University of
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Mathematics Tripos

Part III

Ramsey Theory

Example Sheet I

November 15, 2019

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Introduction

These are written solutions to Ramsey TheoryExample Sheet I. Solutions are written based on those seen in examples classes and may contain errors from the author.

Questions

> **Question** (Question 1)**.** How many combinatorial lines are there in $[m]^n$

1st Solution. Each coordinate can take a fixed value in $\{1, .., m\}$ or it can be an active coordinate. Thus for each coordinate there are $m + 1$ choices, so there are $(m + 1)^n$ active coordinates. □

2nd Solution. Let $f(m, n) = /\#$ of combinatorial lines, then:

$$f(m,n) = \sum_{i=\#\text{active coordinates}} \binom{n}{i} 1^i m^{n-i} = (m+1)^n$$

□

Remark. As well as being a shorter proof, the first is a better way to think of combinatorial lines in $[m]^n$.

> **Question** (Question 2)**.** Show that $HJ(2, k) = k$ for all k

Idea. Two things to show, $HJ(2, k) \leq k$ and $HJ(2, k) > k - 1$. One direction is a proof, the other is a bad colouring.

Solution.

$H(2, k) > k - 1$ (Bad Colouring): Colour x by the number of ones in $\{0, 1\}^{k-1}$ then cannot have any combinatorial line since no adjacent x, y are the same colour.

$HJ(2, k) \leq k$ (Proof): If we can find $k + 1$ things, any two of which form a combinatorial line, then we are done. Consider the set

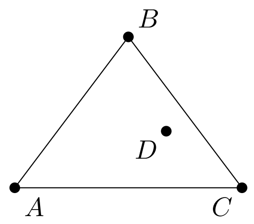
$$\{(0, 0, ..., 0), (0, 0, ..., 1), (0, 0, ..., 1, 1), (0, ..., 1, 1, 1), (1, 1, ..., 1)\}$$

Any two of these form a combinatorial line.

□

> **Question** (Question 3)**.** Let A be an infinite subset of the plane, with no three points collinear. Prove that A contains a set of 2019 points forming a convex 2019-gon.

Solution 1. 2-colour $A^{(4)}$ by colouring a 4-set CONVEX if the points form a convex 4-gon. Else colour them NON-CONVEX.
Ramsey's theorem for r-sets implies that there exists an infinite monochromatic subset of $A^{(4)}$. Suppose this is NON-CONVEX, then it must be of the following form:



Adding a 5th point anywhere leads to a convex 4-gon, this can be shown rigorously by splitting the triangle into six regions. Thus there is an infinite CONVEX subset of $A^{(4)}$. Then we are done since we can take any 2019 points and they must form a convex 2019-gon (Suppose not, then there exists a NON-CONVEX 4-gon by triangulating any 2019 points in our set) $\square$

Solution 2. WLOG assume A is countable. Rotate the points in A such that no two lie on the same vertical line. This is possible since there are uncountably many rotations and only countably many points. Now the points in A represent a function and we can colour the 3-sets of A HAPPY if they form a smile and UNHAPPY if they form a frown. Then we are done by Ramsey for 3-sets. $\square$

Remark. If we were using the finite versions of Ramsey for r-sets and wanted to find a smaller bound on the minimum n such that we can make this work, the second proof would give a lower bound since it requires one less induction to prove Ramsey for 3-sets vs Ramsey for 4-sets.

> **Question** (Question 4)**.** Let c be a 2-colouring of the finite subsets of $\mathbb{N}$. Must there exist an infinite $M \subset \mathbb{N}$ such that, for each r, the colouring c is constant on $M^{(r)}$?

Idea. We want to find a bad colour. Lets ensure one by one that $1 \notin M$, $2 \notin M$,...

Solution. We ensure that $1 \notin M$ by letting:

$$c(i) = \begin{cases} RED & i = 1 \\ BLUE & i \neq 1 \end{cases}$$

Now we ensure $2 \notin M$ by letting:

$$c(ij) = \begin{cases} RED & 2 \in \{i, j\} \\ BLUE & otherwise \end{cases}$$

We continue this by colouring the r-sets $X \in \mathbb{N}^{(r)}$:

$$c(X) = \begin{cases} RED & r \in X \\ BLUE & otherwise \end{cases}$$

Now suppose there is an infinite monochromatic set M with the desired property, then M cannot be RED else it is just $\{1\}$ which is not infinite, thus M must be BLUE. Then since r-set in $M^{(r)}$ contains r, M must be empty. Contradiction. $\square$

> **Question** (Question 5)**.** Prove that $\{0,1\}^{\mathbb{N}}$ (with the product topology) is compact.

Solution. Need to review this, I am unsure of the proof or really what the question is asking. $\square$

> **Question** (Question 6)**.** By mirroring the proof of van der Waerden's theorem for arithmetic progressions of length 3, show that whenever $\mathbb{N}^2$ is finitely coloured there exist a, b, r such that the set $\{(a,b), (a+r,b), (a,b+r)\}$ is monochromatic. Deduce by a product argument that whenever $\mathbb{N}^2$ is 2-coloured there exist a, b, r such that the square $\{(a,b), (a+r,b), (a,b+r), (a+r,b+r)\}$ is monochromatic. Give an explicit n such that whenever $[n]^2$ is 2-coloured there exists a monochromatic square.

Solution. $\square$

> **Question** (Question 7)**.** Show that for every m there is an n with the following property: whenever n^2 is 2-coloured there exists a monochromatic set M of size at least m satisfying $|M| > \min M$.

Idea. Use a compactness argument to get a contradiction.

Solution. Suppose not. Then by compactness we get a 2-colouring of the naturals with no M s.t $|M| > \min M$. However by Infinite Ramsey there is an infinite monochromatic set, say N. Pick a point in this monochromatic set say m, then pick the next m points in N and let these form the set M. However then $|M| > \min M$. $\square$

Remark.

- The exact same argument provides a proof that $\forall m, k, r\ \exists n$ such that when $[m]^{(r)}$ is k-coloured there is a monochromatic M with $|M| > \min M$
- Compactness arguments like these are a very useful tool.
- As an aside, this is an interesting example of a sentence that is true in Peano Arithmetic (PA) but is not provable in PA. This is the Paris-Harrington theorem.

> **Question** (Question 8)**.** Let A be a subset of $\mathbb{N}$ such that, whenever A is finitely coloured, there is a monochromatic arithmetic progression of length m. Must A contain an arithmetic progression of length $m+1$?

Idea. We want a rich enough A such that whenever we k colour A there exists a monochromatic AP of length 3 yet A has no AP of length 4.

Solution. Don't understand this proof, need to go through it. □

> **Question** (Question 9)**.** Let $\mathbb{N}$ be finitely coloured. Must there exist arbitrarily long monochromatic arithmetic progressions having the same common difference?

Idea. Seems like too much to ask for. Can we find a really weird colouring that doesn't work for any integer periodic difference?

Solution. Colour $\mathbb{N}$ as follows:

$$c(n) = \begin{cases} RED & \frac{x}{\sqrt{2}} \mod 1 > \frac{1}{2} \\ BLUE & otherwise \end{cases}$$

Then since $\sqrt{2}$ is irrational there is no periodic nature in the integers, so this colouring suffices. Is there a way to make this more rigorous? □

Remark.
- Nothing special about $\sqrt{2}$ here, anything irrational would have done fine e.g π.
- Could have also found other colouring, like ones involving colouring based on the number of ones. For an example of this, see the Thue-Morse sequence (This is a good example of a sequence that ruins a lot of properties and such examples are useful to know).

> **Question** (Question 10 +)**.** Let A be an uncountable set and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

> **Question** (Question 11 +)**.** Let c be a colouring of $\mathbb{N}$ using (possibly) infinitely many colours. Prove that, for every m, there is an arithmetic progression of length m on which c is either constant or injective.