PHYS3071/7073 - Due Thursday, 17th April 2014 by 11am

Assignment 6 — ODEs, Part 3 – Friedmann equation

In this assignment you will again study the expansion of a homogeneous and isotropic universe as given by Friedmann's differential equation using a Runge-Kutta integrator. If you have not already done so, work first through the worksheet, which solves the case $\Lambda = 0$ and use the program you get as basis for this assignment. In this assignment, we will integrate the full equation allowing for a non-zero cosmological constant. This means that the expansion of the universe is described by the following differential equation:

$$\ddot{a} = \frac{1}{2}a\Lambda c^2 - \frac{1}{2}\frac{\dot{a}^2}{a} \tag{1}$$

where again $a(t)$ is the scale factor of the universe given by $a(t) = R(t)/R(t_0)$. Observations indicate that $\Lambda = 1.25 \cdot 10^{-52}$ m^{-2}. In order to solve eq. 1 numerically, we introduce a new quantity $b(t) = \dot{a}$. We can then split up eqn. 1, which is a second-order differential equation, into the following system of coupled first-order differential equations:

$$\dot{a} = b \tag{2}$$

$$\dot{b} = \frac{1}{2}a\Lambda c^2 - \frac{1}{2}\frac{b^2}{a} \tag{3}$$

The starting conditions for the solution are the current scale factor $a_0 = 1.0$ and the current expansion rate b_0.

1 Problem 1/5

Use the program that you wrote for the worksheet and modify it such that it simultaneously integrates the two differential equations eqs. 2 and 3 from the present time back to the Big Bang. Your program should take as input the time step Δt, an output time step Δt_{out} (both in Gyr), the Hubble constant and the cosmological constant. The user should enter the Hubble constant in units of km/sec/Mpc and the cosmological constant in 1/m^2, but your program should rescale the Hubble constant to 1/Gyr and Λc^2 to 1/Gyr^2 before starting the integration. Use again a function which makes one Runge-Kutta step and takes as input the current values of a_i and b_i together with the time step Δt and calculates a_{i+1} and b_{i+1}. In order to pass these values from the function back to the main program use either pointers (not yet explained in class) or global variables. At each output time your program should print the current values of t, a and b into a file. Stop the integration when either $a < 0.01$ or $b < 0$. When finished your program should print out the age of the universe that it has calculated.

2 Problem 2/5

Run the program that you wrote in part 1 with $\Delta t = 0.001$, $\Delta t_{out} = 0.1$ and the following values for b_0 and Λ: $b_0 = \{70, 100\}$ km/sec/Mpc and $\Lambda = \{0.0, 1.25 \cdot 10^{-52}$ m$^{-2}\}$ and use gnuplot to plot all four curves that you get into a file `as06-problem2-studentname-studentid.ps`.

3 Problem 3/5

Use your program to answer the following questions:

- How does the age of the universe change if the cosmological constant is positive compared to the case $\Lambda = 0$? Can one understand this change from the differential equation?
- Observations show that the ages of the oldest stars are around 13 Gyr. With which of the above cosmologies is this observation compatible?

4 Problem 4/5 (Advanced)

We are now going to use adaptive step size control for the integration. As explained in the lecture, the best way to implement adaptive step size control in a Runge-Kutta integrator is to cover an integration integral twice, once by doing a single step with step size Δt and once by doing two steps with step size $\Delta t/2$. If the resulting difference in the solutions $\Delta = |f^{(1)}_{i+1} - f^{(2)}_{i+1}|$ is smaller than a certain maximum value Δ_{Max} we accept the step and use the solution from the two time steps with size $\Delta t/2$ as the starting point for the next step. We also calculate a new time step Δt_{new} through

$$\Delta t_{new} = 0.995 \Delta t \left(\frac{\Delta_{Max}}{\Delta} \right)^{0.2} \tag{4}$$

If the difference between the two solutions is larger than Δ_{Max} then one has to repeat the step with a stepsize that is given by eq. 4. Use the difference in the scale factor a to determine the accuracy of your integration and use $\Delta_{Max} = 10^{-10}$. Also make sure that during the integration time steps never get larger than the output time step, i.e. use $\Delta t_{new} = max(\Delta t_{out}, \Delta t'_{new})$ where $\Delta t'_{new}$ is calculated according to eq. 4. At each output, your program should print out the current time, the current scale factor a and the current $\dot{a}$. If you overshoot an output time, use an Euler step of size $t - t_{out}$, to print out the above values at the current output time (but keep the original values to continue the integration !!). At the end of the integration your program should print out the final scale factor, the final $\dot{a}$ (in units of km/sec/Gyr), the total number of steps taken and the number of rejected steps.

5 Problem 5/5 (Advanced)

Use the program created for problem 4 and study the future expansion history of the universe. Use the same values for b_0 and Λ as in problem 3 and follow the evolution up to $T = 1000$ Gyr from now. Determine the final scale factor at that time for all four cases and add the values to the file with your answers created for problem 3.

Submission

The full set of files you will have when you are done should be:

- A C program named `as06-problem1-studentname-studentid.c` used for problems 1 to 3.
- A C program named `as06-problem4-studentname-studentid.c` used for problems 4 and 5.
- A plot showing the four a vs. t curves in the same graph in a file named `as06-problem2-studentname-studentid.ps`.
- A textfile `as06-answers-studentname-studentid.txt` with your answers to the problems.

To submit your assignment, create a new directory `as06` and upload the above files to your directory, eg. `s1234567/as06/`. Upload your C programs and the text file with COURSENUMBER AS06 SURNAME STUDENTID (case doesn't matter) at the top. Your postscript figure should have the same information in its title, i.e. use `set title <COURSENUMBER> AS06 <SURNAME> <STUDENTID>`. Make sure all files are named correctly. **This information is important for us when marking the assignments and we will have to deduct marks if it is missing or a file has the wrong name.**

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Grading Sheet – Assignment 6: Friedmann equation

A: /30% **Function**: Does the program compile, run and produce the correct output?

B: /10% **Readability**: Is the program easy to read and comprehend? Is it well-commented? If the code is sufficiently complex, has it been broken up into manageable subroutines, each of which is well-documented?

C: /10% **Usability**: Is the program easy to use? Are the input requirements and output formatting easy to understand?

D: /10% **Efficiency**: Does the program run efficiently? Is the coding clunky or unnecessarily complicated?

E: /10% **Analysis**: Were correct answers given to the questions asked in the assignment, and was the process used to obtain them reasonable and clearly explained?

F: /10% **Presentation (Plots)**: Do the plots clearly convey the results? Does each plot have an appropriate title? Are the axes and the plot items clearly labeled?

G: /20% **Advanced part**: The marking criteria for the advanced part (problems 4 and 5) are the same as detailed under points A-E.

Total Points: $\boxed{\qquad /100}$