Stanford Math 51: Problem Set 7

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1 Problem 19.2

(a) Orthonormalization

Proof. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$, and let $|| \cdot ||$ be the l^2 norm. By Gram-Schmidt orthonormalization,

$$w_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$w_2 = \frac{v_2 - \langle w_1, v_2 \rangle w_1}{||v_2 - \langle w_1, v_2 \rangle w_1||} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_3 \propto v_3 - \langle w_1, v_3 \rangle w_1 - \langle w_2, v_3 \rangle w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which demonstrates $w_3 = 0$. □

(b) Backwards

Proof. Observe that v_2 and v_3 can be expressed as

$$v_2 = -\sqrt{5}w_1 + 3w_2, \quad v_3 = -\sqrt{5}w_1 + w_2$$

The verification is as follows:

$$-\sqrt{5}w_1 + 3w_2 = -\sqrt{5} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \equiv v_2$$

$$-\sqrt{5}w_1 + w_2 = -\sqrt{5} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \equiv v_3$$

□

(c) Linear Combination

Proof. Note that

$$2v_1 - v_2 + 3v_3 = \begin{bmatrix} 0 - 3 + 3 \\ -2 - 1 + 3 \\ 4 + 2 - 6 \end{bmatrix} = 0$$

It is immediate that

$$v_3 = -\frac{2}{3}v_1 + \frac{1}{3}v_2, \quad v_1 = \frac{1}{2}v_2 - \frac{3}{2}v_3$$

The verification is

$$-\frac{2}{3}v_1 + \frac{1}{3}v_2 = -\frac{2}{3}\begin{bmatrix}0\\-1\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}3\\1\\-2\end{bmatrix} = \begin{bmatrix}1\\1\\-2\end{bmatrix} \equiv v_3$$

$$\frac{1}{2}v_2 - \frac{3}{2}v_3 = \frac{1}{2}\begin{bmatrix}3\\1\\-2\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}0\\-1\\2\end{bmatrix} \equiv v_1$$

□

2 Problem 19.3

(a) Gram-Schmidt

Proof. Observe that

$$w_1 = \frac{v_1}{||v_1||} = \frac{1}{\sqrt{14}}\begin{bmatrix}0\\2\\3\\-1\end{bmatrix}$$

$$w_2 = \frac{v_2 - \langle w_1, v_2\rangle w_1}{||v_2 - \langle w_1, v_2\rangle w_1||} = \frac{1}{\sqrt{7}}\begin{bmatrix}-1\\2\\-1\\1\end{bmatrix}$$

$$w_3 = \frac{v_3 - \langle w_1, v_3\rangle w_1 - \langle w_2, v_3\rangle w_2}{||v_3 - \langle w_1, v_3\rangle w_1 - \langle w_2, v_3\rangle w_2||} = \frac{1}{\sqrt{14}}\begin{bmatrix}2\\0\\1\\3\end{bmatrix}$$

Because w_1, w_2, w_3 are all linear combinations of v_1, v_2, v_3 (demonstrated in the next part), they span the same subspace. Thus since all w_i are mutually orthogonal, $\dim \operatorname{span}(v_1, v_2, v_3) = \dim \operatorname{span}(w_1, w_2, w_3) = 3$. It follows the v_i are linearly independent.

□

(b) Linear Combination

Proof. Observe that, if $\operatorname{Unit}(v) := v/||v||$ for $v \in \mathbb{R}^n$, then

$$\begin{aligned}
w_1 &= \frac{v_1}{||v_1||} = \frac{1}{\sqrt{14}}v_1 \\
w_2 &= \operatorname{Unit}\left(v_2 - \langle w_1, v_2\rangle \frac{v_1}{||v_1||}\right) = \frac{1}{2\sqrt{7}}(v_2 - 3v_1) \\
w_3 &= \operatorname{Unit}\left(v_3 - \langle w_1, v_3\rangle w_1 - \langle w_2, v_3\rangle w_2\right) = \operatorname{Unit}(v_3 - v_1 + 2(v_2 - 3v_1)) = \frac{1}{3\sqrt{14}}(v_3 - 7v_1 + 2v_2)
\end{aligned}$$

Then for the reverse direction,

$$\begin{aligned}
v_1 &= \sqrt{14}w_1 \\
v_2 &= 2\sqrt{7}w_2 + 3v_1 = 2\sqrt{7}w_2 + 3\sqrt{14}w_1 \\
v_3 &= 3\sqrt{14}w_3 + 7v_1 - 2v_2 = 3\sqrt{14}w_3 + 7\sqrt{14}w_1 - 2(2\sqrt{7}w_2 + 3\sqrt{14}w_1) = 3\sqrt{14}w_3 + \sqrt{14}w_1 - 4\sqrt{7}w_2
\end{aligned}$$

The computation checks out for w_3, as $(v_3 - 7v_1 + 2v_2)/3\sqrt{14} = 1/\sqrt{14}\begin{bmatrix}2 & 0 & 1 & 3\end{bmatrix}^t$. It also checks out for v_3, as $3\sqrt{14}w_3 + \sqrt{14}w_1 - 4\sqrt{7}w_2 = \begin{bmatrix}10 & -6 & 10 & 4\end{bmatrix}^t$. □

(c) Orthonormal Basis

Observe that w_1, w_2, w_3 are orthonormal by construction. Thus

$$w_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad w_2 = \frac{1}{\sqrt{7}} \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad w_3 = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

3 Problem 19.7

(a) Linear Dependence

For the first relation,

$$3v_1 + 2v_2 - 3v_3 - 2v_4 = 3 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = 0$$

For the second relation,

$$-5v_1 + 6v_2 - 2v_3 + v_4 = -5 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = 0$$

(b) Linear Combinations

Note that by adding the first relation and twice the second relation,

$$-7v_1 + 14v_2 - 7v_3 = 0 \implies v_3 = -v_1 + 2v_2$$

and that by subtracting two times the first relation minus three times the second relation,

$$21v_1 - 14v_2 - 7v_4 = 0 \implies 3v_1 - 2v_2 = v_4$$

(c) Explicit Evaluations

Observe

$$2v_2 - v_1 = 2 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} = v_3$$

$$3v_1 - 2v_2 = 3 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = v_4$$

4 Problem 20.1

(a) Multiplication

Note

$$x^t x = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1 + 4 + 4 = 9$$

(b) Multiplication

Note that $xx^t = \langle x, x\rangle = x^t x = 9$.

(c) Norms

Observe

$$||Ax||^2 = \left(\text{Norm}\begin{bmatrix}2\\6\\9\\0\end{bmatrix}\right)^2 = 4+36+81 = 121$$

(d) Transformed

Observe that $x^t A^t Ax = (Ax)^t(Ax) = ||Ax||^2 = 121$.

5 Problem 20.2

(a) Quadratic Forms

Note that

$$\begin{aligned} x^t Ax &= \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}x_1+2x_2+2x_3\\ 2x_1+2x_2+x_3\\ 2x_1+x_2+3x_3\end{bmatrix}\\ &= x_1^2+2x_2^2+2x_2x_3+3x_3^2+4x_1x_2+4x_1x_3 \end{aligned}$$

(d) Symmetric Matrix

Observe that

$$A = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & 0 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \implies x^t Ax = x_1^2 - x_4^2 + x_2x_3$$

6 Problem 20.5

(a) Finally a Real Proof

Let $A \equiv (A_{ij})_{\{1\leq i,j\leq n\}}$, and $B \equiv (A + A^t)/2$. Observe that $B_{ij} = (A_{ij} + A_{ji})/2 = (A_{ji} + A_{ij})/2 = B_{ji}$, so B is symmetric.

(b) Another Proof!

Observe that

$$x^t Mx = ||Bx||^2 = (Bx)^t(Bx) = x^t(B^tB)x \implies M = B^tB$$

7 Problem 20.9

(a) Not Worth a Name

Positive-definite. Note that for $x, y, z \neq 0$, then $x^2, y^2, z^2 > 0 \implies q(x, y, z) = 3x^2 + 7y^2 + 2z^2 > 0$.

(b) Why Bro

Indefinite. Consider $(x, y, z) = (0, -1, 0) \implies q(x, y, z) = -1$. But similarly, $(x, y, z) = (1, 0, 0) \implies q(x, y, z) = 5$.

(c) Whyyyyy

Negative-definite. Note for $x, y \neq 0$, then $x^2, y^2 > 0 \implies q(x, y) = -(17x^2 + 23y^2) < 0$.

(d) The End

Indefinite. Note that $q(1, 1) = 8$, but $q(-1, 1) = -8$.

8 Problem 21.2

(a) Kernel

Let T be the linear map corresponding to each matrix when necessary.
Matrix A. Observe that $\dim T(V) = 1$ since the columns of A are linearly dependent. Then by Rank-Nullity Theorem, $\dim V = \dim \ker T + \dim T(V) \implies \dim \ker T = 2 - 1 = 1$.

Matrix B. Let the columns of B be $(c_i)_{1 \leq i \leq 4}$. Then $c_1 = -2c_3 + c_4$, and $c_2 = -5c_3 - 3c_4$. However, note c_3 and c_4 are not linearly dependent, since the second entry of each vector is 1 and 0, respectively. Thus $\dim T(V) = 2$, as only two of the mapped basis vectors are linearly independent. Then by Rank-Nullity Theorem, $\dim V = \dim \ker T + \dim T(V) \implies \dim \ker T = 4 - 2 = 2$.

Matrix C. Let the columns of C be $(c_i)_{1 \leq i \leq 2}$, and observe c_1 and c_2 are obviously linearly independent since $c_1 - 3c_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}^t \neq 0$. Then it is clear that by Rank-Nullity Theorem, $\dim V = \dim \ker T + \dim T(V) \implies \dim \ker T = 2 - 2 = 0$.

(b) Null Space

Matrix A. Note that $Ax = 0$ implies

$$x_1 - 2x_2 = 0, 2x_1 - 4x_2 = 0$$

Matrix B. Note that $Bx = 0$ implies

$$x_1 - 3x_2 + x_4 = 0, 2x_1 + 5x_2 - x_3 = 0, 3x_1 + 2x_2 - x_3 + x_4 = 0$$

Matrix C. Observe that $Cx = 0$ implies

$$3x_1 + x_2 = 0, -4x_1 - x_2 = 0$$

(c) Explicit Kernel

Matrix A. All $x = (x_1, x_2) = (2x_2, x_2) = \alpha(2, 1)$.

Matrix B. Since the third row is the sum of the first two rows, it is a linearly dependent condition and may be ignored. All that is left is $x = (x_1, x_2, x_3, x_4) = (x_1, x_2, 2x_1 + 5x_2, -x_1 + 3x_2) = \alpha(1, 0, 2, -1) + \beta(0, 1, 5, 3)$.

Matrix C. Since the dimension of the kernel is 0, the only vector that maps to zero is zero itself.

9 Problem 21.3

(a) Sad Computation

Observe that there are three unknows but only two equations, so the system is underdetermined. According to the rule of thumb, it ought to have infinitely many solutions.

(b) More Work

There are three equations but two unknowns, so the system is overdetermined. By the rule of thumb it has zero solutions. Given that the system is homogeneous, we know that the system has at least one solution of $(x, y) = (0, 0)$ since the zero vector is always part of the kernel. So the rule of thumb is incorrect.

(c) And Mroe Work

There are four unknowns but just two equations, so the system is underdetermined. By the rule of thumb, it ought to have infinitely many solutions. However, this is silly since the first and second rows are identical, so any solution requires $1 = -1$. Thus there are no solutions; the rule of thumb is incorrect.

10 Problem 21.7

(a) More Kernel Bash

Obviously a_1, a_2 are a basis for $C(A)$. To ensure orthogonality, simply take $a_2' = a_2 - \langle a_1, a_2 \rangle a_1 / ||a_1||^2 = (-2, 1, 1)$. Then (a_1, a_2') also form a basis with $a_1 \cdot a_2' = 0$. Because such an orthogonal basis exists, $\dim C(A) = 2$, and thus by Rank-Nullity $\dim N(A) = \dim \ker A = 2$. The two vectors (let them be v, w respectively) must be in the kernel because then

$$Av = \sum_{i=1}^{4} v_i a_i = -3a_1 + 2a_2 - 1(-3a_1 + 2a_2) + 0 = 0$$

and that

$$Aw = \sum_{i=1}^{4} w_i a_i = a_1 - a_2 + 0 - (a_1 - a_2) = 0$$

and therefore both $v, w \in \ker A$. Thus since these are linearly independent, it follows they form a basis for $N(A)$ since $\dim N(A) = 2$.

(b) Projections

This is completed handwritten.

(c) Parametric Form

This is completed handwritten.

b) $b_1 = \begin{bmatrix}1\\6\\8\end{bmatrix}$:

$\text{Proj}_{C(A)}(b_1) =$

$$\text{Proj}_{C(A)}(b_1) = \frac{b_1\cdot a_1}{a_1\cdot a_1} a_1 + \frac{b_1\cdot a_2'}{a_2' a_2'} a_2'$$

$b_1 \cdot a_1 = 1\cdot 1 + 6\cdot 3 + 8(-1) = 1+18-8 = 11$

$a_1\cdot a_1 = 11$

$b_1 \cdot a_2' = 1(-2) + 6\cdot 1 + 8\cdot 1 = -2+6+8 = 12$

$a_2'\cdot a_2' = (-2)^2 + 1^2 + 1^2 = 4+1+1 = 6$

Thus $\text{Proj}_{C(A)}(b_1) = \frac{11}{11}\begin{bmatrix}1\\3\\-1\end{bmatrix} + \frac{12}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}1\\3\\-1\end{bmatrix} + 2\begin{bmatrix}-2\\1\\1\end{bmatrix}$

$= \begin{bmatrix}1\\3\\-1\end{bmatrix} + \begin{bmatrix}-4\\2\\2\end{bmatrix} = \begin{bmatrix}-3\\5\\1\end{bmatrix}$

since $\text{Proj}_{C(A)}(b_1) \neq b_1$, there is no solution for $Ax = b_1$.

For $b_2 = \begin{bmatrix}-5\\-1\\3\end{bmatrix}$:

$$\text{Proj}_{C(A)}(b_2) = \frac{b_2\cdot a_1}{a_1\cdot a_1} a_1 + \frac{b_2\cdot a_2'}{a_2'\cdot a_2'} a_2'$$

$b_2\cdot a_1 = -5\cdot 1 + (-1)\cdot 3 + 3\cdot(-1) = -5-3-3 = -11$

Figure 1: Part b, image 1.

$b_2\cdot a_1 = -5\cdot 1 + (-1)\cdot 3 + 3\cdot(-1) = -5-3-3 = -11$

$b_2\cdot a_2' = -5\cdot(-2) + (-1)\cdot 1 + 3\cdot 1 = 10-1+3 = 12$

$\text{Proj}_{C(A)}(b_2) = \frac{-11}{11}\begin{bmatrix}1\\3\\-1\end{bmatrix} + \frac{12}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix} = -\begin{bmatrix}1\\3\\-1\end{bmatrix}$

$+ 2\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\-3\\1\end{bmatrix} + \begin{bmatrix}-4\\2\\2\end{bmatrix}$

$= \begin{bmatrix}-5\\-1\\3\end{bmatrix} = b_2$

$\text{Proj}_{C(A)}(b_2) = b_2$, $Ax = b_2$ has a solution

$b_2 = -a_1 + 2a_2'$

$\begin{bmatrix}-5\\-1\\3\end{bmatrix} = -\begin{bmatrix}1\\3\\-1\end{bmatrix} + 2\begin{bmatrix}-2\\1\\1\end{bmatrix}$

$x = \begin{bmatrix}-1\\2\\0\\0\end{bmatrix}$

$A\begin{bmatrix}-1\\2\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 2 & 1 & -1\\3 & 13 & 17 & -10\\-1 & -3 & -3 & 2\end{bmatrix}\begin{bmatrix}-1\\2\\0\\0\end{bmatrix} = \begin{bmatrix}-5\\-1\\3\end{bmatrix} = b_2$

Figure 2: Part b, image 2.

$b_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$:

$$\text{Proj}_{C(A)}(b_3) = \frac{b_3 \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b_3 \cdot a_2'}{a_2' \cdot a_2'} a_2'$$

$$b_3 \cdot a_1 = 1 \cdot 1 + (-4) \cdot 3 + 0 \cdot (-1) = 1 - 12 = -11$$

$$b_3 \cdot a_2' = 1(-2) + (-4) \cdot 1 + 0 \cdot 1 = -2 - 4 = -6$$

$$\text{Proj}_{C(A)}(b_3) = \frac{-11}{11}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{-6}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$

$\text{Proj}_{C(A)}(b_3) \neq b_3$, there is no solution for $Ax = b_3$.

Figure 3: Part b, image 3.

c) $Ax = b_2$ has a solution $x = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

basis of $N(A)$ $\left\{ \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 - 3t_1 + t_2 \\ 2 + 2t_1 - t_2 \\ -t_1 \\ -t_2 \end{bmatrix} \quad \text{where } t_1, t_2 \in \mathbb{R}$$

Figure 4: Part c.