PHYS3071/7073 - Due Friday, 30th of May, 2014 by 11am

Assignment 11 —- Traffic Jams on Coronation Drive

Note: I again recommend to solve the problems on worksheet 20 first before working on the assignment problems.

Theory

Consider the traffic flow of cars on a single-lane highway where overtaking is not possible. Instead of modeling the cars individually, we use a density function $\rho(x,t)$ to describe the number of vehicles per kilometer on the road at time $t \geq 0$. We introduce a velocity $v(t,x)$ which describes the velocity of cars at point x and time t. Since the change in the number of cars between two points x_1 and x_2 is just the difference between the number of cars entering at x_1 and the number of cars leaving at x_2, i.e.

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t) \tag{1}$$

the change in car density is described by the following partial-differential equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial v(x,t)\rho(x,t)}{\partial x} = 0 \tag{2}$$

We assume that cars will travel with a maximum velocity $v = v_{max}$ if the highway is empty ($\rho = 0$) and that traffic grinds to a standstill if cars are bumper to bumper. i.e. the density reaches a maximum ρ_{max}. In between these two limits we make a linear interpolation:

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) \quad \text{for} \quad 0 \leq \rho \leq \rho_{max} \tag{3}$$

Insertion of eq. 3 into eq. 2 gives:

$$\frac{\partial \rho}{\partial t} + v_{max}\frac{\partial \rho(x,t)}{\partial x}\left(1 - \frac{2\rho(x,t)}{\rho_{max}}\right) = 0 \tag{4}$$

Problem 1/3

Show that using the Lax-Friedrich scheme with central differences for the $x-$derivatives, one obtains the following equation for the change of the ρ_j^n:

$$\rho_j^{n+1} = \frac{1}{2}(\rho_{j-1}^n + \rho_{j+1}^n) - \frac{\Delta t v_{max}}{2\Delta x}\left(\rho_{j+1}^n - \rho_{j-1}^n\right)\left(1 - \frac{2\rho_j^n}{\rho_{max}}\right) \tag{5}$$

Here we have used the same notation as in class, i.e. the n indices describe time evolution and the j indices describe different points along the x-axis, such that $t = n\Delta t$ and $x = j\Delta x$. Submit your derivation as a PDF file named `as11-problem1-studentname-studentid.pdf`. You can either use programs like Word, TeX etc. to write down your solution, or do it by hand and scan/photograph your solution. In the latter case, you can use `convert` to convert your solution from a standard picture format like gif or jpg to PDF.

Problem 2/3

Write a C program that takes as input:

- The maximum velocity of cars, v_{max}
- The maximum density ρ_{max}
- The timestep size Δt
- The grid size Δx
- The end time t_f until which the PDE should be integrated

Your program should then integrate eq. 4 using the Lax-Friedrich scheme as given by eq. 5. Assume as boundary conditions $\rho_0^n = 0$ and $\rho_J^{n+1} = \rho_{J-1}^n$ for all times and integrate between $0 < x < 4$. Your program should print out the current values of the ρ_j^n every $\Delta t_{out} = 1.0$ into a file with a different name. Your program should also print out to the screen the location of the maximum at each output time. Name your program `as11-problem2-studentname-studentid.c`.

Problem 3/3

Run the program that you wrote for problem 1 with $\Delta x = 0.001$, $\Delta t = 0.002$, $v_{max} = 0.2$, $\rho_{max} = 1.0$ and $t_{fin} = 20.0$. Assume as initial condition $\rho(x,0) = 0.5 \cdot \rho_{max} sin(\pi/2x)$ for $0 < x < 2$ and zero otherwise. Use gnuplot to plot the solution at $T = 0.0$, $T = 2.0$, $T = 5.0$, $T = 10.0$ and $T = 15.0$ and name this plot `as11-problem3-studentname-studentid.ps`. Use either your program or the plot to answer the following two questions: How quickly does the place of maximum density move along the road at later times? How does this compare with the expression for $v(\rho)$ given this density?

Submission

The full set of files you will have when you are done should be:

- `as11-problem1-studentname-studentid.pdf`
- `as11-problem2-studentname-studentid.c`
- `as11-problem3-studentname-studentid.ps`
- `as11-answers-studentname-studentid.txt`

As always, to submit your assignment, mount your directory on the submission server, create a subdirectory called `as11` under your student ID, eg. `s1234567/as11/` and copy your files into this subdirectory. You must submit your files into this subdirectory. Be sure to follow the above submission instructions, otherwise you make the marking of the assignments unnecessary complicated for us.

PHYS3071/7073 - Due Friday, 30th of May, 2014 by 11am

Grading Sheet – Assignment 11 —- Traffic Jams on Coronation Drive

---

A: /20% **Theory**: Was a derivation of eq. 5 submitted with the solution and is this derivation correct?

B: /30% **Function**: Does the program run and produce the correct output?

C: /10% **Usability**: Is the program easy to use? Are the input requirements and output formatting easy to understand?

D: /10% **Readability**: Is the program easy to read and comprehend? Is it well-commented? If the code is sufficiently complex, has it been broken up into manageable subroutines, each of which is well-documented?

E: /10% **Efficiency**: Does the program run efficiently? Is the coding clunky or unnecessarily complicated?

F: /10% **Presentation (Plots)**: Do the plots clearly convey the results? Does each plot have an appropriate title? Are the axes and the plot items clearly labeled? Was the correct style (points or lines) used for each item?

G: /10% **Analysis**: Were correct answers given to the questions asked in the assignment, and was the process used to obtain them reasonable and clearly explained?

Total Points: $\boxed{\quad\quad /100}$