Analytic Number Theory Sheet 1

Lent Term 2020

Hyperbola Method with 1 & τ. Using $\sum_{a\le x^{1/3}}$ instead of $\sum_{a \le x^{1/2}}$.

1. Let $\tau_3(n) = \sum_{a_1a_2a_3=n} 1 = 1 \star \tau(n)$. Prove that

$$\sum_{n\le x} \tau_3(n) = \frac{1}{2}x(\log x)^2 + c_1 x\log x + c_2 x + O(x^{2/3}\log x)$$

for some constants c_1 and c_2.

2. Let $\omega(n)$ count the number of distinct prime divisors of n.

(a) Prove that

$$\sum_{n\le x}\omega(n) = x\log\log x + O(x).$$

$\sum_{n\le x}\sum_{p|n} 1 = \sum_{p\le x}\lfloor \frac{x}{p} \rfloor = x\sum_{p\le x}$

(b) Prove the 'variance bound'

Expand out, only need upperbound not asymptotics.

$$\sum_{n\le x}|\omega(n) - \log\log x|^2 \ll x\log\log x.$$

(c) Deduce that

$$\sum_{n\le x}|\omega(n) - \log\log n|^2 \ll x\log\log x.$$

and hence 'almost all n have $(1+o(1))\log\log n$ distinct prime divisors' in the sense that the number of $n\le x$ such that $|\omega(n)-\log\log n| > (\log\log n)^{3/4}$ is $o(x)$.

3. (a) Show that

$$\sum_{n\le x}\frac{1}{n} = \log x + \gamma - \frac{\{x\}-1/2}{x} + O(x^{-2}).$$

Just take the explicit error from lectures and break up from $x \to \lfloor x \rfloor$.

(b) Let $\Delta(x)$ be the error term in the approximation for the sum of the divisor function, so that

$$\sum_{n\le x}\tau(n) = x\log x + (2\gamma - 1)x + \Delta(x).$$

We proved in lectures that $\Delta(x) = O(x^{1/2})$. Prove the more precise estimate

Look at the error term.

$$\Delta(x) = x^{1/2} - 2\sum_{a\le x^{1/2}}\left\{\frac{x}{a}\right\} + O(1).$$

(c) Deduce that

$$\int_0^x \Delta(t)\,\mathrm{d}t \ll x$$

integrate error term by term, exchange sum & integral.

(so that, 'on average', $\Delta(x) = O(1)$).

4. Prove the following Dirichlet series identities, and give for each a half-plane in which the identity is valid.

For all of these can use EITHER 1) $F_{f*g}(s) = F_f(s)F_g(s)$ $*$ve prop. of dirichlet series

2) f is multiplicative $\Rightarrow F_f(s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \dots\right)$ Euler products.

Can do any part in both ways, try both for each one for revision

"Product of absolutely conv. series is absolutely conv."

~~(a)~~

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

where $\sigma(n) = \sum_{d|n} d$,

~~(b)~~

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

where $\lambda(n)$ is the completely multiplicative function such that $\lambda(p) = -1$ for all primes p,

(c)

$$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)},$$

(d) and

$$\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

where $s(n)$ is the indicator function for the square-full numbers, i.e.

$$s(n) = \begin{cases} 1 & \text{if } p \mid n \text{ implies } p^2 \mid n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

5. ~~Consider the function~~

$$\cancel{\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}}.$$

(a) Show that for $0 < \sigma < 1$

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}\, \mathrm{d}x.$$

Extend $\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^x-1}\,dx$ by pulling out the pole term $\int_0^1 \frac{x^{s-1}}{e^x-1}\,dx$ $\frac{1}{s-1}$.

(b) Show that for $-1 < \sigma < 0$

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1}\, \mathrm{d}x.$$

(c) Deduce the functional equation for $\zeta(s)$, using the identity

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + 2x \sum_{n=1}^{\infty} \frac{1}{4n^2\pi^2 + x^2}.$$

Evaluate using contour.

4. Prove the following Dirichlet series identities, and give for each a half-plane in which the identity is valid.

(a)

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

where $\sigma(n) = \sum_{d|n} d$,

σ is multiplicative. Idea: Show $\sigma = id * 1$ by proving $\sigma * \mu = id$.

Then done as

$$\sum \frac{\sigma(n)}{n^s} = \sum \frac{id * 1(n)}{n^s} = \left(\sum \frac{n}{n^s}\right)\left(\sum \frac{1}{n^s}\right)$$

$$= \zeta(s-1)\zeta(s)$$

$\sigma * \mu$ is multiplicative as σ & μ are. $\sigma * \mu(p^k) = \sum_{ab=p^k} \sigma(a)\mu(b)$

$= \sigma(p) - \sigma(1) = p^k$ for $k \geq 1$.

Thus $\sigma * \mu(n) = n$ so $\sigma * \mu = id$ as required.

(b)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

where $\lambda(n)$ is the completely multiplicative function such that $\lambda(p) = -1$ for all primes p,

Since there are infinitely many primes $\sum_{n=1}^{\infty} \frac{|\lambda(n)|}{n^s} = \sum_p \frac{1}{p^s} < \infty$ iff $\sigma \leq 1$

Thus for $\sigma > 1$ we have

$$\sum \frac{\lambda(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^s}\right)^{-1} = \frac{\prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1}}{\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}} = \frac{\zeta(2s)}{\zeta(s)}$$

by Euler's Product representation.

(c)

$$\sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)},$$

$\tau(p) = 1$

$\tau(n^2) = \sum_{d \mid n^2} 1 = \sum_{\substack{d \mid n^2 \\ d \leq n}}$

$(1 -$

$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} =$

$n = p_1 p_2 \qquad n^2 = p_1^2 p_2^2$

$d = 1, p_1, p_2, p_1p_2 \quad \} \quad 1, p_1, p_2, p_1p_2, p_1^2p_2, p_1p_2^2, p_1^2p_2^2$

$n = p_1^2 p_2$

$d = 1, p_1, p_2, p_1^2, p_1p_2, p_1^2p_2$

(d) and

$$\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

where $s(n)$ is the indicator function for the square-full numbers, i.e.

$$s(n) = \begin{cases} 1 & \text{if } p \mid n \text{ implies } p^2 \mid n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note: $s(n)$ is multiplicative

$s(p^k) = 1$ if $k \geq 2$

Thus since s multiplicative, $\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \prod_p \left(1 + s(p)p^{-s} + s(p^2)p^{-2s} + \cdots\right)$

$= \prod \left(1 + p^{-2s} + p^{-3s} + \cdots\right)$

$s * \mu(p^k) = \sum_{ab=p^k} s(a)\mu(b) = 1 + \mu(p^2) + \cdots \mu(p^k) = 1$

$\Rightarrow \quad s * \mu = I \quad$ so $\quad s = I * \mu$