ANALYTIC NUMBER THEORY

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These are lecture notes for the Part III lecture course given in Lent Term 2020. They are meant to be a faithful copy of the material given in lectures, with some supplementary footnotes and historical notes. The lectures themselves are the guide for what material is examinable, and any additional material in these printed notes will be marked as non-examinable. In the case of any doubt, ask the lecturer.

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CHAPTER 1

Elementary Techniques

Review of asymptotic notation. We write $f(x) = O(g(x))$ if there exists some constant $C > 0$ such that $|f(x)| \le C\,|g(x)|$ for all sufficiently large x. We will also use the Vinogradov notation $f \ll g$ to denote the same thing (so that $f = O(g)$ and $f \ll g$ are equivalent).

We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We write $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Observe that

$$f \sim g \text{ if and only if } f = (1 + o(1))g.$$

1. Arithmetic functions

An arithmetic function is simply a function on the natural numbers[1], $f : \mathbb{N} \to \mathbb{R}$. An arithmetic function is multiplicative if

$$f(nm) = f(n)f(m) \text{ whenever } (n, m) = 1,$$

and is completely multiplicative if $f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{N}$.

An important operation on the space of arithmetic functions is that of multiplicative convolution:

$$f \star g(n) = \sum_{ab=n} f(a)g(b).$$

This is since we are looking primarily at multiplicative functions.

If f and g are both multiplicative functions, then so too is $f \star g$. The most obvious arithmetic function is the constant function:

$$\mathbf{1}(n) = 1.$$

We recall the definition of the Möbius function,

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ where } p_i \text{ are distinct primes and} \\ 0 & \text{otherwise (i.e. if } n \text{ is divisible by a square).} \end{cases}$$

$\delta(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$

A fundamental relationship is that of Möbius inversion, which says that the Möbius function acts as an inverse to multiplicative convolution:

$$\mathbf{1} \star f = g \text{ if and only if } \mu \star g = f.$$

Properties
- $*$ is commutative
- $*$ is associative
- $*$ has an inverse
- f, g multiplicative $\Rightarrow f * g$ multiplicative
- δ is the identity since $\delta * f = f$

A great deal of analytic number theory is concerned with a deep study of the distribution of the prime numbers. For this the 'correct' way to count primes is not, as one might expect, the indicator function

$$1_{\mathbb{P}}(n) = \begin{cases} 1 & \text{if } n \text{ is prime, and} \\ 0 & \text{otherwise,} \end{cases}$$

[1]For the purposes of this course, 0 is not a natural number.

$1 * \mu = \delta$

Proof: $n=1$ clear.
- $\mu, \mathbb{1}$ multiplicative $\Rightarrow \mathbb{1} * \mu$ multiplicative.
- Show that $\mathbb{1} * \mu(p^k) = 0$. Done.

but instead the von Mangoldt function, which firstly also counts prime powers p^k, but also counts them not with weight 1, but with weight $\log p$ instead:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This is a clever idea. Since prime powers are relatively scarce, including them does not affect the analysis much.

The main reason that this function is much easier to work with than $1_{\mathbb{P}}$ directly, is the following identity.

Lemma 1.

$$\mathbf{1} \star \Lambda(n) = \log n \text{ and } \log \star \mu(n) = \Lambda(n).$$

Proof. The second identity follows from the first by Möbius inversion. To establish the first, if we let $n = p_1^{k_1} \cdots p_r^{k_r}$, then

Note that log is not multiplicative, so have to do all the work & check for $n = p_1^{k_1} \cdots p_r^{k_r}$ instead of just for prime powers.

$$\begin{aligned} \mathbf{1} \star \Lambda(n) &= \sum_{i=1}^{r} \sum_{j=1}^{k_i} \log p_i \\ &= \sum_{i=1}^{r} \log p_i^{k_i} \\ &= \log n. \end{aligned}$$

□

2. Summation

A major theme of analytic number theory is understanding the basic arithmetic functions, particularly how large they are on average, which means understanding $\sum_{n \le x} f(n)$. For example, if f is the indicator function of primes, then this summatory function is precisely the prime counting function $\pi(n)$.

We say that f has average order g if

$$\sum_{n \le x} f(n) \sim x g(x).$$

One of the most useful tools in dealing with summations is partial summation, which is a discrete analogue of integrating by parts.

Theorem 1 (Partial summation)**.** *If a_n is any sequence of complex numbers and $f : \mathbb{R}_+ \to \mathbb{R}$ is such that f' is continuous then*

$$\sum_{1 \le n \le x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, \mathrm{d}t,$$

where $A(x) = \sum_{1 \le n \le x} a_n$.

Idea: If we understand $A(x) = \sum_{1 \le n \le x} a_n$ then we understand the weighted sum for a suitably behaved x.

Proof. Let $N = \lfloor x \rfloor$. Using $a_n = A(n) - A(n-1)$

$$\begin{aligned} \sum_{1 \le n \le N} a_n f(n) &= \sum_{n=1}^{N} f(n)(A(n) - A(n-1)) \\ &= f(N)A(N) - \sum_{n=1}^{N-1} A(n)(f(n+1) - f(n)). \end{aligned}$$

We now observe that

$$\int_n^{n+1} f'(x)\,\mathrm{d}x = f(n+1) - f(n),$$

This lets us take the jump from the continuous world to the discrete world.

and so, since $A(x)$ is constant for $x \in [n, n+1)$,

$$\sum_{1\le n\le N} a_n f(n) = f(N)A(N) - \sum_{n=1}^{N-1} \int_n^{n+1} A(x) f'(x)\,\mathrm{d}x,$$

and the result follows since if $N \le x < N+1$ then

$$A(x)f(x) = A(N)f(x) = A(N)\left(f(N) + \int_N^x f'(x)\,\mathrm{d}x\right).$$

□

This is extremely useful even when the coefficients a_n are identically 1, when $A(x) = \lfloor x \rfloor = x + O(1)$.

Lemma 2.

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + O(1/x),$$

where $\gamma = 0.577\cdots$ is a constant, known as Euler's constant.

Proof. By partial summation

$\{t\} = t - \lfloor t \rfloor$

$$\begin{aligned}\sum_{n\le x} \frac{1}{n} &= \underbrace{\frac{\lfloor x \rfloor}{x}}_{=1+O(\frac{1}{x})} + \int_1^x \frac{\lfloor t \rfloor}{t^2}\,\mathrm{d}t \\ &= 1 + \underbrace{\int_1^x \frac{1}{t}\,\mathrm{d}t}_{=\log x} + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t - \int_x^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t + O(1/x) \\ &= \log x + \left(1 + \int_1^\infty \frac{\{t\}}{t^2}\,\mathrm{d}t\right) + O(1/x).\end{aligned}$$

It remains to note that the second term is a constant, since the integral converges. As $\{t\} \le 1$ $\forall t$

□

It is remarkable how little we understand about Euler's constant – it is not even known whether it is irrational or not.

γ measures the difference between continuous and discrete.

Lemma 3.

$$\sum_{1\le n\le x} \log n = x\log x - x + O(\log x).$$

Proof. By partial summation

$$\begin{aligned}\sum_{n\le x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor t \rfloor}{t}\,\mathrm{d}t \\ &= x\log x - x + O(\log x).\end{aligned}$$

□

How to do (most) of ANT:

① Write as a multiple sum/integral

② Change the order of summation

Note: $f(1) = 1$ for any multiplicative function f

3. Divisor function

We now turn our attention to number theory proper, and examine one of those most important arithmetic functions: the divisor function[2]

$$\tau(n) = \mathbf{1} \star \mathbf{1}(n) = \sum_{ab=n} 1 = \sum_{d|n} 1.$$

We first find its average order.

Theorem 2.

$$\sum_{n\le x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^{1/2}).$$

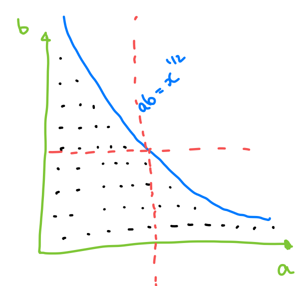
In particular, the average order of $\tau(n)$ is $\log n$.

Proof. A first attempt might go as follows:

$= \sum_{n\le x}\sum_{ab=n} 1$

$$\begin{aligned}\sum_{n\le x}\tau(n) &= \sum_{ab\le x} 1\\ &= \sum_{a\le x}\left\lfloor \frac{x}{a}\right\rfloor\\ &= x\sum_{a\le x}\frac{1}{a} + O(x)\\ &= x\log x + \gamma x + O(x).\end{aligned}$$

$= \sum_{a\le x}\sum_{b\le x/a} 1$



The problem is that the second term γx is lost in the error term $O(x)$. To improve the error term we use what is known as the hyperbola method, which is the observation that when summing over pairs (a, b) such that $ab \le x$ we can express this as the sum over pairs where $a \le x^{1/2}$ and where $b \le x^{1/2}$, and then subtract the contribution where $\max(a, b) \le x^{1/2}$.

since $ab = x \Rightarrow a \le x^{1/2}$ or $b \le x^{1/2}$ but $\max(a,b) \geq x^{1/2}$

Here we are applying the same proof but the Hyperbola method is used to pin down the error term

$$\begin{aligned}\sum_{ab\le x} 1 &= \sum_{a\le x^{1/2}}\left\lfloor\frac{x}{a}\right\rfloor + \sum_{b\le x^{1/2}}\left\lfloor\frac{x}{b}\right\rfloor - \sum_{a,b\le x^{1/2}} 1\\ &= 2x\sum_{a\le x^{1/2}}\frac{1}{a} - \lfloor x^{1/2}\rfloor^2 + O(x^{1/2})\\ &= x\log x + (2\gamma - 1)x + O(x^{1/2}).\end{aligned}$$

These sums are equal.

□

It is a deep and difficult problem to improve the error term here – the truth is probably $O(x^{1/4+\epsilon})$, but this is an open problem, and the best known is $O(x^{0.3149\cdots})$.

We have just shown that the ‘average’ number of divisors of n is $\log n$. The worst case behaviour can differ dramatically from this average behaviour, however.

Theorem 3. *For any $n \ge 1$,*

$$\tau(n) \le n^{O(1/\log\log n)}.$$

In particular, for any $\epsilon > 0$, $\tau(n) = O_\epsilon(n^\epsilon)$.

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[2]Alternative notation used in some places is $d(n)$ or $\sigma_0(n)$.

Using the fact that $\psi(x) = 0$ for any $x \le 1$,

$$\psi(x) = \sum_{k=0}^{\lceil \log_2 x \rceil} (\psi(x/2^k) - \psi(x/2^{k+1})) \le (2\log 2)x + O((\log x)^2),$$

and hence $\psi(x) \ll x$ as required. □

Chebyshev's estimate is the first non-trivial quantitative information we have about the primes, and leads to a host of other facts about the primes – rather surprisingly, not just big-oh behaviour, but precise asymptotic results.

Lemma 5.

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

In particular, $\pi(x) \asymp x/\log x$*, and* $\pi(x) \sim x/\log x$ *if and only if* $\psi(x) \sim x$.

Proof. We first remove the contribution from prime powers by noting that, if $\theta(x) = \sum_{p \le x} \log p$, then

$$\psi(x) - \theta(x) \le \sum_{k \ge 2} \sum_{p \le x^{1/k}} \log p \ll \log x \sum_{k=2}^{\lceil \log x \rceil} x^{1/k} \ll x^{1/2}(\log x)^2.$$

It follows that $\theta(x) = \psi(x) + O(x^{1/2}(\log x)^2)$. In particular, by Chebyshev's estimate, we have $\theta(x) = O(x)$. We apply partial summation with $a_n = \Lambda(n)$ if n is prime, and 0 otherwise, and $f(n) = \frac{1}{\log n}$. This gives

$$\pi(x) = \sum_{p \le x} 1 = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}\,\mathrm{d}t = \frac{\theta(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

We have used $\theta(t) = O(t)$ to bound the contribution from the integral here. □

Lemma 6.

$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1).$$

Proof. Recalling that $\log = \mathbf{1} \star \Lambda$, and using Lemma 3,

$$\begin{aligned} x\log x + O(x) &= \sum_{n \le x} \log n \\ &= \sum_{ab \le x} \Lambda(b) \\ &= x \sum_{b \le x} \frac{\Lambda(b)}{b} + O(\psi(x)). \end{aligned}$$

Using Chebyshev's estimate, this proves that

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

It remains to deal with the contribution from prime powers $p^k \le x$ for $k \ge 2$, which we bound trivially by

$$\sum_{p \le x^{1/2}} \log p \sum_{k \ge 2} \frac{1}{p^k} = \sum_{p \le x^{1/2}} \log p \frac{1}{p^2 - p} \ll 1.$$

□

Lemma 7.
$$\sum_{p\leq x}\frac{1}{p}=\log\log x+b+O(1/\log x),$$
where b is some constant.

Proof. Let $A(x)=\sum_{p\leq x}(\log p)/p=\log x+R(x)$, say, where $R(x)=O(1)$. By partial summation
$$\begin{aligned}\sum_{p\leq x}\frac{1}{p}&=\frac{A(x)}{\log x}+\int_2^x\frac{A(t)}{t(\log t)^2}\,\mathrm{d}t\\&=1+O(1/\log x)+\int_2^x\frac{1}{t\log t}\,\mathrm{d}t+\int_2^x\frac{R(t)}{t(\log t)^2}\,\mathrm{d}t\\&=\log\log x+1-\log\log 2+\int_2^\infty\frac{R(t)}{t(\log t)^2}\,\mathrm{d}t+O(1/\log x).\end{aligned}$$
□

Lemma 8.
$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}=c\log x+O(1)$$
where $c>1$ is some constant.

Proof. We use $\log(1-t)=-\sum_{k=1}^\infty\frac{t^k}{k}$ to deduce that
$$\begin{aligned}\log\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\right)&=-\sum_{p\leq x}\log(1-1/p)\\&=\sum_{k=1}^\infty\sum_{p\leq x}\frac{1}{kp^k}\\&=\sum_{p\leq x}\frac{1}{p}+\sum_{k\geq 2}\sum_{p\leq x}\frac{1}{kp^k}\\&=\sum_{p\leq x}\frac{1}{p}+\sum_{p}\sum_{k\geq 2}\frac{1}{kp^k}+O\left(\sum_{p>x}\sum_{k\geq 2}\frac{1}{p^k}\right).\end{aligned}$$
Note that the infinite sum over p converges to some constant. Furthermore, the error term is
$$\ll\sum_{p>x}\frac{1}{p^2}\ll\sum_{n>x}\frac{1}{n^2}\ll\frac{1}{x}.$$
It follows from Lemma 7 that
$$\log\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\right)=\log\log x+b'+O(1/\log x)$$
for some constant b'. The result follows since $e^x=1+O(x)$ for $|x|\leq 1$. □

It is a little tricky to determine what the constant c in Lemma 8 actually is – it turns out to be $e^\gamma \approx 1.78\cdots$. We can use this fact to point out why the naive probabilistic heuristic can be misleading (and hopefully give some idea why the prime number theorem itself, unlike these simple asymptotics, is hard to prove).

As a heuristic, we might guess that the probability that a given prime number p divides a randomly chosen n is $1/p$. Furtermore, we expect that these probabilities should be independent for distinct primes p. Using the fact that $n \geq 3$ is prime if and only if $p \nmid n$ for all $2 \leq p \leq n^{1/2}$, we might guess that

$$1_{n \text{ is prime}} \approx \mathbb{P}(p \nmid n \text{ for all } 2 \leq p \leq n^{1/2}) \approx \prod_{p \leq n^{1/2}} \left(1 - \frac{1}{p}\right) \approx 2e^{-\gamma}/\log n.$$

This would in turn suggest that

$$\pi(x) = \sum_{n \leq x} 1_{n \text{ is prime}} \approx 2e^{-\gamma} \sum_{n \leq x} \frac{1}{\log n} \approx 2e^{-\gamma} \frac{x}{\log x}.$$

But since $2e^{-\gamma} = 1.12\cdots$, this contradicts the prime number theorem! This shows that, while heuristically thinking about discrete concepts in terms of 'probability' can lead to roughly the right order of magnitude, one must take care not to take the constants obtained too seriously!

Indeed, we can use the elementary estimates already obtained to show, not the prime number theorem itself, but at least the fact that *if* $\frac{\pi(x)\log x}{x}$ *converges to a limit at all*, then this limit must be 1, and hence the prime number theorem is true. The hard part is showing that the limit exists.

Theorem 3 (Chebyshev)**.** *If* $\pi(x) \sim c\frac{x}{\log x}$ *then* $c = 1$.

Proof. By partial summation,

$$\sum_{p \leq x} \frac{1}{p} = \frac{\pi(x)}{x} + \int_1^x \frac{\pi(t)}{t^2}\, \mathrm{d}t.$$

The first term is trivially $O(1)$. If $\pi(x) = c(1 + R(x))\frac{x}{\log x}$, where $R(x) \to 0$ as $x \to \infty$, then

$$\sum_{p \leq x} \frac{1}{p} = c \int_1^x \frac{1 + R(t)}{t \log t}\, \mathrm{d}t + O(1) = c(1 + o(1)) \log\log x + O(1).$$

By Lemma 7 the left hand side is $(1 + o(1)) \log\log x$, and hence $c = 1$. □

CHAPTER 2

Dirichlet series and the Riemann zeta function

We will now begin to harness the power of complex analysis for number theory. The main object of study will the Riemann zeta function. Before we explore the applications to number theory, we will spend some time proving various essential facts about this function.

In the rest of the course, we will use (as is traditional for this topic) the letter s to denote a complex variable, and σ and t to denote its real and imaginary parts respectively, so that $s = \sigma + it$. Before we begin, it's worth pausing to explicitly point out what we mean by n^s, where n is a natural number and $s \in \mathbb{C}$. By definition this is

$$n^s = e^{s \log n} = n^{\sigma} e^{it \log n}.$$

It is easy to check the multiplicative property, that $(nm)^s = n^s m^s$.

A Dirichlet series is an infinite series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for some coefficients $a_n \in \mathbb{C}$. If we denote the coefficients a_n by an arithmetic function $f(n)$ then we may write $F_f(s)$ to denote this dependence.

Lemma 9. *For any sequence a_n there is an abscissa of convergence σ_c such that $F(s)$ converges for all s with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$. If $\sigma > \sigma_c$ then there is a neighbourhood of s in which $F(s)$ converges uniformly. In particular, $\alpha(s)$ is holomorphic at s.*

Proof. It suffices to show that if $F(s)$ converges at $s = s_0$ and we take some s with $\sigma > \sigma_0$ then F converges uniformly in some neighbourhood of s. The lemma then follows by taking $\sigma_c = \inf\{\sigma : F(s) \text{ converges}\}$.

Suppose that $F(s)$ converges at $s = s_0$. If we let $R(u) = \sum_{n>u} a_n n^{-s_0}$ then by partial summation, for any s,

$$\sum_{M<n\leq N} a_n n^{-s} = R(M)M^{s_0-s} - R(N)N^{s_0-s} + (s_0 - s)\int_M^N R(u)u^{s_0-s-1}\,\mathrm{d}u.$$

If $|R(u)| \leq \epsilon$ for all $u \geq M$, and if $\sigma > \sigma_0$, then it follows that

$$\left| \sum_{M<n\leq N} a_n n^{-s} \right| \leq 2\epsilon + \epsilon\,|s - s_0| \int_M^{\infty} t^{\sigma_0-\sigma-1}\,\mathrm{d}t \leq \left(2 + \frac{|s-s_0|}{\sigma - \sigma_0}\right)\epsilon.$$

There is some neighbourhood of s in which $|s - s_0| \ll \sigma - \sigma_0$, and hence by Cauchy's principle the series converges uniformly in this neighbourhood of s. $\square$

Lemma 10. *If $\sum a_n n^{-s} = \sum b_n n^{-s}$ for all s in some half-plane $\sigma > \sigma_0$ (where both series converge) then $a_n = b_n$ for all n.*

Proof. It suffices to show that if $\sum c_n n^{-s} = 0$ for all s with $\sigma > \sigma_0$ then $c_n = 0$ for all n. Suppose that $c_n = 0$ for all $n < N$. We can write

$$c_N = -\sum_{n>N} c_n (n/N)^{-\sigma}.$$

Since the sum here is convergent, the summands tend to 0, and hence $c_n \ll n^{\sigma_0}$. It follows that that this sum is absolutely convergent for $\sigma > \sigma_0 + 1$. Since each term tends to 0 as $\sigma \to \infty$, and the series is absolutely convergent, the right-hand side tends to 0, and hence $c_N = 0$. □

Lemma 11. *If $F_f(s)$ and $F_g(s)$ are two Dirichlet series, both absolutely convergent at s, then*

$$\sum_{n=1}^{\infty} f \star g(n) n^{-s}$$

is absolutely convergent and equals $\alpha(s)\beta(s)$.

Proof. We simply multiply out the product of two series,

$$\left(\sum_n a\frac{a_n}{n^s}\right)\left(\sum_m \frac{b_m}{m^s}\right) = \sum_{n,m} \frac{a_n b_m}{(nm)^s} = \sum_k \left(\sum_{nm=k} a_n b_m\right) k^{-s},$$

which is justified since both series are absolutely convergent. □

We now define the Riemann zeta function in the half-plane $\sigma > 1$ by

$$\zeta(s) = \sum_n \frac{1}{n^s}.$$

Observe that this series diverges at $s = 1$, and the series actually converges absolutely for $\sigma > 1$. By the above, $\zeta(s)$ defines a holomorphic function in this half-plane. For our applications, we need to extend this definition to be able to talk about $\zeta(s)$ for $\sigma > 0$.

Lemma 12. *For $\sigma > 1$,*

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Proof. By partial summation, for any x,

$$\sum_{1\le n\le x} n^{-s} = \frac{\lfloor x\rfloor}{x^s} + s\int_1^x \frac{\lfloor t\rfloor}{t^{s+1}}\,\mathrm{d}t.$$

The integral here is

$$s\int_1^x t^{-s}\,\mathrm{d}t - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t = \frac{s}{s-1} - \frac{s}{s-1}x^{1-s} - s\int_1^x \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t.$$

Since $\sigma > 1$, if we take the limit as $x \to \infty$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{t\}}{t^{s+1}}\,\mathrm{d}t,$$

noting that the integral converges. □

The integral here is convergent for any $\sigma > 0$, and therefore the right hand side defines an analytic function for $\sigma > 0$, aside from a simple pole at $s = 1$ with residue 1. We have therefore given an analytic continuation for $\zeta(s)$ up to $\sigma = 0$.

A crash course on infinite products

3.1. **Euler products.** Since it is a topic not often covered in analysis courses, we first take a brief digression to discuss infinite products. If $a_n \in \mathbb{C}\backslash\{0\}$ then tshe infinite product

$$\prod_{n=1}^{\infty} a_n$$

is defined to be the limit $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$ if this exists and is not zero.

Lemma 13 (Cauchy criterion)**.** *If $a_n \neq 0$ then the infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if for any $\epsilon > 0$ there exists N such that*

$$\left| \prod_{n<k\leq m} a_k - 1 \right| < \epsilon$$

for all $m > n \geq N$.

In particular, $\lim_{n\to\infty} a_n = 1$. For this reason it is often convenient to change variables so that we consider the product $\prod(1 + a_n)$ instead. We say that

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges absolutely if and only if $\prod(1 + |a_n|)$ converges. The following is a simple consequence of the Cauchy criterion.

Lemma 14. *If $a_n \neq -1$ and $\prod(1 + a_n)$ converges absolutely then it converges.*

The final fundamental fact we will require is the following.

Lemma 15. *If $a_n > 0$ for all $n \geq 1$ then $\prod(1 + a_n)$ converges if and only if $\sum a_n$ converges.*

Proof. By the monotone convergence theorem, it suffices to show that the partial sums are bounded above if and only the partial products are. This follows from the inequalities

$$a_1 + \cdots + a_n < (1 + a_1) \cdots (1 + a_n) \leq e^{a_1+\cdots+a_n}.$$

□

All of the infinite products we will encounter in this course will converge absolutely. The previous lemmas have the following useful consequence: if $\sum |a_n|$ converges (and $a_n \neq -1$) then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. In particular, it is not zero!

Lemma 16. *If f is multiplicative and $\sum |f(n)| n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right).$$

Proof. Note that this product is absolutely convergent. By comparison each sum in the product is absolutely convergent. Since a product of finitely many absolutely convergent series can be arbitrarily rearranged,

$$\prod_{p\leq y} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) = \sum_{\substack{n \\ p|n \implies p\leq y}} f(n)n^{-s}.$$

Therefore the difference between the product here and the Dirichlet series is at most

$$\sum_{n>y} |f(n)|\, n^{-\sigma} \to 0 \text{ as } y \to \infty.$$

□

Corollary 1 (Euler product)**.** *If f is completely multiplicative and $\sum |f(n)|\, n^{-\sigma}$ converges then*

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

In particular, we note the Euler product for $\zeta(s)$:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

which is valid for $\sigma > 1$. From this it follows that $\zeta(s) \neq 0$ for $\sigma > 1$. The Euler product leads to the identity

We can play these games to get almost any multiplicative function in terms of $\zeta(s)$.

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_n \frac{\mu(n)}{n^s}.$$

Q: Express the Von Mangoldt function in terms of $\zeta(s)$

e.g. since $F_{f*g} = F_f \cdot F_g$

Furthermore, when $\sigma > 1$, the series is absolutely convergent, and so the derivative can be computed summand by summand, leading to

$$\sum \frac{\tau(n)}{n^s} = \sum \frac{1 * 1(n)}{n^s} = \zeta(s)^2$$

$$\zeta'(s) = -\sum_n \frac{\log n}{n^s}.$$

$$\sum \frac{|\mu(n)|}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right) = \prod_p \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - 1/p^s\right)} = \frac{\zeta(s)}{\zeta(2s)}$$

From the Euler product we have

$$\log \zeta(s) = -\sum_p \log\left(1 - p^{-s}\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} = \sum_n \frac{\Lambda(n)}{\log n} n^{-s}.$$

Finally, taking the derivative of this, we obtain the Dirichlet series with $\Lambda(n)$ as coefficients:

"Any interesting dirichlet series can be expressed in terms of the zeta function"

$$\frac{\zeta'}{\zeta}(s) = -\sum_n \frac{\Lambda(n)}{n^s}.$$

Note: Möbius inversion looks trivial in terms of ζ

$1 * g = f \iff \mu * f = g$

$\zeta \cdot F_f = F_g \iff \frac{1}{\zeta} \cdot F_g = F_f$

4. Gamma function

4.1. The Weierstrass definition. Let

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N\right) = 0.5772157\cdots$$

and define the Gamma function $\Gamma(s) : \mathbb{C} \to \mathbb{C}$ by

Weierstrass canonical product

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} e^{-s/n} \left(1 + \frac{s}{n}\right).$$

* The following proofs about $\Gamma(s)$ are NON-EXAMINABLE however the definition of $\Gamma(s)$, and the statements are. *

This product is analytic for all $s \in \mathbb{C}$, because when $|s| \leq N/2$ the series

i.e. it is entire

$$\sum_{n=N+1}^{\infty} \left(\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right)$$

is absolutely and uniformly convergent, and so its exponential is also an analytic function. This shows that the product is an analytic function for $|s| \leq N/2$, and we then take N arbitrarily large.

Tail converges $\Leftrightarrow$ function converges

It is clear from this expression that $\Gamma(s)$ itself is analytic at all $s \in \mathbb{C}$ apart from simple poles at $s = 0, -1, -2, \ldots$. The residue at $s = -n$ is $(-1)^n/n!$.

4.2. **The Euler definition.** Inserting the definition of γ gives

$$\frac{1}{\Gamma(s)} = s \lim_{N\to\infty} e^{\left(\sum_{m=1}^{N} \frac{1}{m} - \log N\right)s} \prod_{n=1}^{N} \underbrace{e^{-s/n}}\left(1+\frac{s}{n}\right)$$

$$= s \lim_{N\to\infty} N^{-s} \prod_{n=1}^{N}\left(1+\frac{s}{n}\right)$$

these cancel with the respective part of γ.

Convince myself of this $\to$?

$$= s \lim_{N\to\infty} \left(1+\frac{1}{N}\right)^{s} \prod_{n=1}^{N}\left(1+\frac{s}{n}\right)\left(1+\frac{1}{n}\right)^{-s},.$$

whence we have the following formula of Euler,

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1},$$

valid for all $s \in \mathbb{C}$ except $s = 0, -1, -2, \ldots$. It follows that $\Gamma(1) = 1$. Rewriting this, we also get

$$\Gamma(s) = \lim_{N\to\infty} N^{s}\frac{(N-1)!}{s(s+1)\cdots(s+N-1)}.$$

4.3. **The difference equation.** By Euler's formula, if s is not a negative integer,

$$\frac{\Gamma(s+1)}{\Gamma(s)} = \frac{s}{s+1}\lim_{N\to\infty}\prod_{n=1}^{N}\frac{\left(1+\frac{1}{n}\right)(s+n)}{s+n+1}$$

$$= s\lim_{N\to\infty}\frac{N+1}{s+N+1} = s,$$

whence

$$\Gamma(s+1) = s\Gamma(s).$$

"It is not Γ who is one off here, it is the factorial. We should be attached to Γ & not factorials."

In particular, since $\Gamma(1) = 1$, if s is a positive integer then $\Gamma(s) = (s-1)!$.

4.4. **The reflection formula.**

$$\Gamma(s)\Gamma(1-s) = \frac{1}{s(1-s)}\prod_{n=1}^{\infty}\frac{1+1/n}{\left(1+\frac{s}{n}\right)\left(1+\frac{1-s}{n}\right)}$$

$$= \frac{1}{s(s-1)}\prod_{n=1}^{\infty}\frac{1}{\left(1+\frac{s}{n}\right)\left(1-\frac{s}{n+1}\right)}$$

$$= \frac{1}{s}\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right)^{-1}$$

$$= \frac{\pi}{\sin(\pi s)}.$$

Idea: $\sin(\pi s)$ is entire and should equal the product over it's zeros. Exercise: Prove this rigorously.

It follows, for example, that $\Gamma(1/2) = \sqrt{\pi}$.

4.5. **The duplication formula.** Consider the expression

$$\frac{2^{2s}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)}{2\Gamma(2s)}.$$

We claim that this is independent of s. By Euler's formula it is

$$2^{2s-1}\frac{\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^s}{(s)\cdots(s+N-1)}\cdot\lim_{N\to\infty}\frac{1\cdot 2\cdots(N-1)N^{s+1/n}}{\left(s+\frac{1}{n}\right)\cdots\left(s+\frac{1}{n}+N-1\right)}}{\lim_{N\to\infty}\frac{1\cdot 2\cdots(2N-1)(2N)^{2s}}{2s(2s+1)\cdots(2s+2N-1)}}$$

which is

$$\lim_{N\to\infty}\frac{((N-1)!)^2N^{1/2}2^{2N-1}}{(2N-1)!},$$

and in particular independent of s. To evaluate it we set $s=1/2$, yielding

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

We have proved the duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=2^{1-2s}\pi^{1/2}\Gamma(2s).$$

4.6. **Euler's integral expression.** By integration by parts

$$\begin{aligned}\int_0^N\left(1-\frac{t}{N}\right)^N t^{s-1}\,\mathrm{d}t &= N^s\int_0^1(1-t)^Nt^{s-1}\,\mathrm{d}t\\ &= N^s\frac{N!}{s(s+1)\cdots(s+N)}\\ &\to\Gamma(s)\end{aligned}$$

as $N\to\infty$ using Euler's formula. This is valid if $\sigma>0$, whence we have the formula for this region

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t.$$

5. Functional equation

Examination Note:
Only need to learn one of the proofs of the FE, do not need to memorise both.

Theorem 4 (Functional equation)**.** *The zeta function $\zeta(s)$ can be extended a function meromorphic on the whole complex plane, and for all s satisfies the identity*

$$\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Equivalently,

$$\zeta(s)=\chi(s)\zeta(1-s)$$

This is the way to think about this formula. $\zeta(s) = \zeta(1-s) \times$ some analytic noise

where

$$\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)=\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Many interesting facts can be deduced from this identity. We will first use it to study the possible poles of $\zeta(s)$. We know that $\zeta(s)$ has a simple pole at $s=1$, and nowhere else for $\sigma>-1$. Suppose that ζ has a pole at s for $\sigma<0$. Then so too does $\Gamma(1-s)\zeta(1-s)$, but both $\Gamma(s)$ and $\zeta(s)$ are holomorphic for all s with $\Re s>1$, which is a contradiction. It follows that $\zeta(s)$ only has one pole in $\mathbb{C}$, which is a simple pole at $s=1$.

4.5. **The duplication formula.** Consider the expression

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which is

$$\lim_{N\to\infty}\frac{((N-1)!)^2N^{1/2}2^{2N-1}}{(2N-1)!},$$

and in particular independent of s. To evaluate it we set $s=1/2$, yielding

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as $N\to\infty$ using Euler's formula. This is valid if $\sigma>0$, whence we have the formula for this region

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5. Functional equation

Theorem 4 (Functional equation)**.** *The zeta function $\zeta(s)$ can be extended a function meromorphic on the whole complex plane, and for all s satisfies the identity*

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Equivalently,

$$\zeta(s)=\chi(s)\zeta(1-s)$$

where

$$\chi(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)=\pi^{s-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

Many interesting facts can be deduced from this identity. We will first use it to study the possible poles of $\zeta(s)$. We know that $\zeta(s)$ has a simple pole at $s=1$, and nowhere else for $\sigma>-1$. Suppose that ζ has a pole at s for $\sigma<0$. Then so too does $\Gamma(1-s)\zeta(1-s)$, but both $\Gamma(s)$ and $\zeta(s)$ are holomorphic for all s with $\Re s>1$, which is a contradiction. It follows that $\zeta(s)$ only has one pole in $\mathbb{C}$, which is a simple pole at $s=1$.

Alternatively, if we write

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

then

$$\xi(s) = \xi(1-s)$$

and $\xi(s)$ is an entire function, and is real for real s.

We will now consider the zeros of $\zeta(s)$. Suppose that $\zeta(s) = 0$ and $\sigma < 0$. It follows that

$$\sin(\pi s/2)\Gamma(1-s)\zeta(1-s) = 0.$$

Again, neither $\Gamma(1-s)$ nor $\zeta(1-s)$ can be zero or a pole, and so $\sin(\pi s/2)$, which means s must be an even integer. These are called the trivial zeros of $\zeta(s)$, located at $s = -2, -4, -6, \ldots$. Since there are no zeros with $\sigma \geq 1$, there are no other zeros with $\sigma \leq 0$.

Aside from the trivial zeros, then, all zeros of ζ must lie in the critical strip $0 < \sigma < 1$. Furthermore, since the other factors in the functional equation are entire and non-zero in this strip, this implies that if ρ is a zero in the critical strip, then so too is $1-\rho$. There is therefore a symmetry around the critical line $\sigma = 1/2$. The Riemann hypothesis is motivated in part by the belief that this symmetry should collapse so that all the zeros are located exactly on this line.

5.1. **Method One.** We first extend the definition of the zeta function to a larger half-plane. Recall that for $\sigma > 0$ we defined

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,\mathrm{d}u.$$

We will extend the region where this is valid by integrating by parts. First let $f(x) = \frac{1}{2} - \{x\}$, so that

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u.$$

If we let $F(x) = \int_0^x f(u)\,\mathrm{d}u$ then, by integration by parts,

$$\int_1^\infty \frac{f(u)}{u^{s+1}} = [F(u)u^{-s-1}]_0^\infty + (s+1)\int_1^\infty \frac{F(u)}{u^{s+2}}\,\mathrm{d}u.$$

Since $F(x)$ is bounded, the integral here converges for any s with $\sigma > -1$, and hence the left-hand side also converges in this region. We may therefore take

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u$$

as the definition of $\zeta(s)$ in the half-plane $\sigma > -1$. If $-1 < \sigma < 0$ then

$$\int_0^1 \frac{f(u)}{u^{s+1}}\,\mathrm{d}u = \frac{1}{2}\int_0^1 \frac{1}{u^{s+1}}\,\mathrm{d}u - \int_0^1 \frac{1}{u^s}\,\mathrm{d}u = -\frac{1}{2s} + \frac{1}{s-1},$$

and so in this strip

$$\zeta(s) = s\int_0^\infty \frac{f(u)}{u^{s+1}}\,\mathrm{d}u.$$

We now note that $f(x)$ is a periodic function, continuous in $(0,1)$, and so it has a Fourier series, which is

$$f(u) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n u)}{\pi n}.$$

For $-1 < \sigma < 0$ we therefore get

$$\zeta(s) = s \int_0^\infty \frac{1}{u^{s+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n u)}{\pi n} \,\mathrm{d}u = \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty \frac{\sin(2\pi n u)}{u^{s+1}} \,\mathrm{d}u.$$

We should justify the interchange of integral and summation here. We can interchange the infinite sum with any finite integral by the dominated convergence theorem, since the partial sums converge almost-everywhere pointwise, and is bounded above by $O(1)$. We then note that for any λ,

$$\int_\lambda^\infty \frac{\sin(2\pi n x)}{x^{s+1}} \,\mathrm{d}x = \left[-\frac{\cos(2\pi n x)}{2n\pi x^{s+1}} \right]_\lambda^\infty - \frac{s+1}{2n\pi} \int_\lambda^\infty \frac{\cos(2\pi n x)}{x^{s+2}} \,\mathrm{d}x$$

$$= O\left(\frac{1}{n\lambda^{\sigma+1}} \right) + O\left(\frac{1}{n} \int_\lambda^\infty \frac{1}{x^{\sigma+2}} \,\mathrm{d}x \right) = O\left(\frac{1}{n\lambda^{\sigma+1}} \right).$$

It follows that

$$\lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_\lambda^\infty \frac{\sin(2n\pi x)}{x^{s+1}} \,\mathrm{d}x = 0$$

for $-1 < \sigma < 0$.

By change of variable, we have

$$\int_0^\infty \frac{\sin(2\pi n u)}{u^{s+1}} = (2\pi n)^s \int_0^\infty \frac{\sin(u)}{u^{s+1}} \,\mathrm{d}u.$$

Furthermore, writing $\sin(u) = \frac{1}{2i}(e^{iu} - e^{-iu})$

$$\begin{aligned}
\int_0^\infty \frac{\sin u}{u^{s+1}} \,\mathrm{d}u &= \frac{1}{2i} \left(\int_0^\infty u^{-s-1} e^{iu} \,\mathrm{d}u - \int_0^\infty u^{-s-1} e^{-iu} \,\mathrm{d}u \right) \\
&= \frac{1}{2i} \left((-i)^s - i^s \right) \int_0^\infty t^{-s-1} e^{t} \,\mathrm{d}t \\
&= -\sin(\pi s/2) \int_0^\infty t^{-s-1} e^{-t} \,\mathrm{d}t \\
&= -\sin(\pi s/2)\Gamma(-s),
\end{aligned}$$

using the change of variable $iu = t$ and $-iu = t$ respectively for the two integrals. Combining the above we have shown that, for $-1 < \sigma < 0$,

$$\zeta(s) = \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{(2\pi n)^s}{n} \sin(\pi s/2)\Gamma(-s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

The right-hand side is actually analytic for any $\sigma < 1$, and hence we can take the right-hand side to be a definition of $\zeta(s)$ in this region. By analytic continuation it follows that this identity must hold for all $s \in \mathbb{C}$.

5.2. **Method Two.**

Lemma 17. *For $\sigma > 1$*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x.$$

Proof. The key observation is that

$$\frac{\Gamma(s)}{n^s} = \frac{1}{n^s} \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t = \int_0^\infty x^{s-1} e^{-nx} \, \mathrm{d}x.$$

We then sum both sides over n, and note that $\sum e^{-nx} = (e^x - 1)^{-1}$. We can interchange the sum and integral here by absolute convergence, since $\sigma > 1$, as

$$\sum \int_0^\infty x^{\sigma-1} e^{-nx} \, \mathrm{d}x = \Gamma(\sigma)\zeta(\sigma)$$

converges for $\sigma > 1$. □

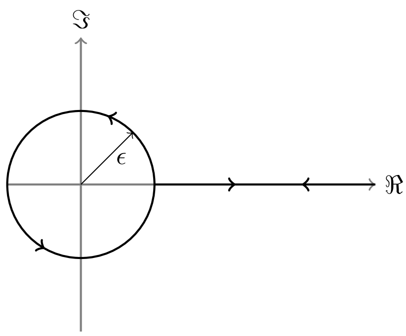

FIGURE 1. The contour C of Lemma 18.

Lemma 18. *For $\sigma > 1$*

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i} \Gamma(1 - s) \int_C \frac{z^{s-1}}{e^z - 1} \, \mathrm{d}z$$

where the contour goes from positive infinity, circles the origin, and returns to infinity, where z^{s-1} is defined as $\exp((s-1)\log z)$ with the logarithm real at the beginning of the contour.

Proof. Suppose the circle part has radius ϵ. On the circle,

$$\left|z^{s-1}\right| = e^{(\sigma-1)\log|z| - t\arg(z)} \leq |z|^{\sigma-1} e^{2\pi|t|}$$

and

$$|e^z - 1| \gg |z|$$

and so the integral around the circle tends to zero as $\epsilon \to 0$. It follows that the integral is

$$-\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^x - 1} \, \mathrm{d}x = (e^{2\pi is} - 1)\Gamma(s)\zeta(s).$$

Since

$$\sin(s) = \frac{e^{is} - e^{-is}}{2i} = \frac{e^{2is} - 1}{2ie^{is}}$$

by the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} = \frac{2\pi i e^{i\pi s}}{e^{2\pi i s} - 1}$$

and the result follows. □

So far we've been working in the half-plane $\sigma > 1$. The integral over C, however, is uniformly convergent in any finite region, and so the right-hand side defines a meromorphic continuation of ζ to the entire complex plane, with the only possible poles those of $\Gamma(1-s)$, which are $s = 1, 2, 3, \ldots$, and hence just at $s = 1$.

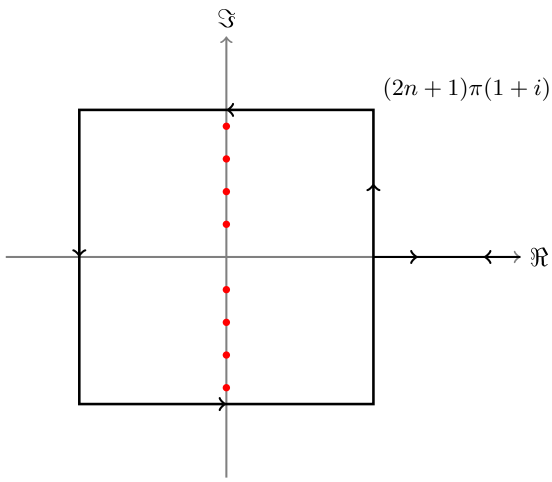

FIGURE 2. The contour C_n with poles at $\pm 2\pi i, \ldots, \pm 2\pi i n$ marked in red.

For the functional equation, now take the integral C_n which is the positive real axis from ∞ to $(2n+1)\pi$, round the square $\pm 1 \pm i$ then back to infinity. Moving from C to C_n we pick up poles at $\pm 2i\pi, \ldots, \pm 2in\pi$. The residues from $\pm 2\pi i m$ are together

$$\begin{aligned}(2m\pi i)^{s-1} + (-2m\pi i)^{s-1} &= (2m\pi)^{s-1} e^{i\pi(s-1)} 2\cos(\pi(s-1)/2)\\ &= -2(2m\pi)^{s-1} e^{i\pi s}\sin(\pi s/2).\end{aligned}$$

It follows that the integral is

$$\int_{C_n} \frac{z^{s-1}}{e^z - 1}\,\mathrm{d}z + 4\pi i e^{i\pi s}\sin(\pi s/2)\sum_{m=1}^{n}(2m\pi)^{s-1}.$$

We now take $\sigma < 0$ and let $n \to \infty$. The function $1/(e^z - 1)$ is bounded on the contours C_n, and $z^{s-1} = O(|z|^{\sigma-1})$, so the integral around C_n tends to 0. It follows that

$$\int_C \frac{z^{s-1}}{e^z - 1}\,\mathrm{d}z = 4\pi i e^{i\pi s}\sin(\pi s/2)(2\pi)^{s-1}\zeta(1-s),$$

so that for $\sigma < 0$

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s) 4\pi i e^{i\pi s}\sin(\pi s/2)(2\pi)^{s-1}\zeta(1-s).$$

This verifies the functional equation for $\sigma < 0$. As before, by analytic continuation, it must in fact hold for all s.

6. Special values of zeta

If we let

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

then B_n are known as the Bernoulli numbers. Multiplying out both sides by $e^z - 1$ it follows that

$$z = \left(\sum_{n\geq 0} \frac{B_n}{n!} z^n\right)\left(\sum_{m\geq 1} \frac{1}{m!} z^m\right) = \sum_{k\geq 1}\left(\sum_{\substack{n+m=k\\ n\geq 0, m\geq 1}} \frac{B_n}{n!m!}\right) z^k$$

and so $B_0 = 1$ and for $k \geq 2$

$$\sum_{0\leq n\leq k-1} \frac{B_n}{n!(k-n)!} = 0 = \sum_{0\leq n\leq k-1} \binom{k}{n} B_n.$$

This shows that each B_n is a rational number, and allows for efficient computation. For example, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, and so on.

Recall that by Lemma 18 we have, for any $s \neq 1, 2, 3, \ldots$,

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)\int_C \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z.$$

When $s = -m \leq 0$ is an integer the contour integral can be evaluated using the theory of residues. Inside C there is only a single pole at $z = 0$, and since

$$\frac{z^{-m-1}}{e^z-1} = z^{-m-2}\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

the residue at $z = 0$ when $s = -m$ is $B_{m+1}/(m+1)!$. It follows that

$$\zeta(0) = -\frac{1}{2} \text{ and } \zeta(-m) = \frac{(-1)^m B_{m+1}}{m+1} \text{ for } m \geq 1.$$

Since we already know that $\zeta(-2k) = 0$ for $k \geq 1$ it follows that $B_n = 0$ whenever $n \geq 3$ is odd, something not at all obvious from the definition! Furthermore, it follows from the functional equation that for $m \geq 1$

$$\begin{aligned}
\zeta(2m) &= 2^{2m}\pi^{2m-1}\sin(\pi m)\Gamma(1-2m)\zeta(1-2m)\\
&= -2^{2m}\pi^{2m-1}\sin(\pi m)\Gamma(1-2m)\frac{B_{2m}}{2m}\\
&= (-1)^{m+1}2^{2m-1}\pi^{2m}\frac{B_{2m}}{(2m)!}.
\end{aligned}$$

Here we have used the fact that $\sin(\pi m)\Gamma(1-2m) = (-1)^m\pi/2$, which is not immediately obvious, since it is the product of a zero and a pole. One way to check this is to note that $\sin(\pi z)$ has a simple zero at $z = m$ around which it can be expanded as $(-1)^m\pi(z-m)+O((z-m)^3)$ and Γ has a simple pole at $s=1-2m$ with residue $(-1)^{2m-1}/(2m-1)!$, and hence near $z=m$ we have

$$\begin{aligned}\sin(\pi z)\Gamma(1-2z) &= \big((-1)^m\pi(z-m)+O((z-m)^3)\big)\left(\frac{(-1)^{2m-1}}{(2m-1)!}\frac{1}{(2m-2z)}+O(1)\right)\\ &= (-1)^m\frac{\pi}{2}+O(z-m).\end{aligned}$$

For example, using the previous values for the Bernoulli numbers B_2 and B_4 we deduce that $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$. Note that if we want to recover the values of $\zeta(2m+1)$ for $m\geq 1$ then things are not so simple – for now the functional equation gives

$$\zeta(2m+1)=(-1)^m 2^{2m+1}\pi^{2m}\Gamma(-2m)\zeta(-2m).$$

Now the pole of Γ at $s=-2m$ is cancelled by the zero of $\zeta(-2m)$ but we have no information about the series expansion of $\zeta(-2m)$. We were lucky in the case of even arguments that the pole of Γ is cancelled by the zero of sin, both of which we understand well.

It is likely, but unknown, that $\zeta(2m+1)/\pi^{2m+1}$ is irrational for all $m\geq 1$. It is even more likely that $\zeta(2m+1)$ is irrational – this is only known for $\zeta(3)$ at the moment.

Finally, in the next chapter we will need to know the value of $\zeta'(0)$. This can be calculated as follows. Recall in the proof that

$$\sum_{n\leq x}\frac{1}{n}=\log x+\gamma+O(1/x)$$

we obtained the explicit representation

$$\gamma=1-\int_1^\infty\frac{\{t\}}{t^2}\,\mathrm{d}t.$$

If we let $s\to 1$ in the expression

$$\zeta(s)=1+\frac{1}{s-1}-s\int_1^\infty\frac{\{t\}}{t^{s+1}}\,\mathrm{d}t$$

then we see that $\zeta'(1)=\gamma$.

Finally, from the functional equation we have

$$(s-1)\zeta(s)=-2^s\pi^{s-1}\Gamma(2-s)\sin(\pi s/2)\zeta(1-s).$$

Differentiating both sides and setting $s=1$ gives

$$\gamma=2\zeta'(0)-2\zeta(0)\log 2\pi+2\zeta(0)\Gamma'(1).$$

Using the fact that $\zeta(0)=-1/2$ and $\Gamma'(1)=-\gamma$ (which can be seen, for example, from the Weierstrass definition), it follows that

$$\zeta'(0)=-\frac{\log 2\pi}{2},$$

and hence

$$-\frac{\zeta'(0)}{\zeta(0)}=-\log(2\pi).$$

7. Counting zeros of zeta

Recall that a holomorphic non-zero function has only a finite number of zeros in any compact region (otherwise there exists an infinite sequence of zeros, and hence an infinite convergent sequence of zeros, and hence the function is zero). It therefore makes sense to count them. We are interested in the zeros of the zeta function, which we know (aside from the trivial zeros) all lie in the rectangular strip $0 \leq \sigma \leq 1$.

To make this a compact region, we introduce some cut-off at height $t = T$. It is a natural question to ask how the number of zeros changes as we increase T. To this end, let $N(T)$ count the number of zeros $\rho = \beta + i\gamma$ in the region $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq T$.

Our main tool is the following useful bound from complex analysis.

Lemma 19 (Jensen's inequality)**.** *Suppose that $f(z)$ is analytic in a domain containing a disc with radius R and centre a, that $|f(z)| \leq M$ in this disc, and that $f(a) \neq 0$. Let $0 < r < R$. The number of zeros of f in the disc with centre a and radius r is at most*

$$\frac{\log(M/\left|f(0)\right|)}{\log(R/r)}.$$

Proof. Without loss of generality we can assume that $a = 0$. By the identity principle, the number of zeros in $|z| \leq R$ is finite . Let these zeros be denoted by $z_1, z_2, \ldots, z_K$. Let

$$g(z) = f(z) \prod_{k=1}^{K} \frac{R^2 - z\overline{z_k}}{R(z - z_k)}.$$

Observe that the kth factor has a pole at z_k, and has modulus 1 on $|z| = R$. It follows that g is an analytic function in $|z| \leq R$, and if $|z| = R$ then $|g(z)| = |f(z)| \leq M$. By the maximum modulus principle,

$$|g(0)| = |f(0)| \prod_{k=1}^{K} \frac{R}{|z_k|} \leq M.$$

Each factor is ≥ 1 and if $|z_k| \leq r$ then the factor is $\geq R/r$, and the bound follows. $\square$

To apply Jensen's inequality to $\zeta(s)$ we first need to give some estimates for how large $\zeta(s)$ can get.

Lemma 20. *When $\delta \leq \sigma \leq 2$ and $|t| \geq 1$*

$$\zeta(s) \ll (1 + |t|^{1-\sigma}) \min\left(\frac{1}{|\sigma - 1|}, \log(|t| + 4)\right).$$

Proof. For any $x \geq 2$, by partial summation, when $\sigma > 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \frac{\{t\}}{t^{s+1}} \, \mathrm{d}t.$$

By analytic continuation this identity continues to hold for all $s \neq 1$ with $\sigma > 0$. The second summand is $O(x^{1-\sigma})$. The third is $O(x^{-\sigma})$. The integral is $O(x^{-\sigma}/\sigma)$.

Since $|s|/\sigma \ll 1 + |t|$ in the given region we have
$$\zeta(s) \ll \sum_{n \leq x} \frac{1}{n^\sigma} + x^{1-\sigma} + \frac{|t|}{x^\sigma}.$$
The sum is
$$\ll 1 + \int_1^x \frac{1}{t^\sigma} \,\mathrm{d}t$$
uniformly for $\sigma \geq 0$. We choose $x = |t| + 4$, say, so that
$$\zeta(s) \ll 1 + |t|^{1-\sigma} + \int_1^x \frac{1}{u^\sigma} \,\mathrm{d}u.$$
If $|\sigma - 1| \leq 1/\log x$ then the integral is $\asymp \log x$. If $0 \leq \sigma \leq 1 - 1/\log x$ this is $< x^{1-\sigma}/(1-\sigma)$. If $\sigma \geq 1 + 1/\log x$ then it is $< 1/(\sigma - 1)$. The result follows. □

Theorem 5. *For any $T \geq 4$*
$$N(T+1) - N(T) \ll \log T.$$

Proof. By the symmetry of zeros implied by the functional equation, it suffices to show that the number of zeros in the rectangle between $1/2 \leq \sigma \leq 1$ and $T \leq t < T+1$ is $O(\log T)$. We apply Jensen's inequality to $\zeta(s)$ to a disc with centre $2 + i(T+1/2)$ and radii $R = 11/6$ and $r = 7/4$, say, which certainly includes this rectangle.

By Lemma 20 we know that $|\zeta(s)| \ll T$ in this disc, and furthermore
$$|\zeta(2 + i(T+1/2)) - 1| \leq \sum_{n \geq 2} \frac{1}{n^2} \leq 3/4$$
and so $|\zeta(2 + i(T+1/2))| \geq 1/4$, say. The result follows from Jensen's inequality. □

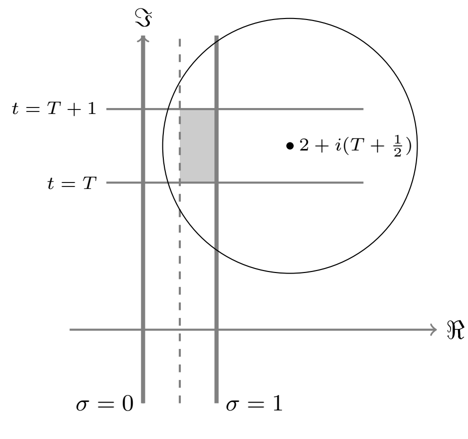

FIGURE 3. We bound the number of zeros in the rectangle $1/2 \leq \sigma \leq 1$ and $T \leq t \leq T+1$ by bounding the number in the circle.

It follows that $N(T) \ll T \log T$. Note that this is only an upper bound – at the moment, we don't know there are any zeros at all the critical strip, so it is possible that $N(T)=0$. We will later show that, not only are there many zeros, but $T\log T$ is the right order of magnitude, establishing the asymptotic formula

$$N(T)\sim\frac{1}{2\pi}T\log T.$$