Positive Mass Theorem

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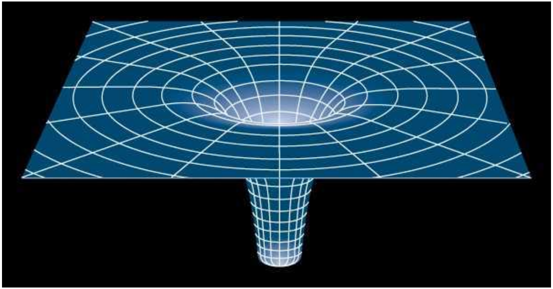
March 9, 2021

Introduction: Positive mass theorem

The Riemannian positive mass theorem (PMT): an asymptotically flat Riemannian manifold M^n with nonnegative scalar curvature has nonnegative ADM mass. The ADM mass is strictly positive unless M^n is isometric to flat $\mathbb{R}^n$.



First proven in 1979 by Schoen and Yau for $n \leq 7$ using minimal surface techniques.

Introduction: Penrose Inequality

Penrose inequality - a generalization of the PMT in the presence of an *area outer minimizing horizon* (a minimal surface such that every other surface enclosing it has greater area).

Theorem ((Weak) Penrose inequality)

Given an asymptotically flat Riemannian manifold M^n with nonnegative scalar curvature, containing an area outer minimizing horizon Σ, the ADM mass m is bounded below in terms of the volume A of Σ and the volume ω_{n-1} of the unit $(n-1)$ sphere:

$$m \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1}$$

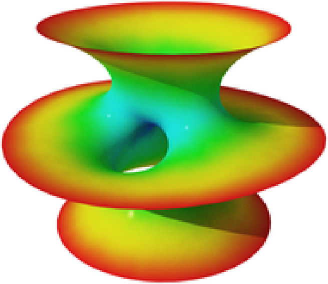
with equality iff M^n is isometric to a Schwarzschild metric.

First proved for $n = 3$: $m \geq \sqrt{A/16\pi}$ (Huisken and Illmanen, 1997). Later for $n \leq 7$ under certain conditions on M (Bray and Lee).

No known proofs for $n \geq 8$ (beyond spherically symmetric cases).

Reminder: minimal surfaces and mean curvature

A minimal surface locally minimizes area $\iff$ mean curvature is zero everywhere



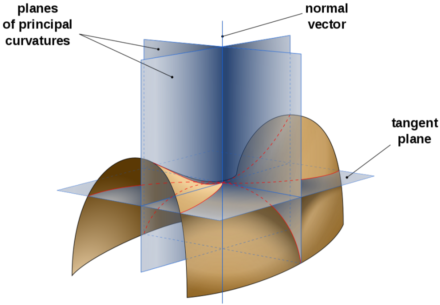


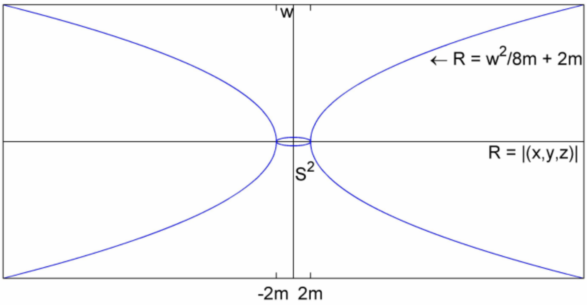
Figure: Left: a minimal surface   Right: principal curvatures

Intro: Isometric embedding of Schwarzschild metric

Equality case of the Penrose inequality is attained by the Schwarzschild metric.

When $n = 3$, $(M^3, g) = (\mathbb{R}^3 \backslash B_{2m}(0), (1 - 2m/r)^{-1} dr^2 + r^2 d\Omega^2)$ can be isometrically embedded as a rotating parabola in $\mathbb{R}^4$:

$$\{(x, y, z, w) \subset \mathbb{R}^4 \quad ; \quad |(x, y, z)| = \frac{w^2}{8m} + 2m\}$$



Intro: Isometric embedding of Schwarzschild metric

The end of the $n = 3$ Schwarzschild metric containing infinity is the graph of the spherically symmetric function $f : \mathbb{R}^3 \backslash B_{2m}(0) \to \mathbb{R}$ given by $f((x, y, z)) = \sqrt{8m(r - 2m)}$, where $r = |(x, y, z)|$.

In this case, one can check directly that the ADM mass of (M^3, g) is the positive constant m by computing a certain boundary integral at infinity involving the function f.

Also, the minimizing surface has 2-volume A equal to that of $\partial B_{2m}(0)$, so indeed $m = \sqrt{A/16\pi}$.

Introduction: Penrose inequality for graphs over $\mathbb{R}^n$

Thus, an end of the $n = 3$ Schwarzschild metric can be isometrically embedded in $\mathbb{R}^4$ as the graph of a function over $\mathbb{R}^3 \setminus B_{2m}(0)$.

If Ω is bounded and open in $\mathbb{R}^n$ and f is a smooth function on $\mathbb{R}^n \setminus \Omega$ such that the graph of f is an asymptotically flat manifold M with nonnegative scalar curvature R and horizon $f(\partial\Omega)$, can we prove the Penrose inequality for M?

Yes!

In the presence of a boundary whose connected components are convex, we can even get a stronger bound than the Penrose inequality.

We now discuss this elementary proof due to Lam (2010).

Asymptotic flatness

Definition (Schoen)

A complete Riemannian manifold (M^n, g) of dimension n is said to be **asymptotically flat** if there is a compact subset $K \subset M^n$ such that $M^n \backslash K$ is diffeomorphic to $\mathbb{R}^n \backslash \{|x| \leq 1\}$, and a diffeomorphism $\Phi : M^n \backslash K \to \mathbb{R}^n \backslash \{|x| \leq 1\}$ such that, in the coordinate chart defined by Φ, $g = g_{ij}(x) dx^i dx^j$, where

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + O(|x|^{-p}) \\ |x||g_{ij,k}(x)| + |x|^2 |g_{ij,kl}(x)| &= O(|x|^{-p}) \\ |R(g)(x)| &= O(|x|^{-q}) \end{aligned}$$

for some $q > n$ and $p > (n-2)/2$.

Simply put: outside a compact set, M^n is diffeomorphic to $\mathbb{R}^n$ minus a closed ball and the metric g decays sufficiently fast to the flat metric at infinity. p and q are chosen so that the ADM mass is finite.

ADM mass

Definition

(Schoen) The **ADM mass** m of a complete, asymptotically flat manifold (M^n, g) is defined to be

$$m = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu_j dS_r,$$

where ω_{n-1} is the volume of the $n-1$ unit sphere, S_r is the coordinate sphere of radius r, ν is the outward unit normal to S_r and dS_r is the area element of S_r in the coordinate chart.

This definition for $n = 3$ was originally due to Arnowitt, Deser and Misner.

The ADM mass is independent of the choice of asymptotically flat coordinates (Bartnik).

Positive mass theorem for graphs over R^n

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the graph of f is a complete Riemannian manifold.

The graph of f with the induced metric from $\mathbb{R}^{n+1}$ is isometric to $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let f_i denote the ith partial derivative of f. We say that f is **asymptotically flat** if

$$f_i(x) = O(|x|^{-p/2})$$
$$|x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| = O(|x|^{-p/2})$$

at infinity for some $p > (n-2)/2$.

Positive mass theorem for graphs over R^n

Theorem (Positive mass theorem for graphs over $\mathbb{R}^n$)

Let (M^n, g) be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ with the induced metric from $\mathbb{R}^{n+1}$. Let R be the scalar curvature and m the ADM mass of (M^n, g). Let ∇f denote the gradient of f in the flat metric and $|\nabla f|$ its norm with respect to the flat metric. Let dV_g denote the volume form on (M^n, g). Then

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

In particular, $R \geq 0$ implies $m \geq 0$.

Positive mass theorem for graphs with horizons

Theorem

Let Ω be a bounded and open set in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$. Let $f : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ be a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level set of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Let (M^n, g) be the graph of f with the induced metric from $\mathbb{R}^n \backslash \Omega \times \mathbb{R}$ and ADM mass m. Let H_0 be the mean curvature of Σ in $(\mathbb{R}^n \backslash \Omega, \delta)$. Then

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 d\Sigma + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g.$$

The mean curvature H of $f(\Sigma)$ in (M^n, g) and the mean curvature H_0 with respect to the flat metric δ are related by

$$H = \frac{1}{\sqrt{1+|\nabla f|^2}} H_0.$$

Thus if $|\nabla f(x)| \to \infty$ as $x \to \Sigma$, then $f(\Sigma)$ is a horizon in (M^n, g).

Corollary (Penrose inequality for graphs on $\mathbb{R}^n$ with convex boundaries)

With the same hypotheses as above, let Ω_i be the connected components of Ω, $i = 1, \ldots, k$, and $\Sigma_i = \partial\Omega_i$. If each Ω_i is convex, then

$$m \geq \sum_{i=1}^{k} \frac{1}{2} \left(\frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g.$$

In particular,

$$R \geq 0 \ \ \textit{implies} \ \ m \geq \sum_{i=1}^{k} \frac{1}{2} \left(\frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

We retrieve the (weak) Penrose inequality with:

$$\sum_{i=1}^{k} \frac{1}{2} \left(\frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \geq \frac{1}{2} \left(\frac{\sum_{i=1}^{k} |\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} = \frac{1}{2} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

Proof of the positive mass theorem for graphs over $\mathbb{R}^n$

Let $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$. Since $g_{ij} = \delta_{ij} + f_i f_j$, the inverse of g_{ij} is

$$g^{ij} = \delta^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2},$$

where the norm of ∇f is taken with respect to the flat metric δ on $\mathbb{R}^n$. We first compute the Christoffel symbols Γ_{ij}^k of (M^n, g):

$$\Gamma_{ij}^k = \frac{1}{2} g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}) = \dots = \frac{f_{ij} f^k}{1 + |\nabla f|^2}$$

and the scalar curvature:

$$\begin{aligned} R &= g^{ij}(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}) \\ &= \frac{1}{1 + |\nabla f|^2}\left(f_{ii} f_{jj} - f_{ij} f_{ij} - \frac{2 f_j f_k}{1 + |\nabla f|^2}(f_{ii} f_{jk} - f_{ij} f_{ik}) \right) \end{aligned}$$

Proof of the positive mass theorem for graphs over $\mathbb{R}^n$

Lemma

The scalar curvature R of the graph $(\mathbb{R}^n, \delta + df \otimes df)$ satisfies

$$R = \nabla \cdot \left(\frac{1}{1 + |\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \partial_j \right).$$

By definition, the ADM mass of $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ is

$$\begin{aligned} m &= \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu_j dS_r \\ &= \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r. \end{aligned}$$

By asymptotic flatness assumption, $1/(1 + |\nabla f|^2)$ goes to 1 at infinity. Thus

$$m = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1 + |\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \nu_j dS_r.$$

Proof of the positive mass theorem for graphs over $\mathbb{R}^n$

Now apply the divergence theorem in $(\mathbb{R}^n, \delta)$ and use the Lemma to get

$$\begin{aligned} m &= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} \nabla \cdot \left(\frac{1}{1+|\nabla f|^2} (f_{ii} f_j - f_{ij} f_i) \partial_j \right) dV_\delta \\ &= \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n} R dV_\delta \\ &= \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g \end{aligned}$$

since

$$dV_g = \sqrt{\det g} dV_\delta = \sqrt{1+|\nabla f|^2} dV_\delta.$$

If (M^n, g) is the graph of a smooth spherically symmetric function $f = f(r)$ on $\mathbb{R}^n$, then the ADM mass m of (M^n, g) is nonnegative even without the nonnegative scalar curvature assumption (simply computation).

Thus, there are no spherically symmetric asymptotically flat smooth functions on $\mathbb{R}^n$ whose graphs have negative scalar curvature everywhere.

Proof of the Penrose Inequality for Graphs over $\mathbb{R}^n$

Let Ω be a bounded open set in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$. If $f : \mathbb{R}^n \backslash \Omega \to \mathbb{R}$ is a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$, then the graph of f, $(M^n, g) = (\mathbb{R}^n \backslash \Omega, \delta + df \otimes df)$, is an asymptotically flat manifold with area outer minimizing horizon Σ.

As before we can write the mass of (M^n, g) as

$$m = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu_j dS_r.$$

But now when we apply the divergence theorem, we get an extra boundary integral:

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{\mathbb{R}^n\backslash\Omega} \nabla \cdot \left(\frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\partial_j\right) dV_\delta$$
$$- \frac{1}{2(n-1)\omega_{n-1}} \int_{\Sigma} \frac{1}{1+|\nabla f|^2}(f_{ii}f_j - f_{ij}f_i)\nu_j d\Sigma$$

Proof of the Penrose Inequality for Graphs over $\mathbb{R}^n$

The outward normal to Σ is $\nu = -\nabla f / |\nabla f|$. Let Δf be the Laplacian of f in (M^n, g) and $\Delta_\Sigma f$ the Laplacian of f along Σ. Let H^f denote the Hessian of f and H_0 the mean curvature of Σ with respect to the flat metric. Using the identity

$$\Delta f = \Delta_\Sigma f + H^f(\nu, \nu) + H_0 \cdot \nu(f)$$

where $\Delta_\Sigma f = 0$ since f is constant on Σ. we get

$$-\frac{1}{1 + |\nabla f|^2}(f_{ii} f_j - f_{ij} f_i)\nu_j = \cdots = \frac{|\nabla f|^2}{1 + |\nabla f|^2} H_0 = H_0.$$

Therefore,

$$m = \frac{1}{2(n-1)\omega_{n-1}} \int_{M^n} R \frac{1}{\sqrt{1 + |\nabla f|^2}} dV_g + \frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 d\Sigma.$$

Proof of the Penrose Inequality for Graphs over $\mathbb{R}^n$

Let Ω_i, $i = 1, \ldots, k$ be the connected components of the bounded open set Ω.

To get a stronger lower bound for the ADM mass in the case of convex Ω_i, we use:

Lemma (special case of Aleksandrov-Fenchel inequality)

If Σ is a convex surface in $\mathbb{R}^n$ with mean curvature H_0 and area $|\Sigma|$, then

$$\frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 \geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} .$$

Proof of the Penrose Inequality for Graphs over $\mathbb{R}^n$

Let $\Sigma \subset \mathbb{R}^n$ be a convex surface with principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$. Let

$$\sigma_j(\kappa_1, \ldots, \kappa_{n-1}) = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n-1} \kappa_{i_1} \cdots \kappa_{i_j}$$

be the jth normalized elementary symmetric functions in $\kappa_1, \ldots, \kappa_{n-1}$ for $j = 1, \ldots, n-1$. In particular,

$$\sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = 1$$
$$\sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i = \frac{1}{n-1} H_0$$
$$\sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \prod_{i=1}^{n-1} \kappa_i.$$

Define

$$V_k = \int_\Sigma \sigma_k(\kappa_1, \ldots, \kappa_{n-1}).$$

Special case of Aleksandrov-Fenchel inequality gives: $V_1^{n-1} \geq V_0^{n-2} V_{n-1}$. (2)

Proof of the Penrose Inequality for Graphs over $\mathbb{R}^n$

Now,

$$\begin{aligned} V_0 &= \int_\Sigma \sigma_0(\kappa_1, \ldots, \kappa_{n-1}) = |\Sigma| \\ V_1 &= \int_\Sigma \sigma_1(\kappa_1, \ldots, \kappa_{n-1}) = \frac{1}{n-1} \int_\Sigma H_0 \\ V_{n-1} &= \int_\Sigma \sigma_{n-1}(\kappa_1, \ldots, \kappa_{n-1}) = \omega_{n-1}. \end{aligned}$$

Thus (2) becomes

$$\begin{aligned} \left(\frac{1}{n-1} \int_\Sigma H_0 \right)^{n-1} &\geq |\Sigma|^{n-2} \omega_{n-1} \\ \frac{1}{n-1} \int_\Sigma H_0 &\geq |\Sigma|^{\frac{n-2}{n-1}} \omega_{n-1}^{\frac{1}{n-1}} \\ \frac{1}{2(n-1)\omega_{n-1}} \int_\Sigma H_0 &\geq \frac{1}{2} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \end{aligned}$$

The paper by Lam: arXiv:1010.4256v1