Notes on Combinatorics Example Sheet 3

Q30

Given $f_j = \left(\sum_{i=1}^{n} (x_i - s_{ij})^2 - d_1^2\right)\left(\sum_{i=1}^{n} (x_i - s_{ij})^2 - d_2^2\right)$

$\in \text{span}\left\{ \left(\sum_{k=1}^{n} x_k^2\right)^2, \; x_i\left(\sum x_k^2\right), x_i x_k, x_i, 1 \right\}$

Want: $n+1$ polys not in the span of the f_j.

Work out coefficients:

$$f_j(x) = \left(\sum x_k^2\right)^2 - \sum_{i=1}^{n} 4 s_{ij} x_i \left(\sum x_k^2\right) + \left(\sum_{i \neq k} 8 s_{ij} s_{kj} x_i x_k\right)$$

$$+ \sum_{i=1}^{n} \left(4 s_{ij} - (d_1^2 + d_2^2)\right) x_i^2 - \sum_{i=1}^{n} 2(d_1^2 + d_2^2) s_{ij} x_i + d_1^2 d_2^2$$

Then add:
- $(1, d_1^2 d_2^2)$ so add in 1
- $(1, -4s_{ij}, 4s_{ij} - (d_1^2 + d_2^2))$ so add in x_i for each i

Q31 Take $x \in \mathbb{R}^{n+1}$ with two 1s and the rest 0's

Q32 Essentially impossible in finite time. Don't bother.

Q33 No one, nor the supervisor knew how to do it.

Quite rude to put on a sheet.

Q34 Atleast one, Take $[n]^{(>\frac{n}{2})} \cup \{A \in [n]^{(\frac{n}{2})} : 1 \in A\}$

Take an element of size $\lfloor \frac{n}{2} \rfloor$ and add that.

For second part WLOG $\mathcal{A}$ is a down-set (closed under subsets)

If $A \in \mathcal{A}$, $|A| > \frac{n}{2}$, then A disjoint from $\leq 2^{\lfloor \frac{n-1}{2} \rfloor}$ elements.

So adding in a set C of size $\lfloor \frac{n-1}{2} \rfloor$ s.t. $C \cap B$ some $|B| = \lfloor \frac{n-1}{2} \rfloor$, $B \in \mathcal{A}$.

doesn't decrease # disjoint pairs. Works for n even.

n odd? No one solved it.

Q35 G. Let H be subgraph. $a_{v,e} = \begin{cases} 1 & \text{if } v \in e, e \in H \\ 0 & \text{otherwise} \end{cases}$ $x_e = \begin{cases} 1 & e \in H \\ 0 & e \notin H \end{cases}$

Work in $\mathbb{Z}_p$ $\prod_{v \in V} \left(1 - \left(\sum_{e \in E} a_{v,e} x_e\right)^{p-1}\right) = \begin{cases} 0 & \text{if any } v \text{ has } d(v) \neq 0, p \\ 1 & \text{otherwise} \end{cases}$

$\prod_{e \in E} (1 - x_e) = \begin{cases} 0 & \text{if } H \text{ has an edge} \\ 1 & \text{otherwise} \end{cases}$

Thus $f(x) = \begin{cases} 1 & \text{if } H \text{ is } p\text{-regular} \\ 0 & \text{otherwise} \end{cases}$

$\deg \prod_{v \in V} = |V|(p-1)$ $\deg \prod_{e \in E} = |E|$ Avg. $\deg > 2p-2 \Rightarrow |E| = |V|(p-1)$

$\deg f = |E| = \sum_{e \in E} 1$ $x \in \{0,1\}^{|E|}$ $\underset{\text{ACNS}}{\Rightarrow}$ $\exists x$ s.t. $f(x) \neq 0$ because coeff. of $\prod_{e \in E} x_e \neq 0$

So $\exists x$ s.t. $f(x) \neq 0$ and thus we are done.

Q36 $d(A,B) \leq k \iff N^k(A) \cap B \neq \emptyset$

Thus If $|N^k(A)| + |B| > 2^n$ then certainly $d(A,B) \leq k$.

Let A' be initial segment of simplicial so $|N^k(A)| \geq |N^k(A')|$ and with $|A| = |A'|$
B' last $|B|$ elements of simplicial order

Then $N^k(A') \cap B' \neq \emptyset \iff |N^k(A')| + |B'| > 2^n$
$\Rightarrow |N^k(A)| + |B'| > 2^n$

Then if $d(A,B) > k \Rightarrow |N^k(A)| + |B| \leq 2^n \Rightarrow |N^k(A')| + |B| \leq 2^n$
$\Rightarrow d(A', B') > k$.

Q37 $\text{diam}(A) \leq 2k \iff d(A, \bar{A}) \geq n - 2k$ with $\bar{A} = \{x^c : x \in A\}$

Q37 (Cont.) Let C be initial segment of simplicial with $|C| = |A|$

Then $d(C, \bar{C}) \geq d(A, \bar{A})$ by Q36. So $|A| = |C| \leq \sum_{i=0}^{k} \binom{n}{i}$

Q38 $A \in \mathcal{P}(n) \longleftrightarrow \underline{v}_A \in \mathbb{F}_2^n$.

$|A \cap B|$ even $\Leftrightarrow$ $v_A \cdot v_B = 0$

odd $\Leftrightarrow$ $v_A \cdot v_B = 1$

Best to start w/ this part

(iii) $v_1, \dots, v_k$ such that $v_i \cdot v_j = \delta_{ij}$. Can find atmost $k \leq n$.

(as $\sum \lambda_i v_i = 0 \Rightarrow \sum \lambda_i v_i \cdot v_j = 0 \Rightarrow \lambda_j = 0$) ... $1, \dots, n$ works

If have a lot of things that get the maximum value, try some linear algebra.

(i) $2^{\lfloor n/2 \rfloor}$. Take $V = \langle v_1, \dots, v_k \rangle$ with basis $w_1, \dots, w_r$ and then $V \subset \ker \alpha$

$\alpha : \underline{x} \mapsto (\underline{x} \cdot w_1, \dots, \underline{x} \cdot w_r)$

α has rank $r \Rightarrow \dim V = n - r \Rightarrow r \leq n - r$, $r \leq \lfloor \frac{n}{2} \rfloor$ so $k \leq |V| \leq 2^{\lfloor n/2 \rfloor}$

(ii) Stick in a dummy element $2^{\frac{n-1}{2}}$

(iv) n if n is odd, $n-1$ if n is even.

Note: Can do parts (iii) & (iv) with polynomials

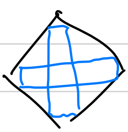
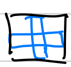
Q39 $n \leq 5$ $n+1$, $n \geq 6$ even $2^{n/2}$, $n \geq 7$ odd $2^{\frac{n-1}{2}} + 1$

Solution was not shown in Examples Class, shares similarity with Q39.

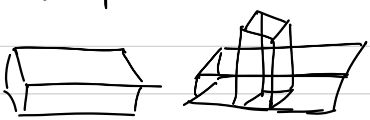
Q40 Need to show closed & bdd. Have bounded since $0 \leq x_A \leq |k_A|$

the closed for $x = |k_A|$. If $x_A = 0$ then $|k|^k \leq \prod_{A \in \mathcal{A}} x_A = 0$

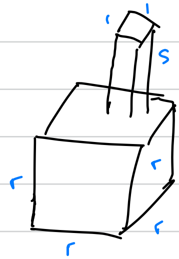
so K empty.

Q41 No. Consider  $L(K)$ must contain then a BTBT box  which is too big.

Q41 Second part is hard.



... No idea. Construct something like.



$|k| = r^3 + s$ Suppose $a \times b \times c$ box works.

$bc \le |k_{23}| = r^2 + s$

$\Rightarrow a, b \le \frac{r^3+s}{r^2+s}$

$c \geq r+s$

$abc \geq \left(\frac{r^3+s}{r^2+s}\right)^2 (r+s) = (r^3+s) \times \frac{(r^3+s)(r+s)}{(r^2+s)^2}$

Q42 $k = r^{\alpha_1} \times \cdots \times r^{\alpha_n}$ and let $r \to \infty$

$\sum \alpha_i > 0$, $\sum_{i \in A} \alpha_i \le 0$ each $A \in \mathcal{A}$.

Note: Can use BTBT to show that if we are going to break it we can break it with a box.

Suppose $\mathcal{A}$ doesn't generate a uniform cover.

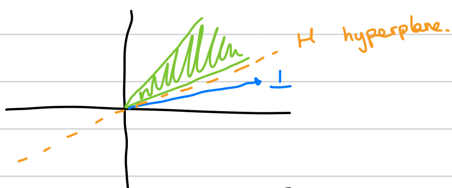
$\underline{v_A}$ as before. Then can't have $\underline{1} = \sum_{A \in \mathcal{A}} \lambda_A \underline{v_A}$ $\lambda_A \geq 0$

Think of as uniform cover

(else $\lambda \in \mathbb{Q}$ and multiplying up gives uniform cover).

Consider $C = \{\underline{x} \in \mathbb{R}^n : x = \sum_{A \in \mathcal{A}} \lambda_A \underline{v_A}$, some $\lambda_A \geq 0\} \subset \mathbb{R}^n_{\geq 0}$.

This looks like a high dimensional cone.

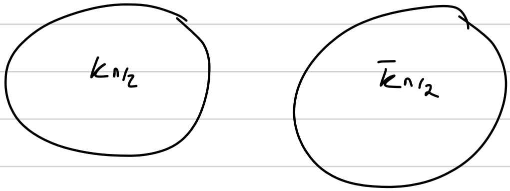


H given by $\sum_{i=1}^n \alpha_i x_i = 0$ for some α_i

Then $\sum_{i=1}^n \alpha_i \underline{1} > 0$ i.e. $\sum_{i=0}^n \alpha_i > 0$

and each $\underline{v_A}$ $(A \in \mathcal{A})$ is in C, so $\sum_{i=1}^n \alpha_i (\underline{v_A})_i < 0$ i.e. $\sum_{i \in A} \alpha_i < 0$.

Q43 Lower Bound



can have any edges between and no 4-cycle $\Rightarrow \geq 2^{\frac{1}{2}\binom{n}{2}}$ graphs with no induced C_4.

Upper Bound Want to find many edge disjoint K_4's. i.e. lots of 4-sets in $[n]^{(4)}$ that intersect in at most 1 point. 'lots' $= O(n^2)$

Order the 4-sets $A_1, \ldots, A_{\binom{n}{4}}$ can construct these inductively.

Q44 One important fact for shattering questions: For any $\mathcal{F}$ can find some down-set $\mathcal{D}$ such that $|\mathcal{D}| = |\mathcal{F}|$ and $|\mathcal{D} \cap \mathcal{F}| \leq |\mathcal{F} \cap S| \ \forall S$.
Thus if we show the result for down-sets then we are done.

Then the question becomes how large can the downsets be given neither contain the same 4-set.
So best possible is $2^n + \sum_{i=0}^{4-1} \binom{n}{i}$ achieved by $\mathcal{A} = P(n)$, $\mathcal{B} = [n]^{(\leq 4-1)}$

Q45 Again: This is saying something we only need to prove for down sets.
WLOG $\mathcal{F}$ down-set.
(i) If $\mathcal{F}$ down-set of size $|\mathcal{F}| = n \Rightarrow$ some element $i \in [n]$ not in any $A \in \mathcal{F}$. Then project along $[n] \setminus \{i\}$.
since $|\{\emptyset\} \cup \{1\} \cup \{2\} \ldots \cup \{n\}| = n+1$
(ii) $2 \leq n \leq m \leq \lceil 3n/2 \rceil$. If for some i, only element containing i is $\{i\}$ then project along $[n] \setminus \{i\}$ and done. Else if for each i have ≥ 2 sets cont. i, then we have $\{i\}$, $\{i, j(i)\} \in \mathcal{F} \ \forall i$.
So we have $\geq \lceil n/2 \rceil$ 2-sets in $\mathcal{F}$, plus singletons and $\emptyset \Rightarrow m > \lceil \frac{3n}{2} \rceil$ #
(iii) Counterexample: $\emptyset, 1, 2, \ldots, n, 12, 34, 56, \ldots$ $\begin{cases} (n-1)n & n \text{ even} \\ (n-2)(n-1) \text{ and } 1n & \text{if } n \text{ odd} \end{cases}$

So $(n,m) \not\to (n-1, m-1)$

Q46 $(n, m) \to (3, 7) \Rightarrow$ Mantel

Mantel: $e(G) \geq \lfloor n^2/4 \rfloor + 1 \Rightarrow \Delta \subset G$

Let $\mathcal{F} = \underbrace{E}_{\lfloor \frac{n^2}{4} \rfloor + 1} \cup \underbrace{V}_{n} \cup \underbrace{\emptyset}_{1} \Rightarrow |\mathcal{F}| = \lfloor n^2/4 \rfloor + n + 2$.

So $\exists S \subset [n]^{(3)}$ s.t. $|\mathcal{F} \cap S| \geq 7$. $\mathcal{F}$ has no 3-sets so $\mathcal{F} \cap S = P(S) \setminus S$ which includes $S^{(2)}$ which is a Δ

Mantel $\Rightarrow (n, m) \to (3, 7)$ WLOG $\mathcal{F}$ is a down-set. If $A \subset \mathcal{F}$, $|A| = 3$ then $|\mathcal{F} \cap A| = 8$ ✓

So $\mathcal{F} \subset [n]^{(2)}$. So $\mathcal{F}$ is $E \cup V \cup \emptyset$ for some graph $G = (V, E)$ and then Mantel $\Rightarrow |\mathcal{F} \cap S| \geq 7$ for $S = V(\Delta)$