Examiner: *N. Berestycki*

Mock Exam:

Mixing Times of Markov chains (Lent 2017).

Duration: two hours.

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Problem 1.

(a) Define the *total variation distance* $\|\mu-\nu\|_{\text{tv}}$ for probability distributions μ, ν on a finite set S. Show that

$$\|\mu-\nu\|_{\text{tv}} = (1/2)\sum_{x\in S}|\mu(x)-\nu(x)| = \sum_{x\in S}(\mu(x)-\nu(x))_+$$

where $a_+ = \max(a,0)$. Show that if P is the transition matrix of an irreudcible, aperiodic Markov chain on a state space S with invariant distribution π, and if $d(t) = \sup_x \|P^t(x,\cdot)-\pi(\cdot)\|_{\text{tv}}$ then $d(t) \leqslant \bar{d}(t) \leqslant 2d(t)$ where $\bar{d}(t) = \sup_x \|P^t(x,\cdot)-P^t(y,\cdot)\|_{\text{tv}}$.

(b) Define what is meant by a *coupling* of μ and ν, and show that if (X,Y) is such a coupling then

$$\|\mu-\nu\|_{\text{tv}} \leqslant \mathbb{P}(X \neq Y).$$

(c) Using a coupling or otherwise, show that $\bar{d}(t+s) \leqslant \bar{d}(t)\bar{d}(s)$. Hence deduce that $\rho = \lim_{t\to\infty} d(t)^{1/t}$ exists. [*Hint: you can use without proof the following lemma: if f is subadditive, i.e., if $f(t+s) \leqslant f(t)+f(s)$ for all $s,t \geqslant 0$ then $\lim_{t\to\infty} f(t)/t$ exists in $\mathbb{R}\cup\{-\infty\}$.*]

(d) Assuming also reversibility in the above setting, what is the value of ρ? [You can use without proof any result from the course, provided it is clearly stated].

Problem 2.

(a) Let P be the transition matrix of an irreducible, aperiodic and reversible Markov chain on a finite state space S of size n with invariant distribution $(\pi(x))_{x\in S}$, with eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_n$. Define the *Dirichlet form* $\mathcal{E}(f,f)$ associated to P, and give without proof an equivalent expression. State and prove the *variational characterisation* of the spectral gap in terms of $\mathcal{E}(f,f)$. State without proof a similar characterisation for higher order eigenvalues.

(b) Let $P, \tilde{P}$ be two transitive Markov chains on S, with corresponding Dirichlet forms $\mathcal{E}, \tilde{\mathcal{E}}$ respectively. Suppose that if $A > 0$ is such that $\tilde{\mathcal{E}}(f,f) \leqslant A\mathcal{E}(f,f)$. State and prove a theorem concerning their respective mixing behaviours in L^2, defining carefully the expressions you introduce. (You can use without proof a relation between eigenvalues and L^2 distance to stationarity, provided that this is stated clearly).

(c) Define the interchange process on a connected graph $\mathcal{G} = (V,E)$. State a theorem giving a bound of mixing time of the interchange on $\mathcal{G}$ in terms of geometric quantities associated with $\mathcal{G}$.

(d) Suppose $\mathcal{G} = (V,E) = [0,n)^2 \cap \mathbb{Z}^2$ is the $n \times n$ square, and E is the set of nearest neighbour edges, so $(u,v) \in E$ if and only if $\|u-v\|_1 = 1$ for $u, v \in V$ (here $\|u\|_1 = |u_1| + |u_2|$ for $u = (u_1, u_2)$). Show that the interchange process (in continuous time) satisfies $t_{\text{mix}} = O(n^4 \log n)$. On the other hand, explain *briefly*, e.g. by considering the position of a single card, why $t_{\text{mix}} \geqslant cn^4$ for some $c > 0$.

Problem 3.

(a) Let $(X_t, t = 0, 1, \ldots)$ be an irreducible, aperiodic and reversible Markov chain on a finite state space S with invariant distribution $\pi(y), y \in S$. Define the notion of *mixing time* $t_{\mathrm{mix}}(\alpha)$ *at level* $\alpha \in (0, 1)$.

Give the definition of the *absolute spectral gap* γ_* of the chain, as well as that of the *relaxation time* t_{rel}, and give without proof the statement of a relation between relaxation time and mixing time at level $\varepsilon > 0$.

(b) Define the *bottleneck* (or isoperimetric) *ratio* Φ_* of an irreducible, reversible Markov chain on a finite state space S. State Cheeger's inequality, and prove that if γ is the spectral gap, then $\gamma \leqslant 2\Phi_*$.

(c) Let $S = \{1, \ldots, n\}$ be the n-cycle and consider the Markov chain on S which is the lazy simple random walk on S. Show that $\Phi_* = (1/n)(1 + o(1))$ as $n \to \infty$. Deduce that $\gamma \geqslant (1 + o(1))2/n^2$, and hence show that $t_{\mathrm{mix}}(1/4) \leqslant O(n^2 \log n)$.

(d) Compute all the eigenvalues of this Markov chain, and compare the estimate above with the actual value of the spectral gap.

Problem 4.

(a) Let $(X_t, t = 0, 1, \ldots)$ be an irreducible, aperiodic and reversible Markov chain on a finite state space S with invariant distribution $\pi(y), y \in S$. Show that if $P^t(x,y)$ denote the t-step transition probabilities of the chain,

$$\frac{P^t(x,y)}{\pi(y)} = \sum_{j=1}^{n} \lambda_j^t f_j(x) f_j(y)$$

where λ_j are the eigenvalues and f_j are functions which you should specify. [You can assume without proof that there exists an orthonormal basis of eigenfunctions for the inner product associated with the ℓ^2 norm $\|f\|_2 = (\sum_x f(x)^2 \pi(x))^{1/2}$].

(b) Define the *relaxation time* t_{rel}, and the ℓ^2 distance $d_2(t)$ to equilibrium. Show that $d(t) \leqslant (1/2) d_2(t)$ where $d(t)$ is the total variation distance to equilibrium, and show that

$$t_{\text{mix}}(\varepsilon) \leqslant \log\left(\frac{1}{2\varepsilon\sqrt{\pi_{\min}}}\right) t_{\text{rel}}$$

where $\pi_{\min} = \min\{\pi(x) : x \in S\}$, and $t_{\text{mix}}(\varepsilon)$ is the mixing time at level ε.

(c) Show that $P^{2t}(x,x)$ is a decreasing sequence (as a function of $t = 0, 1, \ldots$). Show however with an example that $P^t(x,x)$ is not in general monotone, as a function of t.

(d) Show that $d_2(t)$ is a contraction: for all $t, s \geqslant 0$:

$$d_2(t+s) \leqslant d_2(s) e^{-t/t_{\text{rel}}}.$$

[You can use freely without proof the inequality $1 - x \leqslant e^{-x}$, valid for all $x \in \mathbb{R}$].