Unitary representations of space(time) symmetries

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Intuitively, a symmetry in physics is a transformation which leaves the results of experiments unchanged. Figuring out which transformations satisfy such a strong property, and then studying the consequences of the resulting invariance, has been one of the cornerstones of modern physics. In this note, we study two important examples: the three-dimensional Euclidean group and the Poincaré group.

Another driving force of this exposition is Wigner's theorem from quantum mechanics. It tells us that a representation of a symmetry transformation on a Hilbert space must be either a unitary or an anti-unitary (projective) representation. As irreducible representations form the building blocks of any other representation, we are thus naturally led to investigate the irreducible unitary representations of the groups mentioned above.

1 Euclidean groups

Copernicus' insight that the Earth is not the center of the universe was, at least in spirit, a precursor to Galileo's realization that the laws of motion are the same in all inertial frames. The idea has since been pushed to the extreme. Indeed, it appears (empirically) that the laws of physics are invariant under spatial rotations, translations in both space and time, as well as changes of constant velocity of the observer's frame (boosts). We start by studying the spatial symmetries in this section.

Mathematically, we think of physical space as a 3-dimensional Euclidean space $\mathbb{E}^3$ over the reals. We then declare that physical laws are invariant under translations and rotations. These transformations belong to the symmetry group of $\mathbb{E}^3$.

Definition 1.1. The *Euclidean group* $\mathrm{E}(n)$ consists of the group of isometries of n-dimensional Euclidean space $\mathbb{E}^n$.

However, the laws of physics are surprisingly not invariant under the full $\mathrm{E}(3)$ because the weak interactions violate parity - they are not preserved under reflections. Let us hence be more precise. Each $g \in \mathrm{E}(n)$ induces a linear mapping of $\mathbb{R}^n$ onto itself given by $y - x \mapsto g(y) - g(x)$, where $x, y \in \mathbb{E}^n$ and $y - x$ denotes the translation vector in $\mathbb{R}^n$ mapping x to y. If the induced transformation, which is orthogonal, has determinant 1, then we say that g is a direct isometry. These form a subgroup $\mathrm{SE}(n)$ which excludes the reflections, and to which we restrict our attention.

The surjective mapping $\mathrm{SE}(n) \to \mathrm{SO}(n)$ assigning to each direct isometry its induced orthogonal mapping is a group homomorphism. Its kernel $T(n)$ is the abelian normal subgroup of translations, and, if we think of $\mathrm{SO}(n)$ as the direct isometries preserving a fixed point, we obtain $\mathrm{SE}(n) = T(n) \rtimes \mathrm{SO}(n)$. One of the interesting aspects of the special Euclidean group is that rotations and translations fail to commute. We will be particularly interested in SE(2) (mainly for illustrative purposes) and SE(3), which we now introduce.

1.1 Special Euclidean group in two-dimensional space

Since a rotation of the plane is characterized by an angle θ, we can uniquely denote a general isometry in SE(2) as $g(b, \theta)$, where $b \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$. The map

$$g(b,\theta) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & b_1 \\ \sin\theta & \cos\theta & b_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

then gives us a representation of SE(2) on $\mathbb{R}^3$. Since the representation is faithful, we will abuse notation and think of $g(b, \theta)$ as the matrix itself. A general element of the subgroup SO(2) is $R(\theta) \equiv g(0, \theta) = e^{-i\theta J}$, where our inclination for physics leads us to use the generator

$$J = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

On the other hand, a general element of the two-parameter subgroup of translations is $T(b) = e^{-ib_1 P_1} e^{-ib_2 P_2}$, where we have used the observation that the generators

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

commute with each other. Given that $\mathrm{SE}(2) = \mathrm{T}(2) \rtimes \mathrm{SO}(2)$, a general group element of SE(2) can be uniquely written as $g(b, \theta) = T(b)R(\theta)$. To later study how translations and rotations interact, we record the following observation.

Proposition 1.2. *The generators of the Lie algebra* $\mathfrak{ise}(2)$ *satisfy*

$$[P_1, P_2] = 0, \quad [J, P_1] = iP_2, \quad \text{and} \quad [J, P_2] = -iP_1. \tag{4}$$

1.2 Special Euclidean group in three-dimensional space

Generalizing what we have done for the two-dimensional case, we introduce the analogous faithful representation of SE(3) on $\mathbb{R}^4$:

$$g(Q, b) \mapsto \begin{pmatrix} Q & b \\ 0 & 1 \end{pmatrix}, \tag{5}$$

where $Q \in \mathrm{SO}(3)$ and $b \in \mathbb{R}^3$. We can be more explicit. Every rotation can be uniquely decomposed in terms of the Euler angles (α, β, γ) into a product $R_z(\alpha)R_y(\beta)R_z(\gamma)$ of rotations around the fixed y- and z-axes, where $\alpha, \gamma \in [0, 2\pi)$, $\beta \in [0, \pi]$,

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{6}$$

By defining the generators J_1, J_2, and J_3 in analogy with the two-dimensional case, we can write a general element of the subgroup SO(3) as

$$R(\alpha, \beta, \gamma) \equiv e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}. \tag{7}$$

Similarly, the generators for the subgroup T(3) are the direct generalizations of the two-dimensional case. For the sake of completeness, we conclude by noticing the following commutation relations.

Proposition 1.3. *The generators of the Lie algebra* $i\mathfrak{se}(3)$ *satisfy*

$$[P_k, P_l] = 0, \quad [J_k, J_l] = i \sum_m \varepsilon_{klm} J_m, \quad \textit{and} \quad [P_k, J_l] = i \sum_m \varepsilon_{klm} P_m, \qquad k, l = 1, 2, 3 \tag{8}$$

where ε_{klm} *is the 3-dimensional Levi-Civita symbol.*

2 Unitary representations of SE(2) and SE(3)

By passing through the quotient $\mathrm{SE}(2)/\mathrm{T}(2) \cong \mathrm{SO}(2)$, we obtain a family of one-dimensional representations of SE(2) indexed by the integers. Explicitly, for each $m \in \mathbb{Z}$, we have a representation $U_m : \mathrm{SE}(2) \to \mathbb{C}^\times$ given by $U_m(T(b)R(\theta)) = e^{-im\theta}$. Similarly, we can exploit our knowledge of SO(3) to find degenerate unitary irreducible representations of SE(3). We cannot afford to give a full construction from scratch, but we remind the reader that, for each $j \in \frac{1}{2}\mathbb{N}_0$, we have the so-called spin-j representation $D_j : \mathrm{SO}(3) \to \mathrm{GL}(\mathbb{C}^{2j+1})$.

It turns out that these are the only finite-dimensional unitary representations of SE(2) and SE(3). All the faithful representations will be infinite-dimensional.[1] To find them, we will use (mostly for didactic reasons) two substantially different approaches.

2.1 Irreducible representations by the Lie algebra method

To find the irreducible representations of the connected group SE(2), we pass through its Lie algebra $\mathfrak{se}(2)$. Given a representation of the latter, we will then be able to build the representation of the original group by using the exponential map.

Instead of working directly with $\mathfrak{se}(2)$, it will be useful to work with its complexification $\mathfrak{se}(2)_\mathbb{C} = \mathfrak{se}(2) \otimes_\mathbb{R} \mathbb{C}$. Once we are done, it will suffice to reduce to a real form of the algebra. We hence bring our attention to the Lie algebra representations

[1] Most references justify this claim by noting that the groups at hand are not compact. However, because of the normal abelian subgroup of translations, the standard argument using that a representation $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ must have a closed image for simple Lie groups does not go through.

$$\rho : \mathfrak{se}(2)_{\mathbb{C}} \to \mathrm{End}(\mathcal{H}), \tag{9}$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space over $\mathbb{C}$. We introduce the squared momentum operator $\mathbf{P}^2 = \rho(P_1)^2 + \rho(P_2)^2$ as well as the ladder operators $P_\pm = P_1 \pm iP_2$. Then $\{J, P_\pm\}$ generates $\mathfrak{se}(2)_{\mathbb{C}}$, and we have the commutation relations

$$[P_+, P_-] = 0, \quad [J, P_\pm] = \pm P_\pm, \quad \text{and} \quad [\mathbf{P}^2, \rho(J)] = [\mathbf{P}^2, \rho(P_\pm)] = 0. \tag{10}$$

Since we want our representation of SE(2) to be unitary, the operators $\rho(J)$, $\rho(P_1)$, and $\rho(P_2)$ must be self-adjoint. Therefore, $\mathbf{P}^2$ is also self-adjoint with non-negative eigenvalues, and $\rho(P_\pm)^* = \rho(P_\mp)$.

By the spectral theorem, we can write $\mathcal{H} = \oplus_p \mathcal{H}_p$, where each $\mathcal{H}_p$ is the eigenspace corresponding to some eigenvalue $p^2 \geq 0$ of $\mathbf{P}^2$.[2] Since $\mathbf{P}^2$ commutes with $\rho(J)$ and $\rho(P_\pm)$, each subspace $\mathcal{H}_p$ is invariant under the action of $\mathfrak{se}(2)_{\mathbb{C}}$. As a result, each irreducible representation must be contained in one of these eigenspaces, which we now fix.

Since any representation of SE(2) is in particular a representation of its subgroup SO(2), our knowledge of the latter group, along with its generator J, tells us that $\mathcal{H}_p$ has an orthonormal basis $\{e_m\}$ made up of eigenvectors of $\rho(J)$ and labeled by integer eigenvalues (with potential degeneracies). We notice that

$$\begin{aligned} \|\rho(P_\pm)e_m\|^2 &= \langle \rho(P_\pm)e_m, \rho(P_\pm)e_m\rangle = \langle e_m, \rho(P_\pm)^*\rho(P_\pm)e_m\rangle & (11)\\ &= \langle e_m, \rho(P_\mp)\rho(P_\pm)e_m\rangle = \langle e_m, \mathbf{P}^2 e_m\rangle & (12)\\ &= p^2\|e_m\|^2 = p^2. & (13) \end{aligned}$$

Therefore, if $p = 0$, each one-dimensional subspace spanned by an eigenvector e_m is an irreducible representation. These correspond to the degenerate irreducible representations that we have already encountered.

Let us now consider the case where $p > 0$. Our strategy is to pick an arbitrary initial vector e_{m_0} and obtain a basis of an invariant subspace by successive applications of $\rho(P_\pm)$. It is enough to consider these operators because

$$\rho(J)\rho(P_\pm)e_m = (\rho(P_\pm)\rho(J) \pm \rho(P_\pm))e_m = (m \pm 1)\rho(P_\pm)e_m, \tag{14}$$

where we have used the relation $[\rho(J), \rho(P_\pm)] = \rho([J, P_\pm]) = \pm\rho(P_\pm)$. For each $k \in \mathbb{N}$, let us define

$$\tilde{e}_k = (i/p)^k \rho(P_+)^k e_{m_0} \quad \text{and} \quad \tilde{e}_{-k} = (-i/p)^k \rho(P_-)^k e_{m_0}. \tag{15}$$

Then $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of an invariant and irreducible subspace $\tilde{\mathcal{H}}_p \subseteq \mathcal{H}_p$. Therefore, without loss of generality, we may relabel the separable space $\tilde{\mathcal{H}}_p$ as $l^2(\mathbb{Z})$, and define the operators by their action on the canonical basis $\{e_m\}_{m\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$:[3]

$$\rho(J)e_m = me_m, \quad \text{and} \quad \rho(P_\pm)e_m = \mp ip e_{m\pm1}. \tag{16}$$

[2] We put on our physicist hat and ignore any operator compactness issues.

[3] Technically the operator $\rho(J)$ is unbounded, but it is well-defined on the dense subspace $c_{00}(\mathbb{Z})$.

As promised, we now want to use our findings to construct the representations of SE(2). An ingredient that we will (surprisingly?) need is the Bessel functions of the first kind J_n, $n \in \mathbb{Z}$, which are given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n+1)!} \left(\frac{x}{2}\right)^{2k+n} \qquad (\forall n \geq 0), \tag{17}$$

and then extended by $J_{-n}(x) = (-1)^n J_n(x)$. Let us reap the fruits of our labor.

Theorem 2.1. *The faithful unitary irreducible representations of* SE(2) *are characterized by a real number $p > 0$. We may explicitly define a representation $U_p : \mathrm{SE}(2) \to \mathrm{GL}(l^2(\mathbb{Z}))$ by*

$$U_p(g(b,\theta))e_m = \sum_{m' \in \mathbb{Z}} e^{i(m-m')\phi} J_{m-m'}(p\|b\|) e^{-im\theta} e_{m'}, \tag{18}$$

where $(\|b\|, \phi)$ denotes the polar coordinates of $b \in \mathbb{R}^2$.

Proof. Given that $\mathrm{SE}(2) = \mathrm{T}(2) \rtimes \mathrm{SO}(2)$, it suffices to consider rotations and translations separately. For rotations, we simply have

$$U_p(R(\theta))e_m = e^{-i\theta\rho(J)} e_m = e^{-im\theta} e_m. \tag{19}$$

When $b \in \mathbb{R}^2$ parametrizes a translation along the x-axis ($b_2 = 0$), we can use that P_+ and P_- commute with each other to write

$$U_p(T(b)) = e^{-ib_1\rho(P_1)} = e^{-ib_1(\rho(P_+)+\rho(P_-))/2} = \sum_{k,l=0}^{\infty} \frac{(-1)^k}{k!l!} \left(\frac{b_1}{2}\right)^{k+l} (i\rho(P_+))^k (-i\rho(P_-))^l.$$

Given the defining equations (16) (where we are now grateful we introduced the i factors), we obtain

$$\langle e_{m'}, U_p(T(b))e_m \rangle = \sum_{k,l=0}^{\infty} \frac{(-1)^k}{k!l!} \left(\frac{pb_1}{2}\right)^{k+l} \langle e_{m'}, e_{m+k-l} \rangle = \sum_{k-l=m-m'} \frac{(-1)^k}{k!l!} \left(\frac{pb_1}{2}\right)^{k+l}. \tag{20}$$

Manipulation of the summation indices and comparing with (17) then yields

$$\langle e_{m'}, U_p(T(b))e_m \rangle = J_{m-m'}(pb_1). \tag{21}$$

For a general $b \in \mathbb{R}^2$ with polar coordinates $(\|b\|, \phi)$, we can finally use the observation that

$$T(b) = R(\phi) T(\|b\|, 0) R(\phi)^{-1}, \tag{22}$$

to conclude that

$$U_p(T(b))e_m = \sum_{m' \in \mathbb{Z}} e^{i(m-m')\phi} J_{m-m'}(p\|b\|) e_{m'}. \tag{23}$$

□

2.2 Irreducible representations by the induction method

We start by considering the unitary irreducible representations of T(3). Since the subgroup of translations is abelian, such representations must be characters. Given what we know on the generators of T(3), the unitary characters are labeled by $p \in \mathbb{R}^3$ and given explicitly as

$$\chi_p : \mathrm{T}(3) \to \mathbb{S}^1, \quad T(b) \mapsto e^{-i\langle b,p\rangle}. \tag{24}$$

We note that we have an SO(3)-action on the dual $\widehat{\mathrm{T}(3)}$ given by

$$(Q \cdot \chi_p)(T(b)) = \chi_p\left(g(Q,0)T(b)g(Q,0)^{-1}\right) = \chi_p\left(T(Qb)\right). \tag{25}$$

This leads us to the following concept.

Definition 2.2. For a given $p \in \mathbb{R}^3$, we call the stabilizer of χ_p under the previously defined SO(3)-action the *little group* of p and denote it by $\mathrm{SO}(3)_p$.

Using that $Q^T = Q^{-1}$ for all orthogonal matrices Q, we notice that

$$\mathrm{SO}(3)_p = \{Q \in \mathrm{SO}(3) \mid Qp = p\}. \tag{26}$$

When $p = 0$ (i.e., χ_p is the trivial character), we recover all of SO(3). We discard this case because it would merely lead us to the degenerate irreducible representations that we have already encountered. When $p \neq 0$, we recognize the rotations that preserve the axis associated to p, and hence $\mathrm{SO}(3)_p \cong \mathrm{SO}(2)$.

Let us be more precise. If we characterize a fixed $p_0 \neq 0$ by two angles (θ, ϕ) such that $p_0 = R(\theta, \phi, 0)e_3$, then, for each $Q \in \mathrm{SO}(3)_{p_0}$, there exists a unique $\psi \in [0, 2\pi)$ satisfying

$$R(0,0,\psi) = R(\theta,\phi,0)^{-1} Q R(\theta,\phi,0). \tag{27}$$

We identify Q with $R(\psi)$. We can now leverage our knowledge of SO(2). We know that all its irreducible representations $\tilde{U}_\lambda$ are one-dimensional and characterized by an integer $\lambda \in \mathbb{Z}$. We hence obtain a unitary representation $U_\lambda : \mathrm{T}(3) \rtimes \mathrm{SO}(3)_{p_0} \to \mathbb{C}^\times$ given by

$$U_\lambda(T(b)Q) = \chi_{p_0}(T(b))\tilde{U}_\lambda(R(\psi)) = e^{-i\langle b,p_0\rangle} e^{-i\lambda\psi}. \tag{28}$$

Armed with this, we can now make the jump to all of SE(3). Indeed, it suffices to induce the representation $(U_\lambda, \mathbb{C})$ from $\mathrm{SO}(3)_{p_0} \ltimes \mathrm{T}(3)$ to SE(3). If we are careful to use normalized induction, we again obtain a unitary representation. While giving an explicit construction was deemed to be more obscuring than elucidating, the importance lies in realizing that the resulting representation is irreducible because all its basis vectors are generated from a single vector, and no smaller invariant subspace exists.

Proposition 2.3. *The faithful unitary irreducible representations of* SE(3) *are characterized by a real number* $p > 0$ *and an integer* λ. *They can be realized as*

$$\mathrm{Ind}_{\mathrm{T}(3)\rtimes \mathrm{SO}(3)_{p_0}}^{\mathrm{SE}(3)} U_\lambda, \tag{29}$$

where $p_0 \in \mathbb{R}^3$ *satisfies* $\|p_0\| = p$.

3 Poincaré Group

We now study spacetime symmetries. We model spacetime as a Minkowski space, i.e., a 4-dimensional vector space $\mathbb{R}^{3,1}$ endowed with a nondegenerate and symmetric bilinear form of metric signature $(-,+,+,+)$. In analogy to $\mathrm{E}(3)$, we start with the following definition:

Definition 3.1. The Poincaré group is the isometry group of Minkowski space.

The Poincaré group is generated by the spacetime translations and Lorentz transformations, and it can be written as a semi-direct product $\mathrm{T}(4) \rtimes \mathrm{O}(3,1)$. Special relativity postulates that the laws of physics are invariant under $\mathrm{T}(4) \rtimes \mathrm{SO}^{\uparrow}(3,1)$, where we have restricted to Lorentz transformations that preserve the orientation of space and the direction of time.[4] A general element of the group will hence be characterized by a unique restricted Lorentz transformation Λ and a translation $T(b)$ with $b \in \mathbb{R}^{3,1}$.

The Poincaré group is the full symmetry group of any relativistic field theory, and thus elementary particles are irreducible representations of this group. In the following we establish Wigner's classification of the unitary irreducible representations of positive energy, which includes representations corresponding to physical particles, indexed by a nonnegative mass and an integer or half-integer spin, as well as the non-physical "continuous spin" representation for massless states. We also discuss the non-physical negative energy representations, which correspond to tachyons. We include the essential mathematical details but choose here to emphasize physical reasoning.

3.1 The spin double cover

We can identify $\mathbb{R}^{3,1}$ with the real vector space of Hermitian 2×2 matrices via the map

$$x \mapsto X(x) = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}. \tag{30}$$

We conveniently have that $\det X(x)$ is the square of the Minkowski norm of x. We can then define an action $\tau : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(\mathbb{R}^{3,1})$ by $\tau(A) \cdot x = X^{-1}(AX(x)A^*)$. More precisely, we notice that τ lands in $\mathrm{SO}^{\uparrow}(3,1)$ because each $\tau(A)$ preserves the Minkowski norm, and the continuous image of a connected set is connected. One can check that $\ker \tau = \{\pm I_2\}$, and consequently that $\tau : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}^{\uparrow}(3,1)$ is surjective (as an open homomorphism between connected Lie groups of the same dimension). Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected, τ is a universal (double) cover of $\mathrm{SO}^{\uparrow}(3,1)$.

One consequence of this observation is that the unitary faithful representations of the Poincaré group must be infinite dimensional. Indeed, any such unitary faithful representation restricts to a faithful representation $\rho : \mathrm{SO}^{\uparrow}(3,1) \to \mathrm{U}(n)$. This in turn induces a representation $\tilde{\rho}$ on its universal cover $\mathrm{SL}_2(\mathbb{C})$, which is a quasisimple group with finite center $\{\pm I_2\}$. Although the detailed proof is beyond the scope of this paper, the vanishing theorem of Howe-More can then be used to show that the image of $\tilde{\rho}$ (which is the same as that of ρ) must be closed in $\mathrm{U}(n)$. Therefore, by the closed-subgroup theorem and the fact that ρ is injective, the image is an embedded Lie group diffeomorphic to $\mathrm{SO}^{\uparrow}(3,1)$. However, the image is compact as a closed subset of the compact group $\mathrm{U}(n)$, so we reach a contradiction.

[4] In the following, the term Poincaré group will refer to the restricted Poincaré group $\mathrm{T}(4) \rtimes \mathrm{SO}^{\uparrow}(3,1)$.

4 Unitary representations of the Poincaré group

We will construct unitary irreducible representations (irreps from now on) of the Poincaré group by the (now familiar) method of induced representations. The basis vectors will be chosen as eigenvectors p^μ of the generators of translations P^μ. For a given p^μ we identify its *little group*, i.e., the maximal subgroup of the Lorentz group leaving p^μ invariant. From a unitary irrep of the little group on the p^μ-eigenspace, we induce a representation of the Poincaré group by applying Lorentz transformations to basis vectors.

We define the square of the momentum operator

$$C_1 := -P_\mu P^\mu = P_0^2 - \mathbf{P}^2. \tag{31}$$

As the scalar product of 4-vectors, C_1 is Lorentz invariant, and is also invariant under translations, as T(4) is abelian. Hence C_1 is thus a Casimir operator of the Poincaré group. In contrast to the Euclidean group, the eigenvalues c_1 of C_1 are not positive definite. We will classify irreps according to the region of the light-cone in which p^μ lies, determined by the sign of c_1.

4.1 Null vector

The null vector $p_n^\mu \equiv 0$ is Lorentz invariant, and its little group is thus the full Lorentz group. Each unitary irrep of the Lorentz group (which we do not explicitly classify here but recall that they are indexed by two parameters j_0, ν) [5] induces a unitary irrep of the Poincaré group in which the basis consists of common eigenvectors $|0jm\rangle$ of (P^μ, J^2, J_3) with respective eigenvalues $(0, j, m)$. Given a set of unitary matrices $D_{j_0,\nu}(\Lambda)$ representing the Lorentz group, we then have

$$T(b)\,|0jm\rangle = |0jm\rangle\,, \tag{32}$$

$$\Lambda\,|0jm\rangle = |0j'm'\rangle\, D_{j_0,\nu}(\Lambda)^{j'm'}_{jm}\,. \tag{33}$$

This representation describes the vacuum.

4.2 Time-like vector

For a given positive $c_1 = -p_\mu p^\mu := M^2$, we define a "standard vector" $p_t := (p_0, \mathbf{p}) = (M, \mathbf{0})$ corresponding to a state at rest with mas M.[6] The little group of p_t is the group of 3-dimensional rotations SO(3), whose unitary irreps are are the spin $j = s$ representations D_j mentioned in Section 2. Suppressing the implicit indices $p^0 = M$ and $J^2 = s(s+1)$, we denote the basis vectors of the p_t^μ-eigenspace by $|\mathbf{0}, \lambda\rangle$, where $\mathbf{0}$ refers to $\mathbf{p} = 0$. They satsify

$$P^\mu\,|\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle\, p_t^\mu, \tag{34}$$

$$J^2\,|\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle\, s(s+1), \tag{35}$$

$$J_3\,|\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle\, \lambda. \tag{36}$$

[5] With either (i) $v \in i\mathbb{R}$, and $j_0 \in \frac{1}{2}\mathbb{N}_0$, or (ii) $v \in [-1, 1]$ and $j_0 = 0$.

[6] Or energy in units $c = 1$.

A general Lorentz transformation Λ may be written in terms of two rotations and a boost along the z-axis: $\Lambda = R(\alpha,\beta,0)L_3(\zeta)R^{-1}(\phi,\theta,\psi)$, where the rapidity satisfies $v = c\tanh(\zeta)$. We now obtain a complete basis consisting of general eigenvectors of P^μ by applying Lorentz transformations on $|\mathbf{0},\lambda\rangle$. As the p_t^μ-eigenspace is invariant under rotations, it suffices to apply $H(p) = R(\alpha,\beta,0)L_3(\zeta)$, giving $|\mathbf{p}\lambda\rangle := H(p)\,|\mathbf{0}\lambda\rangle$.

Proposition 4.1. *The action of group transformations on the Poincaré-invariant vector space spanned by* $\{|\mathbf{p}\lambda\rangle\}$ *is given by*

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-ib_\mu p^\mu}, \tag{37}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda'\rangle\, D_s(H^{-1}(p)\Lambda H(p))_\lambda^{\lambda'}, \tag{38}$$

where $p'^\mu = \Lambda^\mu_\nu p^\nu$. *This representation, labelled by* (M,s), *is a unitary irrep.*

We omit the verification of (37)-(38), but note that the space is irreducible because the $\{|\mathbf{p}\lambda\rangle\}$ may be generated from a given vector $|\mathbf{0}\lambda = s\rangle$ by applying $\{J_\pm\}$ and the $\{H(p)\}$, so there cannot be a nontrivial invariant subspace. Unitarity follows because the generators are hermitian operators and the representation matrices on the RHS of (37) and (38) are unitary. In the time-like case, basis vectors $|M,s;\mathbf{p}\lambda\rangle$ represent states of mass M, 3-momentum $\mathbf{p}$, and intrinsic spin s. Moreover, λ is the eigenvalue of $J_3 = J\cdot\mathbf{P}/|\mathbf{p}|$, hence it corresponds to the angular momentum along the direction of the motion, or *helicity* of the state.

4.3 The second Casimir operator

We have seen that the "mass" M is the square root of the eigenvalue of the Casimir C_1. In fact, the Poincaré group admits a second Casimir operator related to the "spin" s. Such an operator should be a translationally invariant Lorentz scalar which reduces in the subspace $\mathbf{p} = 0$ to J^2. From the relativistic angular momentum tensor operator $J_{\mu\nu}$, we define the Pauli-Lubanski vector

$$W^\lambda := \frac{1}{2}\epsilon^{\lambda\mu\nu\sigma}J_{\mu\nu}P_\sigma, \tag{39}$$

which is orthogonal to P^λ and satisfies the commutation relations

$$[W^\lambda, P^\mu] = 0, \tag{40}$$

$$[W^\lambda, J^{\mu\nu}] = i(W^\mu g^{\lambda\nu} - W^\nu g^{\mu\lambda}), \tag{41}$$

$$[W^\lambda, W^\sigma] = i\epsilon^{\lambda\sigma\mu\nu}W_\mu P_\nu. \tag{42}$$

We get that W^λ is translationally invariant from (40), and that it transforms as a 4-vector from (41). Thus $C_2 := W^\lambda W_\lambda$ commutes with all generators of the Poincaré group, and is the second Casimir. Relation (42) implies that, on the $\{p^\mu\}$-eigenspace, the components of $\{W^\mu\}$ form a Lie algebra, that of the little group of p^μ. For example, for $p^\mu = p_t^\mu = (M,\mathbf{0})$ we have $W = (0, M\mathbf{J})$ whose components generate SO(3). The unitary irreps of the Poincaré group are characterized by the eigenvalues of the Casimirs $-P^2$ and $-W^2/p^2$. For the following cases we will construct the little group directly from the Lie algebra generated by $\{W^\mu\}$.

4.3.1 Light-Like Vector

We have $p_\mu p^\mu = 0$, which is satisfied by the 4-momenta of photons. We choose the standard light-like vector $p_l^\mu \equiv (\omega_0, 0, 0, \omega_0)$ with $\omega_0 \neq 0$. To obtain a general momentum $p^\mu = (\omega, \mathbf{p})$, where $\mathbf{p} = \omega\hat{\mathbf{p}}$ and the unit vector $\hat{\mathbf{p}}$ is characterized by the angles (θ, ϕ), we apply a boost $L_3(\zeta)$ to transform the energy from ω_0 to ω, then rotate via $R(\phi, \theta, 0)$ the z-axis into the $\hat{\mathbf{p}}$-direction. As in Section 3.2 we denote the transformation from p_l^μ to p^μ by $H(p)$:

$$p^\mu = H(p)^\mu_\nu p_l^\nu = [R(\phi, \theta, 0)L_3(\zeta)]^\mu_\nu p_l^\nu. \tag{43}$$

In terms of the rotations J_i and boosts K_i, the components of W^λ are

$$W^0 = W^3 = \omega_0 J_{12} = \omega_0 J_3, \tag{44}$$

$$W^1 = \omega_0(J_{23} + J_{20}) = \omega_0(J_1 + K_2), \tag{45}$$

$$W^2 = \omega_0(J_{31} - J_{10}) = \omega_0(J_2 - K_1), \tag{46}$$

and the second Casimir is $C_2 = W_1^2 + W_2^2$. The Lie algebra of the little group is given by

$$[W^1, W^2] = 0, \quad [W^2, J_3] = iW^1, \quad \text{and} \quad [W^2, J_3] = -iW^2. \tag{47}$$

We have seen in Section 2.1 that this is the Lie algebra of SE(2), whose unitary irreps are either degenerate ($w = 0$) with basis vectors labeled by the eigenvalues λ of J_3, or non-degenerate and infinite-dimensional ($w > 0$), with basis vectors $\{|w, \lambda\rangle, \lambda = 0, \pm 1, ..\}$. For a given representation, labeled by $M = 0$ and w, we again induce a representation of the Poincaré group by applying the transformations $H(p)$ to the basis vectors of the subspace. The "continuous spin representation" ($M = 0, w > 0$) has not been observed in nature.[7] The ($M = w = 0, \lambda$) representations correspond to the photon/gluons ($\lambda = \pm 1$), and eventually graviton ($\lambda = \pm 2$) states.[8] We consider thus only the degenerate case in the following, for which the little group, generated by J_3 alone, is SO(2). The p_l^μ-eigenspace is one-dimensional and the basis vector $|\mathbf{p_l}\lambda\rangle$ transforms as

$$P^\mu |\mathbf{p_l}\lambda\rangle = |\mathbf{p_l}\lambda\rangle\, p_l^\mu \tag{48}$$

$$J_3 |\mathbf{p_l}\lambda\rangle = |\mathbf{p_l}\lambda\rangle\, \lambda \tag{49}$$

$$W_i |\mathbf{p_l}\lambda\rangle = 0 \quad ; \quad i = 1, 2 \tag{50}$$

When λ is an integer (resp. odd half-integer), we obtain a single (resp. double) valued representation. The general basis vector is obtained as $|\mathbf{p}\lambda\rangle \equiv H(p)\,|\mathbf{p_l}\lambda\rangle$, where $p = \omega_0 e^\zeta$. It is not complicated to show that the light-like analogue of Proposition 4.1 is

Proposition 4.2. *The action of group transformations on the Poincaré-invariant space spanned by* $\{|\mathbf{p}\lambda\rangle\}$ *is given by*

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-ib_\mu p^\mu}, \tag{51}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda\rangle\, e^{-i\lambda\theta(\Lambda,p)}, \tag{52}$$

[7] Although interestingly it has a condensed matter analogue as a massless generalization of the anyon and could potentially be observed in these systems.

[8] Ref. [3] includes the neutrino ($\lambda = -1/2$) and anti-neutrino ($\lambda = 1/2$) in this characterization, although we now know the neutrinos have mass.

where $\theta(\Lambda, p)$ is a real number given by $e^{-i\lambda\theta(\Lambda,p)} = \langle \mathbf{p}_l\lambda| H^{-1}(\Lambda p)\Lambda H(p) |\mathbf{p}_l\lambda\rangle$. This representation, labeled by $(M = 0, \lambda)$, is a unitary irrep.

Once again $|\mathbf{p}\lambda\rangle$ represents a state with momentum $\mathbf{p}$ and helicity λ. Note that under Lorentz transformations the helicity λ is transformed among all $2s + 1$ possible values for massive time-like states but is invariant for light-like states.

4.3.2 Space-like vector

The 4-momenta with $c_1 = -p_\mu p^\mu < 0$ correspond to imaginary mass in the rest frame and thus cannot represent physical particles. This case nonetheless merits study as, e.g, in deep inelastic scattering, a virtual space-like photon is exchanged between the incoming lepton and the incoming hadron. We choose the "standard state" as $p_s^\mu \equiv (0, 0, 0, Q)$ where $Q^2 = -c_1 > 0$. The little group is generated by the components

$$W^0 = QJ_3, \tag{53}$$

$$W_1 = QJ_{20} = QK_2, \tag{54}$$

$$W_2 = QJ_{01} = -QK_1, \tag{55}$$

and coincides with $\mathrm{SO}(2,1)$. The second Casimir becomes

$$C_2 = Q^2(K_1^2 + K_2^2 - J_3^2). \tag{56}$$

The Lie algebra of the little group has the commutation relations

$$[K_2, J_3] = iK_1, \quad [J_3, K_1] = iK_2, \quad \text{and} \quad [K_1, K_2] = -iJ_3. \tag{57}$$

which are the same as those of $\mathfrak{so}(3)$ up to the minus sign in (57), which makes $\mathrm{SO}(2,1)$ non-compact, and hence its unitary irreps are infinite-dimensional.[9] They are obtained similarly to $\mathrm{SO}(3)$ and the Lorentz group, and they are labeled by the eigenvalue c_2, which may (i) vary continuously in $(0, \infty)$, or (ii) assume the discrete negative values $c_2 = -j(j + 1)$, $j \in \mathbb{N}_0$.[10] The basis vectors are labelled by $|p_s\lambda\rangle_{c_2}$, where in case (i) $\lambda \in \mathbb{Z}$, and in case (ii) $\lambda = j + 1, j + 2, \ldots$ or $\lambda = -(j + 1), -(j + 2), \ldots$. Given any of these representations, we obtain the general basis vectors of a unitary irrep of the Poincare group as $|\mathbf{p}\lambda\rangle \equiv H(p) |p_s\lambda\rangle$, where this time $H(p) := R_3(\phi)L_1(\eta)L_3(\zeta)$. As in Proposition 4.1, we have

$$T(b) |\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle e^{-ib_\mu p^\mu}, \tag{58}$$

$$\Lambda |\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda'\rangle D_{c_2}(H^{-1}(\Lambda p)\Lambda H(p))^{\lambda'}_{\lambda}, \tag{59}$$

where D_{c_2} is the matrix of the c_2-representation of $\mathrm{SO}(2,1)$.

References

1. Niklas Beisert, *Symmetries in physics.*
2. Shlomo Sternberg, *Group theory and physics.*
3. Wu-Ki Tung, *Group theory in physics.*

[9] In this case $\mathrm{SO}(2,1)$ is simple, so our difficulties in proving this claim disappear.

[10] If double-valued representations are included, then $j = 1/2, 3/2, \ldots$ are also allowed.