Lecture 7

Question: Let $\mathcal{A} \subset P(n) = \{S : S \subset [n]\}$ be a family of sets of size n. such that $A \not\subset B$ $\forall A, B \in \mathcal{A}$ $A \neq B$ ← Called an antichain

Theorem (Sperner, 1910s) $\mathcal{A} \subset P(n)$ and an antichain $\Rightarrow$ $|\mathcal{A}| \leq \binom{n}{n/2}$

Theorem (Lemma of LYMB, 60s) $\mathcal{A}$ is an antichain $\Rightarrow$ $\displaystyle\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$

¡ Bollobas!

Proof of Sperner using LYMB: $\binom{n}{|A|} \leq \binom{n}{n/2}$ $\overset{\text{LYMB}}{\Rightarrow}$ $1 \geq \displaystyle\sum_{\mathcal{A}} \frac{1}{\binom{n}{|A|}} \geq |\mathcal{A}| \frac{1}{\binom{n}{n/2}}$

$$\Rightarrow \quad |\mathcal{A}| \leq \binom{n}{n/2} \qquad \blacksquare$$

Proof of LYMB: We will prove this by counting something in two ways. Lets count permutations π of $[n]$ in a way that involves $|A|$. We count # pairs (π, A) s.t. π is perm. of $[n]$, $A \in \mathcal{A}$ and $\{\pi(1), \ldots, \pi(|A|)\} = A$.

First fix $A \in \mathcal{A}$. $\#\{\pi$ such that $\{\pi(1), \ldots, \pi(|A|)\} = A\} = |A|!\,(n-|A|)!$

Now fix π perm. of $[n]$. $\# A \in \mathcal{A}$ s.t. $\{\pi(1), \ldots, \pi(|A|)\} = A$ is $\leq$ $\# A \in P(n)$ s.t. $\{\pi(1), \ldots, \pi(|A|)\} = A$ ← i.e. A is an initial segment of π ≤ 1 since $\mathcal{A}$ is an antichain and $\{\pi(1), \ldots, \pi(l)\} \subset \{\pi(1), \ldots, \pi(l+1)\}$

Thus $\displaystyle\sum_{A \in \mathcal{A}} |A|!\,(n-|A|)! = \#$ of pairs (π, A) s.t. π perm. of $[n]$, $A \in \mathcal{A}$ and $\{\pi(1), \ldots, \pi(|A|)\} = A$ $\leq n!$ so done.

Lecture 7 (2)

Question: $\mathcal{A}$ is intersecting if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$.
$\mathcal{A} \subset P(n)$ is intersecting $\Rightarrow$ $|\mathcal{A}| \leq$?

Note: Phrased diff. Let $\mathcal{A}$ be a set of friendship groups, what's the maximum number of friendship groups such that each group shares a friend.

Examples $|\{A : 1 \in A|$ $= 2^{n-1}$

Prop^n $\mathcal{A}$ intersecting $\Rightarrow$ $|\mathcal{A}| \leq 2^{n-1}$

Proof: For each set take $\{A, A^c\}$, can only have one of each.

Theorem (Erdös-Ko-Rado) $\mathcal{A} \subset [n]^{(k)}$ is intersecting $\Rightarrow$ $|\mathcal{A}| \leq \binom{n-1}{k-1}$

Proof: Count cycles. Say two permutations are equivalent if they are the same up to their starting point. e.g $123 = 231 = 312$.

Fix $A \in \mathcal{A}$, $\#\, \pi$ s.t. A is an interval $=$ Can order A in $k!$ ways, then have $(n-k)(n-k-1) \cdots 2 \cdot 1$ choices for the other elements. So there are $k!\,(n-k)!$ choices.

Now fix π a permutation of circle.
$\#\, A \in \mathcal{A}$ s.t. A is an interval $\leq$