Stellar Dynamics of Cold Collapse, N-Body Simulation

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Introduction to the N-body problem

N-body problem

Given the initial positions and velocities of N point masses m_i, $i = 1, 2, \ldots, N$ (in an inertial reference frame) which interact gravitationally, determine their trajectories at all future times.

Introduction to the N-body problem

- $N = 2$ (Kepler's problem): 2 bodies move in the same plane in conic sections around their barycenter (Bernoulli,1734).
- $N = 3$: there is no general analytical solution, and the movement is generally non-planar, non-periodic, and chaotic (Poincaré, 1887).
- N large (e.g galaxies, 10^{12} stars): an explicit solution is impossible, we look for a numerical approximation. A **N-body simulation** gives useful information about the *macroscopic* evolution of the system.

Objectives

- **Objective 1**: construct a simulator for $N > 1000$ bodies with a good runtime-accuracy compromise
- **Objective 2**: modeling of the “cold collapse" of a globular cluster, study of stellar dynamics: evolution towards equilibrium, core collapse, energy exchange via binaries, and evaporation.

Some general properties of an N-body system

From the law of gravitation, $\forall i \in [|1, N|]$:

$$\ddot{\vec{r}_i} = \sum_{i=1, j \neq i}^{N} -Gm_j \frac{\vec{r}_i - \vec{r}_j}{||\vec{r}_i - \vec{r}_j||^3}$$

Potential energy of the system:

$$U = \sum_{1 \leq i < j \leq N} -\frac{Gm_i m_j}{||\vec{r}_i - \vec{r}_j||}$$

Kinetic energy of the system:

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i ||\vec{v}_i||^2$$

For an isolated system, $E = U + K$ is conserved, as well as the momentum of angular momentum.

Simulator construction: code organization

Language: Python.
GUI: Pygame.

For proof of concept: 2D systems. Latest results on 3D systems not discussed in this presentations.

Code structure:

- Initialization
- Main loop
    1. integration, updating parameters of particle objects
    2. simtime increases one step
    3. data collection
    4. display
- Generation of plots

Simulator construction: choice of integrator

Euler (order 1) gives divergent solutions, even for small time steps.

Other possibilities: Runge-Kutta, variable step-size methods, etc

We opt for Verlet's algorithm:

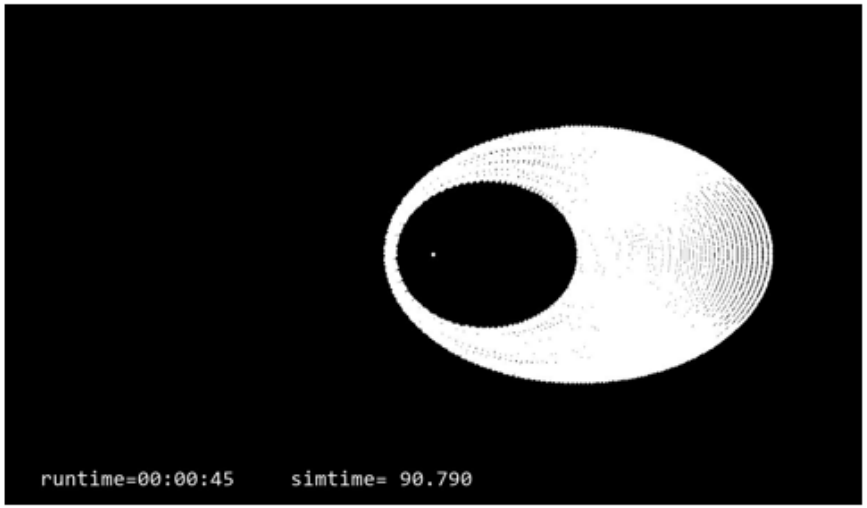
$$\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{v}(t)\,\Delta t + \frac{1}{2}\,\vec{a}(t)\Delta t^2$$

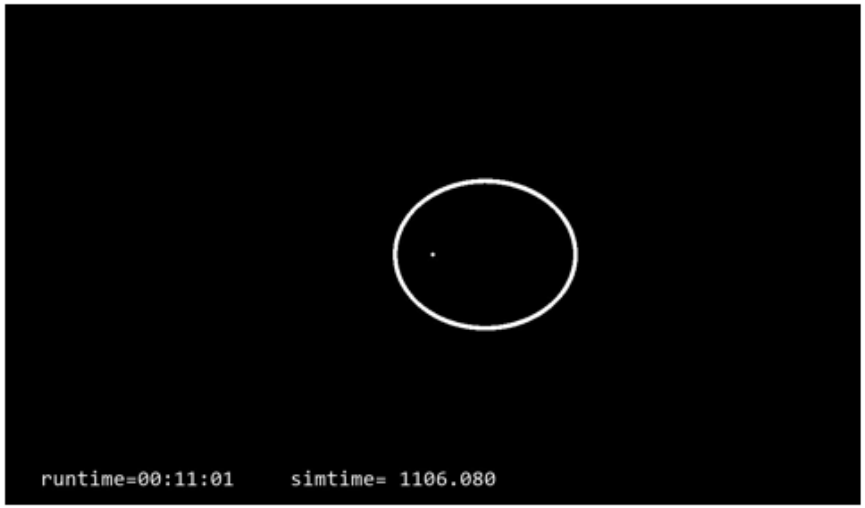
$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{a}(t) + \vec{a}(t+\Delta t)}{2}\Delta t$$

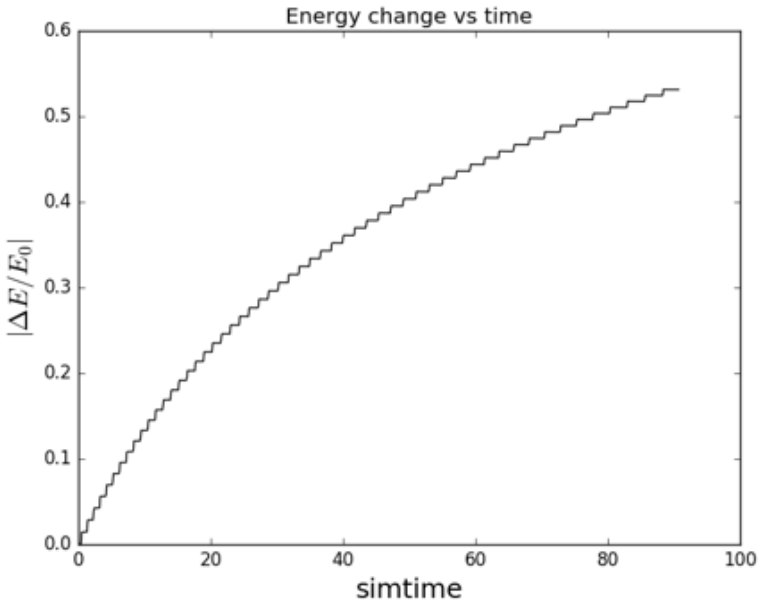
Verlet is second order, time-reversible, and minimizes energy drift.

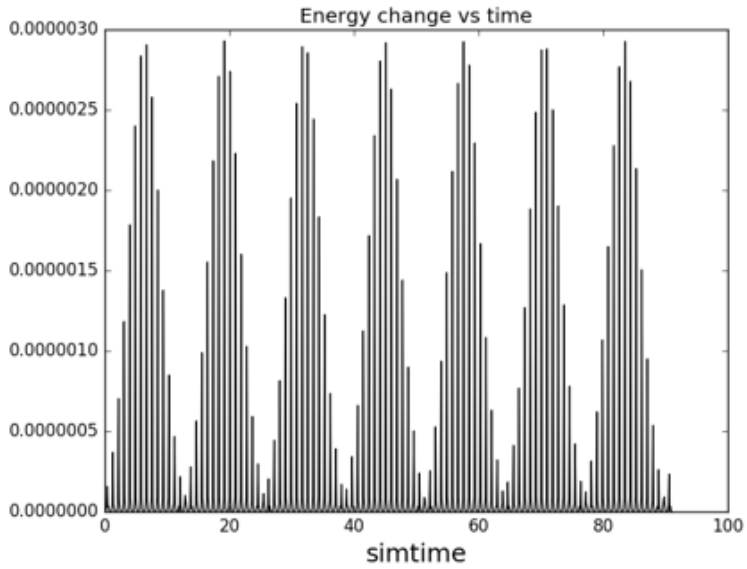
Simulator construction: choice of integrator

Euler vs Verlet integration for simple orbit:









When the distance between two bodies goes to zero, the force between them goes to infinity, introducing numerical errors.

We use the approximation

$$|F_{i,j}| = \frac{G m_i m_j}{r^2 + \epsilon}$$

where ϵ is small and chosen to minimize energy drift.

Simulator construction: collisions

In general we do not take into account collisions, although we can study (very naively) how lead to stable "solar systems".

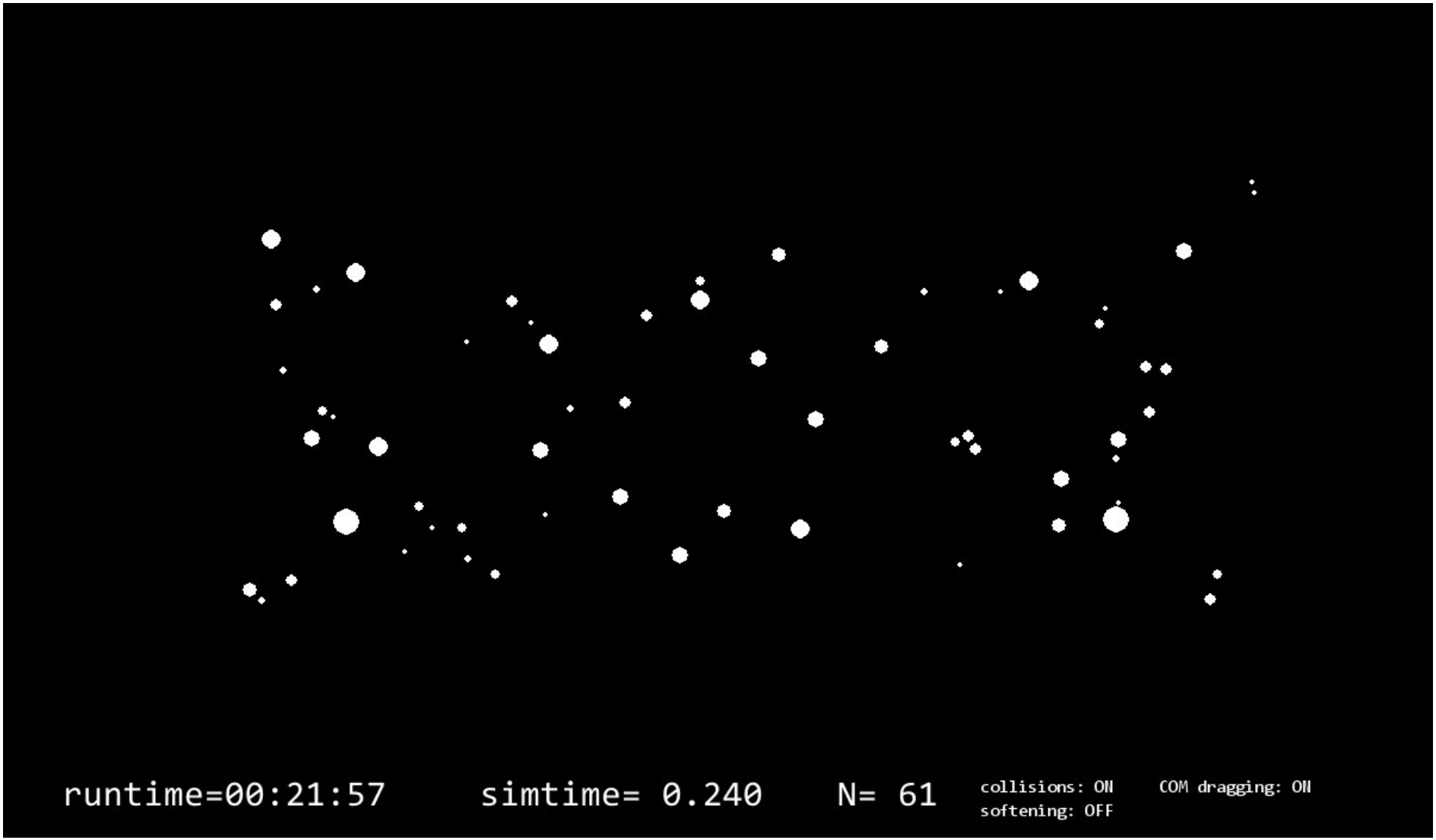
Experiment: simulation of solar system formation

- all the bodies are spherical of the same uniform density
- if two bodies come into contact, they merge
- $N_0 = 200$, particle masses, initial positions and velocities initialized at random.

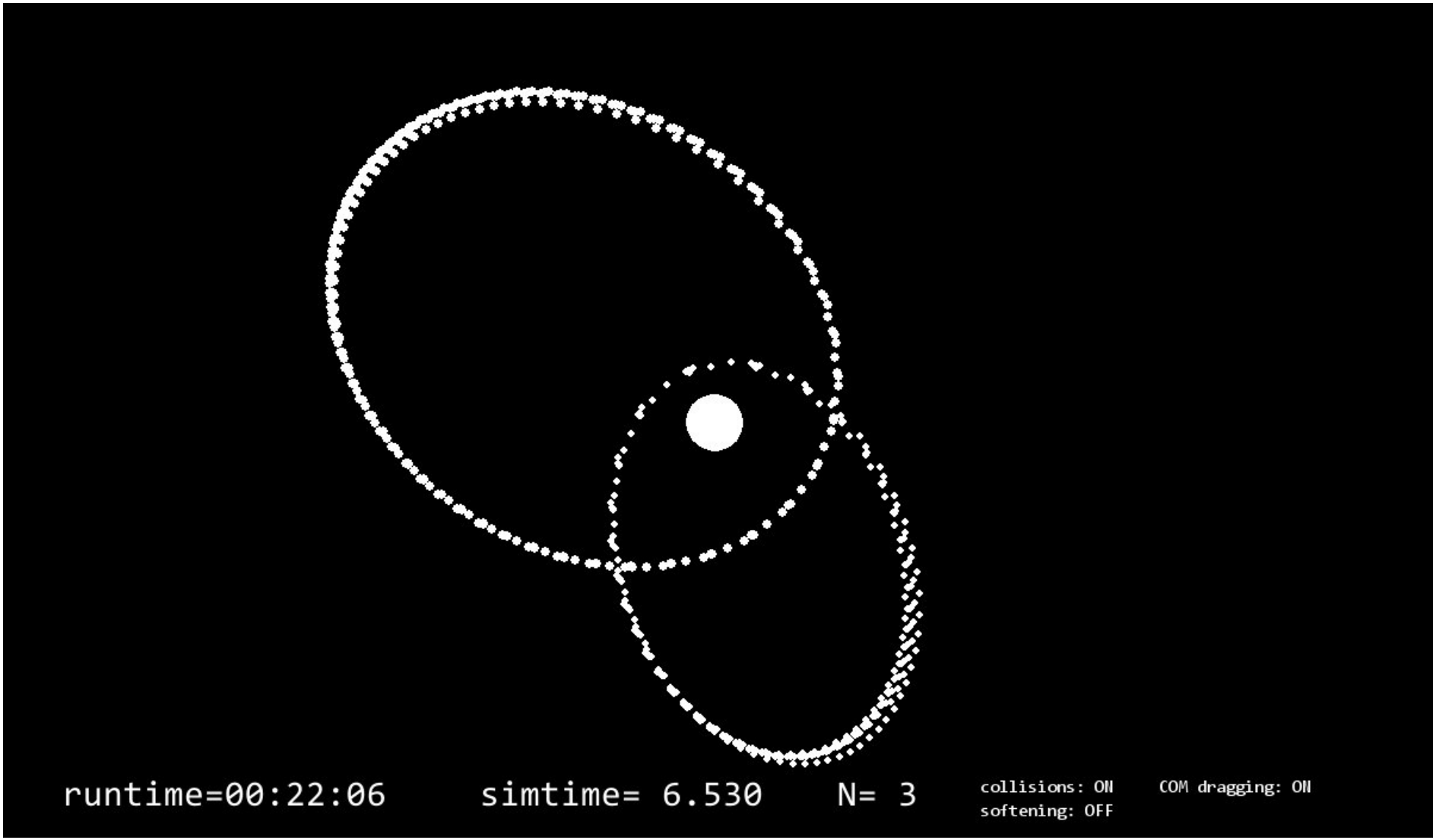
Results: (100 runs)

- In 70 runs, the system is rapidly reduced to a single mass
- In the remaining runs, formation of a "solar system" with between 1 and 5 planets
- In five runs, formation of sun-planet-moon system

Simulator construction: collisions



Simulator construction: collisions



Simulator construction: Barnes-Hut optimization

A direct approach requires $\binom{N}{2}$ calculations per time step, complexity is $O(N^2)$. For $N > 100$ we implement the **Barnes-Hut** algorithm, $O(N \ln(N))$, with a minimal loss of precision.

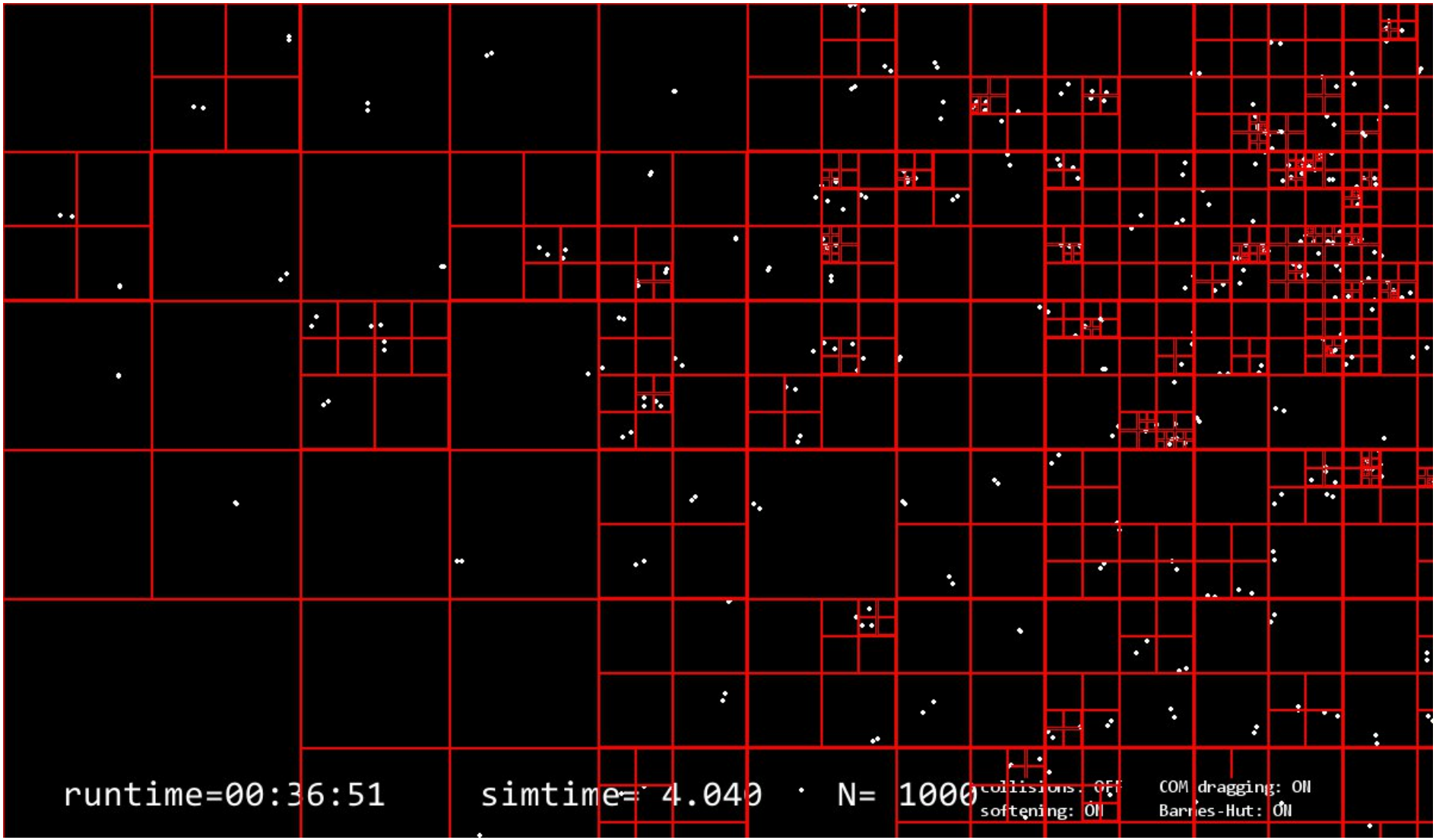
Principle: approximate the force exerted on a particle by a group of bodies close in space by treating them as one body.

The bodies are stored in a quadtree. Each node represents a region of 2D space.

The root represents the whole space, its 4 children the 4 quadrants of this space.

The space is recursively subdivided into quadrants until each subdivision contains 0 or 1 bodies.

Simulator construction: Barnes-Hut optimization



Simulator construction: Barnes-Hut optimization

Each external node represents a single body. Each internal node represents the group of bodies below and stores the barycenter of total mass of each of the bodies of its children.

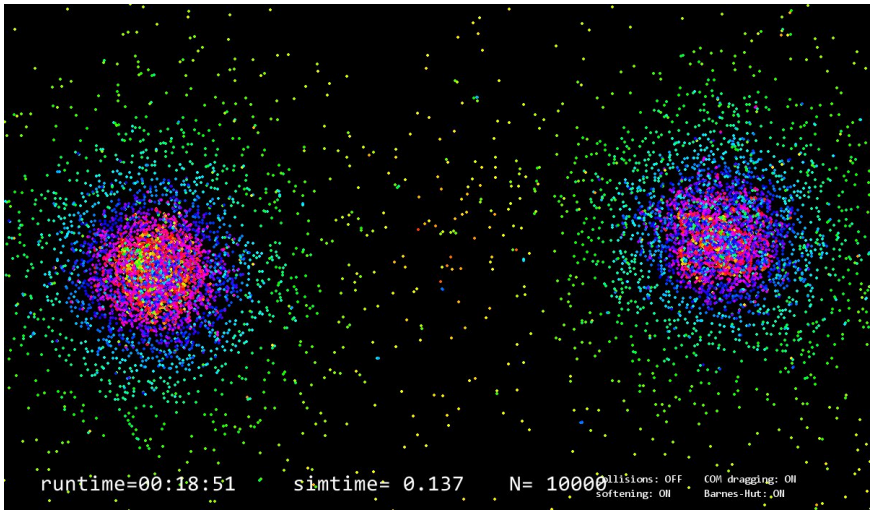
Calculation of force on a body m_i: Iterate through the nodes of tree, starting at the root

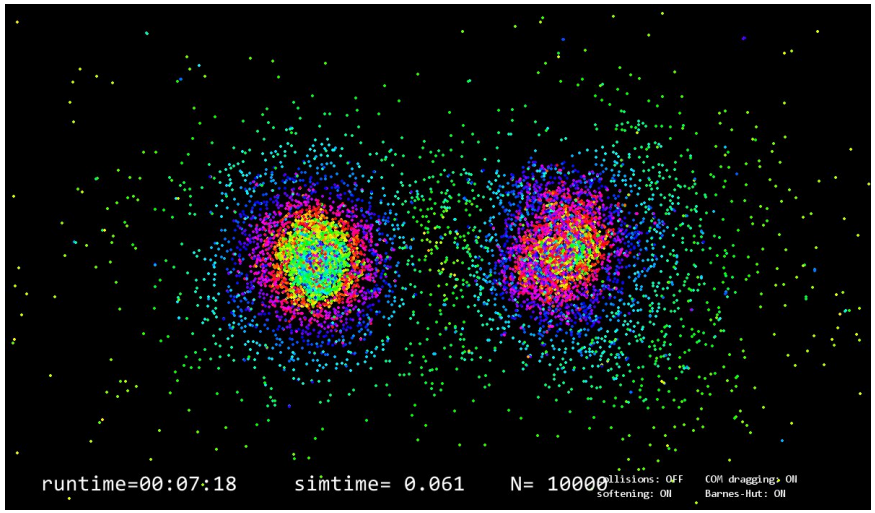
- If the barycenter of an internal node is sufficiently far from m_i (*), treat the bodies contained in this part of the tree as a single body.
- Otherwise, iterate recursively through each of its subtrees

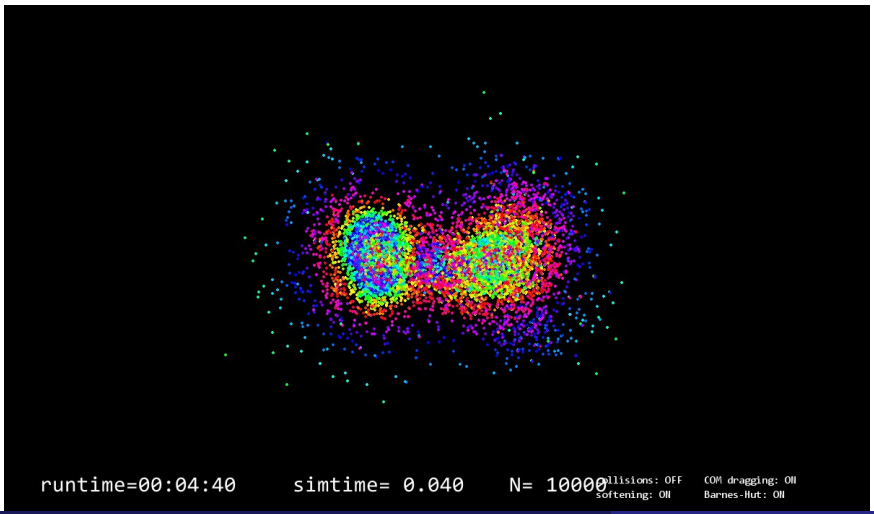
(*)If the angular diameter $s/d < \theta$, where s =size of the region represented by the node, d =distance between m_i and the node's barycenter.

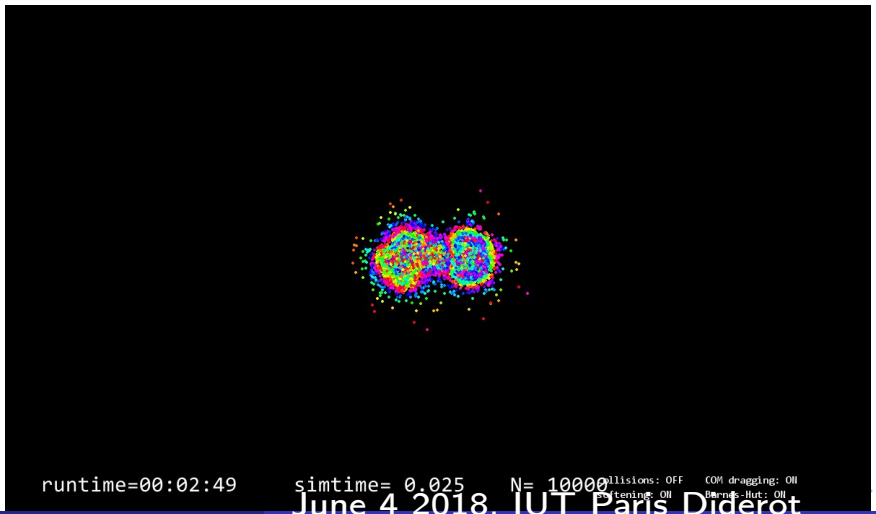
Simulator construction: Barnes-Hut optimization

$N = 10^4$ “naive" simulation of collision of two galaxies (zero initial angular momentum and linear momentum)

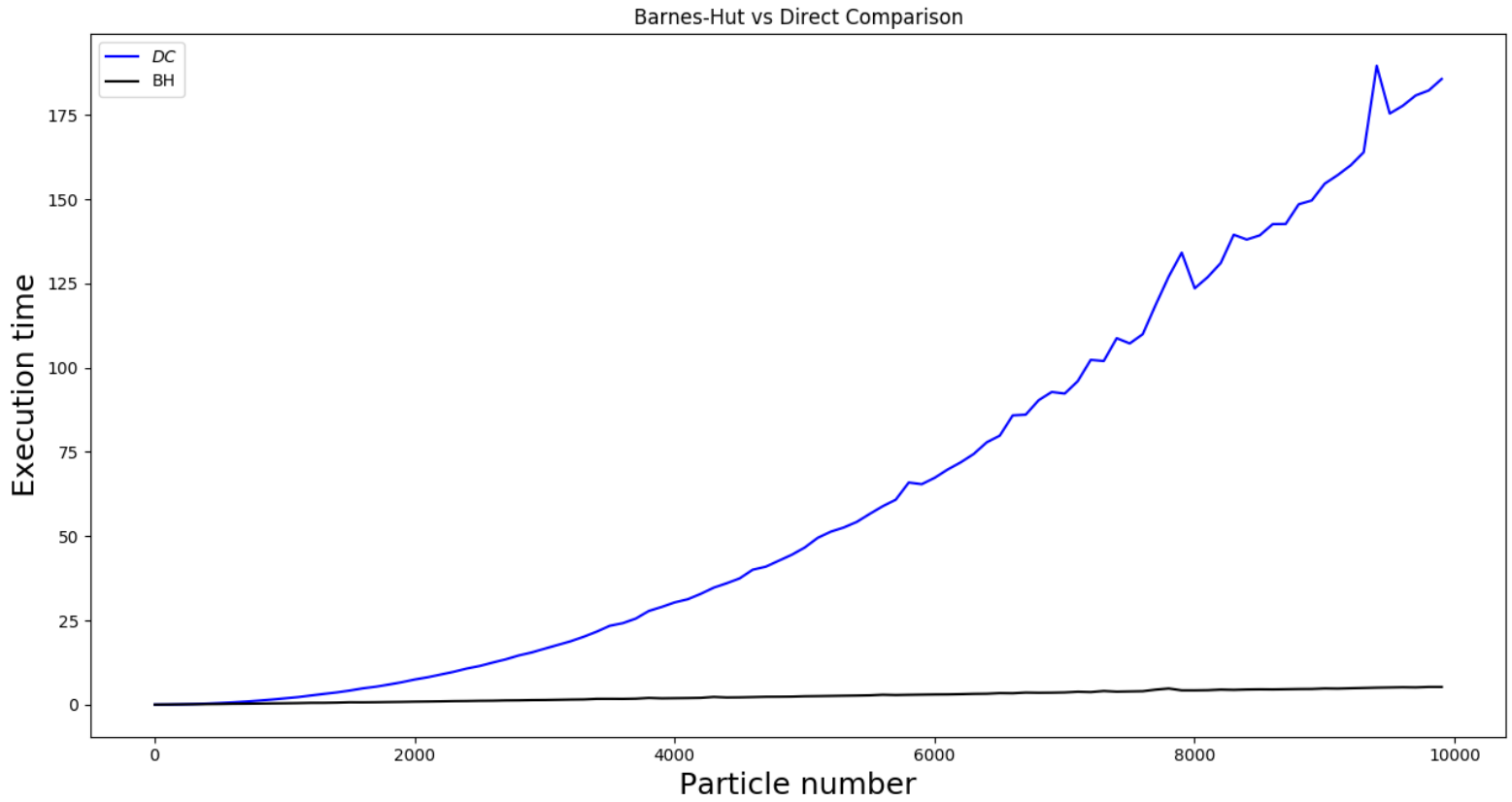








Simulator construction: Barnes-Hut optimization



Stellar dynamics: virial theorem, relaxation

Virial theorem

A *bound* N-body system in equilibrium satisfies: $2 < K > + < U >= 0$.

Thus when the system tends towards equilibrium ("relaxation"), $Q = -K/U \rightarrow 0.5$.

Mechanism of relaxation in a globular cluster:

- Close encounters between two stars, $\delta v \approx v$ (strong, rare)
- Long range interactions, $b >> r_s$ (weak, frequent)

We can show that the long range interactions have a greater effect on the movement of a star in the cluster.

Stellar dynamics: relaxation time

relaxation time (t_r) - the average time for the component $v_\perp$ of a star's velocity to be of the same magnitude as its original velocity

Intuitively, the time for a star to lose memory of its original trajectory.

For a system of numerical density n in which all the bodies have equal mass m and average speed V, a probabilistic calculation shows:

$$t_r = \frac{V^3}{8\pi G^2 m^2 n \ln(b_{max}/b_{min})}$$

Example of a globular cluster: $V = 10\ kms^{-1}$, $n = 10^4 pc^{-3}$, $m = M_s$, $t_r = 10^8 years$

Stellar dynamics: consequences of relaxation

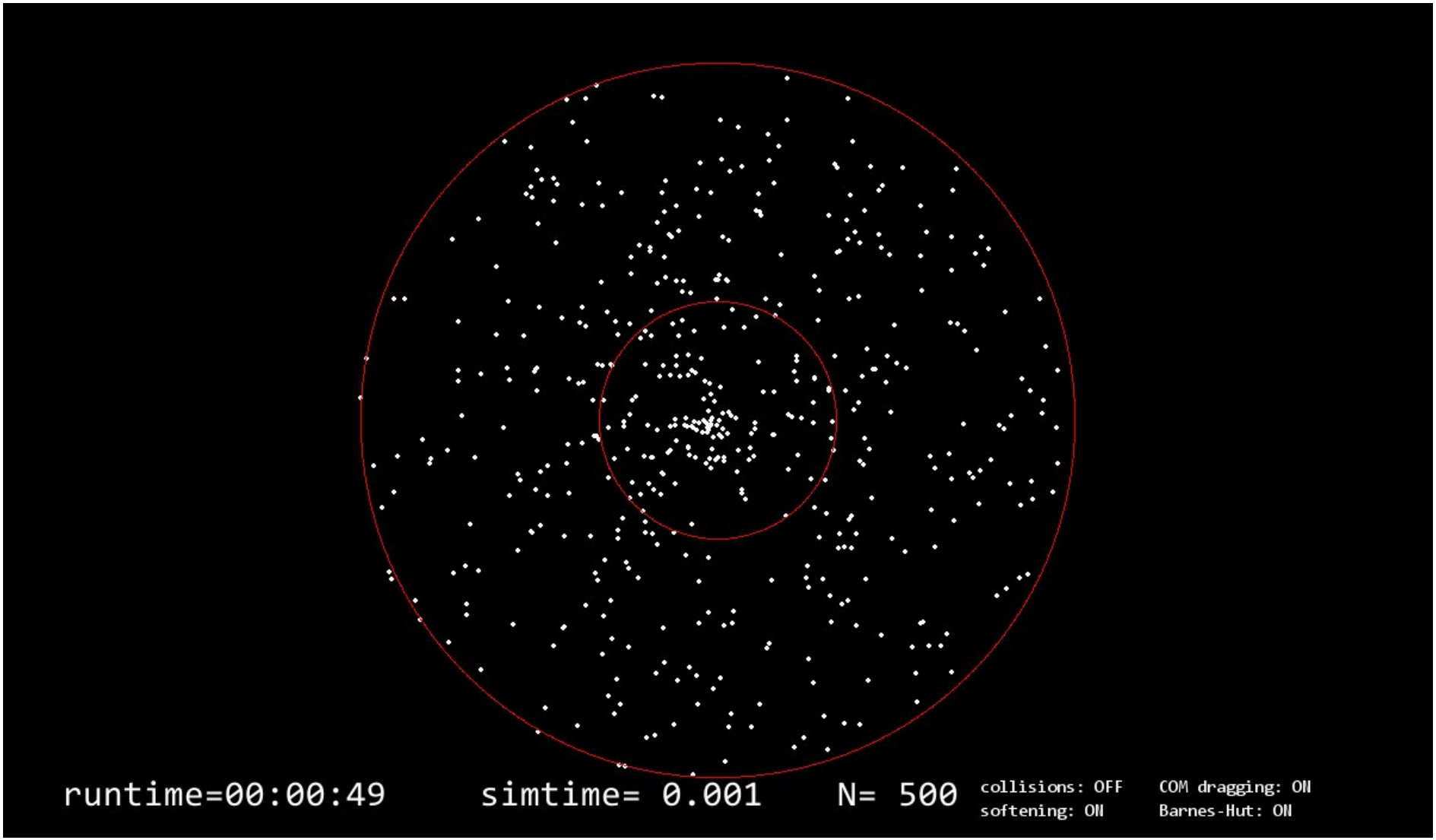
Relaxation creates “conduction" of energy from hot regions (where speeds are high, i.e in the core) towards colder regions (i.e edge)

The stars of the core are cooled down, which makes them fall closer to the center and accelerate; the stars at the edge are heated up, driving them away from the enter of slowing them down.

Thus, the temperature gradient increases an the core becomes more compact (time scale=t_r)

Evaporation: relaxation allows energy exchange between stars. If the total energy of a star becomes positive, it can leave the cluster ($t_{evap} \approx 100 t_r$)

Simulation of cold collapse of a GC



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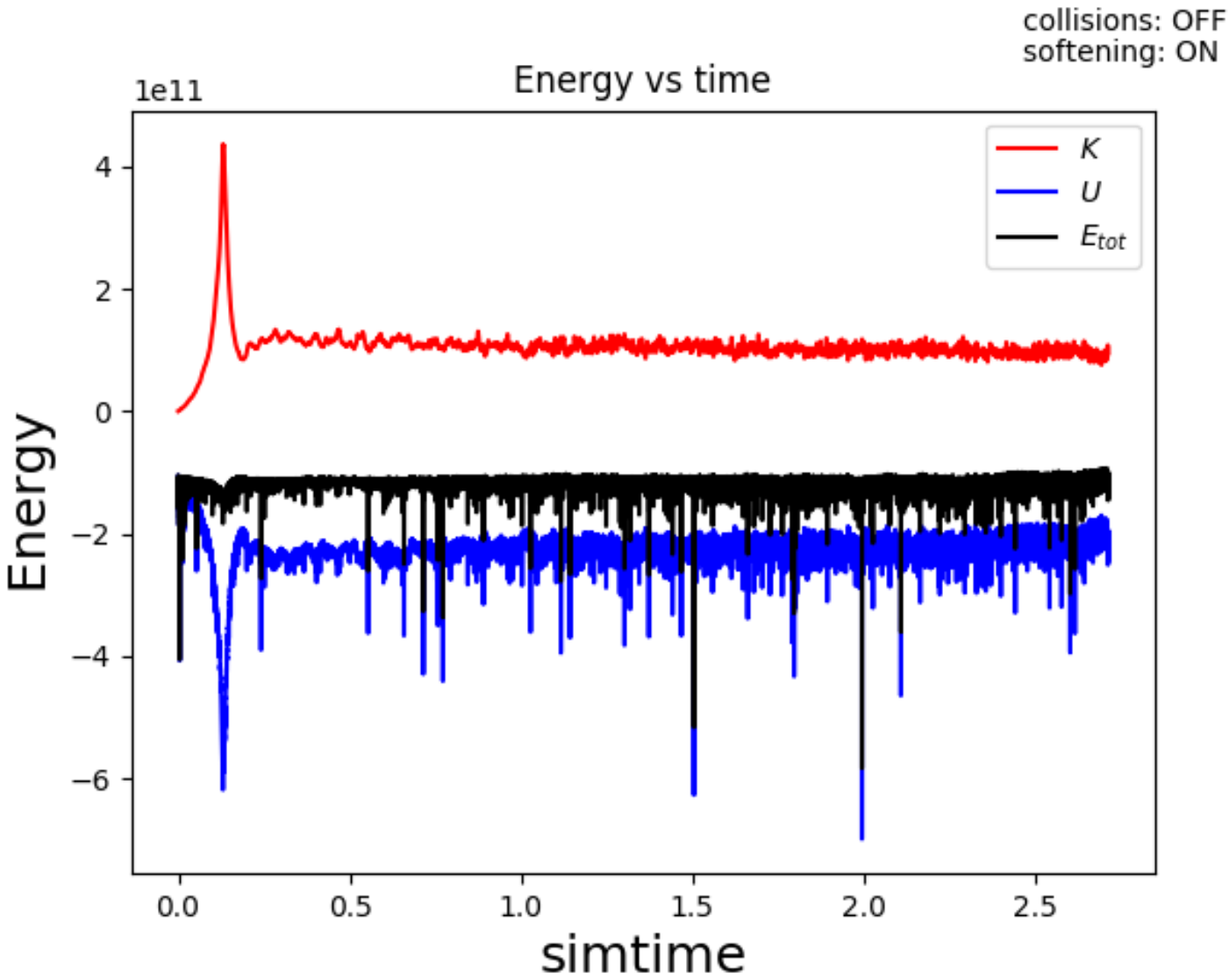

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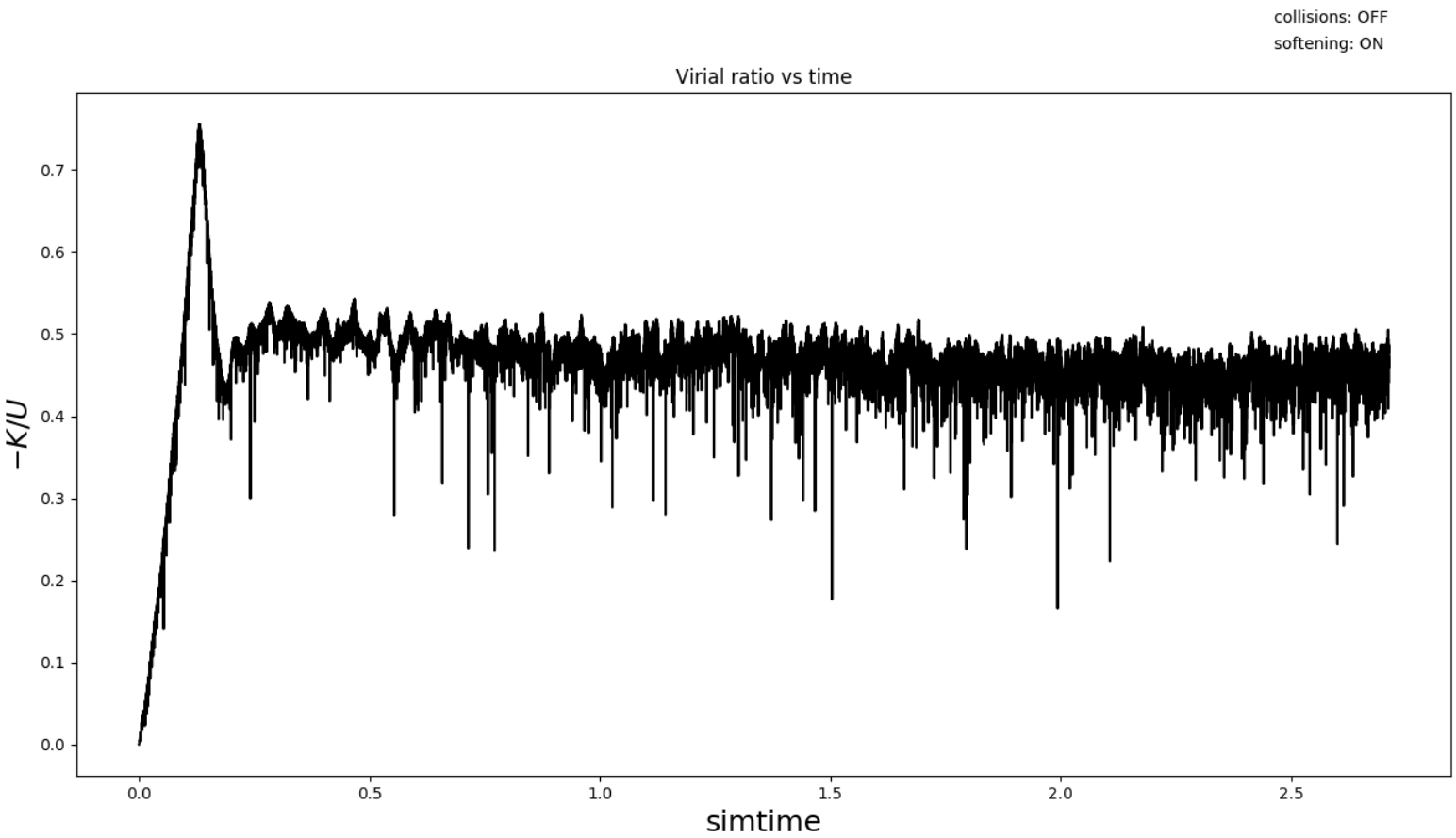
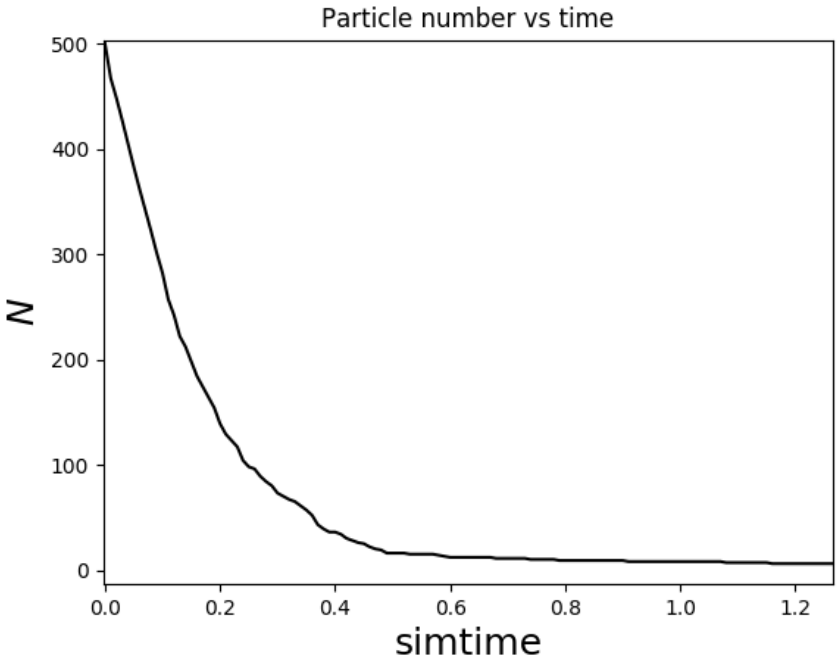
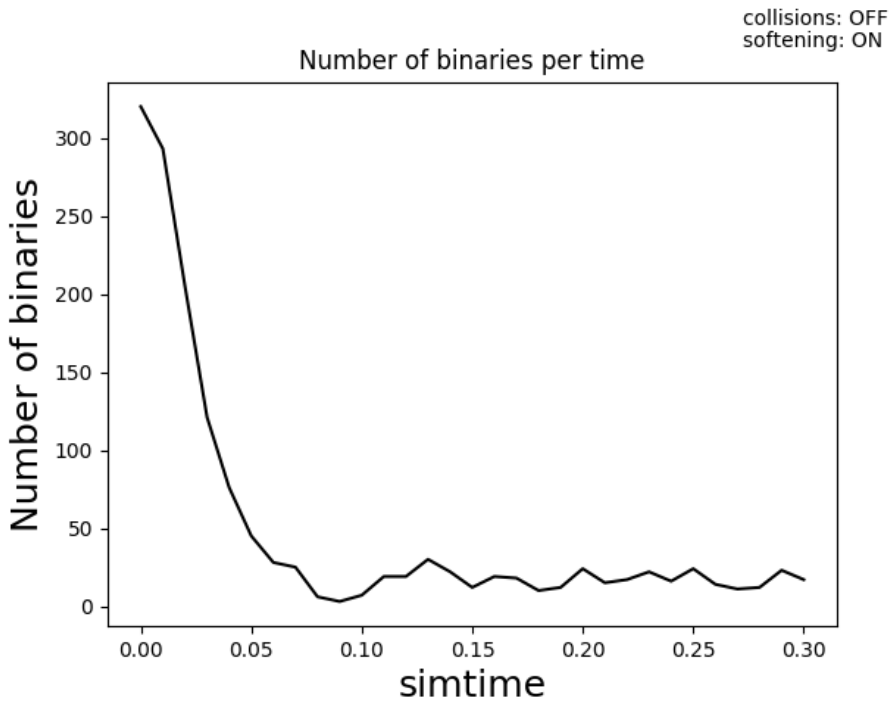


time= 34h, $< \Delta E/E_0 > = 0.072$, t_r **predicted** ≈ 1.3

Simulation of cold collapse of a GC





Future work

Several avenues that may be readily explored:

- Recent 3D results not discussed in this presentation. Preliminary results are in agreement with theory of cold collapse ([4]). Scaling laws also appear to be verified. $\Longrightarrow$ Verify with improved statistics.

- Without complicating our simulation we can already investigate questions of statistical validity raised in Ref. [1] and Ref. [5]

- We also don't need any new tools to explore "choreographies" for the 3 body problem discussed in Ref [2], [3].

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