Analytic Number Theory Sheet 3

Lent Term 2020

1. Using the fact that $\zeta(s)$ has no zeros in the region $\sigma > 1 - c/\log|t|$ and $|t| \geq 2$ prove that, all in this same region,

   (a)
   $$\frac{\zeta'}{\zeta}(s) \ll \log|t|$$
   Hint: Use Dirichlet series to handle $\sigma > 1 + 1/\log|t|$ and apply the formula for $\frac{\zeta'}{\zeta}$ in terms of the zeros of $\zeta(s)$ to handle the remaining region.

   (b)
   $$|\log \zeta(s)| \leq \log\log|t| + O(1),$$

   (c)
   $$\frac{1}{\zeta(s)} \ll \log|t|.$$

2. Show that if $|t| \geq 4$ then the number of zeros of $\zeta(s)$ in the disc of radius r around $1 + it$ is $O(r\log|t|)$ for all $r \leq 3/4$. *Hint: Again, use the formula for $\frac{\zeta'}{\zeta}$ in terms of its zeros and take real parts at $s = 1 + r + it$.*

3. If we arrange the non-trivial zeros of the Riemann zeta function in the upper half-plane as $\rho_n = \sigma_n + i\gamma_n$ where $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ then show that
   $$\gamma_n \sim \frac{2\pi n}{\log n}.$$
   Deduce that $\sum_\rho \frac{1}{|\rho|} = \infty$.

4. Show that there exists some constant $C > 0$ such that there is no vertical gap greater than C between successive zeros of $\zeta(s)$, that is,
   $$N(T + C) - N(T) > 0$$
   for all sufficiently large T.

5. Let $M(x) = \sum_{n \leq x} \mu(n)$.[1] Let $\Theta = \sup\{\sigma : \zeta(\sigma + it) = 0\}$.

   (a) Show that $M(x) = \Omega_\pm(x^{\sigma_0})$ for every $\sigma_0 < \Theta$.

   (b) If there is a simple zero of $\zeta(s)$ at $\rho = \Theta + it$ then show that $M(x) = \Omega_\pm(x^\Theta)$.

   (c) If there is a zero of $\zeta(s)$ of multiplicity $m \geq 2$ at $\rho = \Theta + it$ then show that
   $$M(x) = \Omega_\pm(x^\Theta(\log x)^{m-1}).$$
   In particular, if we could prove that $M(x) = O(x^{1/2})$ then we'd get both the Riemann hypothesis and also that all zeros of $\zeta(s)$ are simple!
   Hint: Consider the function $\frac{1}{s\zeta(s)} - c\frac{(m-1)!}{(s-\Theta)^m}$ for some constant $c > 0$.

[1] Mertens conjectured in 1897 that $|M(x)| \leq x^{1/2}$ for all $x \geq 1$. This was disproved by Odlyzko and te Riele in 1984.

This 'challenge question' outlines a proof that there are infinitely many zeros on the line $\sigma = 1/2$.

6. *CAUTION: DIFFICULT AND LENGTHY.* Recall that $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. For $t \in \mathbb{R}$ define $\Xi(t) = \xi(\frac{1}{2} + it)$.

   (a) Show that
   $$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \int_0^\infty F(x)x^{\frac{s}{2}-1}\,\mathrm{d}x$$
   where
   $$F(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

   (b) Using
   $$2F(x) + 1 = x^{-1/2}\left(2F(1/x) + 1\right) \tag{1}$$
   (which is an application of Poisson summation) show that
   $$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty F(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1}{2}-\frac{s}{2}}\right)\mathrm{d}x$$
   and deduce the functional equation.

   (c) Use (1) to also show that $4F'(s) + F(1) = -1/2$.

   (d) Use integration by parts and set $x = e^{2u}$ to deduce that
   $$\Xi(t) = 2\int_0^\infty \Phi(u)\cos(ut)\,\mathrm{d}u$$
   where
   $$\Phi(u) = e^{\frac{5}{2}u}\left(3F'(e^{2u}) + 2F''(e^{2u})\right).$$

   (e) Deduce that for any $n \geq 1$
   $$\Phi^{(2n)}(u) = \frac{(-1)^n}{\pi}\int_0^\infty \Xi(t)t^{2n}\cos(ut)\,\mathrm{d}t.$$

   (f) Noting that, since $F(x)$ is regular for $\Re x > 0$, we know that $\Phi(u)$ is regular for $-\frac{\pi}{4} < \Im(u) < \frac{\pi}{4}$, deduce that for $|u| < \pi/4$
   $$\Phi(iu) = \sum_{n\geq 0} c_n u^{2n}$$
   where
   $$c_n = \frac{1}{\pi(2n)!}\int_0^\infty \Xi(t)t^{2n}\,\mathrm{d}t.$$

   (g) Deduce from (1) that $\frac{1}{2} + F(x)$ and all its derivatives tend to zero as $x \to i$ provided the argument of $x - i$ is at most $\pi/2$ in absolute value.

   (h) Deduce that $\Phi(iu)$ and all its derivatives tend to 0 as $u \to \pi/4$ along the real axis.

   (i) Deduce that the coefficients c_n must change sign infinitely often.

   (j) Show that if $\Xi(t) > 0$ for $t > T$ then
   $$\int_0^\infty Xi(t)t^{2n}\,\mathrm{d}t > (T+1)^{2n}\int_{T+1}^{T+2}\Xi(t)\,\mathrm{d}t - T^{2n}\int_0^T |\Xi(t)|\,\mathrm{d}t.$$

   (k) Deduce that $\Xi(t)$ has infinitely many real zeros, and hence $\zeta(s)$ has infinitely many zeros on the line $\sigma = 1/2$.