Overlaying a Velocity Profile

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Given a path, we would like to turn it into a trajectory that satisfies kinematic constraints in velocity, acceleration, and jerk. The approach here follows [1] and creates up to 7 segments for an S-curve [2] profile, as shown in Figure 1. This formulation assumes that kinematic constraints $v_{\max}^{\text{user}}$, $a_{\max}$, and $j_{\max}$ are symmetric, i.e., $*_{\min} = -*_{\max}$. Further, we assume that the robots starts at $v = v_0 \leq v_{\max}^{\text{user}}$ and $a = a_0 \leq a_{\max}$.

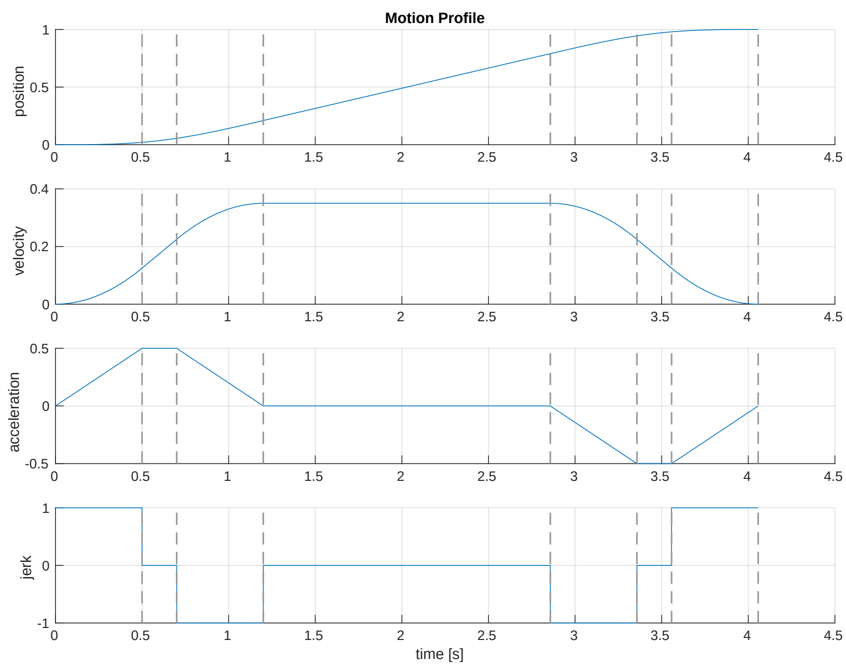

Figure 1: The 7 segments of the produced velocity profile. In the case that the path is short enough, or $v_{\max}$ is large enough, the fourth segment (cruise at $v_{\max}$) is omitted.

1 Kinematic Segments

Here we derive the kinematics of each segment. Note that the value of $v_{\max}$ is distinct from $v_{\max}^{\text{user}}$ provided by the user and must be assigned a value before the kinematic segments are calculated. The calculation of $v_{\max}$ is left for Section 2.

1.1 Maximum Jerk

The amount of time needed to reach $a_{\max}$ from a_0 is

$$\Delta t_1 = \frac{a_{\max} - a_0}{j_{\max}} \tag{1}$$

The path length traversed during this segment and the final velocity at the end of this segment are

$$\Delta s_1 = v_0 \Delta t_1 + \frac{1}{2} a_0 \Delta t_1^2 + \frac{1}{6} j_{\max} \Delta t_1^3 \tag{2}$$

$$v_1 = v_0 + a_0 \Delta t_1 + \frac{1}{2} j_{\max} \Delta t_1^2 \tag{3}$$

1.2 Maximum Acceleration

The robot continues at maximum acceleration until it needs to begin slowing down to hit $v_{\max}$, which depends on the parameters of segment 3 (specifically $j_{\min}$). We can calculate the velocity at the end of segment 2 by subtracting the velocity change during segment 3 from the maximum attainable velocity,

$$v_2 = v_{\max} - \Delta v_3. \tag{4}$$

Then, the amount of time spent in section 2 and the path length traversed are given as

$$\Delta t_2 = \frac{v_2 - v_1}{a_{\max}} \tag{5}$$

$$\Delta s_2 = v_1 \Delta t_2 + \frac{1}{2} a_{\max} \Delta t_2^2. \tag{6}$$

1.3 Minimum Jerk, part 1

During this segment, minimum jerk is applied to null out the acceleration, leading up to $v_{\max}$ and a potential cruise segment. The amount of time needed to null out the acceleration from $a_{\max}$ is

$$\Delta t_3 = \frac{0 - a_{\max}}{j_{\min}}. \tag{7}$$

We then calculate the velocity change during this segment by considering this segment in reverse: what would the "final" velocity be "starting" from $v = v_{\max}$, $a = 0$ and applying $j_{\min}$ for t_3 seconds? Notice that this value is related to v_2 as

$$\begin{aligned} \Delta v_3 &= -\frac{1}{2} j_{\min} \Delta t_3^2 \\ &= v_{3,\mathrm{f}} - v_{3,\mathrm{i}} = v_{\max} - v_2. \end{aligned} \tag{8}$$

The path length traversed is

$$\Delta s_3 = v_2 \Delta t_3 + \frac{1}{2} a_{\max} \Delta t_3^2 + \frac{1}{6} j_{\min} \Delta t_3^3. \tag{9}$$

1.4 Cruise

If there is a cruise segment, its parameters can be easily found using the sum of each other segment's path length. Since it is a constant velocity segment, it is simple to calculate the time spent in this segment.

$$\Delta s_4 = s_{\text{total}} - \sum_{i \neq 4} \Delta s_i \tag{10}$$

$$\Delta t_4 = \frac{\Delta s_4}{v_{\max}} \tag{11}$$

Determining if there is a cruise segment is as simple as checking if $\sum_{i \neq 4} \Delta s_i < s_{\text{total}}$. If there is no cruise section, these values are set to zero.

1.5 Minimum Jerk, part 2

Here, regardless of there being a cruise section or not, the robot applies minimum jerk until maximum decelaration is achieved. The amount of time needed for this (from $a = 0$) is

$$\Delta t_5 = \frac{a_{\min} - 0}{j_{\min}}. \tag{12}$$

The path length traversed and final velocity are

$$\Delta s_5 = v_{\max}\Delta t_5 + \frac{1}{6} j_{\min} t_5^3, \tag{13}$$

$$v_5 = v_{\max} + \frac{1}{2} j_{\min} \Delta t_5^2. \tag{14}$$

1.6 Minimum Acceleration

The robot continues to decelerate at $a_{\min}$ until it needs to apply maximum jerk to hit the terminal constraint of zero velocity and acceleration (similar to segment 2). The velocity at the end of this segment is calculated by subtracting the change in velocity during sectsegment 7 from the terminal velocity

$$v_6 = 0 - \Delta v_7. \tag{15}$$

Then, the amount of time spent in segment 6 and the path length traversed are given as

$$\Delta t_6 = \frac{v_6 - v_5}{a_{\min}} \tag{16}$$

$$\Delta s_6 = v_5 \Delta t_6 + \frac{1}{2} a_{\min} \Delta t_6^2. \tag{17}$$

1.7 Maximum Jerk

To come to rest, a final maximum jerk command is required. The amount of time required to null out the acceleration from $a_{\min}$ is

$$\Delta t_7 = \frac{0 - a_{\min}}{j_{\max}}. \tag{18}$$

Similarly to segment 3, we consider this segment in reverse to calculate the change in velocity over this segment, which is related to v_6,

$$\begin{align} \Delta v_7 &= -\frac{1}{2} j_{\max} \Delta t_7^2 \tag{19} \\ &= v_{7,\mathrm{f}} - v_{7,\mathrm{i}} = 0 - v_6. \tag{20} \end{align}$$

The path length traversed is

$$\Delta s_7 = v_6 \Delta t_7 + \frac{1}{2} a_{\min} \Delta t_7^2 + \frac{1}{6} j_{\max} \Delta t_7^3. \tag{21}$$

2 Calculating Maximum Velocity

In implementation, before all of the kinematic segments are created, we must first ascertain if there is a cruise segment. The existence of a cruise segment depends on the maximum allowable velocity. We first consider what the maximum *attainable* velocity would be under the acceleration and jerk constraints. In this case, note that the time-optimal jerk control would be some sort of bang-bang control and there would be no cruise period. Thus, to determine what this maximum attainable velocity $v_{\max}$ is, we sum up the remaining segment lengths (excluding Δs_4) and determine what $v_{\max}$ must be for the summed segment length to be equal to the input segment length, s_{total}.

First, we express each Δs_i, $1 < i < 7$ in terms of known quantities and the unknown maximum attainable velocity, $v_{\max}$. To simplify notation, we recall that constraints are symmetric, allowing us to express everything in terms of maximum limits. Further, we drop the $\cdot_{\max}$ subscript for $a_{\max}$ and $j_{\max}$.

$$\Delta s_2 = v_1 \Delta t_2 + \frac{1}{2} a_{\max} \Delta t_2^2 \tag{22}$$

$$= v_1 \left(\frac{v_{\max} - \frac{a^2}{2j} - v_1}{a} \right) + \frac{1}{2} a \left(\frac{v_{\max} - \frac{a^2}{2j} - v_1}{a} \right)^2 \tag{23}$$

$$= \frac{v_1}{a} v_{\max} + \frac{v_1}{a} \left(-\frac{a^2}{2j} - v_1 \right) + \frac{1}{2a} \left(v_{\max} - \frac{a^2}{2j} - v_1 \right)^2 \tag{24}$$

$$= \frac{v_1}{a} v_{\max} + \frac{v_1}{a} \left(-\frac{a^2}{2j} - v_1 \right) + \frac{1}{2a} \left(v_{\max}^2 - \frac{a^2}{2j} v_{\max} - v_1 v_{\max} - \frac{a^2}{2j} v_{\max} + \frac{a^4}{4j^2} + \frac{a^2}{2j} v_1 - v_1 v_{\max} + \frac{a^2}{2j} v_1 + v_1^2 \right) \tag{25}$$

$$= \frac{v_1}{a} v_{\max} + \frac{1}{2a} v_{\max}^2 - \frac{a}{4j} v_{\max} - \frac{v_1}{2a} v_{\max} - \frac{a}{4j} v_{\max} - \frac{v_1}{2a} v_{\max} + \frac{1}{2a} \left(\frac{a^4}{4j^2} + \frac{a^2}{2j} v_1 + \frac{a^2}{2j} v_1 + v_1^2 \right) + \frac{v_1}{a} \left(-\frac{a^2}{2j} - v_1 \right) \tag{26}$$

$$= \frac{1}{2a} v_{\max}^2 - \frac{a}{2j} v_{\max} + \frac{1}{2a} \left(\frac{a^4}{4j^2} + \frac{a^2}{2j} v_1 + \frac{a^2}{2j} v_1 + v_1^2 \right) + \frac{v_1}{a} \left(-\frac{a^2}{2j} - v_1 \right) \tag{27}$$

$$= \frac{1}{2a} v_{\max}^2 - \frac{a}{2j} v_{\max} + \frac{a^3}{8j^2} - \frac{v_1^2}{2a} \tag{28}$$

$$\Delta s_3 = v_2 \left(\frac{a}{j} \right) + \frac{1}{2} a \left(\frac{a}{j} \right)^2 + \frac{1}{6} j_{\min} \left(\frac{a}{j} \right)^3 \tag{29}$$

$$= \left(v_{\max} - \frac{a^2}{2j} \right) \left(\frac{a}{j} \right) + \frac{1}{2} a \left(\frac{a}{j} \right)^2 - \frac{1}{6} j \left(\frac{a}{j} \right)^3 \tag{30}$$

$$= \frac{a}{j} v_{\max} - \frac{a^3}{6j^2} \tag{31}$$

$$\Delta s_5 = v_{\max} \Delta t_5 + \frac{1}{6} j_{\min} t_5^3 \tag{32}$$

$$= v_{\max} \left(\frac{a}{j} \right) - \frac{1}{6} j \left(\frac{a}{j} \right)^3 \tag{33}$$

$$= \frac{a}{j} v_{\max} - \frac{a^3}{6j^2} \tag{34}$$

$$\Delta s_6 = v_5 \Delta t_6 + \frac{1}{2} a_{\min} \Delta t_6^2 \tag{35}$$

$$= \frac{1}{a} \left[\frac{1}{2} v_{\max}^2 - v_{\max} \frac{a^2}{2j} \right] \tag{36}$$

$$= \frac{1}{2a} v_{\max}^2 - \frac{a}{2j} v_{\max} \tag{37}$$

The sum of these segments (which all depend on $v_{\max}$) is given by

$$\Delta s_2 + \Delta s_3 + \Delta s_5 + \Delta s_6 = \frac{1}{2a}v_{\max}^2 - \frac{a}{2j}v_{\max} + \frac{a^3}{8j^2} - \frac{v_1^2}{2a} + \frac{a}{j}v_{\max} - \frac{a^3}{6j^2} + \frac{a}{j}v_{\max} - \frac{a^3}{6j^2} + \frac{1}{2a}v_{\max}^2 - \frac{a}{2j}v_{\max} \tag{38}$$

$$= \frac{1}{a}v_{\max}^2 + \left(\frac{a}{j}\right)v_{\max} + \frac{a^3}{8j^2} - \frac{v_1^2}{2a} - \frac{a^3}{3j^2} \tag{39}$$

$$= \frac{1}{a}v_{\max}^2 + \left(\frac{a}{j}\right)v_{\max} + \left(-\frac{5a^3}{24j^2} - \frac{v_1^2}{2a}\right). \tag{40}$$

Then, taking the difference between the known path length and calculated path length with unknown $v_{\max}$, we have

$$s_{\text{total}} - (\Delta s_1 + \Delta s_2 + \Delta s_3 + \Delta s_5 + \Delta s_6 + \Delta s_7) = 0 \tag{41}$$

$$s_{\text{total}} - \Delta s_1 - (\Delta s_2 + \Delta s_3 + \Delta s_5 + \Delta s_6) - \Delta s_7 = 0 \tag{42}$$

$$0 = \Delta s_1 + \Delta s_7 - s_{\text{total}} + (\Delta s_2 + \Delta s_3 + \Delta s_5 + \Delta s_6) \tag{43}$$

$$0 = \Delta s_1 + \Delta s_7 - s_{\text{total}} + \frac{1}{a}v_{\max}^2 + \left(\frac{a}{j}\right)v_{\max} + \left(-\frac{5a^3}{24j^2} - \frac{v_1^2}{2a}\right) \tag{44}$$

$$= \frac{1}{a}v_{\max}^2 + \left(\frac{a}{j}\right)v_{\max} + \left(\Delta s_1 + \Delta s_7 - s_{\text{total}} - \frac{5a^3}{24j^2} - \frac{v_1^2}{2a}\right), \tag{45}$$

which is quadratic in $v_{\max}$. Note that this differs from the quadratic equation used in the implementation of [1]:

$$0 = \frac{1}{a}v_{\max}^2 + \left(\frac{3}{2}\frac{a}{j} + \frac{v_1}{a} - \frac{\frac{a^2}{j} - v_1}{a}\right)v_{\max} + \left(\Delta s_1 + \Delta s_7 - s_{\text{total}} - \frac{7}{3}\frac{a^3}{j^2} - v_1\left(\frac{a}{j} + \frac{v_1}{a}\right) + \frac{1}{2a}\left(\frac{a^2}{j} + \frac{v_1}{a}\right)^2\right) \tag{46}$$

$$= \frac{1}{a}v_{\max}^2 + \left(\frac{1}{2}\frac{a}{j}\right)v_{\max} + \left(\Delta s_1 + \Delta s_7 - s_{\text{total}} - \frac{7}{3}\frac{a^3}{j^2} - \frac{av_1}{j} - \frac{v_1^2}{a} + \frac{a^3}{2j^2} + \frac{v_1}{j} + \frac{v_1^2}{2a^3}\right). \tag{47}$$

If $v_{\max} > v_{\max}^{\text{user}}$, then we limit $v_{\max} := v_{\max}^{\text{user}}$ and there will be a cruise section.

References

[1] https://jwdinius.github.io/blog/2018/eta3traj

[2] https://www.pmdcorp.com/resources/type/articles/get/mathematics-of-motion-control-profiles-article