now assume $n \geq 2$, $1 \leq d < q$.

Write $f = f_1 + f_2$, where f_1 consists of monomials of degree d in f, and f_2 of terms of deg $< d$ in f. By inductive argument, can show that there is some $v \in \mathbb{F}^n$ s.t. $f_1(v) \neq 0$. Consider, for each $w \in \mathbb{F}^n$, sets of the form

$S_w = \{ \underline{w} + t\underline{v} : t \in \mathbb{F} \}$

The $S_w : w \in \mathbb{F}^n$ form a partition of $\mathbb{F}^n$ to $q^n/q = q^{n-1}$ sets.

Consider for some fixed w, $f(\underline{w} + t\underline{v})$. This is a polynomial in t of degree d as coeff. of t^d is $f_1(\underline{v}) \neq 0$. So $f(\underline{w} + t\underline{v})$ has $\leq d$ zeroes, so f has $\leq dq^{n-1}$ zeroes.