Part III **Michaelmas, 2019**

COMBINATORICS

B. B.

Examples Sheet I.

If an exercise seems to make no sense, correct it and then solve it.

1. Let $P = (V, <)$ be a finite poset. Recall that a subset $U \subset V$ is a *chain* if any two elements of U are comparable, and it is an *antichain* if no two elements of U are comparable. Show that the maximal size of an antichain in P is equal to the minimal number of chains in P that cover V.

2. Let $(V, <)$ be a *finite ranked poset* with non-empty level sets $V_0, V_1, \ldots, V_n$. Suppose for $0 < i \leq n$ every $v \in V_i$ dominates exactly $d_i \geq 1$ elements of V_{i-1}, for $0 \leq i < n$ every $v \in V_i$ is dominated by exactly $e_i \geq 1$ elements of V_{i+1}, and the partial order on $V = \cup_0^n V_i$ is induced by these relations.

Show that if $U \subset V$ is an *antichain* then

$$\sum_0^n |U \cap V_i|/|V_i| \leq 1.$$

3. Let $\mathcal{F} \subset \mathcal{P}(n)$ a *Sperner family*, i.e. let $\mathcal{F}$ be such that $A \not\subset B$ whenever $A, B \in \mathcal{F}$, $A \neq B$. Show that

$$\sum_{k=0}^n f_k \Big/ \binom{n}{k} \leq 1,$$

where f_k is the number of k-sets in $\mathcal{F}$.

Can't do this.

4. Let $2 \leq 2r < n$ and let $\mathcal{F} = \mathcal{F}_r \cup \mathcal{F}_{n-r} \subset \mathcal{P}(n)$ be a Sperner family, where $\mathcal{F}_r \subset X^{(r)}$, $\mathcal{F}_{n-r} \subset X^{(n-r)}$ and $|\mathcal{F}_r| = |\mathcal{F}_{n-r}| = m$. At most how large is m?

Erdös ko rado

Most important

5. For $2 \leq r \leq n/2$, let $\mathcal{A} \subset X^{(r)}$ be an *intersecting family.* (Thus $A \cap B \neq \emptyset$, whenever $A, B \in \mathcal{A}$.) *Deduce from the Kruskal–Katona Theorem* that $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

And what is the maximal size of an intersecting family $\mathcal{A} \subset \mathcal{P}(X)$?

And if $\mathcal{A} \subset X^{(\leq r)}$?

6. Prove Lemma 5. Thus, let $\mathcal{A} \subset X^{(r)}$ and $1 \leq i, j \leq n$, $i \neq j$, and prove that

$$\partial(C_{ij}(\mathcal{A})) \subset C_{ij}(\partial\mathcal{A}).$$

Relies on 4

~~7~~. Let $2 \le 2k < n$, and let $\mathcal{A} \subset [n]^{(k)} \cup [n]^{(n-k)}$ be a Sperner system. Set $\mathcal{A}_i = \mathcal{A} \cap [n]^{(i)}$. At most how large is

$$\min\{|\mathcal{A}_k|, |\mathcal{A}_{n-k}|\}?$$

8. Let X be the disjoint union of sets Y and Z with $|Y|$ and $|Z|$ even. What is the maximal cardinality of a set system $\mathcal{A} \subset \mathcal{P}(X)$ if $A, B \in \mathcal{A}$, $A \neq B$ and $A \subset B$ imply that

$$A \cap Y \neq B \cap Y \quad \text{and} \quad A \cap Z \neq B \cap Z?$$

In lectures

Most important

~~9~~. Prove Lemma 7. Thus, for $U, V \subset X$ with $|U| = |V|$ and $U \cap V = \emptyset$, define the *UV-compression* of a set $A \subset X$ as follows:

$$C_{UV}(A) = \begin{cases} (A \cup U) \setminus V & \text{if } V \subset A,\ A \cap U = \emptyset, \\ A & \text{otherwise.} \end{cases}$$

Furthermore, for $\mathcal{A} \subset \mathcal{P}(n)$, set

$$C_{UV}(\mathcal{A}) = \{C_{UV}(A) :\ A \in \mathcal{A}\} \bigcup \{A \in \mathcal{A} :\ C_{UV}(A) \in \mathcal{A}\}.$$

A family $\mathcal{A} \subset \mathcal{P}(X)$ is *UV-compressed* if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Let $U, V \subset X$ with $|U| = |V|$ and $\max U < \max V$. Suppose that $\mathcal{A}$ is $U'V'$-compressed for all $|U'| = |V'| < |U| = |V|$ with $\max U' < \max V'$. Show that $\partial C_{UV}(\mathcal{A}) \subset C_{UV}(\partial \mathcal{A})$.

~~10~~. What is the 100^{th} element of $\mathbb{N}^{(5)}$ in the colex order? And the 100^{th} element of the cube Q_{10} in the simplicial order?

~~11~~. Let $x, x_1, \ldots, x_n$ be positive real numbers. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$ are equal to x.

12. Let $r \ge 1$ and $\mathcal{A} \subset X^{(r)}$ with $|\mathcal{A}| = \binom{y}{r} > 1$ and $|\partial \mathcal{A}| = \binom{y}{r-1}$. Is it true that y is an integer, and $\mathcal{A} = Y^{(r)}$ for some set $Y \in X^{(r)}$?

13. Give lower and upper bounds on the number of intersecting families $\mathcal{A} \subset \mathcal{P}(n)$ consisting of 2^{n-1} sets.

14. Let $Z_1, \ldots, Z_n$ be i.i.d. Bernoulli random variables with mean $p \ge 1/2$. [Thus $\mathbb{P}(Z_i = 1) = p$ and $\mathbb{P}(Z_i = 0) = 1 - p$.] Let $c_1, \ldots, c_n$ be positive reals summing to 1. Show that

$$\mathbb{P}\Big(\sum_{i=1}^{n} c_i Z_i \ge 1/2\Big) \ge p.$$

Hint. Check that wma that n is odd, $c_i > 0$ for every i, and there is no $A \subset [n]$ with $\sum_{i \in A} c_i = 1/2$. Set $\mathcal{A} = \{A \subset [n] :\ \sum_{i \in A} c_i > 1/2\}$,

$\mathcal{A}_k = \mathcal{A} \cap [n]^{(k)}$ and $a_k = |\mathcal{A}_k|$. What can you say about the sequence $(a_k)_1^n$?

15. Let $\mathcal{A} \subset \mathbb{N}^{(<\omega)}$ be an intersecting family of finite subsets of $\mathbb{N}$. Is there a finite set F such that $\{A \cap F : A \in \mathcal{A}\}$ is also intersecting?

16. Let $\mathcal{P}(n) = \bigcup_{i=1}^{k} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an intersecting family. At least how large is k?

17. Is there an integer m such that if $\mathcal{A} \subset \mathbb{N}^{(\leq 3)}$ is a finite intersecting family then there is a set $M \in \mathbb{N}^{(m)}$ such that $A_i \cap A_j \cap M \neq \emptyset$ for all $A_i, A_j \in \mathcal{A}$?

~~Isoperimetric inequalities in $\mathbb{R}^n$~~ - Not related to the course / won't be examined.

1. Let A be a non-empty subset of $\mathbb{R}^n$. Show that for $\varepsilon > 0$ the ε-neighbourhood of A,

$$A_{(\varepsilon)} = \{x \in A : \ d(x, A) < \varepsilon\},$$

is Lebesgue measurable.

2. The *outer measure* of a set $A \subset \mathbb{R}^n$ is

$$\lambda^*(A) = \inf \sum_{i=1}^{\infty} |B_i|,$$

where the infimum is taken over all covers $\bigcup_{i=1}^{\infty} B_i$ of A by *boxes* ('bricks', rectangular parallelepipeds, with sides parallel to the axes), and $| \, . \, |$ stands for the volume of a set, so that if B_i is a $b_1 \times \cdots \times b_n$ box then $|B_i| = \prod_1^n b_i$. Is it true that if $\emptyset \neq A \subset \mathbb{R}^n$ then

$$\lambda^*(A) = \lim_{\varepsilon \to 0+} |A_{(\varepsilon)}|?$$

3. Let a set $A \subset \mathbb{R}^n$ have outer measure m, and let $B \subset \mathbb{R}^n$ be a Euclean ball of volume m. Show that for $\varepsilon > 0$ we have

$$|B_{(\varepsilon)}| \leq |A_{(\varepsilon)}|.$$

4. Let A be a bounded open set in $\mathbb{R}^n$. Given a hyperplane (i.e. 1-codimensional subspace) H in $\mathbb{R}^n$ and a line ℓ perpendicular to H, let $\ell(A)$ be the open segment on ℓ that is symmetric about H and whose length is the measure ('length') of $A \cap \ell$. Define

$$H(A) = \bigcup \{\ell(A) : \ \ell \text{ is perpendicular to } H\},$$

so that $H(A)$ is an open set in $\mathbb{R}^n$ symmetric about H, with $\mathrm{vol}_n(H(A) = \mathrm{vol}_n A$. Show that for $\varepsilon > 0$ we have

$$\mathrm{vol}_n(H(A)_{(\varepsilon)}) \le \mathrm{vol}_n(A_{(\varepsilon)}).$$

5. Prove the following classical isoperimetric inequality in $\mathbb{R}^n$.
Let $A \subset \mathbb{R}^n$ have outer measure m and let $B \subset \mathbb{R}^n$ be a Euclidean ball of volume m. Then for $\varepsilon > 0$ we have

$$\mathrm{vol}_n(A_{(\varepsilon)}) \ge \mathrm{vol}_n(B_{(\varepsilon)}). \tag{1}$$

Hint. You may use the following fact. Let $\mathcal{C}$ be the collection of closed subsets of $[0,1]^n \subset \mathbb{R}^n$, say. For $A, B \in \mathcal{C}$, put $d(A,B) = \inf\{\varepsilon > 0 : A \subset B_{(\varepsilon)}, B \subset A_{(\varepsilon)}\}$. Then $(\mathcal{C}, d)$ is a compact metric space.

6. Give another proof of the classical isoperimetric inequality along the lines below.

Let A be a bounded open set in $\mathbb{R}^n$. Given a 1-dimensional subspace ℓ of $\mathbb{R}^n$ (i.e. a line through the origin) and an *affine* hyperplane orthogonal to ℓ, let $D_H(A)$ be the open (Euclidean) ball in H with centre $H \cap \ell$, whose $(n-1)$-dimensional volume is $\mathrm{vol}_{n-1}(A \cap H)$. Define

$$S_\ell(A) = \bigcup \{D_H(A) : H \text{ is perpendicular to } \ell\}.$$

Show that $\mathrm{vol}_n\big((S_\ell(A))_{(\varepsilon)}\big) \le \mathrm{vol}_n(A_{(\varepsilon)})$ for every $\varepsilon > 0$, and use it to prove the classical isoperimetric inequality (1) in $\mathbb{R}^n$.

7. Let $K \subset \mathbb{R}^n$ be the union of a finite collection of disjoint boxes, each with sides parallel with the axes, and let L be defined similarly. The sum of these sets is

$$K + L = \{x + y : \ x \in K, \ y \in L\}.$$

Show that

$$\big(\mathrm{vol}_n(K+L)\big)^{1/n} \ge \big(\mathrm{vol}_n(K)\big)^{1/n} + \big(\mathrm{vol}_n(L)\big)^{1/n} \tag{2}$$

Hint. If K is made up of at least two boxes, find a coordinate affine hyperplane (i.e. perpendicular to one of the axes that separates the interiors of two of them.

8. Deduce from (2) that the same inequality holds if K and L are open sets. [In fact, the same holds if K and L are Lebesgue measurable.]

9. Use the inequality above to give a third proof of the classical isoperimetric inequality (1).

Reviewed
Known as
Dilworth's Thm

1. Let $P = (V, <)$ be a finite poset. Recall that a subset $U \subset V$ is a *chain* if any two elements of U are comparable, and it is an *antichain* if no two elements of U are comparable. Show that the maximal size of an antichain in P is equal to the minimal number of chains in P that cover V.

Let N_1 = max size of antichain N_2 = minimum number of chains that cover V.

$N_2 \geq N_1$: Given $A_1, \dots, A_{N_2}$ minimal number of chains covering V. Any antichain B can contain atmost one element from each A_i.

$\Rightarrow N_1 \geq |B| \geq N_2$.

$N_1 \geq N_2$: We prove this by induction on n, the size of the partially ordered set. Theorem trivially true if P empty. Assume P has atleast one element, let a be maximal in P. By induction, assume $\exists k$: $P' := P \setminus \{a\}$ can be covered by k-disjoint chains $C_1, \dots, C_k$ and there is an antichain A_0 of size atleast k.

Have $A_0 \cap C_i \neq \emptyset$. Let x_i be the maximal element of C_i belonging to an antichain of length atleast k.

Let $A = \{x_1, \dots, x_k\}$, A is an antichain: Let A_i be an antichain of size k that contains x_i, fix $i \neq j$ arbitrarily. Then $A_i \cap C_j \neq 0$. Suppose $y \in A_i \cap C_j$. Then $y \leq x_j$ since x_j maximal in C_j. Thus $x_i \not\leq x_j$ since $x_i \not\leq y$. Exchanging i, j gives $x_j \not\leq x_i$.

Now suppose $a \geq x_i$ for some $i \in \{1, 2, \dots, k\}$. Let K be the chain $\{a\} \cup \{z \in C_i : z \leq x_i\}$. Then by choice of x_i, $P \setminus K$ does not have an antichain of size k. Induction $\Rightarrow$ $P \setminus K$ can be covered by $k-1$ disjoint chains since $A \setminus \{x_i\}$ is an antichain of size $k-1$ in $P \setminus K$. Thus, P can be covered by k disjoint chains.

Otherwise if $a \not\geq x_i$ $\forall i \in \{1, \dots, k\}$. Then $A \cup \{a\}$ is an antichain of size $k+1$ in P. P can be covered by $k+1$ chains $\{a\}, C_1, \dots, C_k$.

Notes: This is a very long proof, good to understand the key ideas else it can seem intractable

Reviewed

2. Let $(V,<)$ be a *finite ranked poset* with non-empty level sets $V_0, V_1, \dots, V_n$. Suppose for $0 < i \le n$ every $v \in V_i$ dominates exactly $d_i \ge 1$ elements of V_{i-1}, for $0 \le i < n$ every $v \in V_i$ is dominated by exactly $e_i \ge 1$ elements of V_{i+1}, and the partial order on $V = \cup_0^n V_i$ is induced by these relations.

Show that if $U \subset V$ is an *antichain* then

$$\sum_0^n |U \cap V_i|/|V_i| \le 1.$$

There are $|V_0| e_0 e_1 \dots e_{n-1}$ chains of maximal length in V.

Have $|V_i| e_i = |V_{i+1}| d_{i+1}$. $= |V_n| d_n d_{n-1} \dots d_1$

$\forall\ 0 \le i < n$.

For each maximal chain C we have $|C \cap U| \le 1$ as U is an antichain

Every element of V of rank k is contained in exactly $(d_k d_{k-1} \dots d_1)(e_k \dots e_{n-1})$ maximal chains.

Thus $\sum_0^n |U \cap V_k| (d_k \dots d_1)(e_k \dots e_{n-1}) = \#$ maximal chains $= |V_0| e_0 \dots e_{n-1}$

$|V_0| e_0 \dots e_{n-1} = d_1 \dots d_k |V_k| e_k \dots e_{n-1}$ $\forall\ 0 \le k \le n$

$\Rightarrow \sum_0^n \frac{|U \cap V_k|}{|V_k|} \le 1$ ∎

Reviewed

LYM Inequality

3. Let $\mathcal{F} \subset \mathcal{P}(n)$ a *Sperner family*, i.e. let $\mathcal{F}$ be such that $A \not\subset B$ whenever $A, B \in \mathcal{F}$, $A \neq B$. Show that

$$\sum_{k=0}^{n} f_k / \binom{n}{k} \leq 1,$$

where f_k is the number of k-sets in $\mathcal{F}$.

Proof 1 Let $A \subset X^{(r)}$, by Local LYM we have $\frac{|\partial A|}{\binom{n}{r-1}} \geq \frac{|A|}{\binom{n}{r}}$. $\mathcal{F}_r = \mathcal{F} \cap X^{(r)}$.

We have $\frac{|\mathcal{F}_n|}{\binom{n}{n}} \leq 1$ and $\partial \mathcal{F}_n$ & $\mathcal{F}_{n-1}$ are disjoint subsets of $X^{(r-1)}$.

Thus $1 \geq \frac{|\partial \mathcal{F}_n \cup \mathcal{F}_{n-1}|}{\binom{n}{n-1}} = \frac{|\partial \mathcal{F}_n|}{\binom{n}{n-1}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}} \geq \frac{|\mathcal{F}_n|}{\binom{n}{n}} + \frac{|\mathcal{F}_{n-1}|}{\binom{n}{n-1}}$. Repeat this to get the result.

Proof 2

Let $V = P(n)$ as in question 2 ranked by inclusion. Then every set of size k contains k sets of size $k-1$ and is contained in $k+1$ sets of size $k+1$. Thus from Q2, $|V_k| = \binom{n}{k}$ and $|U \cap V_k| = f_k$.

So $\sum_{0}^{n} f_k / \binom{n}{k} \leq 1$

Favourite

Proof 3 This is similar to proof 2, but uses probability.

Pick uniformly at random a chain in Q_n. Take $A \in X^{(r)}$.

$\mathbb{P}(C \text{ meets } A) = 1/\binom{n}{r}$ for any chain C.

$\mathbb{P}(C \text{ meets } \mathcal{F}_r) = f_r / \binom{n}{r}$

$\mathbb{P}(C \text{ meets } \mathcal{F}) = \sum_r f_r / \binom{n}{r} \leq 1$ since it is a probability.

Proof 2 (probability)

Pick unif. at random chain in Q_n.

Take $A \in X^{(r)}$ $\mathbb{P}(C \text{ meets } A) = \frac{1}{\binom{n}{r}}$

$\mathbb{P}(C \text{ meets } \mathcal{F}_r) = f_r / \binom{n}{r}$

$\mathbb{P}(C \text{ meets } \mathcal{F}) = \sum_r f_r / \binom{n}{r}$

Reviewed

Answer may be wrong but method is correct.

4. Let $2 \leq 2r < n$ and let $\mathcal{F} = \mathcal{F}_r \cup \mathcal{F}_{n-r} \subset \mathcal{P}(n)$ be a Sperner family, where $\mathcal{F}_r \subset X^{(r)}$, $\mathcal{F}_{n-r} \subset X^{(n-r)}$ and $|\mathcal{F}_r| = |\mathcal{F}_{n-r}| = m$. At most how large is m?

We have that $|\partial^{n-2r} \mathcal{F}_{n-r}| \geq \binom{n-1}{r}$. Use $\partial^{n-2r}\mathcal{F}_{n-r}$ & $\mathcal{F}_r$ disjoint.

$\frac{|\partial^{n-2r}\mathcal{F}_{n-r}|}{\binom{n}{r-1}} \geq \frac{|\partial^{n-2r+1}\mathcal{F}_{n-r}|}{\binom{n}{r}} \geq \dots \geq \frac{|\mathcal{F}_{n-r}|}{\binom{n}{r}}$ and $\partial^{n-2r}\mathcal{F}_{n-r}$ & $\mathcal{F}_r$ disjoint gives that $|\partial^{n-2r}\mathcal{F}_{n-r}| + |\mathcal{F}_r| \leq \binom{n}{r}$

$\Rightarrow \frac{\binom{n}{r} - |\mathcal{F}_r|}{\binom{n}{r-1}} \geq \frac{|\mathcal{F}_{n-r}|}{\binom{n}{r}}$ so $(r-1)!\,(n-r+1)!\left[\binom{n}{r} - m\right] \geq m\, r!\,(n-r)!$

$\Rightarrow \binom{n}{r} - m \geq \frac{mr}{(n-r+1)}$ so $\frac{m}{(n-r+1)}(r+n-r+1) \leq \binom{n}{r}$

$\Rightarrow m \leq \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r+1)}{n+1} = \frac{(n-r+1)}{n+1}\binom{n}{r}$

Reviewed.

5. For $2 \leq r \leq n/2$, let $\mathcal{A} \subset X^{(r)}$ be an *intersecting family*. (Thus $A \cap B \neq \emptyset$, whenever $A, B \in \mathcal{A}$.) *Deduce from the Kruskal–Katona Theorem* that $|\mathcal{A}| \leq \binom{n-1}{r-1}$.

And what is the maximal size of an intersecting family $\mathcal{A} \subset \mathcal{P}(X)$?

And if $\mathcal{A} \subset X^{(\leq r)}$?

Let $A, B \in \mathcal{A}$, then $A \not\subset B^c$ so $\mathcal{A}, \mathcal{A}^c$ are Sperner family. Then apply Q4.

A best possible solution can be achieved by including all sets containing / not containing 1

For $\mathcal{A} \subset \mathcal{P}(X)$, max size is 2^{n-1} since a set & it's complement cannot both be in a Sperner family, e.g. take $\mathcal{A} = \{A : 1 \in A\}$

For $\mathcal{A} \subset X^{(\leq r)}$: Apply $|\mathcal{A} \cap X^{(s)}| \leq \binom{n-1}{s-1}$ $\forall s \leq r$ and note $\mathcal{A} = \{A : |A| \leq r$ and $1 \in A\}$

5. For $2 \le r \le n/2$, let $\mathcal{A} \subset X^{(r)}$ be an *intersecting family*. (Thus $A \cap B \neq \emptyset$, whenever $A, B \in \mathcal{A}$.) *Deduce from the Kruskal–Katona Theorem* that $|\mathcal{A}| \le \binom{n-1}{r-1}$.

Kruskal Katona $\Rightarrow$ $|\mathcal{B}| \geq \binom{n}{r} \Rightarrow |\partial \mathcal{B}| \geq \binom{n-1}{r-1}$

$A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A} \Rightarrow A \Delta B \neq \emptyset \ \forall A, B$.

There exists $\mathcal{A}$ intersecting with $|\mathcal{A}| = \binom{n-1}{r-1}$. Just pick $a \in [n]$ and pick $\mathcal{A} = \{A \in X^{(r)} : a \in A\}$.

This is Erdős – Ko – Rado

Define $B_s := \{s, s+1, \dots, s+r-1\} \subseteq X$ for $s \in X$, where addition is modulo n.

Claim: Atmost r of the sets B_s can be in $\mathcal{A}$.

Proof: Since modulo n doesn't matter what value of s we start at. $B_s \in \mathcal{A}$ and $B_{s+r} \in \mathcal{A} \Rightarrow$ if $\mathcal{A}$ maximal $B_{s+j} \in \mathcal{A} \ \forall 1 \le j < r$. Thus best choice of $(r+1)$ sets $B_s, B_{s+1}, \dots, B_{s+r}$ however

$B_s \cap B_{s+r} = \emptyset$ so can have atmost r sets $\{B_s, \dots, B_{s+r-1}\}$

(or split into disjoint sets)

Idea: Double count pairs (π, s) such that π permutation of S and $\pi(B_s) = (\pi(s), \pi(s+1), \dots, \pi(s+r-1)) \in \mathcal{A}$

Let L be # of such pairs.

Fixing B_s gives $r!(n-r)!$ to permute B_s so $L = |\mathcal{A}|(r!)(n-r)!$

Also there are $(n-1)!$ cyclic permutations of X, each of which contains at most r of B_s. So $L \le r(n-1)!$

$\Rightarrow |\mathcal{A}| r! (n-r)! \le r(n-1)! \Rightarrow |\mathcal{A}| \le \binom{n-1}{r-1}$

OR Bubble down using Kruskal - Katona

6. Prove Lemma 5. Thus, let $\mathcal{A} \subset X^{(r)}$ and $1 \leq i, j \leq n$, $i \neq j$, and prove that

$$\partial(C_{ij}(\mathcal{A})) \subset C_{ij}(\partial \mathcal{A}).$$

Let $\mathcal{A}' = C_{ij}(\mathcal{A})$. Will show that if $B \in \partial\mathcal{A}' - \partial\mathcal{A}$ then $i \in B$, $j \notin B$ and $B \cup \{j\} - \{i\} \in \partial\mathcal{A} - \partial\mathcal{A}'$. This gives injection.

Let $B \in \partial\mathcal{A}' - \partial\mathcal{A}$ then $B \cup \{x\} \in \mathcal{A}'$ for some $x \notin A$ (since $B \notin \partial\mathcal{A}$).

Have $i \in B \cup \{x\}$, $j \notin B \cup \{x\}$ and $(B \cup \{x\}) \cup \{j\} - \{i\} \in \mathcal{A}$

$x \neq i$ else $B \cup \{j\} \in \mathcal{A}$. Have $B \cup \{j\} - \{i\} \in \partial\mathcal{A}$.

Claim: $B \cup \{j\} - \{i\} \notin \partial\mathcal{A}'$

Proof: Suppose $(B \cup \{y\}) \cup \{j\} - \{i\} \in \mathcal{A}'$. $y \neq i$ else $B \cup \{j\} \in \mathcal{A}' \Rightarrow B \cup \{j\} \in$

$\Rightarrow B \in \partial\mathcal{A}$ #

Thus $j \in (B \cup \{y\}) \cup \{j\} - \{i\}$, $i \notin (B \cup \{y\}) \cup \{j\} - \{i\}$.

so $(B \cup \{j\} - \{i\}) \cup \{y\} \in \mathcal{A}$ and $B \cup \{y\} \in \mathcal{A}$

contradicting fact that $B \in \partial\mathcal{A}' - \partial\mathcal{A}$.

Thus done.

Struggling to understand this proof.

7. Let $2 \leq 2k < n$, and let $\mathcal{A} \subset [n]^{(k)} \cup [n]^{(n-k)}$ be a Sperner system. Set $\mathcal{A}_i = \mathcal{A} \cap [n]^{(i)}$. At most how large is

$$\min\{|\mathcal{A}_k|, |\mathcal{A}_{n-k}|\}?$$

$\min\{|A_k|, |A_{n-k}|\} =$ max m from Q4

This is since we can just remove extra elements from the larger set.

This is a very difficult problem!

8. Let X be the disjoint union of sets Y and Z with $|Y|$ and $|Z|$ even. What is the maximal cardinality of a set system $\mathcal{A} \subset \mathcal{P}(X)$ if $A, B \in \mathcal{A}$, $A \neq B$ and $A \subset B$ imply that

$$A \cap Y \neq B \cap Y \quad \text{and} \quad A \cap Z \neq B \cap Z?$$

9. Prove Lemma 7. Thus, for $U, V \subset X$ with $|U| = |V|$ and $U \cap V = \emptyset$, define the *UV-compression* of a set $A \subset X$ as follows:

$$C_{UV}(A) = \begin{cases} (A \cup U) \setminus V & \text{if } \ V \subset A, \ A \cap U = \emptyset, \\ A & \text{otherwise.} \end{cases}$$

Furthermore, for $\mathcal{A} \subset \mathcal{P}(n)$, set

$$C_{UV}(\mathcal{A}) = \{C_{UV}(A) : \ A \in \mathcal{A}\} \bigcup \{A \in \mathcal{A} : \ C_{UV}(A) \in \mathcal{A}\}.$$

A family $\mathcal{A} \subset \mathcal{P}(X)$ is *UV-compressed* if $C_{UV}(\mathcal{A}) = \mathcal{A}$.

Let $U, V \subset X$ with $|U| = |V|$ and $\max U < \max V$. Suppose that $\mathcal{A}$ is $U'V'$-compressed for all $|U'| = |V'| < |U| = |V|$ with $\max U' < \max V'$. Show that $\partial C_{UV}(\mathcal{A}) \subset C_{UV}(\partial \mathcal{A})$.

Proved in lectures Fiddly proof to learn by heart.

Reviewed
Correct method

10. What is the 100th element of $\mathbb{N}^{(5)}$ in the colex order? And the 100th element of the cube Q_{10} in the simplicial order?

$$100 = \binom{a_5}{5} + \binom{a_4}{4} + \binom{a_3}{3} + \binom{a_2}{2} + \binom{a_1}{1}$$

$a_5 = 8$, $a_4 = 7$, $a_3 = 4$, $a_2 = 3$, $a_1 = 1$

$$56 + 35 + 4 + 3 + 2 = 99$$

Thus 100^{th} element is 24589 which has $\binom{8}{5} + \binom{7}{4} + \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$
$= 99$ elements less than it.

Simplicial on Q_{10} : $0, 1, \dots, 9, 01, 02, 12, 03, 13, 23, \dots$

$$\binom{10}{1} + \binom{10}{2} = 10 + \frac{10 \times 9}{2} = 10 + 45 = 55.$$

Want 45^{th} element of $\binom{[10]}{3}$ in lex.

Suppose it is $a_1 a_2 a_3$. $x_1 x_2 x_3 < y_1 y_2 y_3$ iff $x_3 < y_3$ or $x_3 = y_3 \wedge x_2 < y_2$
or $x_3 = y_3 \wedge x_2 = y_2 \wedge x_1 < y_1$

Then $44 = \# \; x_1 x_2 x_3 \in \binom{[10]}{3}$ s.t. $x_1 x_2 x_3 < a_1 a_2 a_3$

$$= \binom{a_3 - 1}{3} + \binom{a_2 - 1}{2} + \binom{a_1 - 1}{1}$$

$\Rightarrow$ $a_3 = 8$ $a_2 = 4$ $a_1 = 5$ (colex) $\Rightarrow$ $\boxed{548}$

Reviewed

11. Let $x, x_1, \ldots, x_n$ be positive real numbers. Show that at most $\binom{n}{\lfloor n/2 \rfloor}$ of the sums $\sum_{i \in A} x_i$ are equal to x.

Idea: Use sperners theorem by making an antichain $\mathcal{A} \subseteq P(n)$ such that $A \in \mathcal{A}$ iff $\sum_{i \in A} x_i = x$.

Let $\mathcal{A} = \{A : \sum_{i \in A} x_i = x\}$ then $\forall A, B \in \mathcal{A}$ $A \subseteq B \Rightarrow A = B$ since the $x_i > 0$ $\forall i$ so if $A \subset B$ with $A \neq B$ and WLOG $|A| < |B|$ then $\exists j : \{x_j\} \cup A \subseteq B$ $\Rightarrow x = \sum_{i \in A} x_i + x_j = x + x_j \Rightarrow x_j = 0$ #.

Thus $\mathcal{A}$ is an antichain so by Sperner, $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ and we are done.

Note: LYM $\Rightarrow$ $|\mathcal{A}| \leq \max_r \binom{n}{r} = \binom{n}{\lfloor n/2 \rfloor}$ which gives Sperner.

12. Let $r \geq 1$ and $\mathcal{A} \subset X^{(r)}$ with $|\mathcal{A}| = \binom{y}{r} > 1$ and $|\partial \mathcal{A}| = \binom{y}{r-1}$. Is it true that y is an integer, and $\mathcal{A} = Y^{(r)}$ for some set $Y \in X^{(r)}$?

What would this mean if y was not an integer? This question was not understood by anyone.

13. Give lower and upper bounds on the number of intersecting families $\mathcal{A} \subset \mathcal{P}(n)$ consisting of 2^{n-1} sets.

Any intersecting family can be extended to an intersecting family of 2^{n-1} subsets by adding either a set or its compliment.

Each intersecting family contains atmost one of a set or its compliment so there are atmost $\binom{2^n}{2^{n-1}}$ intersecting families of size 2^{n-1}.

$$\binom{2^n}{2^{n-1}} \approx \frac{\sqrt{2\pi 2^n}\left(\frac{2^n}{e}\right)^{2^n}}{\left(\sqrt{2\pi 2^{n-1}}\right)^2\left(\frac{2^{n-1}}{e}\right)^{2^n}} = \frac{1}{\sqrt{2\pi}}\frac{2^{n/2}2^{2^n}}{2^{n-1}} = \frac{1}{\sqrt{2\pi}}2^{1-n/2}2^{2^n} = \frac{1}{\sqrt{\pi}}2^{2^n-\frac{n}{2}+1/2}$$

This question was not done in the examples class

14. Let $Z_1, \ldots, Z_n$ be i.i.d. Bernoulli random variables with mean $p \geq 1/2$. [Thus $\mathbb{P}(Z_i = 1) = p$ and $\mathbb{P}(Z_i = 0) = 1 - p$.] Let $c_1, \ldots, c_n$ be positive reals summing to 1. Show that

$$\mathbb{P}\left(\sum_{i=1}^{n} c_i Z_i \geq 1/2\right) \geq p.$$

Hint. Check that wma that n is odd, $c_i > 0$ for every i, and there is no $A \subset [n]$ with $\sum_{i \in A} c_i = 1/2$. Set $\mathcal{A} = \{A \subset [n] : \sum_{i \in A} c_i > 1/2\}$,

$\mathcal{A}_k = \mathcal{A} \cap [n]^{(k)}$ and $a_k = |\mathcal{A}_k|$. What can you say about the sequence $(a_k)_1^n$?

Removing all the $c_i = 0$ does not effect the value of the sum $\sum_{i=1}^{n} c_i Z_i$ or $\sum_{i=1}^{n} c_i$, so will assume wlog $c_i > 0$ $\forall i$.

Now suppose n is even, then define $c_i' = c_i$ $\forall i <$
$c_n' = \frac{1}{2} c_n$ $\quad c_{n+1}' = \frac{1}{2} c_n$.

Then $\mathbb{P}\left(\sum_{i=1}^{n+1} c_i Z_i \geq p\right) = \mathbb{P}\left(\sum_{i=1}^{n-1} c_i Z_i + \frac{1}{2} c_n (Z_n + Z_{n+1}) \geq \frac{1}{2}\right)$

$$\geq \mathbb{P}\left(\sum_{i=1}^{n} c_i Z_i \geq \frac{1}{2}\right)$$

Hmm. How to show wma n odd?

Suppose $\exists$ $A \subset [n]$ s.t. $\sum_{i \in A} c_i = 1/2$.

Then $\mathbb{P}\left(\sum_{i=1}^{n} c_i Z_i \geq \frac{1}{2}\right) = \mathbb{P}\left(\sum_{i \in A} c_i Z_i + \sum_{i \notin A} c_i Z_i \geq \frac{1}{2}\right)$

This is not true!

$$\geq \mathbb{P}\left(\max\left\{\sum_{i \in A} c_i Z_i, \sum_{i \notin A} c_i Z_i\right\} \geq \frac{1}{4}\right)$$

$$= \mathbb{P}\left(\sum_{i \in A} c_i Z_i \geq \frac{1}{4}\right) + \mathbb{P}\left(\sum_{i \notin A} c_i Z_i \geq \frac{1}{4}\right) - \mathbb{P}(\text{Both are})$$

ind. hyp.

$$= 2p - p^2 = p + (p - p^2) \geq p$$

Let $\mathcal{A} = \{ A \subset [n] : \sum_{i \in A} c_i > 1/2 \}$ and let $\mathcal{A}_k = \mathcal{A} \cap [n]^{(k)}$ and $a_k = |\mathcal{A}_k|$

What can we say about $(a_k)_{k=1}^{n}$:

Note that for $A \subseteq B$ and $A \in \mathcal{A}$, $B \in \mathcal{A}$.

This question was not done in the examples class

15. Let $\mathcal{A} \subset \mathbb{N}^{(<\omega)}$ be an intersecting family of finite subsets of $\mathbb{N}$. Is there a finite set F such that $\{A \cap F : A \in \mathcal{A}\}$ is also intersecting?

No. Take $\{1,3,4,5\}$, $\{2,4,5,6\}$, $\{1,3,5,6,7\}$, $\{2,4,6,7,8\}$, $\{1\ 3\ 5\ 7\ 8\ 9\}$, ...

16. Let $\mathcal{P}(n) = \bigcup_{i=1}^{k} \mathcal{A}_i$, where each $\mathcal{A}_i$ is an intersecting family. At least how large is k?

$P(n) = \{\emptyset\} \cup \bigcup_{i=1}^{n} \{A : i \in A\}$ so $k \geq n+1$, I think. Can I show this is impossible with $k = n$?