Question 1

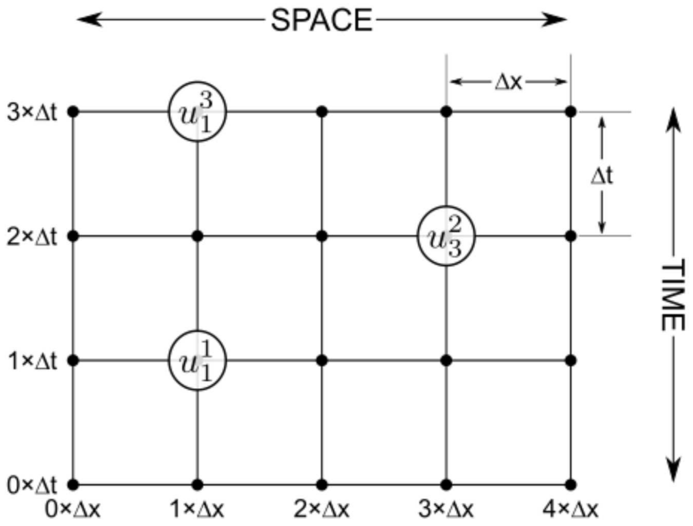

Figure 1: Visualisation of the FTCS scheme.

Usig the notation $u_j^n = u(t_n, x_j)$ for $x_j = x_0 + j\Delta x$, $j = 0, 1, 2..., J$ and $t_n = t_0 + n\Delta t$, $n = 0, 1, 2, ...N$ to approximate the PDE $\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x}$. Starting with the approximations:

$$\left.\frac{\partial u}{\partial t}\right|_{j,n} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$\left.\frac{\partial u}{\partial x}\right|_{j,n} \approx \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

we rearrange our PDE, $\dfrac{\partial \rho}{\partial t} + v_{max}\dfrac{\partial \rho}{\partial x}\left(1 - \dfrac{2\rho}{\rho_{max}}\right) = 0$, where $\rho = \rho(x,t)$ to the form $\dfrac{\partial \rho}{\partial t} = -v_{max}\dfrac{\partial \rho}{\partial x}\left(1 - \dfrac{2\rho}{\rho_{max}}\right)$.

Now we can approximate our PDE using the listed approximations above to:

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} = -v_{max}\left(\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x}\right)\left(1 - \frac{2\rho_j^n}{\rho_{max}}\right)$$

$$\rho_j^{n+1} = \rho_j^n - v_{max}\Delta t\left(\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x}\right)\left(1 - \frac{2\rho_j^n}{\rho_{max}}\right)$$

And using the Lax-Friedrich approximation $u_j^n = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$ and bringing $\frac{1}{2\Delta x}$ outside of the bracket, we end up with:

$$\rho_j^{n+1} = \frac{1}{2}\left(\rho_{j+1}^n + \rho_{j-1}^n\right) - \frac{v_{max}\Delta t}{2\Delta x}\left(\rho_{j+1}^n - \rho_{j-1}^n\right)\left(1 - \frac{2\rho_j^n}{\rho_{max}}\right)$$