Analytic Number Theory Sheet 1 - Solutions

Lent Term 2020

1. Let $\tau_3(n) = \sum_{a_1a_2a_3=n} 1 = 1 \star \tau(n)$. Prove that

$$\sum_{n\leq x} \tau_3(n) = \frac{1}{2}x(\log x)^2 + c_1 x\log x + c_2 x + O(x^{2/3}\log x)$$

for some constants c_1 and c_2.

Solution: In this proof the letter c will be used to denote some fixed absolute constant, which may change from line to line (just to avoid introducing lots of new subscripts) – it's easy to actually follow the constants through and check what they all are.

For the hyperbola method we need to make a choice of where to split the summation – it makes sense to have all 3 factors given the same weight, so we will divide the sum over

$$\tau_3(n) = \sum_{ab=n} \tau(b)$$

according to whether $a \leq x^{1/3}$ or $b \leq x^{2/3}$. This gives

$$\begin{aligned}\sum_{n\leq x} \tau_3(n) &= \sum_{a\leq x^{1/3}} \sum_{b\leq x/a} \tau(b) + \sum_{b\leq x^{2/3}} \tau(b) \left\lfloor \frac{x}{b} \right\rfloor - \sum_{b\leq x^{2/3}} \tau(b) \left\lfloor x^{1/3} \right\rfloor \\ &= \Sigma_1 + \Sigma_2 - \Sigma_3,\end{aligned}$$

say. We will evaluate each of these three summands in turn. Recall that we showed in lectures that

$$\sum_{n\leq y} \tau(n) = y\log y + cy + O(y^{1/2})$$

and

$$\sum_{n\leq y} \frac{1}{n} = \log n + c + O(1/y).$$

It follows that

$$\begin{aligned}\Sigma_1 &= \sum_{a\leq x^{1/3}} \left(\frac{x}{a}\log(x/a) + c\frac{x}{a} + O\left(\frac{x^{1/2}}{a^{1/2}}\right)\right) \\ &= (x\log x + cx)\sum_{a\leq x^{1/3}} \frac{1}{a} - x\sum_{a\leq x^{1/3}} \frac{\log a}{a} + O\left(x^{1/2}\sum_{a\leq x^{1/3}} a^{-3/2}\right) \\ &= (x\log x + cx)(\tfrac{1}{3}\log x + c + O(x^{-1/3})) - x\sum_{a\leq x^{1/3}} \frac{\log a}{a} + O(x^{2/3}) \\ &= \tfrac{1}{3}x(\log x)^2 + cx\log x + cx + O(x^{2/3}\log x) - x\sum_{a\leq x^{1/3}} \frac{\log a}{a},\end{aligned}$$

where we have bounded the sum in the error term by the integral $\int_1^{x^{1/3}} t^{-3/2}\,\mathrm{d}t = O(x^{1/6})$. To evaluate the final sum, we use partial summation, along with $\sum_{a\le t}\frac{1}{a} = \log t + c + O(1/t)$ to obtain

$$\sum_{a\le y}\frac{\log a}{a} = \frac{1}{2}(\log y)^2 + c\log y + c + O\left(\frac{\log y}{y}\right).$$

Note, importantly, in this calculation that is it *not* true that if $R(t) = O(1/t)$ then

$$\int_1^y \frac{R(t)}{t}\,\mathrm{d}t = O(1/y);$$

indeed, depending on the behaviour of the $O(1/t)$ function it could be constant, since the range of integration is from 1. What *is* true, however, is that for any $R(t) = O(1/t)$ we have

$$\int_1^y \frac{R(t)}{t}\,\mathrm{d}t = \int_1^\infty \frac{R(t)}{t}\,\mathrm{d}t - \int_y^\infty \frac{R(t)}{t}\,\mathrm{d}t = c + O(1/y),$$

for some constant c depending on R, which is an acceptable substitute.

It follows that

$$\Sigma_1 = \left(\frac{1}{3} - \frac{1}{2\cdot 3^2}\right)x(\log x)^2 + cx\log x + cx + O(x^{2/3}\log x).$$

We now evaluate

$$\begin{aligned}
\Sigma_2 &= x\sum_{b\le x^{2/3}}\frac{\tau(b)}{b} + O\left(\sum_{b\le x^{2/3}}\tau(b)\right)\\
&= x^{1/3}\sum_{b\le x^{2/3}}\tau(b) + x\int_1^{x^{2/3}}\frac{\sum_{n\le t}\tau(n)}{t^2}\,\mathrm{d}t + O\left(x^{\frac{2}{3}}\log x\right)\\
&= x^{1/3}\left(\tfrac{2}{3}x^{2/3}\log x + cx^{2/3} + O(x^{1/3})\right) + x\int_1^{x^{2/3}}\frac{t\log t + ct + O(t^{1/2})}{t^2}\,\mathrm{d}t + O\left(x^{\frac{2}{3}}\log x\right)\\
&= \frac{1}{2}x(\log x^{2/3})^2 + cx\log x + cx + O(x^{2/3}\log x)
\end{aligned}$$

by partial summation. Finally, we note that

$$\Sigma_3 = x^{1/3}\sum_{b\le x^{2/3}}\tau(b) + O\left(\sum_{b\le x^{2/3}}\tau(b)\right) = cx\log x + cx + O(x^{2/3}\log x).$$

Combining these three estimates,

$$\sum_{n\le x}\tau_3(n) = \left(\frac{1}{3} - \frac{1}{2\cdot 3^2} + \frac{2}{9}\right)x(\log x)^2 + cx\log x + cx + O(x^{2/3}\log x).$$

The leading coefficient here is $1/2$ and we're done.

2. Let $\omega(n)$ count the number of distinct prime divisors of n.

(a) Prove that

$$\sum_{n\le x}\omega(n) = x\log\log x + O(x).$$

(b) Prove the 'variance bound'

$$\sum_{n\le x}|\omega(n) - \log\log x|^2 \ll x\log\log x.$$

(c) Deduce that

$$\sum_{n\le x} |\omega(n) - \log\log n|^2 \ll x\log\log x.$$

and hence 'almost all n have $(1+o(1))\log\log n$ distinct prime divisors' in the sense that the number of $n \le x$ such that $|\omega(n) - \log\log n| > (\log\log n)^{3/4}$ is $o(x)$.

Solution: We have

$$\begin{aligned}\sum_{n\le x}\omega(n) &= \sum_{n\le x}\sum_{p} 1_{p|n}\\ &= \sum_{p\le x}\sum_{n\le x} 1_{p|n}\\ &= \sum_{p\le x}\left\lfloor \frac{x}{p}\right\rfloor\\ &= x\sum_{p\le x}\frac{1}{p} + O(x)\\ &= x\log\log x + O(x).\end{aligned}$$

For part (b), expanding out the left hand side gives

$$\sum_{n\le x}\omega(n)^2 - 2\log\log x\sum_{n\le x}\omega(n) + \lfloor x\rfloor(\log\log x)^2.$$

The second summand is $-2x(\log\log x)^2 + O(x\log\log x)$, and the third is $x(\log\log x)^2 + O((\log\log x)^2)$. It therefore suffices to show that

$$\sum_{n\le x}\omega(n)^2 \le x(\log\log x)^2 + O(x\log\log x).$$

For this we note

$$\begin{aligned}\sum_{n\le x}\omega(n)^2 &= \sum_{p,q\le x}\sum_{n\le x} 1_{p|n}1_{q|n}\\ &= \sum_{p\ne q\le x}\left\lfloor\frac{x}{pq}\right\rfloor + O(x\log\log x)\\ &\le x\left(\sum_{p\le x}\frac{1}{p}\right)^2 + O(x\log\log x)\\ &= x(\log\log x)^2 + O(x\log\log x).\end{aligned}$$

Finally, for part (c) we note that by the triangle inequality it suffices to show that

$$\sum_{n\le x}|\log\log n - \log\log x|^2 \ll x\log\log x.$$

The range $n \le x^{1/2}$ trivially contributes $O(x^{1/2}(\log\log x)^2)$ and if $x^{1/2}\le n\le x$ then $|\log\log n - \log\log x| \ll 1$, and so this range contributes $O(x)$, and hence the sum overall is certainly $O(x\log\log x)$. Finally, if we let $r(x)$ count the number of $n\le x$ such that $|\omega(n)-\log\log n| > \frac{1}{2}(\log\log x)^{3/4}$ then

$$r(x)(\log\log x)^{3/2} \le 4\sum_{n\le x}|\omega(n)-\log\log n|^2 \ll x\log\log x,$$

and hence $r(x) = o(x)$, and the final claim follows since for all $n \ge x^{1/2}$, say, $(\log\log n)^{3/4} \ge \frac{1}{2}(\log\log x)^{3/4}$.

3.

(a) Show that

$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma-\frac{\{x\}-1/2}{x}+O(x^{-2}).$$

(b) Let $\Delta(x)$ be the error term in the approximation for the sum of the divisor function, so that

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+\Delta(x).$$

We proved in lectures that $\Delta(x)=O(x^{1/2})$. Prove the more precise estimate

$$\Delta(x)=x^{1/2}-2\sum_{a\le x^{1/2}}\left\{\frac{x}{a}\right\}+O(1).$$

(c) Deduce that

$$\int_0^x \Delta(t)\,\mathrm{d}t \ll x$$

(so that, 'on average', $\Delta(x)=O(1)$).

Solution: By partial summation as in lectures,

$$\sum_{n\le x}\frac{1}{n}=\frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor t\rfloor}{t^2}\,\mathrm{d}t=\log x+\gamma-\frac{\{x\}}{x}+\int_x^\infty\frac{\{t\}}{t^2}\,\mathrm{d}t.$$

Since $\int_x^\infty \frac{1}{2t^2}\,\mathrm{d}t=-\frac{1}{2x}$ it suffices to show that

$$\int_x^\infty\frac{\{t\}-1/2}{t^2}\,\mathrm{d}t=O(x^{-2}).$$

Note that, for any $n\ge 1$, if $t\in[n,n+1)$, then $1/n^2=1/t^2+O(1/n^3)$, and hence

$$\int_n^{n+1}\frac{\{t\}-1/2}{t^2}\,\mathrm{d}t=\frac{1}{n^2}\int_0^1(t-\tfrac{1}{2})\,\mathrm{d}t+O(n^{-3})=O(n^{-3}).$$

It follows that for any $m\ge 1$

$$\int_m^\infty\frac{\{t\}-1/2}{t^2}\,\mathrm{d}t=\sum_{n\ge m}O(n^{-3})=O(m^{-2}).$$

The required bound follows letting $m=\lceil x\rceil$, since $\int_x^{\lceil x\rceil}t^{-2}\,\mathrm{d}t\ll x^{-2}$.

For part (b), the hyperbola method argument from lectures shows that

$$\begin{aligned}
\sum_{n\le x}\tau(n)&=2\sum_{a\le x^{1/2}}\left\lfloor\frac{x}{a}\right\rfloor-\lfloor x^{1/2}\rfloor^2\\
&=2x\left(\log x^{1/2}+\gamma-\frac{\{x^{1/2}\}-1/2}{x^{1/2}}+O(x^{-1})\right)-2\sum_{a\le x^{1/2}}\{x/a\}-(x^{1/2}-\{x^{1/2}\})^2\\
&=x\log x+(2\gamma-1)x-2x^{1/2}(\{x^{1/2}\}-\tfrac{1}{2})-2\sum_{a\le x^{1/2}}\left\{\frac{x}{a}\right\}+2x^{1/2}\{x^{1/2}\}+O(1)\\
&=x\log x+(2\gamma-1)x+x^{1/2}-2\sum_{a\le x^{1/2}}\left\{\frac{x}{a}\right\}+O(1).
\end{aligned}$$

By definition then

$$\Delta(x) = x^{1/2} - 2 \sum_{a \le x^{1/2}} \left\{ \frac{x}{a} \right\} + O(1).$$

For part (c) we integrate this over $[0, x]$ term by term. The first gives $\frac{2}{3}x^{3/2}$. Integrating the $O(1)$ term trivially gives at most $O(x)$. It therefore remains to show that

$$\int_0^x \sum_{a \le t^{1/2}} \left\{ \frac{t}{a} \right\} \mathrm{d}t = \tfrac{1}{3}x^{3/2} + O(x).$$

Interchanging the summation and integral, the left-hand side is

$$\sum_{a \le x^{1/2}} \int_{a^2}^x \left\{ \frac{t}{a} \right\} \mathrm{d}t.$$

The integral of $\{t/a\}$ over any interval of the form $[ka, (k+1)a]$, where k is an integer, is $a/2$. It follows that

$$\int_{a^2}^x \left\{ \frac{t}{a} \right\} \mathrm{d}t = \frac{x - a^2}{2} + O(a),$$

since the interval $[a^2, x]$ can be divided into $\lfloor \frac{x-a^2}{a} \rfloor$ intervals of the shape $[ka, (k+1)a)$, with a remainder $O(a)$ in length leftover, and so

$$\sum_{a \le x^{1/2}} \int_{a^2}^x \left\{ \frac{t}{a} \right\} \mathrm{d}t = \frac{x}{2} \lfloor x^{1/2} \rfloor - \frac{1}{2} \sum_{a \le x^{1/2}} a^2 + O(x) = \frac{x^{3/2}}{2} - \frac{x^{3/2}}{6} + O(x),$$

using $\sum_{n \le y} n^2 = \frac{1}{3}y^3 + O(y^2)$, and the proof is complete.

4. Prove the following Dirichlet series identities, and give for each a half-plane in which the identity is valid.

(a)

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

where $\sigma(n) = \sum_{d|n} d$,

(b)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

where $\lambda(n)$ is the completely multiplicative function such that $\lambda(p) = -1$ for all primes p,

(c)

$$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)},$$

(d) and

$$\sum_{n=1}^{\infty} \frac{s(n)}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$$

where $s(n)$ is the indicator function for the square-full numbers, i.e.

$$s(n) = \begin{cases} 1 & \text{if } p \mid n \text{ implies } p^2 \mid n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Solution: For each identity there are two ways to proceed: either use the fact that $F_f F_g = F_{f\star g}$, and verify the corresponding identity for $f \star g$, or work with Euler products, if the coefficients of the Dirichlet series are multiplicative. We will demonstrate each identity with one of these methods, but I encourage you to try and reprove each with the other method.

Note that the Dirichlet series for $\iota(n) = n$ is

$$\sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1)$$

and converges absolutely for $\sigma > 2$. Therefore since $\sigma = 1 \star \iota$, the Dirichlet series is equal to $\zeta(s)\zeta(s-1)$ and converges absolutely for $\sigma > 2$ (and this identity is valid in the same region).

For $\lambda(n)$, we instead argue using Euler products. The Dirichlet series converges absolutely for $\sigma > 1$. In this region

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{p^{2s}} - \cdots\right) = \prod_p \left(1 + p^{-s}\right)^{-1}.$$

Using

$$\left(1 + p^{-s}\right)^{-1} = \frac{(1 - p^{-s})}{(1 - p^{-2s})}$$

we have

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 - p^{-2s}\right)^{-1} \prod_p \left(1 - p^{-s}\right) = \zeta(2s)\zeta(s)^{-1},$$

which is valid for $\sigma > 1$.

For (c), we will first express the multiplicative function $\tau(n)^2$ as a convolution. Suppose that $\tau(n)^2 = f \star g(n)$, where f and g are both multiplicative. We can actually cheat a little because we're told in advance what the Dirichlet series is. Let $f(n) = \tau_4(n) = \tau \star \tau(n)$, and let $g(n) = \mu(m)$ if $n = m^2$ and $g(n) = 0$ otherwise. Our claim is that $\tau(n)^2 = f \star g(n)$. Since both sides are multiplicative, it suffices to check this identity for prime powers. Note that $f(p^k)$ counts the number of $n_1 + n_2 + n_3 + n_4 = k$ with $n_i \geq 0$, which is $\binom{k+3}{k}$. Thus

$$f \star g(p) = f(p) = \binom{4}{1} = 4 = \tau(p)^2$$

and for $k \geq 2$

$$f \star g(p^k) = f(p^k) + g(p^2) f(p^{k-2}) = \binom{k+3}{k} - \binom{k+1}{k-2} = (k+1)^2.$$

Since the Dirichlet series coefficients of $\zeta(s)^4$ are $f(n)$ and those of $\zeta(2s)^{-1}$ are $g(n)$, it follows that

$$\sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^s} = \zeta(s)^4 \cdot \frac{1}{\zeta(2s)},$$

valid for $\sigma > 1$.

Finally, we use Euler products again, noting that $s(n)$ is multiplicative. The Dirichlet series converges absolutely for $\sigma > 1$, where

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{s(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^{2s} - p^s}\right).\end{aligned}$$

The Dirichlet series identity now follows since

$$1 + \frac{1}{x^2 - x} = \frac{1 - x^{-6}}{(1 - x^{-2})(1 - x^{-3})},$$

and it is valid for $\sigma > 1$.

5.

(a) Show that for $0 < \sigma < 1$

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}\, \mathrm{d}x.$$

(b) Show that for $-1 < \sigma < 0$

$$\zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1}\, \mathrm{d}x.$$

(c) Deduce the functional equation for $\zeta(s)$, using the identity

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + 2x \sum_{n=1}^{\infty} \frac{1}{4n^2\pi^2 + x^2}.$$

Solution: Recall that in lectures we proved that for $\sigma > 1$,

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, \mathrm{d}x.$$

The integral here does not converge for $0 < \sigma < 1$, because the integrand blows up as we approach $x = 0$. To remove this pole, note that for $\sigma > 1$,

$$\int_0^1 x^{s-2}\, \mathrm{d}x = \frac{1}{s-1},$$

and hence

$$\zeta(s)\Gamma(s) = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}\, \mathrm{d}x + \int_1^\infty \frac{x^{s-1}}{e^x - 1}\, \mathrm{d}x + \frac{1}{s-1},$$

which is valid for $\sigma > 1$. Since $|e^x - x - 1| \leq |e^x - 1|$ for $0 \leq x \leq 1$, however, the first integral actually converges for all $\sigma > 0$, and hence the right-hand side defines an analytic continuation of $\zeta(s)\Gamma(s)$ to the half-plane $\sigma > 0$, with a simple pole at $s = 1$. The identity in (a) follows from the fact that

$$\frac{1}{s-1} = -\int_1^\infty x^{s-2}\, \mathrm{d}x$$

when $0 < \sigma < 1$. For the identity in (b) we do the same again, trying to remove the pole at $s = 0$ from the integral between 0 and 1. To this end, note that

$$\int_0^1 \frac{1}{2x} \cdot x^{s-1}\, \mathrm{d}x = \frac{1}{2s},$$

and hence

$$\int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}\, \mathrm{d}x = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1}\, \mathrm{d}x - \frac{1}{2s}.$$

This is valid for $\sigma > 0$, but again, the integral on the right-hand side actually converges for $\sigma > -1$. To see why (and to explain the mysterious $1/2$ factor) note that we showed in lectures that

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{B_2}{2!}x^2 + \cdots$$

and hence

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + O(x).$$

We have shown that for $-1 < \sigma < 1$

$$\zeta(s)\Gamma(s) = \int_0^1 \left(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} \right) x^{s-1}\, \mathrm{d}x - \frac{1}{2s} + \int_1^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1}\, \mathrm{d}x.$$

The identity in (b) follows from the fact that

$$\int_1^\infty \frac{1}{2} x^{s-1}\, \mathrm{d}x = -\frac{1}{2s}$$

when $-1 < \sigma < 0$.

We will sketch a proof of (c). Substituting in the given series to the integral in part (b), and changing the order of summation and integration (which can be justified by an appeal to uniform convergence of the series in $[\epsilon, \infty)$ for any $\epsilon > 0$), we see that for $-1 < \sigma < 0$

$$\zeta(s)\Gamma(s) = 2 \sum_{n=1}^\infty \int_0^\infty \frac{x^s}{4n^2\pi^2 + x^2}\, \mathrm{d}x.$$

This integral can be explicitly calculate as

$$\int_0^\infty \frac{x^s}{4n^2\pi^2 + x^2}\, \mathrm{d}x = \frac{(\pi/2)(2n\pi)^{s-1}}{\cos(\pi s/2)}.$$

Substituting this in we get

$$\zeta(s)\Gamma(s) = \frac{2^{s-1}\pi^s}{\cos(s\pi/2)} \zeta(1-s).$$

The conventional form of the functional equation follows using the reflection formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ and the identity $2\cos(z)\sin(z) = \sin(2z)$. Finally, we sketch how to do the integral above. By a change of variable it suffices to show that

$$I = \int_0^\infty \frac{y^s}{1+y^2}\, \mathrm{d}y = \frac{\pi/2}{\cos(\pi s/2)}.$$

Consider the semi-circular contour C of radius R, with a circular indentation around $z = 0$ of radius ϵ. We define z^s by taking the branch of the logarithm which is positive on the real axis. Since $1+z^2 = i(z-i)(1+iz)$ the integrand has a pole at $z = i$ of residue $\frac{-i^{s+1}}{2}$, and no other poles on or inside C. By the residue theorem

$$\int_C \frac{z^s}{1+z^2}\, \mathrm{d}z = 2\pi i \cdot \frac{-i^{s+1}}{2} = \pi i^s.$$

The contribution from the two straight lines is

$$(1 + e^{i\pi s}) \int_\epsilon^R \frac{y^s}{1+y^2}\, \mathrm{d}y.$$

The contribution from the large semicircle of radius R is $O(\frac{R^\sigma}{1+R^2} \cdot R) \to 0$ as $R \to \infty$ since $\sigma < 0$, and the contribution from the small semicircle of radius ϵ is $O(\frac{\epsilon^\sigma}{1+\epsilon^2} \cdot \epsilon) \to 0$ as $\epsilon \to 0$ since $\sigma > -1$. Taking the limits it follows that

$$(1 + e^{i\pi s}) \int_0^\infty \frac{y^s}{1+y^2}\, \mathrm{d}y = \pi i^s,$$

and the result follows after rearranging.