ANALYTIC NUMBER THEORY

THOMAS F. BLOOM

These are lecture notes for the Part III lecture course given in Lent Term 2020. They are meant to be a faithful copy of the material given in lectures, with some supplementary footnotes and historical notes. The lectures themselves are the guide for what material is examinable, and any additional material in these printed notes will be marked as non-examinable. In the case of any doubt, ask the lecturer.

My principle sources in preparing these notes were Montgomery and Vaughan's Multiplicative Number Theory and Titchmarsh's Theory of the Riemann Zeta Function.

If you have any questions, concerns, corrections, please contact the lecturer at `tb634@cam.ac.uk`.

CHAPTER 1

Elementary Techniques

Review of asymptotic notation. We write $f(x) = O(g(x))$ if there exists some constant $C > 0$ such that $|f(x)| \leq C\,|g(x)|$ for all sufficiently large x. We will also use the Vinogradov notation $f \ll g$ to denote the same thing (so that $f = O(g)$ and $f \ll g$ are equivalent). Occasionally we will use subscript notation to denote dependence of the constants. For example, $f \ll_\delta g$ means there exists some constant $C(\delta)$ depending on δ such that $|f(x)| \leq C(\delta)\,|g(x)|$ for all sufficiently large x (where sufficiently large may also depend on δ).

We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We write $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Observe that

$$f \sim g \text{ if and only if } f = (1 + o(1))g.$$

We will also write $f \asymp g$ to mean $f \ll g \ll f$.

1. Arithmetic functions

An arithmetic function is simply a function on the natural numbers[1], $f : \mathbb{N} \to \mathbb{R}$. An arithmetic function is multiplicative if

$$f(nm) = f(n)f(m) \text{ whenever } (n, m) = 1,$$

and is completely multiplicative if $f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{N}$. Some important examples of multiplicative functions are

(1) the constant function $\mathbf{1}(n) = 1$ for all n,
(2) the delta function

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and
(3) the Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_i \neq p_j \text{ and} \\ 0 & \text{if some } p^2 \mid n. \end{cases}$$

Note that every multiplicative function satisfies $f(1) = 1$.

An important operation on the space of arithmetic functions is that of multiplicative convolution:

$$f \star g(n) = \sum_{ab=n} f(a)g(b).$$

If f and g are both multiplicative functions, then so too is $f \star g$. We list some basic facts about this operation:

[1] For the purposes of this course, 0 is not a natural number.

(1) it is commutative,
(2) it is associative,
(3) δ acts as identity, so that $\delta \star f = f$,
(4) if f and g are both multiplicative then so is $f \star g$, and
(5) (Möbius inversion) the Möbius function acts as an inverse, in that

$$\mu \star f = g \text{ if and only if } \mathbf{1} \star g = f.$$

All facts are easy to verify, and we will prove only the last. It suffices to show that $\mathbf{1} \star \mu = \delta$, that is, for every n,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since both sides are multiplicative functions (the left-hand side by fact (4) above) it suffices to check this identity when n is a power of a prime, say p^k. If $k = 0$ then the left-hand side is just $\mu(1) = 1$, and if $k \geq 1$, the left-hand side is

$$\mu(1) + \mu(p) + \cdots + \mu(p^k) = \mu(1) + \mu(p) = 1 + (-1) = 0.$$

A great deal of analytic number theory is concerned with a deep study of the distribution of the prime numbers. For this the 'correct' way to count primes is not, as one might expect, the indicator function

$$1_{\mathbb{P}}(n) = \begin{cases} 1 & \text{if } n \text{ is prime, and} \\ 0 & \text{otherwise,} \end{cases}$$

but instead the von Mangoldt function, which firstly also counts prime powers p^k, but also counts them not with weight 1, but with weight $\log p$ instead:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The main reason that this function is much easier to work with than $1_{\mathbb{P}}$ directly, is the following identity.

Lemma 1.

$$\mathbf{1} \star \Lambda(n) = \log n \textit{ and } \log \star \mu(n) = \Lambda(n).$$

Proof. The second identity follows from the first by Möbius inversion. To establish the first, if we let $n = p_1^{k_1} \cdots p_r^{k_r}$, then

$$\begin{aligned} \mathbf{1} \star \Lambda(n) &= \sum_{i=1}^{r} \sum_{j=1}^{k_i} \log p_i \\ &= \sum_{i=1}^{r} \log p_i^{k_i} \\ &= \log n. \end{aligned}$$

□

2. Summation

A major theme of analytic number theory is understanding the basic arithmetic functions, particularly how large they are on average, which means understanding $\sum_{n\leq x} f(n)$. For example, if f is the indicator function of primes, then this summatory function is precisely the prime counting function $\pi(n)$.

One of the most useful tools in dealing with summations is partial summation, which is a discrete analogue of integrating by parts.

Theorem 1 (Partial summation)**.** *If a_n is any sequence of complex numbers and $f : \mathbb{R}_+ \to \mathbb{R}$ is such that f' is continuous then*

$$\sum_{1\leq n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t,$$

where $A(x) = \sum_{1\leq n\leq x} a_n$.

Proof. Let $N = \lfloor x \rfloor$. Using $a_n = A(n) - A(n-1)$

$$\begin{aligned}\sum_{1\leq n\leq N} a_n f(n) &= \sum_{n=1}^{N} f(n)(A(n) - A(n-1)) \\ &= f(N)A(N) - \sum_{n=1}^{N-1} A(n)(f(n+1) - f(n)).\end{aligned}$$

We now observe that

$$\int_n^{n+1} f'(x)\,\mathrm{d}x = f(n+1) - f(n),$$

and so, since $A(x)$ is constant for $x \in [n, n+1)$,

$$\sum_{1\leq n\leq N} a_n f(n) = f(N)A(N) - \sum_{n=1}^{N-1} \int_n^{n+1} A(x)f'(x)\,\mathrm{d}x,$$

and the result follows since if $N \leq x < N+1$ then

$$A(x)f(x) = A(N)f(x) = A(N)\left(f(N) + \int_N^x f'(x)\,\mathrm{d}x\right).$$

□

This is extremely useful even when the coefficients a_n are identically 1, when $A(x) = \lfloor x \rfloor = x + O(1)$.

Lemma 2.

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O(1/x),$$

where $\gamma = 0.577\cdots$ is a constant, known as Euler's constant.

Proof. By partial summation

$$\begin{aligned}\sum_{n\leq x}\frac{1}{n} &= \frac{\lfloor x\rfloor}{x} + \int_1^x \frac{\lfloor t\rfloor}{t^2}\,\mathrm{d}t\\ &= 1 + \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t - \int_x^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t + O(1/x)\\ &= \log x + \left(1 + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t\right) + O(1/x).\end{aligned}$$

It remains to note that the second term is a constant, since the integral converges. □

It is remarkable how little we understand about Euler's constant – it is not even known whether it is irrational or not.

Lemma 3.

$$\sum_{1\leq n\leq x}\log n = x\log x - x + O(\log x).$$

Proof. By partial summation

$$\begin{aligned}\sum_{n\leq x}\log n &= \lfloor x\rfloor \log x - \int_1^x \frac{\lfloor t\rfloor}{t}\,\mathrm{d}t\\ &= x\log x - x + O(\log x).\end{aligned}$$

□

We now give an application to a more number-theoretic function, the divisor function[2]

$$\tau(n) = \mathbf{1}\star\mathbf{1}(n) = \sum_{ab=n} 1 = \sum_{d|n} 1.$$

Lemma 4.

$$\sum_{n\leq x}\tau(n) = x\log x + O(x).$$

More precisely,

$$\sum_{n\leq x}\tau(n) = x\log x + (2\gamma - 1)x + O(x^{1/2}).$$

In particular, $\sum_{n\leq x}\tau(n) \sim \sum_{n\leq x}\log n$, *so* $\tau(n)$ *is 'on average' roughly* $\log n$.

Proof. The proof of the first is a simple change in the order of summation:

$$\begin{aligned}\sum_{n\leq x}\tau(n) &= \sum_{ab\leq x} 1\\ &= \sum_{a\leq x}\left\lfloor \frac{x}{a}\right\rfloor\\ &= x\sum_{a\leq x}\frac{1}{a} + O(x)\\ &= x\log x + O(x),\end{aligned}$$

where we used Lemma 2 as $\sum_{a\leq x}\frac{1}{a} = \log x + \gamma + O(1/x) = \log x + O(1)$.

[2]Alternative notation used in some places is $d(n)$ or $\sigma_0(n)$.

To improve the error term we use what is known as the hyperbola method, which is the observation that when summing over pairs (a, b) such that $ab \leq x$ we can express this as the sum over pairs where $a \leq x^{1/2}$ and where $b \leq x^{1/2}$, and then subtract the contribution where $\max(a, b) \leq x^{1/2}$.

$$\begin{aligned}\sum_{ab\leq x} 1 &= \sum_{a\leq x^{1/2}} \left\lfloor \frac{x}{a} \right\rfloor + \sum_{b\leq x^{1/2}} \left\lfloor \frac{x}{b} \right\rfloor - \sum_{a,b\leq x^{1/2}} 1 \\ &= 2x \sum_{a\leq x^{1/2}} \frac{1}{a} - \lfloor x^{1/2} \rfloor^2 + O(x^{1/2}) \\ &= x\log x + (2\gamma - 1)x + O(x^{1/2}).\end{aligned}$$

□

It is a deep and difficult problem to improve the error term here – the truth is probably $O(x^{1/4+\epsilon})$, but this is an open problem, and the best known is $O(x^{0.3149\cdots})$.

3. Estimates on prime numbers

The prime number theorem is the statement that

$$\pi(x) \sim \frac{x}{\log x},$$

or, equivalently (we will justify this equivalence soon),

$$\psi(x) = \sum_{n\leq x} \Lambda(n) \sim x.$$

The proof of this is surprisingly involved, and we will return to it later in the course when we examine the Riemann zeta function. It is much easier to show, if not an asymptotic formula, at least that this is the correct rate of growth of the function. This was proved in 1850 by Chebyshev.

Theorem 2 (Chebyshev)**.**

$$\psi(x) \asymp x.$$

Proof. We will first prove the lower bound. This relies on the observation that, for any $y \geq 0$, $\lfloor 2y \rfloor \leq 2\lfloor y \rfloor + 1$.

$$\begin{aligned}\psi(x) &= \sum_{n\leq x} \Lambda(n) \\ &\geq \sum_{n\leq x} \Lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor \right) \\ &= \sum_{nm\leq x} \Lambda(n) - 2 \sum_{nm\leq x/2} \Lambda(n) \\ &= \sum_{n\leq x} \log n - 2 \sum_{n\leq x/2} \log n.\end{aligned}$$

Here we have used Lemma 1, that $\mathbf{1} \star \Lambda(n) = \log(n)$. By Lemma 3,

$$\psi(x) \geq x\log x - x + O(\log x) - 2\left(\frac{x}{2} \log(x/2) - \frac{x}{2} + O(\log x) \right) = (\log 2)x + O(\log x).$$

It follows that, for any $c > 0$ and x sufficiently large, $\psi(x) \geq (\log 2 - c)x$, and hence $\psi(x) \gg x$.

For the upper bound, we do something very similar, except we note that for $y \in [1/2, 1)$ we have equality $\lfloor 2y \rfloor = 2\lfloor y \rfloor + 1$. Furthermore, for any $y \geq 0$, we have the lower bound $\lfloor 2y \rfloor \geq 2\lfloor y \rfloor$. It follows that

$$\begin{aligned}\sum_{x/2<n\leq x} \Lambda(n) &= \sum_{x/2<n\leq x} \Lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - 2\left\lfloor \frac{x}{2n} \right\rfloor \right) \\ &\leq \sum_{n\leq x} \Lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - 2\left\lfloor \frac{x}{2n} \right\rfloor \right) \\ &= (\log 2)x + O(\log x)\end{aligned}$$

by the above calculation. The left hand side is $\psi(x) - \psi(x/2)$, and so we have shown that

$$\psi(x) - \psi(x/2) \leq (\log 2)x + O(\log x).$$

Using the fact that $\psi(x) = 0$ for any $x \leq 1$,

$$\psi(x) = \sum_{k=0}^{\lceil \log_2 x \rceil} (\psi(x/2^k) - \psi(x/2^{k+1})) \leq (2\log 2)x + O((\log x)^2),$$

and hence $\psi(x) \ll x$ as required. $\square$

Chebyshev's estimate is the first non-trivial quantitative information we have about the primes, and leads to a host of other facts about the primes – rather surprisingly, not just big-oh behaviour, but precise asymptotic results.

Lemma 5.

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2} \right).$$

In particular, $\pi(x) \asymp x/\log x$*, and* $\pi(x) \sim x/\log x$ *if and only if* $\psi(x) \sim x$*.*

Proof. We first remove the contribution from prime powers by noting that, if $\theta(x) = \sum_{p\leq x} \log p$, then

$$\psi(x) - \theta(x) \leq \sum_{k\geq 2} \sum_{p\leq x^{1/k}} \log p \ll \log x \sum_{k=2}^{\lceil \log x \rceil} x^{1/k} \ll x^{1/2}(\log x)^2.$$

It follows that $\theta(x) = \psi(x) + O(x^{1/2}(\log x)^2)$. In particular, by Chebyshev's estimate, we have $\theta(x) = O(x)$. We apply partial summation with $a_n = \Lambda(n)$ if n is prime, and 0 otherwise, and $f(n) = \frac{1}{\log n}$. This gives

$$\pi(x) = \sum_{p\leq x} 1 = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}\,\mathrm{d}t = \frac{\theta(x)}{\log x} + O\left(\frac{x}{(\log x)^2} \right).$$

We have used $\theta(t) = O(t)$ to bound the contribution from the integral here. $\square$

Lemma 6.

$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1).$$

Proof. Recalling that $\log = \mathbf{1} \star \Lambda$, and using Lemma 3,

$$\begin{aligned} x\log x + O(x) &= \sum_{n\leq x} \log n \\ &= \sum_{ab\leq x} \Lambda(b) \\ &= x\sum_{b\leq x} \frac{\Lambda(b)}{b} + O(\psi(x)). \end{aligned}$$

Using Chebyshev's estimate, this proves that

$$\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

It remains to deal with the contribution from prime powers $p^k \leq x$ for $k \geq 2$, which we bound trivially by

$$\sum_{p\leq x^{1/2}} \log p \sum_{k\geq 2} \frac{1}{p^k} = \sum_{p\leq x^{1/2}} \log p \frac{1}{p^2-p} \ll 1.$$

□

Lemma 7.

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + b + O(1/\log x),$$

where b is some constant.

Proof. Let $A(x) = \sum_{p\leq x}(\log p)/p = \log x + R(x)$, say, where $R(x) = O(1)$. By partial summation

$$\begin{aligned} \sum_{p\leq x} \frac{1}{p} &= \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t(\log t)^2}\,\mathrm{d}t \\ &= 1 + O(1/\log x) + \int_2^x \frac{1}{t\log t}\,\mathrm{d}t + \int_2^x \frac{R(t)}{t(\log t)^2}\,\mathrm{d}t \\ &= \log\log x + 1 - \log\log 2 + \int_2^\infty \frac{R(t)}{t(\log t)^2}\,\mathrm{d}t + O(1/\log x). \end{aligned}$$

□

Lemma 8.

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} = c\log x + O(1)$$

where $c > 1$ is some constant.

Proof. We use $\log(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k}$ to deduce that

$$\begin{aligned}
\log\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\right) &= -\sum_{p\leq x}\log(1-1/p)\\
&= \sum_{k=1}^{\infty}\sum_{p\leq x}\frac{1}{kp^k}\\
&= \sum_{p\leq x}\frac{1}{p}+\sum_{k\geq 2}\sum_{p\leq x}\frac{1}{kp^k}\\
&= \sum_{p\leq x}\frac{1}{p}+\sum_{p}\sum_{k\geq 2}\frac{1}{kp^k}+O\left(\sum_{p>x}\sum_{k\geq 2}\frac{1}{p^k}\right).
\end{aligned}$$

Note that the infinite sum over p converges to some constant. Furthermore, the error term is

$$\ll \sum_{p>x}\frac{1}{p^2} \ll \sum_{n>x}\frac{1}{n^2} \ll \frac{1}{x}.$$

It follows from Lemma 7 that

$$\log\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\right) = \log\log x + b' + O(1/\log x)$$

for some constant b'. The result follows since $e^x = 1 + O(x)$ for $|x| \leq 1$. □

It is a little tricky to determine what the constant c in Lemma 8 actually is – it turns out to be $e^{\gamma} \approx 1.78\cdots$. We can use this fact to point out why the naive probabilistic heuristic can be misleading (and hopefully give some idea why the prime number theorem itself, unlike these simple asymptotics, is hard to prove).

As a heuristic, we might guess that the probability that a given prime number p divides a randomly chosen n is $1/p$. Furtermore, we expect that these probabilities should be independent for distinct primes p. Using the fact that $n \geq 3$ is prime if and only if $p \nmid n$ for all $2 \leq p \leq n^{1/2}$, we might guess that

$$1_{n \text{ is prime}} \approx \mathbb{P}(p \nmid n \text{ for all } 2 \leq p \leq n^{1/2}) \approx \prod_{p\leq n^{1/2}}\left(1-\frac{1}{p}\right) \approx 2e^{-\gamma}/\log n.$$

This would in turn suggest that

$$\pi(x) = \sum_{n\leq x} 1_{n \text{ is prime}} \approx 2e^{-\gamma}\sum_{n\leq x}\frac{1}{\log n} \approx 2e^{-\gamma}\frac{x}{\log x}.$$

But since $2e^{-\gamma} = 1.12\cdots$, this contradicts the prime number theorem! This shows that, while heuristically thinking about discrete concepts in terms of 'probability' can lead to roughly the right order of magnitude, one must take care not to take the constants obtained too seriously! (Essentially what's going wrong here is that divisibility by primes is not independent, especially for large primes – if I know that $p_1p_2 \mid n$ where both p_1 and p_2 are primes $\approx n^{1/3}$ then it's impossible that $q \mid n$ for any prime $q \approx n^{1/2}$, for example.)

Indeed, we can use the elementary estimates already obtained to show, not the prime number theorem itself, but at least the fact that *if* $\frac{\pi(x)\log x}{x}$ *converges to a* *limit at all*, then this limit must be 1, and hence the prime number theorem is true. The hard part is showing that the limit exists.

Theorem 3 (Chebyshev). *If* $\pi(x) \sim c\frac{x}{\log x}$ *then* $c = 1$.

Proof. By partial summation,

$$\sum_{p\leq x} \frac{1}{p} = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\,\mathrm{d}t.$$

The first term is trivially $O(1)$. If $\pi(x) = c(1 + R(x))\frac{x}{\log x}$, where $R(x) \to 0$ as $x \to \infty$, then

$$\sum_{p\leq x} \frac{1}{p} = c\int_1^x \frac{1 + R(t)}{t\log t}\,\mathrm{d}t + O(1) = c(1 + o(1))\log\log x + O(1).$$

By Lemma 7 the left hand side is $(1 + o(1))\log\log x$, and hence $c = 1$. □

CHAPTER 2

Dirichlet series and the Riemann zeta function

We will now begin to harness the power of complex analysis for number theory. The main object of study will the Riemann zeta function. Before we explore the applications to number theory, we will spend some time proving various essential facts about this function.

In the rest of the course, we will use (as is traditional for this topic) the letter s to denote a complex variable, and σ and t to denote its real and imaginary parts respectively, so that $s = \sigma + it$. Before we begin, it's worth pausing to explicitly point out what we mean by n^s, where n is a natural number and $s \in \mathbb{C}$. By definition this is

$$n^s = e^{s \log n} = n^\sigma e^{it \log n}.$$

It is easy to check the multiplicative property, that $(nm)^s = n^s m^s$.

A Dirichlet series is an infinite series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for some coefficients $a_n \in \mathbb{C}$. If we denote the coefficients a_n by an arithmetic function $f(n)$ then we may write $F_f(s)$ to denote this dependence.

Lemma 9. *For any sequence a_n there is an abscissa of convergence σ_c such that $F(s)$ converges for all s with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$. If $\sigma > \sigma_c$ then there is a neighbourhood of s in which $F(s)$ converges uniformly. In particular, $\alpha(s)$ is holomorphic at s.*

Proof. It suffices to show that if $F(s)$ converges at $s = s_0$ and we take some s with $\sigma > \sigma_0$ then F converges uniformly in some neighbourhood of s. The lemma then follows by taking $\sigma_c = \inf\{\sigma : F(s) \text{ converges}\}$.

Suppose that $F(s)$ converges at $s = s_0$. If we let $R(u) = \sum_{n>u} a_n n^{-s_0}$ then by partial summation, for any s,

$$\sum_{M<n\leq N} a_n n^{-s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s)\int_M^N R(u)u^{s_0-s-1}\,\mathrm{d}u.$$

If $|R(u)| \leq \epsilon$ for all $u \geq M$, and if $\sigma > \sigma_0$, then it follows that

$$\left|\sum_{M<n\leq N} a_n n^{-s}\right| \leq 2\epsilon + \epsilon\,|s - s_0| \int_M^\infty t^{\sigma_0-\sigma-1}\,\mathrm{d}t \leq \left(2 + \frac{|s-s_0|}{\sigma-\sigma_0}\right)\epsilon.$$

There is some neighbourhood of s in which $|s - s_0| \ll \sigma - \sigma_0$, and hence by Cauchy's principle the series converges uniformly in this neighbourhood of s. ☐

Lemma 10. *If $\sum a_n n^{-s} = \sum b_n n^{-s}$ for all s in some half-plane $\sigma > \sigma_0$ (where both series converge) then $a_n = b_n$ for all n.*

Proof. It suffices to show that if $\sum c_n n^{-s} = 0$ for all s with $\sigma > \sigma_0$ then $c_n = 0$ for all n. Suppose that $c_n = 0$ for all $n < N$. We can write

$$c_N = -\sum_{n>N} c_n (n/N)^{-\sigma}.$$

Since the sum here is convergent, the summands tend to 0, and hence $c_n \ll n^{\sigma_0}$. It follows that that this sum is absolutely convergent for $\sigma > \sigma_0 + 1$. Since each term tends to 0 as $\sigma \to \infty$, and the series is absolutely convergent, the right-hand side tends to 0, and hence $c_N = 0$. □

Lemma 11. *If $F_f(s)$ and $F_g(s)$ are two Dirichlet series, both absolutely convergent at s, then*

$$\sum_{n=1}^{\infty} f \star g(n) n^{-s}$$

is absolutely convergent and equals $\alpha(s)\beta(s)$.

Proof. We simply multiply out the product of two series,

$$\left(\sum_n a\frac{a_n}{n^s}\right)\left(\sum_m \frac{b_m}{m^s}\right) = \sum_{n,m} \frac{a_n b_m}{(nm)^s} = \sum_k \left(\sum_{nm=k} a_n b_m\right) k^{-s},$$

which is justified since both series are absolutely convergent. □

We now define the Riemann zeta function in the half-plane $\sigma > 1$ by

$$\zeta(s) = \sum_n \frac{1}{n^s}.$$

Observe that this series diverges at $s = 1$, and the series actually converges absolutely for $\sigma > 1$. By the above, $\zeta(s)$ defines a holomorphic function in this half-plane. For our applications, we need to extend this definition to be able to talk about $\zeta(s)$ for $\sigma > 0$.

Lemma 12. *For $\sigma > 1$,*

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Proof. By partial summation, for any x,

$$\sum_{1\leq n\leq x} n^{-s} = \frac{\lfloor x\rfloor}{x^s} + s\int_1^x \frac{\lfloor t\rfloor}{t^{s+1}}\,\mathrm{d}t.$$

The integral here is

$$s\int_1^x t^{-s}\,\mathrm{d}t - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t = \frac{s}{s-1} - \frac{s}{s-1}x^{1-s} - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Since $\sigma > 1$, if we take the limit as $x \to \infty$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t,$$

noting that the integral converges. □

The integral here is convergent for any $\sigma > 0$, and therefore the right hand side defines an analytic function for $\sigma > 0$, aside from a simple pole at $s = 1$ with residue 1. We have therefore given an analytic continuation for $\zeta(s)$ up to $\sigma = 0$.

3.1. **Euler products.** Since it is a topic not often covered in analysis courses, we first take a brief digression to discuss infinite products. If $a_n \in \mathbb{C}\backslash\{0\}$ then tshe infinite product

$$\prod_{n=1}^{\infty} a_n$$

is defined to be the limit $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$ if this exists and is not zero.

Lemma 13 (Cauchy criterion)**.** *If $a_n \neq 0$ then the infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if for any $\epsilon > 0$ there exists N such that*

$$\left| \prod_{n<k\leq m} a_k - 1 \right| < \epsilon$$

for all $m > n \geq N$.

In particular, $\lim_{n\to\infty} a_n = 1$. For this reason it is often convenient to change variables so that we consider the product $\prod(1 + a_n)$ instead. We say that

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges absolutely if and only if $\prod(1 + |a_n|)$ converges. The following is a simple consequence of the Cauchy criterion.

Lemma 14. *If $a_n \neq -1$ and $\prod(1 + a_n)$ converges absolutely then it converges.*

The final fundamental fact we will require is the following.

Lemma 15. *If $a_n > 0$ for all $n \geq 1$ then $\prod(1 + a_n)$ converges if and only if $\sum a_n$ converges.*

Proof. By the monotone convergence theorem, it suffices to show that the partial sums are bounded above if and only the partial products are. This follows from the inequalities

$$a_1 + \cdots + a_n < (1 + a_1)\cdots(1 + a_n) \leq e^{a_1+\cdots+a_n}.$$

□

All of the infinite products we will encounter in this course will converge absolutely. The previous lemmas have the following useful consequence: if $\sum |a_n|$ converges (and $a_n \neq -1$) then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. In particular, it is not zero!

Lemma 16. *If f is multiplicative and $\sum |f(n)|\, n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right).$$

Proof. Note that this product is absolutely convergent. By comparison each sum in the product is absolutely convergent. Since a product of finitely many absolutely convergent series can be arbitrarily rearranged,

$$\prod_{p\leq y} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) = \sum_{\substack{n \\ p|n \implies p\leq y}} f(n)n^{-s}.$$

Therefore the difference between the product here and the Dirichlet series is at most

$$\sum_{n>y} |f(n)| n^{-\sigma} \to 0 \text{ as } y \to \infty.$$

$\square$

Corollary 1 (Euler product)**.** *If f is completely multiplicative and $\sum |f(n)| n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

In particular, we note the Euler product for $\zeta(s)$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

which is valid for $\sigma > 1$. From this it follows that $\zeta(s) \neq 0$ for $\sigma > 1$. The Euler product leads to the identity

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_n \frac{\mu(n)}{n^s}.$$

Furthermore, when $\sigma > 1$, the series is absolutely convergent, and so the derivative can be computed summand by summand, leading to

$$\zeta'(s) = -\sum_n \frac{\log n}{n^s}.$$

From the Euler product we have

$$\log \zeta(s) = -\sum_p \log\left(1 - p^{-s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} = \sum_n \frac{\Lambda(n)}{\log n} n^{-s}.$$

Finally, taking the derivative of this, we obtain the Dirichlet series with $\Lambda(n)$ as coefficients:

$$\frac{\zeta'}{\zeta}(s) = -\sum_n \frac{\Lambda(n)}{n^s}.$$

4. Gamma function

4.1. **The Weierstrass definition.** Let

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N\right) = 0.5772157\cdots$$

and define the Gamma function $\Gamma(s) : \mathbb{C} \to \mathbb{C}$ by

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} e^{-s/n} \left(1 + \frac{s}{n}\right).$$

This product is analytic for all $s \in \mathbb{C}$, because when $|s| \leq N/2$ the series

$$\sum_{n=N+1}^{\infty} \left(\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right)$$

is absolutely and uniformly convergent, and so its exponential is also an analytic function. This shows that the product is an analytic function for $|s| \leq N/2$, and we then take N arbitrarily large.

It is clear from this expression that $\Gamma(s)$ itself is analytic at all $s \in \mathbb{C}$ apart from simple poles at $s = 0, -1, -2, \ldots$. The residue at $s = -n$ is $(-1)^n/n!$.

4.2. **The Euler definition.** Inserting the definition of γ gives

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= s \lim_{N\to\infty} e^{\left(\sum_{m=1}^{N} \frac{1}{m} - \log N\right)s} \prod_{n=1}^{N} e^{-s/n}\left(1+\frac{s}{n}\right) \\
&= s \lim_{N\to\infty} N^{-s} \prod_{n=1}^{N}\left(1+\frac{s}{n}\right) \\
&= s \lim_{N\to\infty} \left(1+\frac{1}{N}\right)^{s} \prod_{n=1}^{N}\left(1+\frac{s}{n}\right)\left(1+\frac{1}{n}\right)^{-s},.
\end{aligned}$$

whence we have the following formula of Euler,

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1},$$

valid for all $s \in \mathbb{C}$ except $s = 0, -1, -2, \ldots$. It follows that $\Gamma(1) = 1$. Rewriting this, we also get

$$\Gamma(s) = \lim_{N\to\infty} N^{s} \frac{(N-1)!}{s(s+1)\cdots(s+N-1)}.$$

4.3. **The difference equation.** By Euler's formula, if s is not a negative integer,

$$\begin{aligned}
\frac{\Gamma(s+1)}{\Gamma(s)} &= \frac{s}{s+1}\lim_{N\to\infty}\prod_{n=1}^{N}\frac{\left(1+\frac{1}{n}\right)(s+n)}{s+n+1} \\
&= s\lim_{N\to\infty}\frac{N+1}{s+N+1} = s,
\end{aligned}$$

whence

$$\Gamma(s+1) = s\Gamma(s).$$

In particular, since $\Gamma(1) = 1$, if s is a positive integer then $\Gamma(s) = (s-1)!$.

4.4. **The reflection formula.**

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \frac{1}{s(1-s)}\prod_{n=1}^{\infty}\frac{1+1/n}{\left(1+\frac{s}{n}\right)\left(1+\frac{1-s}{n}\right)} \\
&= \frac{1}{s(s-1)}\prod_{n=1}^{\infty}\frac{1}{\left(1+\frac{s}{n}\right)\left(1-\frac{s}{n+1}\right)} \\
&= \frac{1}{s}\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right)^{-1} \\
&= \frac{\pi}{\sin(\pi s)}.
\end{aligned}$$

It follows, for example, that $\Gamma(1/2) = \sqrt{\pi}$.

4.5. **The duplication formula.** Consider the expression

$$\frac{2^{2s}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)}{2\Gamma(2s)}.$$

We claim that this is independent of s. By Euler's formula it is

$$2^{2s-1}\frac{\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^s}{(s)\cdots(s+N-1)}\cdot\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^{s+1/n}}{\left(s+\frac{1}{n}\right)\cdots\left(s+\frac{1}{n}+N-1\right)}}{\lim_{N\to\infty}\frac{1\cdot 2\cdots(2N-1)(2N)^{2s}}{2s(2s+1)\cdots(2s+2N-1)}}$$

which is

$$\lim_{N\to\infty}\frac{((N-1)!)^2N^{1/2}2^{2N-1}}{(2N-1)!},$$

and in particular independent of s. To evaluate it we set $s=1/2$, yielding

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

We have proved the duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=2^{1-2s}\pi^{1/2}\Gamma(2s).$$

4.6. **Euler's integral expression.** By integration by parts

$$\begin{aligned}\int_0^N\left(1-\frac{t}{N}\right)^N t^{s-1}\,\mathrm{d}t &= N^s\int_0^1(1-t)^Nt^{s-1}\,\mathrm{d}t\\ &= N^s\frac{N!}{s(s+1)\cdots(s+N)}\\ &\to\Gamma(s)\end{aligned}$$

as $N\to\infty$ using Euler's formula. This is valid if $\sigma>0$, whence we have the formula for this region

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t.$$

5. Functional equation

Theorem 4 (Functional equation)**.** *The zeta function $\zeta(s)$ can be extended a function meromorphic on the whole complex plane, and for all s satisfies the identity*

$$\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Equivalently,

$$\zeta(s)=\chi(s)\zeta(1-s)$$

where

$$\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)=\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Many interesting facts can be deduced from this identity. We will first use it to study the possible poles of $\zeta(s)$. We know that $\zeta(s)$ has a simple pole at $s=1$, and nowhere else for $\sigma>-1$. Suppose that ζ has a pole at s for $\sigma<0$. Then so too does $\Gamma(1-s)\zeta(1-s)$, but both $\Gamma(s)$ and $\zeta(s)$ are holomorphic for all s with $\Re s>1$, which is a contradiction. It follows that $\zeta(s)$ only has one pole in $\mathbb{C}$, which is a simple pole at $s=1$.

Alternatively, if we write

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

then

$$\xi(s) = \xi(1-s)$$

and $\xi(s)$ is an entire function, and is real for real s.

We will now consider the zeros of $\zeta(s)$. Suppose that $\zeta(s) = 0$ and $\sigma < 0$. It follows that

$$\sin(\pi s/2)\Gamma(1-s)\zeta(1-s) = 0.$$

Again, neither $\Gamma(1-s)$ nor $\zeta(1-s)$ can be zero or a pole, and so $\sin(\pi s/2)$, which means s must be an even integer. These are called the trivial zeros of $\zeta(s)$, located at $s = -2, -4, -6, \ldots$. Since there are no zeros with $\sigma \geq 1$, there are no other zeros with $\sigma \leq 0$.

Aside from the trivial zeros, then, all zeros of ζ must lie in the critical strip $0 < \sigma < 1$. Furthermore, since the other factors in the functional equation are entire and non-zero in this strip, this implies that if ρ is a zero in the critical strip, then so too is $1-\rho$. There is therefore a symmetry around the critical line $\sigma = 1/2$. The Riemann hypothesis is motivated in part by the belief that this symmetry should collapse so that all the zeros are located exactly on this line.

5.1. **Method One.** We first extend the definition of the zeta function to a larger half-plane. Recall that for $\sigma > 0$ we defined

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,\mathrm{d}u.$$

We will extend the region where this is valid by integrating by parts. First let $f(x) = \frac{1}{2} - \{x\}$, so that

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u.$$

If we let $F(x) = \int_0^x f(u)\,\mathrm{d}u$ then, by integration by parts,

$$\int_1^\infty \frac{f(u)}{u^{s+1}} = [F(u)u^{-s-1}]_0^\infty + (s+1)\int_1^\infty \frac{F(u)}{u^{s+2}}\,\mathrm{d}u.$$

Since $F(x)$ is bounded, the integral here converges for any s with $\sigma > -1$, and hence the left-hand side also converges in this region. We may therefore take

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u$$

as the definition of $\zeta(s)$ in the half-plane $\sigma > -1$. If $-1 < \sigma < 0$ then

$$\int_0^1 \frac{f(u)}{u^{s+1}}\,\mathrm{d}u = \frac{1}{2}\int_0^1 \frac{1}{u^{s+1}}\,\mathrm{d}u - \int_0^1 \frac{1}{u^s}\,\mathrm{d}u = -\frac{1}{2s} + \frac{1}{s-1},$$

and so in this strip

$$\zeta(s) = s\int_0^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u.$$

We now note that $f(x)$ is a periodic function, continuous in $(0,1)$, and so it has a Fourier series, which is

$$f(u)=\sum_{n=1}^{\infty}\frac{\sin(2\pi nu)}{\pi n}.$$

For $-1<\sigma<0$ we therefore get

$$\zeta(s)=s\int_0^\infty \frac{1}{u^{s+1}}\sum_{n=1}^{\infty}\frac{\sin(2\pi nu)}{\pi n}\,\mathrm{d}u=\frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty\frac{\sin(2\pi nu)}{u^{s+1}}\,\mathrm{d}u.$$

We should justify the interchange of integral and summation here. We can interchange the infinite sum with any finite integral by the dominated convergence theorem, since the partial sums converge almost-everywhere pointwise, and are bounded above by $O(1)$. We then note that for any λ,

$$\int_\lambda^\infty\frac{\sin(2\pi nx)}{x^{s+1}}\,\mathrm{d}x=\left[-\frac{\cos(2\pi nx)}{2n\pi x^{s+1}}\right]_\lambda^\infty-\frac{s+1}{2n\pi}\int_\lambda^\infty\frac{\cos(2\pi nx)}{x^{s+2}}\,\mathrm{d}x$$
$$=O\left(\frac{1}{n\lambda^{\sigma+1}}\right)+O\left(\frac{1}{n}\int_\lambda^\infty\frac{1}{x^{\sigma+2}}\,\mathrm{d}x\right)=O\left(\frac{1}{n\lambda^{\sigma+1}}\right).$$

It follows that

$$\lim_{\lambda\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\int_\lambda^\infty\frac{\sin(2n\pi x)}{x^{s+1}}\,\mathrm{d}x=0$$

for $-1<\sigma<0$. It follows that (in brief) for any $\lambda>0$

$$\sum_{n=1}^{\infty}\int_0^\infty=\sum_{n=1}^{\infty}\int_0^\lambda+\sum_{n=1}^{\infty}\int_\lambda^\infty=\int_0^\lambda\sum_{n=1}^{\infty}+\sum_{n=1}^{\infty}\int_\lambda^\infty\to\int_0^\infty\sum_{n=1}^{\infty}$$

as $\lambda\to\infty$.

By change of variable, we have

$$\int_0^\infty\frac{\sin(2\pi nu)}{u^{s+1}}=(2\pi n)^s\int_0^\infty\frac{\sin(u)}{u^{s+1}}\,\mathrm{d}u.$$

Furthermore, writing $\sin(u)=\frac{1}{2i}(e^{iu}-e^{-iu})$

$$\begin{aligned}\int_0^\infty\frac{\sin u}{u^{s+1}}\,\mathrm{d}u&=\frac{1}{2i}\left(\int_0^\infty u^{-s-1}e^{iu}\,\mathrm{d}u-\int_0^\infty u^{-s-1}e^{-iu}\,\mathrm{d}u\right)\\&=\frac{1}{2i}\left((-i)^s-i^s\right)\int_0^\infty t^{-s-1}e^{-t}\,\mathrm{d}t\\&=-\sin(\pi s/2)\int_0^\infty t^{-s-1}e^{-t}\,\mathrm{d}t\\&=-\sin(\pi s/2)\Gamma(-s),\end{aligned}$$

NO, CONTOUR INTEGRAL using the change of variable $iu=t$ and $-iu=t$ respectively for the two integrals. Combining the above we have shown that, for $-1<\sigma<0$,

$$\zeta(s)=\frac{s}{\pi}\sum_{n=1}^{\infty}\frac{(2\pi n)^s}{n}\sin(\pi s/2)\Gamma(-s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

The right-hand side is actually analytic for any $\sigma<1$, and hence we can take the right-hand side to be a definition of $\zeta(s)$ in this region. By analytic continuation it follows that this identity must hold for all $s\in\mathbb{C}$.

5.2. **Method Two.**

Lemma 17. *For $\sigma > 1$*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x.$$

Proof. The key observation is that

$$\frac{\Gamma(s)}{n^s} = \frac{1}{n^s} \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t = \int_0^\infty x^{s-1} e^{-nx} \, \mathrm{d}x.$$

We then sum both sides over n, and note that $\sum e^{-nx} = (e^x - 1)^{-1}$. We can interchange the sum and integral here by absolute convergence, since $\sigma > 1$, as

$$\sum \int_0^\infty x^{\sigma-1} e^{-nx} \, \mathrm{d}x = \Gamma(\sigma)\zeta(\sigma)$$

converges for $\sigma > 1$. □

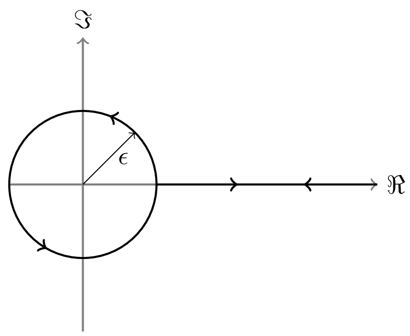

FIGURE 1. The contour C of Lemma 18.

Lemma 18. *For $\sigma > 1$*

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1 - s) \int_C \frac{z^{s-1}}{e^z - 1} \, \mathrm{d}z$$

where the contour goes from positive infinity, circles the origin, and returns to infinity, where z^{s-1} is defined as $\exp((s-1)\log z)$ with the logarithm real at the beginning of the contour.

Proof. Suppose the circle part has radius ϵ. On the circle,

$$\left|z^{s-1}\right| = e^{(\sigma-1)\log|z| - t\arg(z)} \le |z|^{\sigma-1} e^{2\pi|t|}$$

and

$$|e^z - 1| \gg |z|$$

and so the integral around the circle tends to zero as $\epsilon \to 0$. It follows that the integral is

$$-\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^x - 1} \, \mathrm{d}x = (e^{2\pi i s} - 1)\Gamma(s)\zeta(s).$$

Since

$$\sin(s) = \frac{e^{is} - e^{-is}}{2i} = \frac{e^{2is} - 1}{2ie^{is}}$$

by the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = \frac{2\pi i e^{i\pi s}}{e^{2\pi i s}-1}$$

and the result follows. □

So far we've been working in the half-plane $\sigma > 1$. The integral over C, however, is uniformly convergent in any finite region, and so the right-hand side defines a meromorphic continuation of ζ to the entire complex plane, with the only possible poles those of $\Gamma(1-s)$, which are $s = 1, 2, 3, \ldots$, and hence just at $s = 1$.

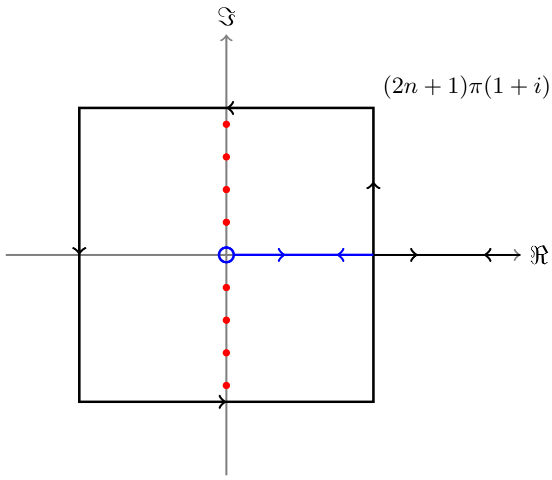

FIGURE 2. The contour C_n with poles at $\pm 2\pi i, \ldots, \pm 2\pi i n$ marked in red, and the previous contour C marked in blue.

For the functional equation, now take the integral C_n which is the positive real axis from ∞ to $(2n+1)\pi$, round the square $\pm 1 \pm i$ then back to infinity. Moving from C to C_n we pick up poles at $\pm 2i\pi, \ldots, \pm 2in\pi$. The residues from $\pm 2\pi i m$ are together

$$\begin{aligned}(2m\pi i)^{s-1} + (-2m\pi i)^{s-1} &= (2m\pi)^{s-1} e^{i\pi(s-1)} 2\cos(\pi(s-1)/2)\\ &= -2(2m\pi)^{s-1} e^{i\pi s}\sin(\pi s/2).\end{aligned}$$

It follows that the integral is

$$\int_{C_n} \frac{z^{s-1}}{e^z - 1}\,\mathrm{d}z + 4\pi i e^{i\pi s}\sin(\pi s/2)\sum_{m=1}^{n}(2m\pi)^{s-1}.$$

We now take $\sigma < 0$ and let $n \to \infty$. The function $1/(e^z - 1)$ is bounded on the contours C_n, and $z^{s-1} = O(|z|^{\sigma-1})$, so the integral around C_n tends to 0. It follows that

$$\int_C \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z = 4\pi i e^{i\pi s}\sin(\pi s/2)(2\pi)^{s-1}\zeta(1-s),$$

so that for $\sigma < 0$

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s) 4\pi i e^{i\pi s}\sin(\pi s/2)(2\pi)^{s-1}\zeta(1-s).$$

This verifies the functional equation for $\sigma < 0$. As before, by analytic continuation, it must in fact hold for all s.

6. Special values of zeta

If we let

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

then B_n are known as the Bernoulli numbers. Multiplying out both sides by $e^z - 1$ it follows that

$$z = \left(\sum_{n \geq 0} \frac{B_n}{n!} z^n \right) \left(\sum_{m \geq 1} \frac{1}{m!} z^m \right) = \sum_{k \geq 1} \left(\sum_{\substack{n+m=k \\ n \geq 0, m \geq 1}} \frac{B_n}{n! m!} \right) z^k$$

and so $B_0 = 1$ and for $k \geq 2$

$$\sum_{0 \leq n \leq k-1} \frac{B_n}{n!(k-n)!} = 0 = \sum_{0 \leq n \leq k-1} \binom{k}{n} B_n.$$

This shows that each B_n is a rational number, and allows for efficient computation. For example, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, and so on.

Recall that by Lemma 18 we have, for any $s \neq 1, 2, 3, \ldots$,

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1-s) \int_C \frac{z^{s-1}}{e^z - 1} \, \mathrm{d}z.$$

When $s = -m \leq 0$ is an integer the contour integral can be evaluated using the theory of residues. Inside C there is only a single pole at $z = 0$, and since

$$\frac{z^{-m-1}}{e^z - 1} = z^{-m-2} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

the residue at $z = 0$ when $s = -m$ is $B_{m+1}/(m+1)!$. It follows that

$$\zeta(0) = -\frac{1}{2} \text{ and } \zeta(-m) = \frac{(-1)^m B_{m+1}}{m+1} \text{ for } m \geq 1.$$

Since we already know that $\zeta(-2k) = 0$ for $k \geq 1$ it follows that $B_n = 0$ whenever $n \geq 3$ is odd, something not at all obvious from the definition! Furthermore, it follows from the functional equation that for $m \geq 1$

$$\begin{aligned}
\zeta(2m) &= 2^{2m} \pi^{2m-1} \sin(\pi m) \Gamma(1-2m) \zeta(1-2m) \\
&= -2^{2m} \pi^{2m-1} \sin(\pi m) \Gamma(1-2m) \frac{B_{2m}}{2m} \\
&= (-1)^{m+1} 2^{2m-1} \pi^{2m} \frac{B_{2m}}{(2m)!}.
\end{aligned}$$

Here we have used the fact that $\sin(\pi m)\Gamma(1-2m) = (-1)^m \pi/2$, which is not immediately obvious, since it is the product of a zero and a pole. One way to check this is to note that $\sin(\pi z)$ has a simple zero at $z = m$ around which it can be expanded as $(-1)^m\pi(z-m)+O((z-m)^3)$ and Γ has a simple pole at $s=1-2m$ with residue $(-1)^{2m-1}/(2m-1)!$, and hence near $z=m$ we have

$$\begin{aligned}\sin(\pi z)\Gamma(1-2z) &= \left((-1)^m\pi(z-m)+O((z-m)^3)\right)\left(\frac{(-1)^{2m-1}}{(2m-1)!}\frac{1}{(2m-2z)}+O(1)\right)\\ &= (-1)^m\frac{\pi}{2}+O(z-m).\end{aligned}$$

For example, using the previous values for the Bernoulli numbers B_2 and B_4 we deduce that $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$. Note that if we want to recover the values of ζ at positive odd integers then things are not so simple – for now the functional equation gives

$$\zeta(2m+1)=(-1)^m 2^{2m+1}\pi^{2m}\Gamma(-2m)\zeta(-2m).$$

Now the pole of Γ at $s=-2m$ is cancelled by the zero of $\zeta(-2m)$ but we have no idea what $\Gamma(-2m)\zeta(-2m)$ is because we don't know what $\zeta'(-2m)$ is. We were lucky in the case of even arguments that the pole of Γ is cancelled by the zero of sin, both of which we understand well.

Unlike the case for even integers it is believed that $\zeta(2m+1)/\pi^{2m+1}$ is irrational for all $m\geq 1$. This is not known for any m. It is even more likely that $\zeta(2m+1)$ is irrational – this is only known for $\zeta(3)$ at the moment, which was shown by Roger Apéry in 1978.

Finally, in the next chapter we will need to know the value of $\zeta'(0)$. This can be calculated as follows. Recall in the proof that

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O(1/x)$$

we obtained the explicit representation

$$\gamma=1-\int_1^\infty\frac{\{t\}}{t^2}\,\mathrm{d}t.$$

If we let $s\to 1$ in the expression

$$\zeta(s)=1+\frac{1}{s-1}-s\int_1^\infty\frac{\{t\}}{t^{s+1}}\,\mathrm{d}t$$

then we see that $\zeta(s)=\frac{1}{s-1}+\gamma+O(s-1)$ near $s=1$.

Finally, from the functional equation we have, near $s=1$,

$$1+\gamma(s-1)+\cdots=(s-1)\zeta(s)=-2^s\pi^{s-1}\Gamma(2-s)\sin(\pi s/2)\zeta(1-s).$$

Differentiating both sides and setting $s=1$ gives

$$\gamma=2\zeta'(0)-2\zeta(0)\log 2\pi+2\zeta(0)\Gamma'(1).$$

Using the fact that $\zeta(0)=-1/2$ and $\Gamma'(1)=-\gamma$ (which can be seen, for example, from the Weierstrass definition), it follows that

$$\zeta'(0)=-\frac{1}{2}\log(2\pi),$$

and hence

$$-\frac{\zeta'(0)}{\zeta(0)}=-\log(2\pi).$$

7. Counting zeros of zeta

Recall that a holomorphic non-zero function has only a finite number of zeros in any compact region (otherwise there exists an infinite sequence of zeros, and hence an infinite convergent sequence of zeros, and hence the function is zero by the Identity Theorem). It therefore makes sense to count them. We are interested in the zeros of the zeta function, which we know (aside from the trivial zeros) all lie in the rectangular strip $0 \leq \sigma \leq 1$.

To make this a compact region, we introduce some cut-off at height $t = T$ on the imaginary axis. It is a natural question to ask how the number of zeros changes as we increase T. To this end, let $N(T)$ count the number of zeros $\rho = \beta + i\gamma$ in the region $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq T$.

Our main tool is the following useful bound from complex analysis.

Lemma 19 (Jensen's inequality)**.** *Suppose that $f(z)$ is analytic in a domain containing a disc with radius R and centre a, that $|f(z)| \leq M$ in this disc, and that $f(a) \neq 0$. Let $0 < r < R$. The number of zeros of f in the disc with centre a and radius r is at most*

$$\frac{\log(M/|f(a)|)}{\log(R/r)}.$$

Proof. Without loss of generality we can assume that $a = 0$. As above, the number of zeros in $|z| < R$ is finite . Let these zeros be denoted by $z_1, z_2, \ldots, z_K$. Let

$$g(z) = f(z) \prod_{k=1}^{K} \frac{R^2 - z\overline{z_k}}{R(z - z_k)}.$$

Observe that the kth factor has a pole at z_k, and has modulus 1 on $|z| = R$. It follows that g is an analytic function in $|z| \leq R$, and if $|z| = R$ then $|g(z)| = |f(z)| \leq M$. By the maximum modulus principle,

$$|g(0)| = |f(0)| \prod_{k=1}^{K} \frac{R}{|z_k|} \leq M.$$

Each factor is ≥ 1 and if $|z_k| \leq r$ then the factor is $\geq R/r$, and the bound follows. □

To apply Jensen's inequality to $\zeta(s)$ we first need to give some estimates for how large $\zeta(s)$ can get.

Lemma 20. *When $\delta \leq \sigma \leq 2$ and $|t| \geq 1$*

$$\zeta(s) \ll (1 + |t|^{1-\sigma}) \min\left(\frac{1}{|\sigma - 1|}, \log(|t| + 4)\right).$$

Proof. For any $x \geq 2$, by partial summation, when $\sigma > 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

By analytic continuation this identity continues to hold for all $s \neq 1$ with $\sigma > 0$. The second summand is $O(x^{1-\sigma})$. The third is $O(x^{-\sigma})$. The integral is $O(x^{-\sigma}/\sigma)$. Since $|s|/\sigma \ll |t|$ in the given region we have

$$\zeta(s) \ll \sum_{n \leq x} \frac{1}{n^\sigma} + x^{1-\sigma} + \frac{|t|}{x^\sigma}.$$

The sum is
$$\ll 1 + \int_1^x \frac{1}{u^\sigma}\,\mathrm{d}u$$
uniformly for $\sigma \geq 0$. We choose $x = \lvert t\rvert + 4$, say, so that
$$\zeta(s) \ll 1 + \lvert t\rvert^{1-\sigma} + \int_1^x \frac{1}{u^\sigma}\,\mathrm{d}u.$$
If $\lvert\sigma - 1\rvert \leq 1/\log x$ then the integral is $\asymp \log x$. If $0 \leq \sigma \leq 1 - 1/\log x$ this is $< x^{1-\sigma}/(1-\sigma)$. If $\sigma \geq 1 + 1/\log x$ then it is $< 1/(\sigma - 1)$. The result follows. □

Theorem 5. *For any $T \geq 4$*
$$N(T+1) - N(T) \ll \log T.$$

Proof. By the symmetry of zeros implied by the functional equation, it suffices to show that the number of zeros in the rectangle between $1/2 \leq \sigma \leq 1$ and $T \leq t < T+1$ is $O(\log T)$. We apply Jensen's inequality to $\zeta(s)$ to a disc with centre $2 + i(T + 1/2)$ and radii $R = 11/6$ and $r = 7/4$, say, which certainly includes this rectangle.

By Lemma 20 we know that $\lvert\zeta(s)\rvert \ll T$ in this disc, and furthermore
$$\lvert\zeta(2 + i(T + 1/2)) - 1\rvert \leq \sum_{n\geq 2} \frac{1}{n^2} \leq 3/4$$
and so $\lvert\zeta(2 + i(T + 1/2))\rvert \geq 1/4$, say. The result follows from Jensen's inequality. □

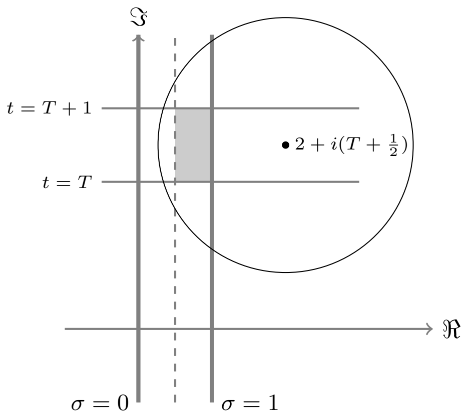

FIGURE 3. We bound the number of zeros in the rectangle $1/2 \leq \sigma \leq 1$ and $T \leq t \leq T + 1$ by bounding the number in the circle.

It follows that $N(T) \ll T\log T$. Note that this is only an upper bound – at the moment, we don't know there are any zeros at all the critical strip, so it is possible that $N(T) = 0$. We will later show that, not only are there many zeros, but $T\log T$ is the right order of magnitude, establishing the asymptotic formula
$$N(T) \sim \frac{1}{2\pi} T\log T.$$

CHAPTER 3

Explicit formula

It is time to see some number-theoretic rewards for all our work trying to understand $\zeta(s)$. The goal of this chapter is to prove the Riemann-von Mangoldt explicit formula, which expresses $\psi(x)$ as a sum over the zeros of $\zeta(s)$:

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1-1/x^2),$$

where ρ runs over all the zeros of $\zeta(s)$ in the critical strip $0 \leq \sigma \leq 1$. This was done by Riemann in his 1859 paper, and is the first great demonstration of how complex analysis can be used to understand number theory.

8. Perron's formula

By partial summation one can show that if

$$F(s) = \sum_n \frac{a_n}{n^s} \text{ and } A(x) = \sum_{n\leq x} a_n$$

then

$$F(s) = s\int_1^\infty \frac{A(t)}{t^{s+1}}\,\mathrm{d}t.$$

That is, $\alpha(s)$ can be expressed as a function of $A(x)$. We are more interested in the converse – for example, if $a_n = \Lambda(n)$, then $F(s) = -\frac{\zeta'}{\zeta}(s)$, which we hope we can understand via analysis, and $A(x) = \sum_{n\leq x}\Lambda(n) = \psi(x)$, the asymptotics of which are the subject of the prime number theorem.

Lemma 21. *If $\sigma_0 > 0$ then*

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \frac{y^s}{s}\,\mathrm{d}s = \begin{cases} 1 & \text{if } y > 1 \text{ and} \\ 0 & \text{if } 0 < y < 1 \end{cases} + O\left(\frac{y^{\sigma_0}}{T\log y}\right).$$

Proof. Let $y > 1$, $u < 0$, and let C be the rectangular contour with corners at $u \pm iT$ and $\sigma_0 \pm iT$. The function y^s/s is analytic apart from a simple pole at $s = 0$, where the residue is 1. It follows that

$$\frac{1}{2\pi i}\int_C \frac{y^s}{s}\,\mathrm{d}s = 1.$$

We can bound the contribution from the top and bottom by

$$\int_{u\pm iT}^{\sigma_0 - \pm iT} \frac{y^s}{s}\,\mathrm{d}s = \int_u^{\sigma_0} \frac{y^{\sigma\pm iT}}{\sigma \pm iT}\,\mathrm{d}\sigma \ll \frac{1}{T}\int_u^{\sigma_0} y^\sigma\,\mathrm{d}\sigma \leq \frac{1}{T}\int_{-\infty}^{\sigma_0} y^\sigma\,\mathrm{d}\sigma = \frac{y^{\sigma_0}}{T\log y}.$$

Note that here we used that $y > 1$ for the final part, to ensure that $y^u \to 0$ as $u \to -\infty$. The contribution from the left-hand side of the rectangle is

$$\ll \int_{-T}^{T} \frac{y^u}{|u - it|} \, \mathrm{d}t \ll T\frac{y^u}{u} \to 0 \text{ as } u \to -\infty.$$

The case $0 < y < 1$ is similar, but we need to take $u \to +\infty$. ☐

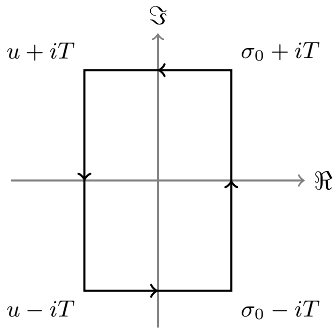

FIGURE 4. The contour C used in the proof of Lemma 21.

Theorem 6 (Perron's formula)**.** *Suppose that $F(s) = \sum \frac{a_n}{n^s}$ is absolutely convergent for $\sigma > \sigma_a$. If $\sigma_0 > \max(0, \sigma_a)$ and $x > 0$ is not an integer then, for any $T \geq 1$,*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} F(s) \frac{x^s}{s} \, \mathrm{d}s + O\left(2^{\sigma_0} \frac{x}{T} \sum_{x/2 < n < 2x} \frac{|a_n|}{|x - n|} + \frac{x^{\sigma_0}}{T} \sum_n \frac{|a_n|}{n^{\sigma_0}} \right).$$

Proof. Since $\sigma_0 > 0$, by Lemma 21 we can write

$$1_{n<x} = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{(x/n)^s}{s} \, \mathrm{d}s + O\left(\frac{(x/n)^{\sigma_0}}{T \log(x/n)} \right).$$

It follows that, since the series converges absolutely, and hence uniformly on the contour we're integrating over, so we can interchange it with the integral,

$$\begin{aligned} \sum_{n<x} a_n &= \sum_{n=1}^{\infty} a_n 1_{n<x} \\ &= \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \left(\frac{1}{n^s} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{(x/n)^{\sigma_0}}{T \log(x/n)} \right) \right) \\ &= \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^s}{s} F(s) + O\left(\frac{x^{\sigma_0}}{T} \sum_n \frac{|a_n|}{n^{\sigma_0} \, |\log(x/n)|} \right) \end{aligned}$$

To simplify the error term, write $\log(x/n) = |\log(1 + (n - x)/x)|$ and use the fact that $|\log(1 + \delta)| \asymp |\delta|$ uniformly for $-1/2 \leq \delta \leq 1$, so that $|\log(x/n)| \asymp \frac{n-x}{x}$ uniformly for $x/2 < n < 2x$. For other values of n, we have $|\log(x/n)| \gg 1$. The error term is therefore

$$\ll \frac{2^{\sigma_0} x}{T} \sum_{x/2 < n < 2x} \frac{|a_n|}{|x - n|} + \frac{x^{\sigma_0}}{T} \sum_n \frac{|a_n|}{n^{\sigma_0}}.$$

☐

9. Estimates for ζ'/ζ

To apply Perron's formula to obtain an explicit formula we will need to evaluate a contour integral of $\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$, which in turn will require some estimates on how large $\frac{\zeta'}{\zeta}$ can get. These will be established in this section.

First, a useful tool from complex analysis that allows us to control the absolute value of an analytic function if we can give a one-sided bound on its real part.

Lemma 22 (Borel-Carathéodory Lemma)**.** *Let f be holomorphic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. For any $r < R$,*

$$\sup_{|z|\leq r} (|f(z)|, |f'(z)|) \ll_{r,R} M.$$

Proof. Let

$$g(z) = \frac{f(z)}{z(2M - f(z))},$$

so that g is holomorphic for $|z| \leq R$. Observe that, using $\Re f(z) \leq M$,

$$|f(z)|^2 = \Re(f(z))^2 + \Im(f(z))^2 \leq (2M - \Re(f(z)))^2 + \Im(f(z))^2 = |2M - f(z)|^2,$$

and so $|2M - f(z)| \geq |f(z)|$ for $|z| \leq R$. In particular, if $|z| = R$ then $|g(z)| \leq 1/R$. By the maximum modulus principle, if $|z| = r$, then

$$|g(z)| = \frac{|f(z)|}{r|2M - f(z)|} \leq \frac{1}{R},$$

and hence

$$R|f(z)| \leq |2Mr - rf(z)| \leq 2Mr + r|f(z)|,$$

or

$$|f(z)| \leq \frac{2r}{R-r}M.$$

This shows that $|f(z)| \ll M$. To deduce the same bound for $f'(z)$, we use Cauchy's formula

$$f'(z) = \frac{1}{2\pi i}\int_{r'} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w,$$

where the integral is taken over some circle of radius $r < r' < R$, say. $\square$

We now apply this to be able to give an approximate formula for $\frac{f'}{f}$ in terms of the zeros of f. To see what kind of formula we should expect, recall the heuristic that a holomorphic function behaves approximately like a polynomial. If it has zeros $z_1, \ldots, z_k$ in a disc then we expect $f(z) \approx \prod(z - z_k)$, and hence $\log f(z) \approx \sum \log(z - z_k)$, and so taking derivatives, we get $\frac{f'}{f}(z) \approx \sum \frac{1}{z - z_k}$. The following lemma makes this precise.

Lemma 23. *Suppose that $f(z)$ is analytic on the disc of radius R centred at a, that $|f(z)| \leq M$ in this disc, and that $f(a) \neq 0$. Let $0 < r < R$. Then in the disc of radius r centred at a*

$$\frac{f'}{f}(z) = \sum_{k=1}^{K} \frac{1}{z - z_k} + O\left(\log \frac{M}{|f(a)|}\right)$$

where the sum is over all zeros z_k of f in the disc of radius R centred at a.

Proof. Without loss of generality, we may suppose that $a = 0$ and $f(a) = 1$. Let the zeros in the disc $|z| \leq R$ be denoted by $z_1, \ldots, z_K$. Note that by Jensen's inequality $K \ll \log M$. As in the proof of Jensen's inequality, let

$$g(z) = f(z) \prod_{k=1}^{K} \frac{R^2 - z\overline{z_k}}{R(z - z_k)},$$

so that g is an analytic function in $|z| \leq R$, and if $|z| = R$ then $|g(z)| = |f(z)| \leq M$. Furthermore,

$$|g(0)| = \prod_{k=1}^{K} \frac{R}{|z_k|} \geq 1.$$

We define an analytic function for $|z| \leq R$ by

$$h(z) = \int_0^z \frac{g'(w)}{g(w)} \,\mathrm{d}w,$$

which is permissible since g has no zeros in this disc, so g'/g is analytic. The derivative is $h'(z) = g'(z)/g(z)$. If we differentiate $e^{-h(z)}g(z)$ then we get

$$e^{-h(z)}(-h'(z))g(z) + e^{-h(z)}g'(z) = 0,$$

and hence $e^{-h(z)}g(z)$ is constant on $|z| \leq R$, so setting $z = 0$ we see that

$$e^{h(z)} = \frac{g(z)}{g(0)}.$$

Taking absolute values then logarithms it follows that, since $|g(0)| \geq 1$,

$$\Re h(z) = \log |g(z)| - \log |g(0)| \leq \log M$$

for all $|z| \leq R$. By the Borel-Carathéodory lemma,

$$|h'(z)| = \left| \frac{g'}{g}(z) \right| \ll \log M.$$

We also have

$$\frac{g'}{g}(z) = \frac{f'}{f}(z) - \sum_{k=1}^{K} \frac{1}{z - z_k} + \sum_{k=1}^{K} \frac{1}{z - R^2/\overline{z_k}}.$$

If $|z| \leq r$ then $\left|z - R^2/\overline{z_k}\right| \geq \left|R^2/\overline{z_k}\right| - |z| \geq R - r$, and so the final sum is $\ll \log M$. $\square$

Lemma 24. *For $t \geq 4$*

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho \\ |t-\gamma| \leq 1}} \frac{1}{s - \rho} + O(\log t), \tag{1}$$

uniformly for $-1 \leq \sigma \leq 2$.

Proof. We apply Lemma 23 to the function $f(s) = \zeta(s)$, on some disc with centre at $3/2 + it$ and radius 3, say, so that in particular the line from $-1 + it$ to $2 + it$ is covered, and the disc stays away from 1, and $f(3/2 + it) \neq 0$. It follows that in the disc with radius $3 - \frac{1}{8}$, say,

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s - \rho} + O(\log t)$$

where the sum ranges over all zeros ρ inside this disc. Notice that every zero ρ with $|t-\gamma| \leq 1$ is included in this disc. It remains to show that the contribution from ρ with $|t-\gamma| > 1$ is $O(\log t)$. For such ρ, when $\Im(s) = t$, $1/|s-\rho| \ll 1$, and hence the contribution is $O(\log t)$ since $N(T+1) - N(T) \ll \log T$.

□

Lemma 25. *For each $T \geq 2$ there is $T \leq T_1 \leq T+1$ such that*

$$\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll (\log T)^2$$

uniformly for $-1 \leq \sigma \leq 2$.

Proof. Since $N(T+1) - N(T) \ll \log T$ there is $T_1 \in [T, T+1]$ such that $|T_1 - \gamma| \gg 1/\log T$ for all zeros ρ. The result follows from (1) since each summand is $\ll \log T$ and there are $O(\log T)$ many summands. □

Finally, we will need to have a good upper bound for $\frac{\zeta'}{\zeta}(s)$ when the real part of s is far to the left of zero. For this we turn, once again, to our old friend, the functional equation. To deduce bounds from this we need to know the size of Γ. Recall Stirling's approximation that $n! \approx \sqrt{2\pi n}(n/e)^n$, or in other words, $\log n! = n \log n - n + O(\log n)$. Reassuringly something similar also holds for complex values.

Lemma 26 (Stirling's formula)**.** *For $|s| \geq \delta$ and $|\arg(s)| < \pi - \delta$*

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1) \text{ and } \log \Gamma(s) = s \log s + O(s).$$

Lemma 27. *If $\sigma \leq -1$ and $|s + 2k| \geq 1/4$ for all positive integers k then*

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s|+1).$$

Proof. Logarithmically differentiating the functional equation gives

$$\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'}{\zeta}(1-s) + \log 2\pi - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot(\pi s/2).$$

The first two terms contribute $O(1)$. By Stirling's formula, the second is $O(\log(|s|+1))$. Finally,

$$\cot(\pi s/2) = i + \frac{2i}{e^{i\pi s} - 1} \ll 1,$$

because we have bounded s away from the possible poles of the left-hand side. □

10. Explicit formula

We can finally prove the explicit formula for $\psi(x)$. We first state a more precise version, with an error bound in terms of some parameter T.

Theorem 7. *If x is not an integer then, for any $T \geq 1$,*

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{\substack{\rho \\ |\gamma| \leq T}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - 1/x^2) + O\left(\frac{x}{T}\left(\log(xT)^2 + \frac{\log x}{\langle x \rangle}\right)\right),$$

where $\langle x \rangle$ denotes the distance from x to the nearest prime power.

Proof. Let $T \leq T_1 \leq T+1$ be some number to be chosen later. By Perron's formula, for any $\sigma_0 > 1$,

$$\sum_{n\leq x} \Lambda(n) = -\frac{1}{2\pi i}\int_{\sigma_0 - iT_1}^{\sigma_0+iT_1} \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s + R$$

where

$$R \ll 2^{\sigma_0}\frac{x}{T}\sum_{x/2<n<2x}\frac{\Lambda(n)}{|x-n|} + \frac{x^{\sigma_0}}{T}\frac{\zeta'}{\zeta}(\sigma_0).$$

We choose $\sigma_0 = 1 + 1/\log x$ so that $x^{\sigma_0} \asymp x$. Since ζ has a simple pole at $s = 1$ so does $\frac{\zeta'}{\zeta}$, and hence $-\frac{\zeta'}{\zeta}(\sigma_0) \asymp \frac{1}{\sigma_0 - 1} = \log x$, and the second sum in the error term is $O(\frac{x\log x}{T})$. Furthermore,

$$\sum_{x+1\leq n<2x}\frac{\Lambda(n)}{|x-n|} \ll \log x \sum_{m=1}^{x}\frac{1}{m} \ll (\log x)^2.$$

The contribution from $x/2 < n \leq x-1$ is similarly bounded. The only remaining contribution is from the n closest to x, which contributes at most $O(x\log x/T\langle x\rangle)$. It follows that

$$R \ll \frac{x(\log x)^2}{T} + \frac{x\log x}{T\langle x\rangle}.$$

We will evaluate the integral by replacing it with that over the contour between $\sigma_0 \pm iT_1$ and $-K \pm iT_1$, for some odd positive integer K. In this contour, say C, the integrand has poles at $s = 0, 1$, and the zeros of $\zeta(s)$, which are either count by ρ or are negative even integers $-K < 2k \leq -2$. Therefore

$$\frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s = -x + \sum_{\substack{\rho\\|\gamma|\leq T}}\frac{x^\rho}{\rho} + \sum_{1\leq k<K/2}\frac{x^{-2k}}{-2k} + \frac{\zeta'}{\zeta}(0).$$

We divide the integral up as

$$\int_C = \int_{\sigma_0-iT_1}^{\sigma_0+iT_1} + \int_{\sigma_0+iT_1}^{-1+iT_1} + \int_{-1+iT_1}^{-K+iT_1} + \int_{-K+iT_1}^{-K-iT_1} + \int_{-K-iT_1}^{-1-iT_1} + \int_{-1-iT_1}^{\sigma_0-iT_1}.$$

The first integral is the main term that we're after. The second we can bound by

$$\ll \int_{\sigma_0}^{-1}\left|\frac{\zeta'}{\zeta}(\sigma+iT_1)\right|\frac{x^\sigma}{T}\,\mathrm{d}\sigma \ll \frac{(\log T)^2}{T}\int_{-1}^{\sigma_0} s x^\sigma\,\mathrm{d}\sigma \ll \frac{(\log T)^2}{T\log x}x,$$

choosing a suitable value for T_1. The third we bound, using $\frac{\log|\sigma+it|}{|\sigma+it|} \ll \frac{\log t}{t}$, by

$$\ll \int_{-1}^{-K}\left|\frac{\zeta'}{\zeta}(\sigma+iT_1)\right|\frac{x^\sigma}{|\sigma+iT_1|}\,\mathrm{d}\sigma \ll \frac{\log T}{T}\int_{-1}^{-K} x^\sigma\,\mathrm{d}\sigma \ll \frac{\log T}{Tx\log x}.$$

The fourth is

$$\ll \frac{\log(KT)}{Kx^K}.$$

The fifth and sixth can be bounded the same way as the third and second, respectively. Putting all this together we have

$$\sum_{n\leq x}\Lambda(n) = x - \sum_{\substack{\rho\\|\gamma|\leq T}}\frac{x^\rho}{\rho} + \sum_{1\leq k<K/2}\frac{x^{-2k}}{2k} - \frac{\zeta'}{\zeta}(0) +$$

$$O\left(\frac{x}{T}((\log x)^2 + (\log T)^2) + \frac{x\log x}{T\langle x\rangle} + \frac{T\log KT}{Kx^K}\right).$$

We now let $K \to \infty$. The result follows since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, whence $\frac{\zeta'}{\zeta}(0) = \log(2\pi)$. □

If we let $T \to \infty$ we obtain the following corollary.

Corollary 2. *If x is not an integer then*

$$\sum_{n\le x}\Lambda(n) = x - \lim_{T\to\infty}\sum_{\substack{\rho\\|\gamma|\le T}}\frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-1/x^2).$$

The mysterious sum over zeros is hard for us to control at the moment, since we don't have much information about where they lie. If we assume the Riemann Hypothesis, however, we can control it very well.

Corollary 3. *If the Riemann Hypothesis is true, so that $\Re\rho = 1/2$ for all ρ, then*

$$\sum_{n\le x}\Lambda(n) = x + O(x^{1/2}(\log x)^2).$$

Proof. We can assume that $x \ge 2$ is not an integer, and that $\langle x\rangle \ge 1$, say, both of which incur a cost of at most $O(\log x)$, which can be absorbed into the error term. Let $T \ge 1$ be some parameter to be chosen later. The explicit formula gives

$$\sum_{n\le x}\Lambda(n) = x + O\left(1 + \sum_{\substack{\rho\\|\gamma|\le T}}\frac{x^\rho}{\rho} + \frac{x}{T}\log(xT)^2\right).$$

Assuming the Riemann hypothesis,

$$\sum_{\substack{\rho\\|\gamma|\le T}}\frac{x^\rho}{\rho} \ll x^{1/2}\sum_{\substack{\rho\\|\gamma|\le T}}\frac{1}{|\rho|}.$$

The number of zeros in the region $n \le t \le n+1$ is $O(\log n)$, and in such a region, $|\rho| \gg n$. It follows that

$$\sum_{\substack{\rho\\|\gamma|\le T}}\frac{1}{|\rho|} \ll 1 + \sum_{2\le n\le T+1}\frac{\log n}{n} \ll (\log T)^2.$$

The error term is therefore bounded by

$$\ll 1 + x^{1/2}(\log T)^2 + \frac{x}{T}\log(xT)^2.$$

If we choose $x = T$, for example, then the result follows. □

Littlewood has shown that if $\psi(x) = x + E(x)$ then the error term cannot be bounded better than $x^{1/2}\log\log\log x$ either way (so it oscillates both positive and negative around this value). That is, both

$$\limsup_{x\to\infty}\frac{E(x)}{x^{1/2}\log\log\log x} > 0$$

and

$$\liminf_{x\to\infty}\frac{E(x)}{x^{1/2}\log\log\log x} < 0.$$

CHAPTER 4

Zeros of Zeta

The explicit formula gives a clear relationship between the distribution of the primes and the distribution of the zeros of $\zeta(s)$ in the critical strip $0 \leq \sigma \leq 1$. We've already seen how, assuming the Riemann Hypothesis, we can deduce a very strong form of the prime number theorem. Unfortunately the Riemann Hypothesis is just that at the moment, so we should ask what we can actually *prove* about the zeros of $\zeta(s)$. In this chapter we will begin this study.

11. Zero-free region

We begin with an 'easy' zero-free region, taking advantage of the pole of $\zeta(s)$ at $s = 1$ to repel zeros.

Theorem 8. *If $\sigma > (1+t^2)/2$ then $\zeta(s) \neq 0$. In particular, $\zeta(s) \neq 0$ if $\frac{8}{9} \leq \sigma \leq 1$ and $|t| \leq \frac{7}{8}$. Furthermore,*

$$\zeta(s) = \frac{1}{s-1} + O(1) \text{ and } -\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

uniformly for $\frac{8}{9} \leq \sigma \leq 2$ and $|t| \leq \frac{7}{8}$.

Proof. We recall the identity

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,\mathrm{d}u.$$

In particular,

$$\left|\zeta(s) - \frac{s}{s-1}\right| \leq \frac{|s|}{\sigma},$$

which proves the first claim and the second. The final claim follows since if $\zeta(s) = (s-1)^{-1} + f(s)$ then

$$\frac{\zeta'}{\zeta}(s) = \frac{-(s-1)^{-2} + f'(s)}{(s-1)^{-1} + f(s)} = \frac{-1}{s-1} + \frac{f(s) + f'(s)(s-1)}{1 + f(s)(s-1)} = \frac{-1}{s-1} + O(1).$$

□

Unfortunately, this is not enough to prove the prime number theorem. For this, recall that we get an asymptotic for $\psi(x)$ we needed to consider all zeros up height aroound $T \approx x$, or else the error term would dominate the main term. We therefore need to rule out zeros too close to the line $\sigma = 1$ for arbitrarily large imaginary parts t. This is provided by the following classical zero-free region, first proved by de la Vallée Poussin in 1899.

Theorem 9. *There is a constant $c > 0$ such that*

$$\zeta(s) \neq 0 \text{ for } \sigma \geq 1 - \frac{c}{\log(|t|+4)}.$$

Before proving it we note the immediate corollary, the prime number theorem (at last!).

Theorem 10 (Prime Number Theorem)**.** *There is $c > 0$ such that*

$$\psi(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

In particular, $\psi(x) \sim x$.

Proof. Without loss of generality, we can assume that x is not an integer and $\langle x \rangle \geq 1$. Let $T \geq 1$ be chosen later. By the explicit formula

$$\psi(x) = x + O\left(1 + \sum_{\substack{\rho \\ |\gamma| \leq T}} \frac{x^\rho}{\rho} + \frac{x}{T}(\log(xT))^2\right).$$

By the zero-free region in Theorem 9 we know that if $\rho = \beta + i\gamma$ is a zero then $\beta \leq 1 - c/\log T$ for some constant $c > 0$. Therefore the error term is

$$\ll 1 + x^{1-c/\log T} \sum_{\substack{\rho \\ |\gamma| \leq T}} \frac{1}{|\rho|} + \frac{x}{T}(\log xT)^2.$$

As before, the sum over zeros we can bound by $O(\log T^2)$. It remains to choose the optimal T. For this we try to balance the two main error terms, wanting to choose T such that

$$x^{1-c/\log T} \approx \frac{x}{T}.$$

Cancelling the x, taking logarithms, and rearranging, this suggests that $T \approx \exp(\sqrt{\log x})$ is an appropriate choice. Indeed, plugging this choice into our error bounds gives the claimed result. $\square$

Before we prove the zero-free region above, let's reexamine how to establish there are no zeros with $\sigma > 1$. We've seen already that this is a consequence of the Euler product representation,

$$\zeta(s) = \prod_p \left(\frac{p^s}{p^s - 1}\right),$$

which is valid for $\sigma > 1$, since the right-hand side is a convergent infinite product with no zero factors, hence cannot be zero. This is slightly unsatisfactory, however, since it relies on some (easy, but not immediate) facts about the nature of infinite products. A more direct proof is to note that

$$\left(\sum_{n=1}^\infty \frac{\mu(n)}{n^s}\right)\zeta(s) = \sum_{n=1}^\infty \frac{1 \star \mu(n)}{n^s} = 1$$

for $\sigma > 1$, where all we have used is that both series are absolutely convergent in this region, so we can multiply them and obtain another Dirichlet series, and that $1 \star \mu(n) = 0$ for $n > 1$, which is an easily checked elementary identity. This implies that $\zeta(s) \neq 0$ in this region.

What's going on here is that we have a nice representation of $\frac{1}{\zeta(s)}$ as a Dirichlet series when $\sigma > 1$, and so it can't have a pole in this region, and hence $\zeta(s)$ can't have a zero. We've also just seen another way of establishing a zero-free region – using the pole at $s = 1$ to 'repel' nearby zeros. To get a zero-free region that covers the entire line $\sigma = 1$ we will need to use a combination of methods, both the existence of Dirichlet series and the presence of a pole at $s = 1$.

We will first sketch the idea behind the argument. Instead of working with $\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$ we will use $-\frac{\zeta'}{\zeta}(s) = \sum \frac{\Lambda(n)}{n^s}$. The advantage of this is that the coefficients $\Lambda(n)$ are now non-negative, but this still has a pole at zeros of $\zeta(s)$.

Suppose then that $\zeta(s)$ has a simple zero at $s = 1 + it$, say. It follows that $\frac{\zeta'}{\zeta}(s)$ has a simple pole of residue 1 at s, and hence

$$-\frac{\zeta'}{\zeta}(\sigma + it) \approx \frac{-1}{\sigma - 1} \text{ as } \sigma \to 1^+.$$

On the other hand, we know that $\zeta(s)$ has a simple pole at $s = 1$, whence $\frac{\zeta'}{\zeta}$ has a simple pole of residue -1 at s, and hence

$$-\frac{\zeta'}{\zeta}(\sigma) \approx \frac{1}{\sigma - 1} \text{ as } \sigma \to 1^+.$$

Taking real parts of both identities and using the Dirichlet series represention of $\frac{\zeta'}{\zeta}$ we deduce that

$$\sum_n \frac{\Lambda(n)}{n^\sigma} \cos(t \log n) \approx \frac{-1}{\sigma - 1} \approx -\sum_n \frac{\Lambda(n)}{n^\sigma}$$

as $\sigma \to 1^+$. Heuristically, this suggests that $\cos(t \log p) \approx -1$ for most primes p. It follows that we should also expect $\cos(2t \log p) \approx 1$ for most primes p, and so

$$\sum_n \frac{\Lambda(n)}{n^\sigma} \cos(2t \log n) \approx \Re - \frac{\zeta'}{\zeta}(\sigma + 2it) \approx \frac{1}{\sigma - 1}.$$

This means that $\zeta(s)$ would have a pole at $s = 1 + 2it$, which is a contradiction, since the only pole of ζ is at $s = 1$. Note the importance of choosing a Dirichlet series with non-negative coefficients here – this proof would not work with $\Lambda(n)$ replaced by $\mu(n)$.

It remains to make this proof sketch rigorous. There are a number of ways of doing this. We present one of the simplest, using ideas we have already seen in the proof of the explicit formula. Note the following immediate corollary of Lemma 23.

Corollary 4. *If $|t| \geq 7/8$ and $5/6 \leq \sigma \leq 2$ then*

$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s - \rho} + O(\log |t|),$$

where the sum is over all zeros ρ of $\zeta(s)$ in the region $|\rho - (3/2 + it)| \leq 5/6$.

Proof of Theorem 9. Let $\rho = \sigma + it$ be such that $\zeta(\rho) = 0$, and let $\delta > 0$ be something to be chosen later. By Corollary 4,

$$-\Re\frac{\zeta'}{\zeta}(1 + \delta + it) = -\frac{1}{1 + \delta - \sigma} - \Re \sum_{\rho' \neq \rho} \frac{1}{1 + \delta + it - \rho'} + O(\log t)$$

Since $\Re\rho' \leq 1$ for all zeros ρ', it follows that $\Re(1/(1 + \delta + it - \rho')) > 0$, provided $\delta > 0$. In particular,

$$-\Re\frac{\zeta'}{\zeta}(1 + \delta + it) \leq -\frac{1}{1 + \delta - \sigma} + O(\log t).$$

Similarly,

$$-\Re\frac{\zeta'}{\zeta}(1+\delta+2it) \ll \log t.$$

Finally,

$$-\frac{\zeta'}{\zeta}(1+\delta) = \frac{1}{\delta} + O(1).$$

We now note that

$$\Re\left(-3\frac{\zeta'}{\zeta}(1+\delta) - 4\frac{\zeta'}{\zeta}(1+\delta+it) - \frac{\zeta'}{\zeta}(1+\delta+2it)\right)$$

is

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}}\left(3+4\cos(t\log n)+\cos(2t\log n)\right).$$

Since $3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0$, the entire sum is ≥ 0. It follows that

$$\frac{3}{\delta} - \frac{4}{1+\delta-\sigma} + O(\log t) \geq 0.$$

First note that we can immediately deduce that $\sigma \neq 1$, or else we'd have $0 \leq \frac{-1}{\delta} + O(\log t)$, which is a contradiction as $\delta \to 0$. Suppose then $\sigma < 1$ and choose $\delta = 4(1-\sigma)$, say, so

$$0 \leq \left(\frac{3}{4} - \frac{4}{5}\right)\frac{1}{1-\sigma} + O(\log t).$$

Rearranging this implies that there is some constant $c > 0$ such that $1-\sigma \geq c/\log t$, and the proof is complete. □

The idea at the heart of this proof – use the assumption of a zero at $\sigma + it$ together with a pole at $s = 1$ to deduce a pole near $\sigma + 2it$ – has not really been bested in over 120 years. The most substantial quantitative improvements have come from providing better upper bounds for $\zeta(s)$ near $\sigma = 1$, allowing for the $O(\log t)$ error term we carried throughout the proof to be reduced. Using improved exponential sum upper bounds this leads to the current best zero-free region (due to Korobov-Vinogradov-Richert) of $\sigma > 1 - c(\log t)^{-2/3}(\log\log t)^{-1/3}$.

12. Asymptotic formula for $N(T)$

We now return to our study of $N(T)$, the count of the number of zeros of $\zeta(s)$ in the region $0 < \sigma < 1$ and $0 < t < T$. In the second chapter of these notes we showed that $N(T) \ll T\log T$. We now give an asymptotic formula for $N(T)$ showing that this is the correct order of magnitude. Incidentally, this also gives the first proof that we've seen so far that there are any zeros of the zeta function at all in the critical strip. (Though that there are infinitely many zeros is also obvious from the explicit formula since $\psi(x)$ is not a continuous function of x).

Theorem 11.

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Proof. We will use what is often called 'the argument principle' to count zeros. This is just the observation that if f has a zero of order k at ρ then $\frac{f'}{f}$ has a simple pole of residue k at ρ as well, and therefore the number of zeros inside a contour C

is exactly $\frac{1}{2\pi i}\int_C \frac{f'}{f}(z)\,\mathrm{d}z$ by the residue theorem. For this to work we need that f has no zeros on the contour C itself, and also that f has no poles on or inside C.

To avoid dealing with the pole of $\zeta(s)$ at $s=1$, we will instead work with a function that has the same zeros as $\zeta(s)$ but is entire. The traditional choice is

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Note that the functional equation is saying precisely that $\xi(s)=\xi(1-s)$. Furthermore, we see that $\xi(s)$ has no poles anywhere, and if $\xi(s)=0$ then $0\leq\sigma\leq 1$ and $\zeta(s)=0$, so $\xi(s)$ has zeros only at the non-trivial zeros of $\zeta(s)$.

Without loss of generality we can assume that there is no zero at height exactly T. The argument principle then gives that

$$N(T)=\frac{1}{2\pi i}\int_C \frac{\xi'}{\xi}(s)\,\mathrm{d}s,$$

where the contour C is a rectangle between $-1,2,2+iT,-1+iT$. We first split the contour by a vertical line at $\sigma=1/2$. The functional equation implies that $\frac{\xi'}{\xi}(s)=-\frac{\xi'}{\xi}(1-s)$, and hence the integral along the left-hand piece is equal to the integral on the contour from $1/2-iT$ to $2-iT$ to 2 to $1/2$. The integrals along $[1/2,2]$ go in both directions so they cancel, and we are left with

$$N(T)=\frac{1}{2\pi i}\int_{C'} \frac{\xi'}{\xi}(s)\,\mathrm{d}s$$

where C' consists of three line segments, from $1/2-iT$ to $2-iT$ to $2+iT$ to $1/2+iT$.

We now use the definition of $\xi(s)$ to expand

$$\frac{\xi'}{\xi}(s)=\frac{1}{s}+\frac{1}{s-1}+\frac{\zeta'}{\zeta}(s)+\frac{1}{2}\frac{\Gamma'}{\Gamma}(s/2)-\frac{s}{2}\log\pi.$$

Note that $\frac{f'}{f}$ is the derivative of $\log f(s)$, provided we can take branch cuts avoiding all the possible zeros of $f(s)$. Our contour C' avoids all such branch cuts, since we can take lines from each zero going left to $-\infty$. Therefore in some open region containing our contour the integrand is the derivative of some single-valued holomorphic function, and we can use the fundamental theorem of calculus. This gives

$$N(T)=\frac{1}{2\pi i}\left[\log s+\log(s-1)+\log\zeta(s)+\log\Gamma(s/2)-\frac{s}{2}\log\pi\right]_{1/2-iT}^{1/2+iT}.$$

The first summand contributes $\frac{1}{2\pi i}\cdot i\pi=\frac{1}{2}$, and similarly for the second. The final summand contributes $-\frac{\log\pi}{2\pi}T$.

We now recall that by Stirling's formula at $s=\frac{1}{4}\pm\frac{iT}{2}$ we have the approximation

$$\log\Gamma(s)=s\log s-s+O(\log s),$$

and so, since the argument of $\frac{1}{4} + \frac{iT}{2}$ is $\pi/2 + O(1/T)$

$$\begin{aligned}\frac{1}{2\pi i}\left[\log\Gamma(s/2)\right]_{1/2-iT}^{1/2+iT} &= \frac{1}{2\pi i}\left(iT\log\left|\tfrac{1}{4}+\tfrac{iT}{2}\right| - iT + O(1)\right)\\ &= \frac{T}{4\pi}\log(\tfrac{1}{16}+\tfrac{T^2}{4}) - \frac{T}{2\pi} + O(1)\\ &= \frac{T}{2\pi}\log(T/2) - \frac{T}{2\pi} + O(1),\end{aligned}$$

since $\log(1/16 + T^2/4) = \log(T^2/4) + O(1/T^2)$. Combining what we have so far, we have shown that

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(1) + S(T),$$

where

$$S(T) = \frac{1}{2\pi i}\left[\log\zeta(s)\right]_{1/2-iT}^{1/2+iT}.$$

It remains to show that this (which is $\frac{1}{\pi}$ multiplied by the argument of $\zeta(1/2 + iT)$) is $O(\log T)$. Note that the real parts cancel, so it is enough to show that $\Im\log\zeta(1/2 \pm iT) = O(\log T)$. For this, recall that uniformly for $-1 \leq \sigma \leq 2$, we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\substack{\rho\\ |\gamma-T|\leq 1}}\frac{1}{s-\rho} + O(\log T).$$

Since there is no zero at height T, we can write

$$\Im\int_{1/2}^{2}\frac{\zeta'}{\zeta}(\sigma+iT)\,\mathrm{d}\sigma = \Im\left(\log\zeta(2+iT) - \log\zeta(1/2+iT)\right).$$

The first part is $O(1)$, since when $\sigma > 1$ we have $\log\zeta(s) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-s}$. The left-hand side is

$$-\sum_{\substack{\rho\\ |\gamma-T|\leq 1}}\int_{1/2}^{2}\Im\frac{1}{\sigma+iT-\rho}\,\mathrm{d}\sigma + O(\log T).$$

Each summand is trivially bounded above by $O(1)$, and there are $O(\log T)$ summands, and therefore

$$\Im\log\zeta(1/2+iT) = O(\log T).$$

The proof is complete. $\square$

To quote Montogomery and Vaughan, "It is remarkable that [Theorem 11] is perhaps the only theorem on the Riemann zeta function that has not seen some significant improvement in the last 100 years."

13. Zeros and error terms

From the explicit formula we see, heuristically at least, that a zero with real part σ should give an error term of size $\gg x^{\sigma}$. To make this precise we need the following lemma of Landau.

Lemma 28 (Landau)**.** *Supppose that A is an integrable function bounded in any finite interval, $A(x) \geq 0$ for all large $x \geq X$, and let*

$$\sigma_c = \inf\left\{\sigma : \int_X^\infty A(x)x^{-\sigma}\,\mathrm{d}x < \infty\right\}.$$

The function

$$F(s) = \int_1^\infty A(x)x^{-s}\,\mathrm{d}x$$

is analytic in $\sigma > \sigma_c$ but not at $s = \sigma_c$.

We will use the following very useful consequence of this lemma. Suppose that

(1) $F(s)$ is defined in some half plane $\sigma > \sigma_1$ by the absolutely convergent integral described in the lemma (and hence defines an analytic function in this half-plane), that
(2) $F(s)$ can be continued to some function (possibly with poles) to some half-plane $\sigma > \sigma_0$, and
(3) there are no poles of $F(s)$ on the real line $s = \sigma$ with $\sigma > \sigma_0$.

Then in fact there are no poles of $F(s)$ at all in the half-plane $\sigma > \sigma_0$! This is a very useful consequence of the non-negativity of the integral defining $F(s)$.

Proof. Divide the integral in the definition of F to $[1, X]$ and $[X, \infty)$, given a corresponding decomposition into $F = F_1 + F_2$, say. The function F_1 is entire. For $\sigma > \sigma_c$, the integral converges absolutely, and hence F_2 also defines an entire function. Suppose that F_2 is analytic at $s = \sigma_c$. We may expand $F_2(s)$ as a power series at $s = \sigma_c + 1$, so that

$$F_2(s) = \sum_{k=0}^\infty c_k(s - 1 - \sigma_c)^k,$$

where

$$c_k = \frac{F_2^{(k)}(1+\sigma_c)}{k!} = \frac{1}{k!}\int_X^\infty A(x)(-\log x)^k x^{-1-\sigma_c}\,\mathrm{d}x.$$

The radius of convergence of this power series is the distance from $1 + \sigma_c$ to the nearest singularity of $F_2(s)$, and hence by assumption is at least $1 + \delta$ for some $\delta > 0$, say. If we consider $s = \sigma_c - \delta/2$, then

$$F_2(s) = \sum_{k=0}^\infty \frac{(1+\sigma_c - s)^k}{k!}\int_X^\infty A(x)(\log x)^k x^{-1-\sigma_c}\,\mathrm{d}x.$$

This is a convergent series with all non-negative terms, and hence by the monotone convergence theorem we can interchange the integral and summation, to find

$$F_2(s) = \int_X^\infty A(x)x^{-1-\sigma_c}\exp((1+\sigma_c - s)\log x)\,\mathrm{d}x = \int_X^\infty A(x)x^{-s}\,\mathrm{d}x,$$

and so the integral must converge at $s = \sigma_c - \delta/2$, which contradicts the definition of σ_c. □

When discussing lower bounds for error terms, the following notation is useful. We say that $f = \Omega_\pm(g)$ if

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} \geq c > 0$$

and

$$\liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq -c < 0,$$

for some absolute constant $c > 0$. That is, not only does $f(x)$ exceed (some constant multiple of) $g(x)$ infinitely often, but it does so both positively and negatively.

Theorem 12. *If σ_0 is the supremum of the real parts of the zeros of $\zeta(s)$ then, for any $\sigma_1 < \sigma_0$,*

$$\psi(x) = x + \Omega_\pm(x^{\sigma_1}).$$

If there is a zero ρ with $\Re\rho = \sigma_0$, then

$$\psi(x) = x + \Omega_\pm(x^{\sigma_0}).$$

Proof. We can certainly assume that $\sigma_1 > 0$. We will prove the Ω_+ statement; the Ω_- statement is an identical proof but with the signs reversed. Suppose for a contradiction that $\psi(x) - x \neq \Omega_+(x^{\sigma_1})$. This means by definition that for every $c > 0$ there exists some $X = X(c)$ such that if $x \geq X$ then $\psi(x) - x \leq cx^{\sigma_1}$. For this part of the proof any finite value of c will work just as well.

We will consider the function

$$F(s) = \int_1^\infty (cx^\sigma - \psi(x) + x)x^{-s-1}\,\mathrm{d}x.$$

Note that since the integrand is trivially $O(x^{-\sigma})$ this integral converges absolutely in the half-plane $\sigma > 1$, and the integrand is non-negative eventually, so the conditions of Lemma 28 are satisfied.

In the half-plane $\sigma > 1$ then we can calculate that

$$F(s) = \frac{c}{s-\sigma_1} + \frac{\zeta'(s)}{s\zeta(s)} + \frac{1}{s-1}.$$

This has a pole at $s = \sigma_1$, but is analytic for real $s > \sigma_1$ – there are no zeros of $\zeta(s)$ on the real line $s > \sigma_1 > 0$ and the pole of $1/(s-1)$ is cancelled out by the pole of $\zeta(s)$ at $s = 1$. It follows by Lemma 28, as discussed after the statement of that lemma, that there are no poles of $F(s)$ for any s with $\sigma > \sigma_1$. This contradicts the choice of σ_1, however, since there is a zero of $\zeta(s)$ with $\rho = \sigma' + it$ for some $\sigma' > \sigma_1$, which will be a pole of $F(s)$.

For the second, stronger, conclusion, we need to argue a little more carefully. Suppose that there is a zero $\rho = \sigma_0 + it$. Consider instead

$$F(s) + \frac{e^{i\theta}F(s+it) + e^{-i\theta}F(s-it)}{2} = \int_1^\infty (cx^\sigma - \psi(x) + x)\,(1+\cos(\theta - t\log x))x^{-s-1}\,\mathrm{d}x.$$

The coefficients here are still non-negative real numbers. The left-hand side has a pole at $s = \sigma$ with residue

$$c + \frac{me^{i\theta}/\rho + me^{-i\theta}/\overline{\rho}}{2},$$

where m is the multiplicity of the zero ρ. We have freedom to choose θ to be whatever we like, in particular so that this expression is $c - m/\lvert\rho\rvert$. The lim inf of the right-hand side is $> -\infty$ as s approaches σ from the right along the real axis. We must therefore have

$$c - \frac{m}{\lvert\rho\rvert} \geq 0,$$

and hence $c \geq m/|\rho|$. In particular we have a contradiction if we choose $c < m/|\rho|$, which establishes the Ω_+ aspect of the theorem. For Ω_- we use the same argument with signs reversed, so that we start with

$$F(s) = \int_1^\infty (-cx^\sigma + \psi(x) - x)x^{-s-1}\,\mathrm{d}s,$$

and so on. □

We can now rigorously prove the following equivalence of the Riemann hypothesis.

Corollary 5. *The Riemann hypothesis is equivalent to the statement that*

$$\psi(x) = x + O_\epsilon\left(x^{1/2+\epsilon}\right)$$

for every $\epsilon > 0$.

Proof. We have already seen that assuming the Riemann hypothesis the error bound $O(x^{1/2}(\log x)^2)$ is possible. Assume then that the Riemann hypothesis is false, and let ρ be a zero with real part $1/2 < \sigma < 1$ (note that such must exist by the functional equation). If we now choose $\epsilon > 0$ such that take some $1/2 + 2\epsilon < \sigma$ then Theorem 12 shows

$$\psi(x) = x + \Omega_\pm(x^{1/2+2\epsilon}),$$

which would contradict $\psi(x) = x + O_\epsilon(x^{1/2+\epsilon})$ for large enough x. □

CHAPTER 5

Zero density results

The dream of analytic number theorists is, of course, to prove the Riemann hypothesis. We have already seen the close connection between the primes and the zeros of $\zeta(s)$. In the absence of proving the Riemann hypothesis itself, we instead to aim to prove weaker results about the distribution of zeros:

(1) Zero-free regions, that is, half-planes of the shape $\sigma > 1-f(t)$ in which there are no zeros at all. In this course we have proved this for $f(t) \gg 1/\log t$. The best known bound is due to Korobov and Vinogradov, which allows for some $f(t) \gg 1/(\log t)^{2/3}(\log\log t)^{1/3}$.
(2) More precise estimates of how many zeros are in the critical strip $0 \leq \sigma \leq 1$, perhaps without knowledge of how their real parts are distributed. In this course we have shown that the number of zeros up to height T is asymptotically $\frac{1}{2\pi}T\log T$.
(3) Some estimates on how many zeros are on the critical line $\sigma = 1/2$. It was first proved by Hardy that there are infinitely many zeros on this line (an alternative proof is sketched on the third examples sheet). If we let $N_0(T)$ be the number of zeros on the line $\sigma = 1/2$ up to height T then one weak form of the Riemann hypothesis would be $N_0(T) \sim \frac{1}{2\pi}T\log T$ (that is, '100% of the zeros are on the critical line').

    One of the great achievements of 20th century analytic number theory is showing that a positive proportion of zeros lie on the critical line – that is, that $N_0(T) \gg T\log T$. This was first established by Selberg, then improved by Levinson (who showed that at least $\frac{1}{3}$ of all zeros are on the critical line) then by Conrey. The best result so far is due to Bui-Conrey-Young who in 2011 showed that over 41% of the zeros are on the critical line.
(4) Finally, we come to zero density estimates, which are the focus of this final chapter of the course. In this we stop trying to show that there aren't *any* zeros with $\sigma > 1/2$ and just try to show that least there aren't *too many.* For example, we will prove an estimate, due to Ingham, that if $N(\sigma, T)$ counts the number of zeros with real part $\geq \sigma$ and imaginary part at most T, then

$$N(\sigma, T) \ll T^{3(1-\sigma)}(\log T)^{O(1)}.$$

Of course, this is trivial for $\sigma \leq 2/3$ as stated, but for example, shows that there are at most $O(T^{3/4+\epsilon})$ many zeros with real part $\geq 3/4$, a tiny proportion of the total number of $\asymp T\log T$ zeros.

14. Approximate functional equation

We begin by proving a very useful tool in deeper study of the zeta function. Recall that the functional equation states that

$$\zeta(s) = \chi(s)\zeta(1-s)$$

where

$$\chi(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s).$$

In particular, since for $\sigma > 1$ the zeta function is given by a Dirichlet series, it states that

$$\zeta(s) = \sum_n \frac{1}{n^s} \text{ for } \sigma > 1$$

and

$$\zeta(s) = \chi(s)\sum_n \frac{1}{n^{1-s}} \text{ for } \sigma < 0.$$

These make study of the zeta function in either region quite straightforward. The critical strip $0 \leq \sigma \leq 1$ is much more mysterious. We might hope that perhaps one can 'interpolate' these two identities, and replace them with finite sums with some small error term. This is indeed possible, as first shown by Hardy and Littlewood.[3]

Theorem 13. *Whenever* $0 \leq \sigma \leq 1$

$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + \chi(s)\sum_{n\leq y}\frac{1}{n^{1-s}} + O\left(x^{-\sigma} + |t|^{-\frac{1}{2}}x^{1-\sigma}\right).$$

for any $x, y \geq 1/2$ *such that* $xy = t/2\pi$*.*

Proof. Recall that in our second method of proving the functional equation we first established the identity

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,\mathrm{d}x$$

for $\sigma > 1$, which followed from

$$\frac{\Gamma(s)}{n^s} = \frac{1}{n^s}\int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t = \int_0^\infty x^{s-1}e^{-nx}\,\mathrm{d}x$$

and then summing over n. If instead we summed over just those $n > m$ then the same proof gives the identity

$$\zeta(s) = \sum_{n\leq m}\frac{1}{n^s} + \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-mx}}{e^x-1}\,\mathrm{d}x.$$

We then transformed this to

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)\int_C \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z$$

[3] Although it was later discovered by Siegel that actually Riemann was well-aware of this equation, and used it extensively in private, unpublished, calculations.

where the contour goes from positive infinity, circles the origin, and returns to infinity, where z^{s-1} is defined as $\exp((s-1)\log z)$ with the logarithm real at the beginning of the contour. Again, the same argument yields

$$\zeta(s) = \sum_{n\le m} \frac{1}{n^s} + \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)\int_C \frac{z^{s-1}e^{-mz}}{e^z-1}\,\mathrm{d}z,$$

and this expression now holds for all s except positive integers.

Without loss of generality let $t>0$ and $xy = t/2\pi$. We first claim that it suffices to prove the approximate functional equation in the case the $x \le y$. For suppose that instead $y < x$. We know then that

$$\zeta(1-s) = \sum_{n\le y}\frac{1}{n^{1-s}} + \chi(1-s)\sum_{n\le x}\frac{1}{n^s} + O\left(y^{\sigma-1}\right).$$

By the functional equation:

$$\begin{aligned}\zeta(s) &= \chi(s)\zeta(1-s)\\ &= \chi(s)\left(\sum_{n\le y}\frac{1}{n^{1-s}} + \chi(1-s)\sum_{n\le x}\frac{1}{n^s} + O(y^{\sigma-1})\right).\end{aligned}$$

Since $\chi(s)\chi(1-s) = 1$ and $|\chi(s)| \ll t^{\frac{1}{2}-\sigma}$ the result follows. We will therefore assume that $x \le y$ in the rest of the proof.

We move the contour C to one consisting of four straight lines as depicted in Figure 14, where the turning points are at

$$\begin{aligned}P_1 &= \pi y + i3\pi y\\ P_2 &= -\pi y + i\pi y\\ P_3 &= -\pi y + i\pi(2\lfloor y\rfloor + 1).\end{aligned}$$

Since y is not an integer this contour does not cross any poles of the integrand, and the poles that are now inside the contour are at $\pm 2i\pi, \ldots, \pm 2i\pi\lfloor y\rfloor$. The residues from $\pm 2\pi i n$ are together

$$\begin{aligned}(2n\pi i)^{s-1} + (-2n\pi i)^{s-1} &= (2n\pi)^{s-1}e^{i\pi(s-1)}2\cos(\pi(s-1)/2)\\ &= -2(2n\pi)^{s-1}e^{i\pi s}\sin(\pi s/2).\end{aligned}$$

It follows that

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + e^{-i\pi s}\Gamma(1-s)\left(\sum_{n\le y}2(2n\pi)^{s-1}e^{i\pi s}\sin(\pi s/2)\right) + O(e^{-i\pi s}\Gamma(1-s)I)$$

where I is the sum of the integrals over C_1, C_2, C_3, C_4. Simplifying the second summand gives

$$\zeta(s) = \sum_{n\le x}\frac{1}{n^s} + 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\sum_{n\le y}\frac{1}{n^{1-s}} + O(e^{-i\pi s}\Gamma(1-s)I).$$

By Stirling's formula $|\Gamma(1-s)| \ll t^{\frac{1}{2}-\sigma}e^{-\frac{\pi}{2}t}$ and hence

$$e^{-i\pi s}\Gamma(1-s) \ll t^{\frac{1}{2}-\sigma}e^{\frac{\pi}{2}t}.$$

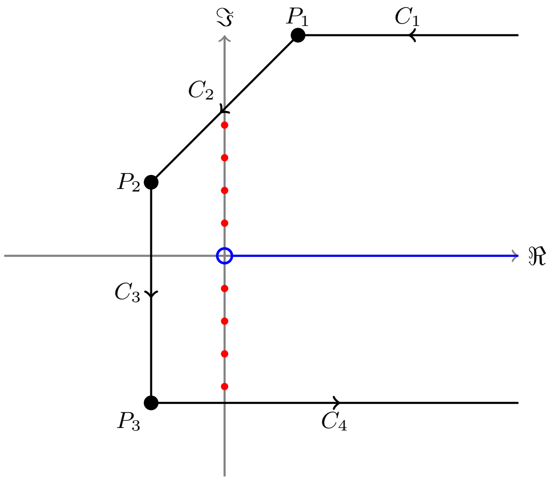

FIGURE 5. The contour used in the proof of the approximate functional equation. The poles at $\pm 2\pi i, \ldots, \pm 2\pi i\lfloor y\rfloor$ are marked in red, and the previous contour C is marked in blue.

It remains to show that the contribution from each integral is

$$\ll e^{-\frac{\pi}{2}t}\left(t^{\sigma-\frac{1}{2}}x^{-\sigma}+y^{\sigma-1}\right).$$

Let $z=u+iv=\rho e^{i\theta}$ for $0<\theta<2\pi$, so that

$$\left|z^{s-1}\right|=\rho^{\sigma-1}e^{-t\theta}.$$

In particular, by the triangle inequality, we can bound the integral over C_i by

$$\leq \int_{C_i}\frac{\rho^{\sigma-1}e^{-t\theta}e^{-mu}}{|e^z-1|}\,\mathrm{d}z.$$

On C_4 we see that $\theta\geq\frac{5}{4}\pi$ and $\rho\gg y$, and furthermore $|e^z-1|\gg 1$. Hence the integral over C_4 is

$$\ll y^{\sigma-1}e^{-\frac{5}{4}\pi t}\int_{-\pi y}^{\infty}e^{-mu}\,\mathrm{d}u\ll e^{m\pi y-\frac{5}{4}\pi t}\ll y^{\sigma-1}e^{t(\frac{1}{2}-\frac{5}{4}\pi)}$$

which is good enough since $\frac{5}{4}\pi-\frac{1}{2}\geq\frac{\pi}{2}$. On C_3, we see that $\theta\geq\frac{3}{4}\pi$, and still $|e^z-1|\gg 1$, so the integral over C_3 is

$$\ll y^{\sigma}e^{-t\frac{5}{4}\pi+\pi ym}\ll y^{\sigma}e^{t(\frac{1}{2}-\frac{5}{4}\pi)}\ll y^{\sigma-1}e^{-\frac{\pi}{2}t}$$

since $y\ll t$.

[The estimations of the integral over C_1 and C_2 are a bit more tricky, and were not covered in lectures, so they do not form part of the examinable content of the course. Some more details are included here for those interested.]

On C_1 we now use $|e^z-1|\gg e^u$, and so the integral over C_1 is

$$\ll y^{\sigma-1}\int_{\pi y}^{\infty}e^{-t\theta}e^{-(m+1)u}\,\mathrm{d}u,$$

where θ is the argument of $u + 3\pi iy$, that is, $\theta = \tan^{-1}(3\pi y/u)$. We divide the integral into the integral over $(\pi y, 2\pi^2 y)$ and $(2\pi^2 y, \infty)$. Since $m + 1 \geq x = t/2\pi y$ the first integral is

$$\ll \int_{\pi y}^{2\pi^2 y} e^{-t(\theta + \frac{u}{2\pi y})}\, \mathrm{d}u.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\tan^{-1}\frac{3\pi y}{u} + \frac{u}{2\pi y}\right) = -\frac{3\pi y}{u^2 + 9\pi^2 y^2} + \frac{1}{2\pi y} > 0$$

we always have $\theta + \frac{u}{2\pi y}$ is at least its value at $u = \pi y$, where it is $c_1 = \tan^{-1}(3) + \frac{1}{2} \approx 1.74$, and therefore this first integral is

$$\ll y e^{-c_1 t}.$$

The second integral is

$$\ll \int_{2\pi^2 y}^{\infty} e^{-xu}\, \mathrm{d}u \ll e^{-\pi t}.$$

Overall then the contribution from C_1 is

$$y^{\sigma} e^{-c_1 t} + y^{\sigma - 1} e^{-\pi t} \ll y^{\sigma - 1} e^{-\frac{\pi}{2} t}$$

as required, since $c_1 > \pi/2$. Finally, we have to deal with C_2. Here $z = 2\pi iy + \lambda e^{\pi i/4}$ where $\lambda \in \mathbb{R}$ and $|\lambda| \leq \sqrt{2}\pi y$. Expanding out the $\log(2\pi y + \lambda e^{-\pi i/4})$ gives

$$\left|z^{s-1}\right| \ll y^{\sigma-1} \exp\left(t\left(-\frac{\pi}{2} + \frac{\lambda}{2^{3/2}\pi y} - \frac{\lambda^2}{8\pi^2 y^2} + O(\lambda^3/y^3)\right)\right).$$

It follows that the integral over C_2 is

$$\ll y^{\sigma-1} e^{-\frac{\pi}{2} t} \int_{C_4} \frac{e^{-f(z)t - mu}}{|e^z - 1|}\, \mathrm{d}z$$

where

$$f(z) = \theta - \frac{\lambda}{2^{3/2}\pi y} + \frac{\lambda^2}{8\pi^2 y^2} + O(\lambda^3/y^3).$$

If $|u| > \pi/2$ then $\left|e^{(x-m)u}\right| \ll |e^z - 1|$ and $e^{-xu} = e^{-\lambda t/2^{3/2}\pi y}$, and therefore this part of the integral contributes

$$\ll y^{\sigma-1} e^{-\frac{\pi}{2}} \int_{-\pi y\sqrt{2}}^{\pi y\sqrt{2}} e^{-\frac{\lambda^2}{8\pi^2 y^2} t + O(\lambda^3 t/y^3)}\, \mathrm{d}\lambda \ll y^{\sigma} t^{-1/2} e^{-\frac{\pi}{2} t}.$$

A similar estimate applies if $|u| \leq \pi/2$ and $|e^z - 1| \gg 1$. Finally, if the contour goes through somewhere close to $z = 2\pi i\lfloor y \rfloor$ then move it to an arc of the circle centred at $2\pi i\lfloor y \rfloor$ with radius $\pi/2$. On this circle $\left|z^{s-1} e^{-mu}\right| \ll y^{\sigma-1} e^{-\frac{\pi}{2} t}$, and the proof is complete. □

15. Mean value estimates

To show that an analytic function does not have too many zeros in a given region, it is important to know that it doesn't get too large. This apparently paradoxical requirement has already appeared in Jensen's inequality, which essentially says that if $|f(s)| \leq M$ in a disc then there are $O(\log M)$ many zeros in this disc (which is less magical once one recalls that if a polynomial satisfies $|P(z)| \ll z^K$ then it has at most K zeros).

We want to count zeros not in a disc, but in a rectangle. Unfortunately there is nothing quite so clean as Jensen's inequality available here – a different tool is required, which we will come to in the next section. We will still need to show that our function (in this case $\zeta(s)$) does not grow too quickly. The information we will use is an upper bound on the average value of $|\zeta|^2$ (or $|\zeta|^4$) over the line $\frac{1}{2}+it$.

First we give a couple of examples of mean value estimates for finite Dirichlet polynomials (the approximate functional equation which enable us to apply these to the zeta function).

Lemma 29. *For any $a_n \in \mathbb{C}$*

$$\int_0^T \left|\sum_{n\leq x} a_n n^{it}\right|^2 \mathrm{d}t = (T+O(x))\sum_{n\leq x}|a_n|^2.$$

Proof. The temptation is to immediately expand out the square and integrate, but this will not be sufficient. Instead, we first use the non-negativity of the integrand to note that

$$\int_0^T \left|\sum_{n\leq x} a_n n^{it}\right|^2 \mathrm{d}t \leq \int f(t)\left|\sum_{n\leq x} a_n n^{it}\right|^2 \mathrm{d}t$$

for any integrable function $f:\mathbb{R}\to[0,\infty)$ such that $f(t)=1$ for $0<t\leq T$. Now we expand the square, to get

$$\sum_{n,m\leq x} a_n\overline{a_m}F(m/n)$$

where

$$F(u)=\int f(t)u^{it}\,\mathrm{d}t.$$

It is desirable then to choose $f(u)$ to have reasonable Fourier properties. For example, in this proof, we will choose to f to be the piecewise linear function which is 1 on $(0,T]$, 0 for $t\leq -x$ or $t>T+x$, and otherwise decays linearly. It follows that $F(1)=T+x$ and $F(u)\ll x^{-1}(\log u)^{-2}$ if $u\neq 1$, which follows after integrating by parts. Furthermore, if $n\leq m\leq x$, say, then

$$\log(m/n)=\log\left(1+\frac{m-n}{n}\right)\gg\frac{m-n}{n}\gg\frac{m-n}{x}.$$

It follows that when $n\neq m$,

$$F(m/n)\ll\frac{1}{x}(\log m/n)^{-2}\ll\frac{x}{(m-n)^2}.$$

Furthermore,

$$\sum_{1\leq n\neq m\leq x}|a_m a_n|\,(m-n)^{-2}\ll\sum_{n\leq x}|a_n|^2$$

since $|a_m a_n|\ll|a_m|^2+|a_n|^2$. It follows that

$$\int_0^T \left|\sum_{n\leq x} a_n n^{it}\right|^2 \mathrm{d}t \leq (T+O(x))\sum_{n\leq x}|a_n|^2.$$

Similarly we can derive the same lower bound, by choosing f to be 0 outside $(0,T]$ and triangular inside this interval with a peak at 1. □

Combining this lemma with the approximate functional equation we can prove the following.

Theorem 14.
$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^2 \, \mathrm{d}t \sim T\log T.$$

Proof. We apply the approximate functional equation with $\sigma = 1/2$ and $y = \sqrt{\log t}$ for some $t \geq 4$, say. Note that by Stirling's formula
$$\left|\chi(\tfrac{1}{2}+it)\right| \ll \left|\sin(\pi(\tfrac{1}{4}+i\tfrac{t}{2}))\Gamma(\tfrac{1}{2}-it)\right| \ll 1$$
for any $|t| \geq 4$. It follows by the approximate functional equation that when $s = \frac{1}{2}+it$
$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + O\left((\log t)^{1/4}\right)$$
where $x = \frac{t}{2\pi(\log t)^{1/2}}$. It follows that, if $Z = \sum_{n\leq x} n^{-\frac{1}{2}-it}$, then
$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^2 \, \mathrm{d}t = \int_0^T |Z|^2 \, \mathrm{d}t + O\left(T(\log T)^{1/2} + T^{1/2}(\log T)^{1/4}\left(\int_0^T |Z|^2 \, \mathrm{d}t\right)^{1/2}\right).$$
It therefore suffices to show that
$$\int_0^T \left|\sum_{n\leq x} n^{-\frac{1}{2}-it}\right|^2 \mathrm{d}t \sim T\log T.$$
Lemma 29 applied with $a_n = n^{-1/2}$ implies that this integral is
$$= (T+O(x))\sum_{n\leq x}\frac{1}{n} = (T+O(x))(\log x + O(1)) = (1+o(1))T\log T$$
as required. □

We will also need a stronger bound on the L^4 moment of the zeta function. For this we use the following mean value estimate.

Lemma 30.
$$\int_0^T \left|\sum_{n\leq x} a_n n^{it}\right|^4 \mathrm{d}t \ll (T+x^2)\left(\sum_{n\leq x} |a_n|^2 \tau(n)\right)^2.$$

Proof. Expanding out the square we note that
$$\left(\sum_{n\leq x} a_n n^{it}\right)^2 = \sum_{n,m\leq x} a_n a_m (nm)^{it} = \sum_{k\leq x^2} b_k k^{it}$$
where
$$b_k = \sum_{\substack{n,m\leq x\\ nm=k}} a_n a_m.$$

The claimed bound now follows from Lemma 29 applied to b_k, since

$$\begin{aligned}\sum_{k\leq x^2} |b_k|^2 &= \sum_k \left|\sum_{n,m\leq x} a_n a_m 1_{nm=k}\right|^2 \\ &= \sum_{n_1,n_2,m_1,m_2\leq x} a_{n_1}a_{n_2}\overline{a_{m_1}a_{m_2}}1_{n_1m_1=n_2m_2} \\ &\leq \sum_{n,m\leq x} |a_n a_m|^2 \tau(nm)\end{aligned}$$

where we have used the fact that $2|uv| \leq |u|^2 + |v|^2$ in the last inequality. The stated bound follows from the fact that $\tau(nm) \leq \tau(n)\tau(m)$, which can be proved either directly or by noting that it suffices to check it for prime powers, when it becomes the trivial inequality

$$1 + k_1 + k_2 \leq (1 + k_1)(1 + k_2).$$

□

Theorem 15.

$$\int_0^T \left|\zeta(\tfrac{1}{2} + it)\right|^4 \ll T(\log T)^4.$$

Proof. We once again apply the approximate functional equation, but now with $x = y = (t/2\pi)^{1/2}$, so that when $s = \frac{1}{2} + it$,

$$\begin{aligned}\zeta(s) &= \sum_{n\leq x} \frac{1}{n^s} + \chi(s)\sum_{n\leq x}\frac{1}{n^{1-s}} + O(t^{-1/4}) \\ &\ll \left|\sum_{n\leq x} n^{-\frac{1}{2}+it}\right| + O(t^{-1/4}).\end{aligned}$$

It therefore suffices to show that

$$\int_0^T \left|\sum_{n\leq x} n^{-1/2+it}\right|^4 \mathrm{d}t \ll T(\log T)^4.$$

For this we use Corollary 30 with $a_n = n^{-1/2}$. It gives the upper bound

$$\ll T\left(\sum_{n\leq x}\frac{\tau(n)}{n}\right)^2.$$

Using $\sum_{n\leq x}\tau(n) = x\log x + O(x)$ and partial summation implies that

$$\sum_{n\leq x}\frac{\tau(n)}{n} \ll (\log x)^2$$

and the proof is complete. Note how important it was that the length of the sum here was only $O(t^{1/2})$ – without the approximate functional equation we'd have to take a sum of length $O(t)$ and then the final bound would look more like $O(T^{2+\epsilon})$. □

This upper bound was proved by Hardy and Littlewood. By refining their method Ingham showed that in fact

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^4 \,\mathrm{d}t \sim \tfrac{1}{2\pi^2} T(\log T)^4.$$

It is conjectured that for all $k \geq 0$

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} \,\mathrm{d}t \sim c_k T(\log T)^{k^2}$$

for some constant $c_k > 0$. The cases $k = 0, 1, 2$ are the only ones for which an asymptotic is known, although lower bounds of the correct order of magnitude were proved by Ramachandra for all k. Upper bounds for $k \geq 3$ remain an open problem, even ones of the shape $O(T^{1+\epsilon})$.

For a long time it was a mystery what the correct conjecture even was - remarkably, in 1998 Keating and Snaith found that the characteristic polynomial of a large random matrix (for a given value of random) experimentally provides a very close model to the Riemann zeta function. Using this heuristic, they conjectured that

$$c_k = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \prod_p \left(\left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{r=0}^{k-1} \binom{k-1}{r}^2 p^{-r} \right),$$

which so far looks like the right guess. One of the most interesting aspects to the recent story of the zeta function has been this nascent connection to random matrices, which is still being explored.