Unitary representations of space(time) symmetries

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Euclidean groups

It appears, empirically, that the laws of physics are invariant under spatial rotations and translations.

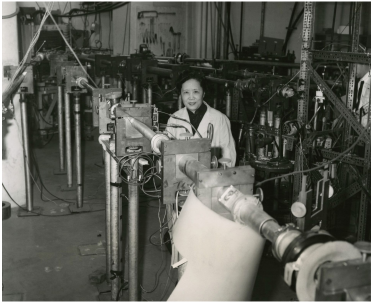
Definition

The **Euclidean group** $\mathrm{E}(n)$ is the isometry group of the Euclidean space $\mathbb{E}^n$.

We restrict our attention to the special Euclidean group

$$\mathrm{SE}(n) := \mathrm{T}(n) \rtimes \mathrm{SO}(n).$$

We also choose to study the cases $n = 2$ and $n = 3$.



Wu experiment, 1956

Special Euclidean group in two-dimensional space

Faithful representation on $\mathbb{R}^3$ given by

$$g(b,\theta) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & b_1 \\ \sin\theta & \cos\theta & b_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta \in [0, 2\pi)$ and $b \in \mathbb{R}^2$.

A general element of the subgroup SO(2) is $R(\theta) := g(0,\theta) = e^{-i\theta J}$, with

$$J = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The subgroup of translations needs two generators P_1 and P_2.

Proposition

The generators of the Lie algebra $i\mathfrak{se}(2)$ satisfy

$$[P_1, P_2] = 0, \quad [J, P_1] = iP_2, \quad \textit{and} \quad [J, P_2] = -iP_1.$$

Unitary irreps by the Lie algebra method

We have uninteresting irreps that factor through the quotient SE(2)/T(2).

1. We give ourselves a Lie algebra representation $\rho : \mathfrak{se}(2)_{\mathbb{C}} \to \mathrm{End}(\mathcal{H})$.
2. We define $\mathbf{P}^2 = \rho(P_1)^2 + \rho(P_2)^2$ and the ladder operators $P_\pm = P_1 \pm iP_2$ satisfying $[P_+, P_-] = 0$, $[J, P_\pm] = \pm P_\pm$. Then $\{J, P_\pm\}$ generates $\mathfrak{se}(2)_{\mathbb{C}}$.
3. We write $\mathcal{H} = \oplus_p \mathcal{H}_p$, where each $\mathcal{H}_p$ is the eigenspace corresponding to some eigenvalue $p^2 \geq 0$ of $\mathbf{P}^2$.
4. Since $[\mathbf{P}^2, \rho(J)] = [\mathbf{P}^2, \rho(P_\pm)] = 0$, each subspace $\mathcal{H}_p$ is invariant under the action of $\mathfrak{se}(2)_{\mathbb{C}}$. We hence fix $p \geq 0$.
5. We pick a $\rho(J)$-eigenbasis $\{e_m\}$ for $\mathcal{H}_p$ labelled by integers. Then

$$\begin{aligned}
\|\rho(P_\pm)e_m\|^2 &= \langle \rho(P_\pm)e_m, \rho(P_\pm)e_m\rangle = \langle e_m, \rho(P_\pm)^*\rho(P_\pm)e_m\rangle \\
&= \langle e_m, \rho(P_\mp)\rho(P_\pm)e_m\rangle = \langle e_m, \mathbf{P}^2 e_m\rangle \\
&= p^2\|e_m\|^2 = p^2.
\end{aligned}$$

If $p = 0$, each subspace $\langle e_m\rangle$ is an irrep.

Unitary irreps by the Lie algebra method (cont'd)

1. Pick an initial vector e_{m_0} in $\mathcal{H}_p$ with $p > 0$.
2. Apply $\rho(P_\pm)$ successively. No need to consider $\rho(J)$ because
$$\rho(J)\rho(P_\pm)e_m = (\rho(P_\pm)\rho(J) \pm \rho(P_\pm))e_m = (m \pm 1)\rho(P_\pm)e_m.$$
3. For each $k \in \mathbb{N}$, define
$$\tilde{e}_k = (i/p)^k \rho(P_+)^k e_{m_0} \quad \text{and} \quad \tilde{e}_{-k} = (-i/p)^k \rho(P_-)^k e_{m_0}.$$
Then $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of an irrep $\tilde{\mathcal{H}}_p \subseteq \mathcal{H}_p$.
4. Relabel separable space $\tilde{\mathcal{H}}_p$ as $\ell^2(\mathbb{Z})$ and define
$$\rho(J)e_m = me_m, \quad \text{and} \quad \rho(P_\pm)e_m = \mp ipe_{m\pm 1}.$$

Proposition

The faithful unitary irreducible representations of SE(2) *are characterized by a real number* $p > 0$. *With exponential map and* $\rho(J)$, $\rho(P_\pm)$ *as above, we may explicitly define a representation* $U_p : \mathrm{SE}(2) \to \mathrm{GL}(\ell^2(\mathbb{Z}))$.

Special Euclidean group in three-dimensional space

As before, we have a faithful representation, this time on $\mathbb{R}^4$, given by

$$g(Q, b) \mapsto \begin{pmatrix} Q & b \\ 0 & 1 \end{pmatrix},$$

where $Q \in \mathrm{SO}(3)$ and $b \in \mathbb{R}^3$.

Every rotation in SO(3) can be uniquely decomposed in terms of Euler angles into a product

$$R_z(\alpha)R_y(\beta)R_z(\gamma)$$

around the fixed y- and z-axes, where $\alpha, \gamma \in [0, 2\pi)$ and $\beta \in [0, \pi]$.

Again, we have the uninteresting unitary irreducible representations that factor through $\mathrm{SE}(3)/\mathrm{T}(3)$. These are the spin-$j$ representations

$$D_j : \mathrm{SO}(3) \to \mathrm{GL}(\mathbb{C}^{2j+1}).$$

Unitary irreps by the induction method

We follow the recipe to construct a representation for a group $G = N \rtimes H$ if N is a normal abelian subgroup.

1. Pick a χ character of N. The unitary characters of T(3) are labelled by $p \in \mathbb{R}^3$ and given by

$$\chi_p : \mathrm{T}(3) \to \mathbb{S}^1, \quad T(b) \mapsto e^{-i\langle b,p\rangle}.$$

2. Consider the orbit of χ in $\widehat{N}$ for the H-action $h \cdot \chi = \chi \circ \mathrm{adj}(h)^{-1}$.

   Here the SO(3)-action on the dual $\widehat{\mathrm{T}(3)}$ is given by

$$(Q \cdot \chi_p)(T(b)) = \chi_p\left(g(Q,0)^{-1}T(b)g(Q,0)\right) = \chi_p\left(T(Q^{-1}b)\right).$$

3. Pick a unitary irrep of the stabilizer in H of χ for this action. For $p \in \mathbb{R}^3$, we call the stabilizer $\mathrm{SO}(3)_p$ of χ_p its **little group**. Since $Q^T = Q^{-1}$,

$$\mathrm{SO}(3)_p = \{Q \in \mathrm{SO}(3) \mid Qp = p\}.$$

Unitary irreps by the induction method (cont'd)

3. When $p = 0$, we have $\mathrm{SO}(3)_p = \mathrm{SO}(3)$. Otherwise, p is characterized by two angles (θ, ψ), and $\mathrm{SO}(3)_p \cong \mathrm{SO}(2)$. Hence for each $\lambda \in \mathbb{Z}$ we have a unitary irrep $Q \mapsto e^{-i\lambda\psi}$, where $\psi \in [0, 2\pi)$ satisfies
$$R(0,0,\psi) = R(\theta,\phi,0)^{-1} Q R(\theta,\phi,0).$$
4. Make $N \rtimes H_\chi$ act on the previous representation space. We obtain a unitary representation $U_\lambda : \mathrm{T}(3) \rtimes \mathrm{SO}_p(3) \to \mathbb{C}^\times$ given by
$$U_\lambda(T(b)Q) = \chi_p(T(b))e^{-i\lambda\psi} = e^{-i\langle b,p\rangle} e^{-i\lambda\psi}.$$
5. Induce from $N \rtimes H_\chi$ to G.

Proposition

The faithful unitary irreducible representations of $\mathrm{SE}(3)$ *are characterized by a real number* $p > 0$ *and an integer* λ. *They can be realized as*
$$\mathrm{Ind}_{\mathrm{T}(3)\rtimes \mathrm{SO}(3)_{p_0}}^{\mathrm{SE}(3)} U_\lambda,$$
where $p_0 \in \mathbb{R}^3$ *satisfies* $\|p_0\| = p$.

Poincaré Group

Definition

The Poincaré group is the isometry group of Minkowski space $\mathbb{R}^{3,1}$.

The generators are the spacetime translations $T(b)$, $b \in \mathbb{R}^{3,1}$, and Lorentz transformations $\Lambda \in O(3,1)$

The (restricted) Poincaré group $\mathrm{T}(4) \rtimes \mathrm{SO}^{\uparrow}(3,1)$ is the full symmetry group of any relativistic field theory, and thus **elementary particles are irreps of this group**.

We establish Wigner's classification of the unitary irreps of positive energy:

- reps. corresponding to particles, indexed by mass $M \geq 0$ and spin $s \in \frac{1}{2}\mathbb{N}$
- non-physical "continuous spin" representation for massless states.

Also: non-physical negative energy representations (tachyons).

Unitary representations of the Poincaré group

Faithful unitary reps. of the Poincaré group are infinite dimensional.

We use method of induced representations to find them:

1. Choose basis from eigenvectors p^μ of the generators of translations P^μ.
2. Given p^μ, identify the maximal subgroup of Lorentz group leaving p^μ invariant ("little group").
3. From a unitary irrep of the "little group" on the p^μ-eigenspace, induce one for the Poincaré group by Lorentz-transforming the basis vectors.

Proposition

$C_1 \equiv -P_\mu P^\mu = P_0^2 - \mathbf{P}^2$ *is a Casimir operator of the Poincaré group.*

We will classify irreps by region of the light-cone in which p^μ (sign of eigenvalue c_1).

Null vector $c_1 = 0, p^\mu = 0$

The null vector $p^\mu \equiv 0$ is Lorentz invariant so its little group is the Lorentz group.

Unitary irreps of the Lorentz group are indexed by two parameters j_0, ν.

We induce a unitary irrep of the Poincaré group in which the basis consists of common eigenvectors $|0jm\rangle$ of (P^μ, J^2, J_3) with resp. eigenvalues $(0, j, m)$.

Given matrices $D_{j_0,\nu}(\Lambda)$ from unitary rep. of the Lorentz group, we have:

$$T(b)\,|0jm\rangle = |0jm\rangle\,, \tag{1}$$

$$\Lambda\,|0jm\rangle = |0j'm'\rangle\, D_{j_0,\nu}(\Lambda)_{jm}^{j'm'}\,. \tag{2}$$

This representation describes the vacuum.

Time-like vector $c_1 = -p_\mu p^\mu = M^2$

We define a “standard vector”: $p_t^\mu \equiv (p_0, \mathbf{p}) = (M, \mathbf{0})$.

The little group of p_t is SO(3) whose unitary irreps are the spin $j = s$ reps. D_j.

The basis vectors of the p_t^μ-eigenspace $|\mathbf{0}, \lambda\rangle$ satisfy

$$P^\mu |\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle \, p_t^\mu \qquad ; p_t^\mu = (M, \mathbf{0}), \tag{3}$$

$$J^2 |\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle \, s(s+1), \tag{4}$$

$$J_3 |\mathbf{0}, \lambda\rangle = |\mathbf{0}, \lambda\rangle \, \lambda. \tag{5}$$

A general Lorentz transformation is obtained as two rotations and a boost along the z axis: $\Lambda = R(\alpha, \beta, 0) L_3(\zeta) R^{-1}(\phi, \theta, \psi)$, where $v = c \, \mathtt{tanh}(\zeta)$.

We obtain a complete basis of general eigenvectors by Lorentz-transforming, giving $|\mathbf{p}\lambda\rangle \equiv H(P) \, |\mathbf{0}\lambda\rangle$ where $H(p) = R(\alpha, \beta, 0) L_3(\zeta)$.

Time-like vector $c_1 = -p_\mu p^\mu = M^2$ (cont'd)

Proposition

The Poincaré group acts on the span of $\{|\mathbf{p}\lambda\rangle\}$ *as*

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-ib_\mu p^\mu} \tag{6}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\mathbf{p}'\lambda'\rangle\, D_s[R(\Lambda, p)]^{\lambda'}_{\lambda} \tag{7}$$

where $p'^{\mu} = \Lambda^\mu_\nu p^\nu$ *and* $R(\Lambda, p) \equiv H^{-1}(p)\Lambda H(p)$. *This rep., labelled by* (M, s), *is a unitary irrep.*

Irreducibility: $\{|\mathbf{p}\lambda\rangle\}$ may be generated from e.g $|\mathbf{0}\lambda = s\rangle$ by applying $\{J_\pm\}$ and the $\{H(p)\}$, there is no nontrivial invariant subspace.

Unitarity: the rep. matrices on the RHS of (6)- (7) are unitary.

The $|M, s; \mathbf{p}\lambda\rangle$ represent states of **mass** M, 3-momentum $\mathbf{p}$, and **intrinsic spin** s. The eigenvalue λ is of $J_3 = \mathbf{J}\cdot\mathbf{P}/|\mathbf{p}|$ corresponds to the *helicity*.

The second Casimir operator

As the "mass" M is related to $C_1 = -P_\mu P^\mu$, the "spin" s is related to a second Casimir of the Poincaré group:

Definition

The Pauli-Lubanski Vector is $W^\lambda = \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma / 2$.

We have $W_\lambda P^\lambda = 0$ and the commutation relations

$$[W^\lambda, P^\mu] = 0 \tag{8}$$

$$[W^\lambda, J^{\mu\nu}] = i(W^\mu g^{\lambda\nu} - W^\nu g^{\mu\lambda}) \tag{9}$$

$$[W^\lambda, W^\sigma] = i\epsilon^{\lambda\sigma\mu\nu} W_\mu P_\nu \tag{10}$$

The second Casimir operator (cont'd)

Proposition

$C_2 \equiv W^\lambda W_\lambda$ is a Casimir of the Poincaré group. On the $\{p^\mu\}$-eigenspace, the components of $\{W^\mu\}$ form the Lie algebra of little group of p^μ. The unitary irreps of the Poincaré group are indexed by the eigenvalues of the $-P^2$ and $-W^2/p^2$.

Example: for $p^\mu = p_t^\mu = (M, \mathbf{0})$, $W = (0, M\mathbf{J})$ whose components generate SO(3).

For the remaining cases we will construct the little group from the Lie algebra generated by $\{W^\mu\}$.

Light-Like Vector $p_\mu p^\mu = 0$

We define the standard light-like vector: $p_l^\mu \equiv (\omega_0, 0, 0, \omega_0)$ with $\omega_0 \neq 0$.

We obtain a general momentum $p^\mu = (\omega, \mathbf{p})$, where $\mathbf{p} = \omega\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}$ has angles (θ, ϕ), as:

$$p^\mu = H(p)^\mu_\nu p_l^\nu = [R(\phi, \theta, 0)L_3(\zeta)]^\mu_\nu p_l^\nu \tag{11}$$

The components of $W^\lambda = \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} p_{l\sigma}/2$ are, in terms of rotations and boosts:

$$W^0 = W^3 = \omega_0 J_{12} = \omega_0 J_3 \tag{12}$$

$$W^1 = \omega_0(J_{23} + J_{20}) = \omega_0(J_1 + K_2) \tag{13}$$

$$W^2 = \omega_0(J_{31} - J_{10}) = \omega_0(J_2 - K_1) \tag{14}$$

and $C_2 = W_1^2 + W_2^2$. Lie algebra of the little group:

$$[W^1, W^2] = 0, \quad [W^2, J_3] = iW^1, \quad \text{and} \quad [W^2, J_3] = -iW^2 \tag{15}$$

This is the same Lie algebra as for SE(2)!

Light-Like Vector $p_\mu p^\mu = 0$ (cont'd)

Unitary irreps of SE(2) are either:

- degenerate ($w = 0$), basis vectors labeled by eigenvalues λ of J_3
- non-degenerate and infinite-dimensional ($w > 0$), basis vectors $\{|w, \lambda\rangle, \lambda = 0, \pm 1, ..\}$

The "continuous spin representation" ($M = 0, w > 0$) has not yet observed in nature.

($M = w = 0, \lambda$) reps. correspond to photon/gluons ($\lambda = \pm 1$), and graviton ($\lambda = \pm 2$) states.

We consider only the degenerate case, with little group SO(2).

Light-Like Vector $p_\mu p^\mu = 0$ (cont'd)

p_l^μ-eigenspace is one-dimensional and the basis vector $|\mathbf{p_l}\lambda\rangle$ transforms as

$$P^\mu |\mathbf{p_l}\lambda\rangle = |\mathbf{p_l}\lambda\rangle\, p_l^\mu \tag{16}$$

$$J_3 |\mathbf{p_l}\lambda\rangle = |\mathbf{p_l}\lambda\rangle\, \lambda \tag{17}$$

$$W_i |\mathbf{p_l}\lambda\rangle = 0 \quad ; \quad i = 1, 2 \tag{18}$$

When λ is an integer (resp. odd half-integer), we obtain a single (resp. double) valued rep.

The general basis vector is obtained as $|\mathbf{p}\lambda\rangle \equiv H(p)\,|\mathbf{p_l}\lambda\rangle$, where $p = \omega_0 e^\zeta$.

Proposition

The Poincaré group acts on the span of $\{|\mathbf{p}\lambda\rangle\}$ as

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-ib_\mu p^\mu} \tag{19}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda\rangle\, e^{-i\lambda\theta(\Lambda,p)} \tag{20}$$

where $e^{-i\lambda\theta(\Lambda,p)} = \langle\mathbf{p_l}\lambda|\, H^{-1}(\Lambda p)\Lambda H(p)\,|\mathbf{p_l}\lambda\rangle$. This rep., labeled by $(M = 0, \lambda)$, is a unitary irrep.

Space-like vector $c_1 = -p_\mu p^\mu < 0$

Space-like vectors cannot represent physical particles, but are relevant for deep inelastic scattering.

We define the standard vector: $p_s^\mu \equiv (0, 0, 0, Q)$ where $Q^2 = -c_1 > 0$.

Little group is generated by the components:

$$W^0 = QJ_3 \tag{21}$$

$$W_1 = QJ_{20} = QK_2 \tag{22}$$

$$W_2 = QJ_{01} = -QK_1 \tag{23}$$

and coincides with SO(2, 1). The second Casimir is

$$C_2 = Q^2(K_1^2 + K_2^2 - J_3^2) \tag{24}$$

The Lie algebra of the little group is

$$[K_2, J_3] = iK_1, \quad [J_3, K_1] = iK_2, \quad \text{and} \quad [K_1, K_2] = -iJ_3. \tag{25}$$

SO(2, 1) is simple and non-compact (contains boosts). Its unitary irreps are thus infinite-dimensional.

Space-like vector $c_1 = -p_\mu p^\mu < 0$ (cont'd)

Unitary irreps of SO$(2,1)$ come in two classes, labeled by c_2, with either $c_2 \in (0,\infty)$ or $c_2 = -j(j+1)$, where $j \in \mathbb{N}_0$.

For a given c_2, we label basis vectors as $|p_s\lambda\rangle$.

We obtain a general basis of a unitary irrep of the Poincaré group is obtained as: $|\mathbf{p}\lambda\rangle \equiv H(p)\,|p_s\lambda\rangle$ where $H(p) \equiv R_3(\phi)L_1(\eta)L_3(\zeta)$.

Proposition

The Poincaré group acts on the span of $\{|\mathbf{p}\lambda\rangle\}$ *as*

$$T(b)\,|\mathbf{p}\lambda\rangle = |\mathbf{p}\lambda\rangle\, e^{-ib_\mu p^\mu}, \tag{26}$$

$$\Lambda\,|\mathbf{p}\lambda\rangle = |\Lambda\mathbf{p}\lambda'\rangle\, D_{c_2}[H^{-1}(\Lambda p)\Lambda H(p)]^{\lambda'}_{\lambda}, \tag{27}$$

where D_{c_2} *is the matrix in the* c_2*-representation of* $SO(2,1)$.