KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

CLASSICAL MECHANICS I

Department Of Physics
Faculty Of Physical and Computational Sciences
College Of Science
Kwame Nkrumah University of Science and Technology

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[1]CLASSICAL MECHANICS I. PHY. 253

Name	Index Number
Adu-Forson Ebenezer McFaya	4267520
Andy Antwi Boasiako	4269920
Armah Amartey Caleb	4268320
Georgina Owusu Ansah	4280020
Illias Mohamed	4277420
Je-chie Amaning Joshua Kofi	4274420
Kudadze Emmanuel Kay	4275020
Larry Jamal Ganihou	4275620
Osei-Mensah Divina	4279120
Salifu Amidu	4280820

[2]

[2] Names of group members.

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1 Introduction to Classical Mechanics

The fundamental principles of classical mechanics were first laid down by Galileo then subsequently followed Newton's work. Classical mechanics is the study of the motion of bodies (including the special case in which bodies remain at rest according to the general principles of first enunciated by Sir Isaac Newton in his famous book Philosophiae Naturalis Principia Mathematica (1687), commonly referred to as the Principia. Newton propounded three laws of motion, one law of gravity and pretended he did not know calculus. Probably the single greatest scientific achievement in history, you might think this pretty much wraps it up for classical mechanics. And in a sense, it does.

Classical mechanics was the first branch of Physics to be discovered, and is the basis upon which all other branches of Physics depends. Classical mechanics has many important applications in other areas of science, such as Astronomy (celestian mechanics), Chemistry (the dynamics of molecular collisions) Geology (the propagation of of seismic waves, generated by earthquakes, through the Earth's crust) and Engineering (the equilibrium and stability of structures). Classical mechanics is also of great significance outside the realm of science. After all, the sequence of events leading to the discovery of classical mechanics-starting with the ground-breaking work of Copernicus, continuing with the researches of Galileo, Keppler, Descartes and culminating in the monumental achievement of Newton-involved the complete overthrow of the Aristotelian picture of the Universe, which had previously prevailed for more than a milennium, and its replacement by a recognizably modern picture in which humankind no longer played a privileged role.

Given a collection of particles, acted upon by a collection of forces, you have to draw a nice diagram, with the particles as points and the forces as arrows. The forces are then added up and Newton's famous "$F = ma$" is employed to figure out where the particles velocities are heading next. All you need is enough patience and a big computer and you are done.

From a modern perspective, this is a little unsatisfactory on several levels: It's messy and inelegant; it is hard to deal with problems that involve extended objects rather than points particles; it obscures certain features of dynamics so that concepts such as chaos theory took over 200 years to discover; and its not all clear what the relationship is between Newton's classical laws and quatum physics.

Classical Mechanics covers the following

- The case in which bodies remain at rest.
- Translational motion : by which a body shifts from one point in space to another.
- Oscillatory motion: example; the motion of a pendulum or spring.
- Circular motion: example; the motion of the earth around the sun.
- More general rotational motion: orbits of the planets or bodies that are spinning.
- Particle collisions (elastic and inelastic)

It is also concerned with:

- Motion of objects: velocity and acceleration.
- Causes of the motion: force and energy.
- Newton's three laws of motion.

1.1 Application of Classical Mechanics in areas of science

1. **Astronomy**: How planets move around the sun, motion of stars.

2. **Molecular and Nuclear Physics**: Collision of atomic and subatomic particles, how an electron moves around the nucleus of an atom.

3. **Geology**: The propagation of seismic waves.

4. **Engineering**: Structure of bridges and buildings, how a skier moves down the slope.

2 Space and Time

Newton's three laws are formulated in terms of four crucial underlying concepts: the motion of space, time, mass and force.

2.0.1 Space

Each point **P** of three dimensional space in which we live can be labelled by a position vector r which specifies the distance and direction of **P** from a chosen origin **O**, as seen in figure 1.

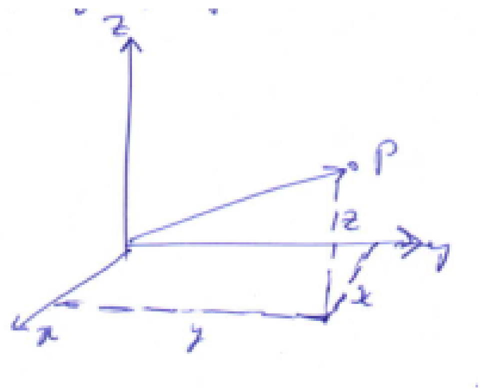


Figure 1

There are many different ways to identify a vector. The most natural way to identify a vector is to give its components (x, y, z) in the direction of three chosen perpendicular axes. One particular way to express this is to introduce three unit vectors $\hat{x}, \hat{y}, \hat{z}$, pointing along the three axes and to write $r = x\hat{x} + y\hat{y} + z\hat{z}$ Our popular choice is to use i, j, k for $(\hat{x}, \hat{y}, \hat{z})$

$$r = (x, y, z) = (r_1, r_2, r_3) = (e_1, e_2, e_3)$$

$$r_1 = x, r_2 = y, r_3 = z$$

$$e_1 = \hat{x}, e_2 = \hat{y}, e_3 = \hat{z}$$

The symbol e is commonly used for unit vectors. Since e stands for the Germa "eins" or one,

$$r = r_1e_1 + r_2e_2 + r_3e_3 = \sum_{i=1} r_ie_i$$

2.0.2 Vector operations

In our study of mechanics, we shall make repeated use of the various operations that can be performed with vectors. If r and s are vectors with the components $r = (r_1, r_2, r_3) \quad and \quad s = (s_1, s_2, s_3)$, their sum is by adding corresponding components. Thus,

$$r + s = (r_1 + s_1, r_2 + s_2), r_3 + s_3$$

An important example of a vector is the resultant force on an object.

When two forces F_a and F_b act on an object, the effect is the same as single force. The resultant force, which is just the vector sum $F = F_a + F_b$. c is a scalar and r is a vector, the product cr is given by:

$$cr = (cr_1, cr_2, cr_3)$$

This means cr is a vector in the same direction as r with magnitude equal to c times the magnitude of r.

Illustration

An object of mass m(a scalar) has an acceleration a (a vector), Newton's second law asserts that the resultant force F on the object will always equal the product $F = ma$

Two important pair of product of pair vectors; the scalar product(or dot product)

$$r \cdots = rs \cos \theta$$

$$= r_1 s_1 + r_2 s_2 + r_3 s_3$$
$$= \sum_{n=1}^{3} r_n s_n$$

Illustration 2

If a force F acts on an object that moves through a displacement dr, the workdone by its force is the scalar product $F \cdot dr$. The magnitude of a vector(length) r is denoted by $|r|$. By Pythagoras theorem

$$|r| = \sqrt{r_1^2 + r_2^2 + r_3^2}$$

. The second product of two vectors r and s is the vector product (or cross product) which defines the product $p = r \times s$

$$p_x = r_y s_z - r_z s_y$$

$$p_y = r_z s_x - r_x s_z$$

$$p_z = r_x s_y - r_y s_x$$

or

$$r \times s = det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix}$$

The vector product plays an important role in the disccussion of rotational motion. For example, the tendency of a force F acting at a point r to cause a body to rotate about the origin is given by the torque of F about O depends on the vector product

$$r = r \times F$$

2.0.3 Vector Differenciation

Many (maybe most) of the laws of physics involve vectors, and most of these involve derivatives of vectors. For instance, the velocity of a particle is the time derivative of the particle's position $r(t)$; that is

$$V = \frac{dr}{dt}$$

similarly, the acceleration

$$a = \frac{dv}{dt}$$

We define derivative as

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

where

$$\Delta x = x(t + \Delta z) - x(t)$$

$$\Delta x \to change \quad of \quad x$$
$$\Delta t \to change \quad in \quad time$$
$$\frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$$
$$\Delta r = r(t + \Delta t) - r(t)$$

All vectors we shall encounter are differencials and hence you can take for granted the limit exists.

If $r(t)$ and $s(t)$ are two vectors that depend on t then the derivative of their sum is first what you would expect:

$$\frac{d}{dt}(r+s) = \frac{dr}{dt} + \frac{ds}{dt}$$

similarly, if $r(t)$ is a vector and $f(t)$ is a scaler, then the derivative of the product $f(t)r(t)$ is given by

$$\frac{d}{dt}(fr) = f\frac{dr}{dt} + \frac{df}{dt}r$$

2.0.4 Time

The classical view is that time is a single universal parameter t on which all observers agree. That is, if all observers are eqipped with accurate clocks, all properly synchronized, then they will all agree on to the time at which at which any given event occured. This view is not exactly correct; According to the theory of relativity, two observers in relative motion do not agree on all times! Nevertheless, in the domain of classical mechanics, with all speed much less than the speed of light c, the differences among the measured times are entirely negligible and therefore adopt the classical assumption of a single universal time. Apart from the obvious ambiguity in the choice of origin of time(the time that we choose to label as $t = 0$), all observers agree on the times of all events.

2.0.5 Reference Frames

Every problem in classical mechanics involves a choice of a reference frame. That is, a choice of spatial origin and axes to label positions as in fig.1, and a choice of temporal origin to measure times.

The difference between two frames maybe quite minor. For instance, they may differ only in their choice of the origin of time – what one frame labels $t = 0$, the other may label $t' = t_o \neq 0$. Or the two frames may have the same origins of space and time, but have different orientations of the three spatial axes. By carefully choosing your reference frame, taking advantage of these different possibilities, you can sometimes simplify your work. A more important difference arises when two frames are in relative motion; that is; when one origin is moving relative to the other.

We shall find that later, not all such frames are physically equivalent. In certain special frames, called *inertial frames*, the basic laws hold time in their standard, simple form(it is because one of these basic laws is Newton's first law, the law of inertial, that these frames are called inertial).

If a second frame is accelerating or rotating relative to an inertial frame, then this second frame is noninertial, and the basic laws – in particular, Newton's law do not hold in their standard form in this second frame. We shall find that the distinction between inertial and noninertial frames is central to our discussion of classical mechanics. It plays an even more explicit role in the theory of relativity.

2.0.6 Galilean Transformation

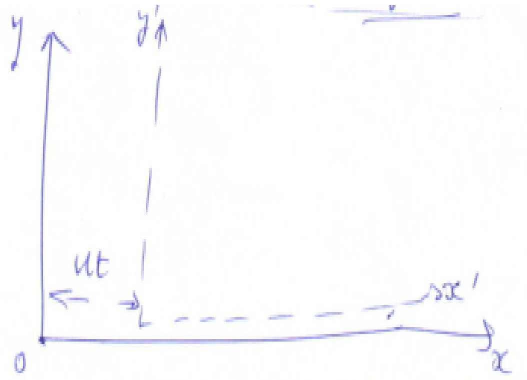


Figure 2

S-frame	S'- frame
$x = x' + ut$	$x' = x - ut$
$y = y'$	$y' = y$
$z' = z$	$z' = z$

This coordinate of transformation was discovered by Galileo Galilei, hence they are called *Galilean transformation*.
instantaneous velocity of an object in the first

$$v = \frac{dr}{dt}$$

$$V'_x = V_x - u$$

$$V'_y = V_y$$

$$V'_z = V_z$$

$$V' = V - u$$

$$a = \frac{dv}{dt} = \left(\frac{dV_x}{dt}, \frac{dV_y}{dt}, \frac{dV_z}{dt}\right)$$

$$a'_x = a_x$$

$$a'_y = a_y$$

$$a'_z = a_z$$

In a more compact form

$$a' = a \tag{2.1}$$

If an object is moving in a straight line with constant speed in our original inertial frame(ie $y' = 0$), then it also moves in a (different) straight line with (a different) constant speed in the second frame of reference(i.e. $a' = 0$). Hence, we conclude that the second frame of reference is also an initial frame. A simple extension of the above argument allows us to conclude that there are an infinit number of different inertial frames moving with constant velocities with respect to another. Newton thought one of these inertial frames was special, and defined an absolute standard of rest. i.e A static object in this frame was in a state of absolute rest. However, Einstein showed that this is not the case. In fact there is no absolute standard of rest, i.e, all motion is relative – hence, the name "*relativity*" for Einstein theory. Consequently, one inertial frame is fast as another as far as Newtonian dynamics is concerned.

But, what happens if the second frame of reference accelerates with respect to the first? In this case, it is not hard to see that (2.1) generalises to

$$a' = a - \frac{du}{dt}$$

where $u(t)$ is the instantaneous velocity of the second frame with respect to the first. According to the above equation, if an object is moving in a straight–line with constant speed in the first frame(i.e $a' = 0$), then it does not move in a straight–line with constant speed in the second frame (i.e $a' \neq 0$). Hence if the first frame is an inertial frame, then the second is not. A simple extention of the above argument allows us to conclude that any frame of reference which accelerates with respect to a given initial frame is not itself an inertial frame.

For most practical purposes, when studing the motions of object close to the Earth's surface, a reference frame which is fixed with respect to this surface is approximately inertial. Alternatively; If a second frame is accelerating or rotating relative to an inertial frame, then this second frame is noninertial, and the basic laws–Newton's law do not hold in their standard form in the second frame.

We shall find that, the distinction between inertial and noninertial frames is the central point of the discussion of classical mechanics.

3 Validity of the first two laws

Quantum mechanics and relativity shows that Newton's laws is not universally valid.
Nevertheless, there is an immense range of phenomena–the phenomena of classical physics–where the first two laws are for all practical purposes exact. We can assume that the first two laws are universally and precisely valid. The classical model of the natural world. The model is logically and is such a good representation of many phenomena that is worthy of our study.

3.0.1 The third law and conservation of momentum

Newton's first two laws concern the response of a single object to applied forces.
The third law addresses a quite different issue. Every force on an object inevitably involves a second object–the object that exerts the force. The nail that is hit by the hammer, the cart is pulled by the horse and so on.

3.0.2 The third law–Newton

If object 1 exerts a force F_{21} on obtect 2, then object 2 always exerts a reaction force F_{12} on object 1 given by

$$F_{12} = -F_{21}$$

Newton's third law assets that the reaction force exerted on object 1 by object 2 is equal and opposite to the force exerted on 2 by 1. That is

$$F_{12} = -F_{21} \tag{3.1}$$

Think of it as force of the Earth on the moon and the reaction force of the moon on the Earth(or a proton on electron and the electron on the proton).
(i) Note that the forces acts along the line joining them. Forces with this extra property are called central forces(act along the line centres).
(ii)The law does not actually require that the forces be central, but most of the forces we encounter(gravity, the electrostatic force between two charges etc.) have this property.
(iii)The law is intimately related to the law of conservation of momentum.

Consider the figure below:

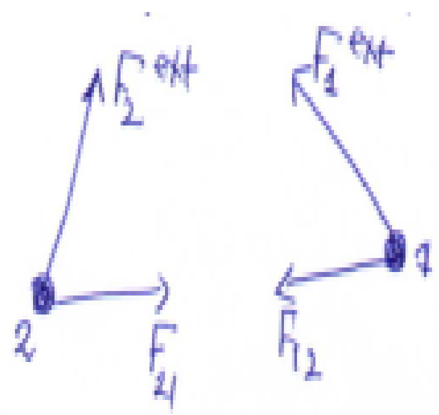


Figure 3: Two objects exerts forces on each other and also be subject to additional "external" forces from other objects.

Let the two objects as shown be say the Earth and the moon or two skaters on the ice.

In addition to the force of each object on the other, there may be "external" forces exerted by other bodies.
The Earth and the moon experience forces exerted by the sun, and both skaters could experience the external force of the wind. We have shown the net external forces on the two objects as

$$F_1^{ext} \quad and \quad F_2^{ext}$$

The total force on object 1 is then

$$(Net \quad force \quad on \quad 1) \equiv F_1 = F_{12} + F_1^{ext}$$

$$similarly \quad \equiv F_2 = F_{21} + F_2^{ext}$$

Let us compare the rate of change of the particle's momentum using Newton's second law:

$$\dot{P} = \frac{dP}{dt} = F_1 = F_{12} + F_1^{ext} \tag{3.2}$$

$$\dot{P}_2 = F_2 = F_{21} + F_2^{ext} \tag{3.3}$$

The momental of the two objects(total) is defined as $P = P_1 + P_2$; and the rate of change is $\dot{P} = \dot{P}_1 + \dot{P}_2$

$$\Rightarrow \quad \dot{P} = F_1 + F_2$$

$$= F_{12} + F_1^{ext} + F_{21} + F_2^{ext}$$

$F_{12} = F_{21}$; according to the third law

$$\dot{P} = F_1^{ext} + F_2^{ext} \equiv F^{ext} \tag{3.4}$$

Where F^{ext} is the total external force on the two- particle system
(i)Basis for the construction of theory of many-particle systems from the basic laws of a single particle.
(ii)It assets that, as far as the total momentum of a system is conserved, the internal forces have no effect.
(iii)If there is no external forces($F^{ext} = 0$), then $\dot{P} = 0$; important results: If $F^{ext} = 0$, then P = constant.

Multiple system

We have proved the equation of momentum in (3.4) for a system of two particles. The extention of the results to any number of particles is straight forward in principle. The particle α is subjected to four internal forces denoted by solid line and denoted by $F_{\alpha\beta}$(the force on α by β). In addition, particle may be subjected to external force shown in dash line with arrow.
Lets consider a system of N- particles labelled either α or β and each take any value $1, 2...N$. The mass is $m\alpha$, its momemtum $P\alpha$

Each of the other $(N-1)$ particles can exert a force $F_\alpha\beta$, net external force on particle α is F_α^{ext}. The net force on particle α is

$$(net \quad force \quad on \quad particle \quad \alpha \quad) = F_\alpha = \sum_{\beta\neq\alpha} F_\alpha\beta + F_\alpha^{ext} \tag{3.5}$$

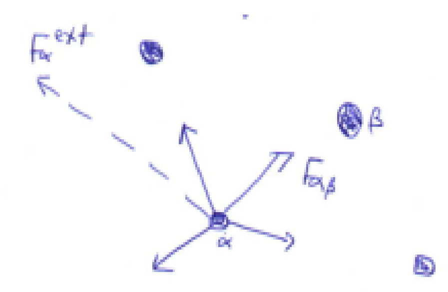


Figure 4: A five-particle system with particles labelled by α or $\beta = 1, 2, ...5$

NB: There is no force like $F_\alpha\alpha$ because the particle cannot exert force on itself. The rate of change of P_α

$$\dot{P}_\alpha = \sum_{\alpha\neq\beta} F_\alpha\beta + F_\alpha^{ext}; \quad results \quad hold \quad for \quad each \quad particle \tag{3.6}$$

The total momentum

$$P = \sum_\alpha P_\alpha$$

; Substituding (3.6), we get

$$\dot{P} = \sum_\alpha \{ \sum_{\alpha\neq\beta} F_\alpha\beta + F_\alpha^{ext} \}$$

$$= \sum_\alpha \sum_{\alpha\neq\beta} F_\alpha\beta + \sum_\alpha F_\alpha^{ext} \tag{3.7}$$

Each term $F_\alpha\beta$ in this sum can be paired with a second term $F_\beta\alpha$ (i.e F_{12} paired with F_{21}), and so on. So that:

$$\sum_\alpha \sum_{\beta\neq\alpha} F_\alpha\beta = \sum_\alpha \sum_{\beta>\alpha} (F_\alpha\beta + F_\beta\alpha) \tag{3.8}$$

$\sum_\alpha \sum_{\beta\neq\alpha} F_\alpha\beta$ includes values of α and β with $\alpha < \beta$ and has as many terms on the left. But each term is the sum of two forces $(F_\alpha\beta + F_\beta\alpha)$ and by the third law, each such sum is zero.

$$F_\alpha\beta = -F_\beta\alpha$$

Therefore, the whole of the double sum in (3.8) is zero.

Therefore (3.7) then becomes

$$\dot{P} = \sum_\alpha F_\alpha^{ext} \equiv F^{ext}$$

The result is exactly the two-particle one. The internal forces have no effect on the evolution of the total momentum P– the rate of change of P is determined by the net external force on the system. In particular, if the net external force is zero, we have the system's total momentum is constant. This is one of the most important result in classical physics and in fact also time in relativity and quantum mechanics.

3.0.3 Validity of Newton's third Law

Within the domain of classical physics, the third law, like the second, is valid with such as accuracy that it can be taken to be exact.

As speed approaches the speed of light, it is easy to see that the third law can not hold: The point is that the law asserts the action and reaction forces $F_{12}(t)$ and $F_{21}(t)$, measured at time t are equal and opposite. Once relativity becomes important, the concept of a single universal time has to be abondoned– two events that are seen as simultaneous by one observer are, in general, not simultaneous as seen by a second observer. Thus; even if the equality $F_{12}(t) = -F_{21}(t)$(with both time the same) where time for one observer, it would generally be force for another. This means the third law can not be valid once relativity becomes important.
The magnetic force between two moving charges– for the third law is not exactly true even at slow speed.

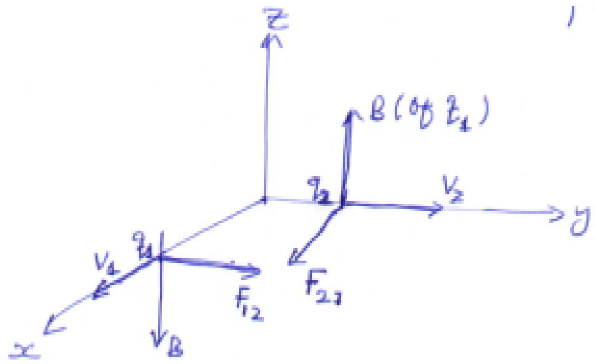


Figure 5: q_1, q_2 are moving charges with velocity V_1 and V_2

Each of the positive charges q_1 and q_2 produces a magnetic field that exerts a force on the other charge. The resulting magnetic forces F_{12} and F_{21} do not obey Newton's third law. This mean the total momentum of the two charges is not conserved.

3.0.4 Newton's second Law in Cartesian coordinates

The most used three laws of Newton is the second law–often described as equation of motion. The second law is

$$F = m\ddot{x} \quad or \quad m\ddot{r} \quad or \quad m\frac{dx^2}{dt^2}$$

where r = vector of position.

$$F = F_x\hat{x} + F_y\hat{y} + F_z\hat{z}$$

The position vector r is:

$$r = x\hat{x} + y\hat{y} + z\hat{z} \tag{3.9}$$

When we differenciate (3.9);

$$\ddot{r} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\Rightarrow \quad F = m\ddot{r} \Leftrightarrow \begin{cases} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{cases} \tag{3.10}$$

Example

Analyse a block sliding down an incline plane: A block slides down a slope of incline θ. The three forces on the block are its weight, $w = mg$, the normal force of the incline, N and the frictional force f, whose magnitude is $f = \mu N$ The x-component of the second law, $F_x = m\ddot{x}$

$$w_x - \rho = m\ddot{x}$$

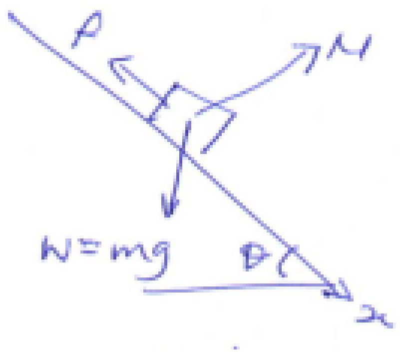


Figure 6

$$mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$
$$\ddot{x} = g(\sin\theta - \mu\cos\theta)$$

$$\dot{x} = g(\sin\theta - \mu\cos\theta)t$$

$$x(t) = \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2$$

3.1 Linear momentum and Angular momentum

We found that as long as the internal forces obey Newton's third law, the rate of change of the system's total linear momentum

$$P = P_1 + P_2 + ...P_N = \sum P_\alpha$$

is determined entirely by the external forces on the system.

$$\dot{P} = F^{ext} \quad total \quad external \quad force \quad on \quad the \quad system \tag{3.11}$$

Because of the third law, the internal forces all cancel out of the rate of change of total momentum. If the net external force F^{ext} on the N- particle system is zero, the system's total mechanical momentum $P = \sum m_\alpha V_\alpha$ is constant.
Consider two bodies of mass m_1 and m_2 moving with velocity V_1 and V_1. They collide and
(i) Stick together and move with a velocity V(perfectly elastic). Assuming that any external forces are negligible during the brief moment of collision

$$P_{in} = m_1V_1 + m_2V_2, \quad initial \tag{3.12}$$

$$P_{fin} = m_1V + m_2V = (m_1 + m_2)V$$

By conservation of momentum:

$$P_{in} = P_{fin}$$
$$\Rightarrow m_1V_1 + m_2V_2 = V(m_1 + m_2)$$

$$V = \frac{m_1V_1 + m_2V_2}{m_1 + m_2} \tag{3.13}$$

If $m_1 = m_2$, then

$$V = \frac{m_1(V_1 + V_2)}{m_1 + m_2}$$
$$= \frac{1}{2}(V_1 + V_2)$$

Rockets

A beautiful example of the use of momentum conservation is the analysis of rocket propulsion. How does the rocket gets moving?

Analysis

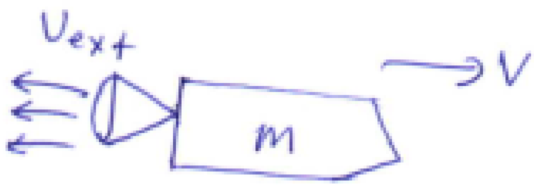


Figure 7

Consider the rocket's motion quanlitatively, use momentum conservation Fuel ejection speed $= V_{ext}$
Velocity of rocket $= V$
Mass of rocket $= m$(this keeps decreasing steadily.)
At time t; momentum $P(t) = mV$ At short time later $t + dt$; rocket mass $= (m + dm)$ This is negative if its moving in opposite direction.

$$P(t + dt) = (m + dm)(V + dV) \tag{3.14}$$

The fuel ejected in time dt has mass $(-dm)$ and velocity $V - V_{ext}$ relative to the ground. The total momentum(rocket + fuel just ejected) at $t + dt$ is

$$P(t + dt) = (m + dm)(V + dV) - dm(V - V_{ext})$$
$$= mv + mdv + dmV_{ext} \tag{3.15}$$

Change in total momentum:

$$dp = P(t + dt) - P(t)$$

$$(mv + mdv + dmV_{ext}) - (mV)$$

$$= mdv + dmV_{ext} \tag{3.16}$$

If there is a net external force F^{ext}(gravity for instance), this change of momentum is F^{ext}.

$$\Rightarrow dp = F^{ext} = mdv + dmV_{ext} \tag{3.17}$$

For simplicity, we shall assume that there is no external force, moment P is constant, $dP = 0$

$$mdV = -dmV_{ext} \tag{3.18}$$

$$= m\frac{dV}{dt} = -\frac{dm}{dt}V_{ext}$$

$$= m\dot{V} = -\dot{m}V_{ext}$$

Where $= m\dot{V}$ = thrust

$$Thrust = -\dot{m}V_{ext} \tag{3.19}$$

$\dot{m}$ is the rate at which the rocket's engine ejects mass. This is negative, this defines the thrust to be positive.

From (3.19), it can be solved by separation of variables and dividing both sides by m and integrating

$$\int_{V_o}^{V} dV = -V_{ext}\int_{m_o}^{m}\frac{dm}{m}$$

$$V - V_o = -V_{exp}\left\{ \log m\Big|_{m_o}^{m} \right\}$$

$$Note, \quad \int \frac{dx}{x} = y, \quad \log x = y$$

$$= -V_{ext}\log(m - m_o)$$

$$= V_{exp}\log(m_o - m)$$

$$V - V_o = -V_{exp}\log\left(\frac{m_o}{m}\right) \tag{3.20}$$

V_o = initial velocity m_o = initial mass of the rocket including fuel and payload. This result puts a significant restriction on the maximum speed of the rocket. $\frac{m_o}{m}$ is largest when all the fuel is burned and m is just the mass of the rocket plus payload.
For example, even if 90$\%$ of m_o is fuel only, the ratio would be

$$\log\left(\frac{100\%}{100\%}\right) = \log 10 = 2.3$$

This means the speed gain, $V - V_o$, cannot exceed $2.3 times V_{ext}$. For this reason, rocket's engines try to make V_{ext} as big as possible and also design multistage rockets, which can jeltison the heavy fuel tanks of the early stages to reduce the total mass for later stages.
jeltisoning the fuel tanks of stage 1 reduces the initial and final masses of stage 2 by the same amount. This increases the ratio $\frac{m_o}{m}$ when we apply (3.20) to stage 2.

3.1.1 The center of mass

Consider N- particles $\alpha = 1, 2...N$ with masses m_α and positions r_α measured relative to an origin O.

The center of mass CM of this system is defined to be positive(relative to the origin O).

$$R = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha r_\alpha = \frac{m_1 r_1 + ... + m_N r_N}{M} \tag{3.21}$$

$$M = M_1 + M_2 + ...M_\alpha = \sum m_\alpha$$

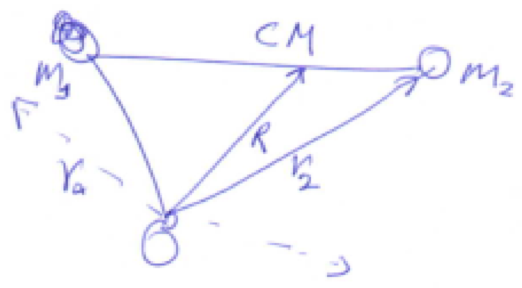


Figure 8

The CM position is a vector R with three components(X,Y.Z) and (3.21) is equivalent to three equations of the components as

$$X = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha X_\alpha, \quad Y = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha y_\alpha \quad z = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha z_\alpha$$

Let us consider the case of first two particles, $N = 2$. Therefore (1.22) becomes

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_1}$$

We can now write the total momentum P of any N- particle system in terms of the system's CM as follows

$$P = \sum_\alpha P_\alpha = \sum_\alpha m_\alpha \dot{r}_\alpha = MR \tag{3.22}$$

This results says that the total momentum of N- particles system is exactly the same as that of a single particle of mass M and velocity equal to that of the CM. We get even more striking results when we differenciate (3.22). We have seen that

$$\dot{P} = \frac{dP}{dt} = F^{ext}$$

$$\therefore P = \sum m_\alpha \dot{r}_\alpha = M\dot{R}$$

$$\dot{P} = M\frac{d\dot{R}}{dt} = M\ddot{R} = F^{ext} \tag{3.23}$$

That is the center of mass R moves exactly as if it were a single particle of mass M, subject to the net external force on the system. This result is the main reason why we can often treat external bodies, such as baseballs and planets, as if they were point particles. Provided a body is small compared to the scale of its trajectory, its CM position R is a good representative of its overall location, and (3.23) implies that R moves just like a point particle.

Note that (3.21) can be re-written as

$$R = \frac{1}{M}\int r dm = \frac{1}{M}\int \delta r dV$$

3.1.2 Angular Momentum for a single particle

The angular momentum l of a single particle is defined as the vector

$$l = r \times P \tag{3.24}$$

P = momentum
l: depends on the choice of origin. Strictly speaking, it refers to or depends on the choice of origin.

$$\dot{l} = \frac{d}{dt}(r \times P) = \dot{r} \times P + (r \times \dot{P}) \tag{3.25}$$

$$= \frac{dr}{dt} \times P + r \times \frac{dP}{dt}$$

$$\dot{r} \times m\dot{r} + r \times \dot{P}$$

$$m\{\dot{r} \times \dot{r}\} + r \times \dot{P}$$

$$m(0) + r \times \dot{P}$$

Note, $\dot{r} \times \dot{r} \longrightarrow$ parallel.

$$\dot{l} = r \times \dot{P} \tag{3.26}$$

$$\dot{K} = r \times F; \quad \frac{dP}{dt} = \dot{P} = F^{ext}$$

$$\tau = r \times F; \quad torque$$

$$\tau = r \times F = \dot{K} \tag{3.27}$$

The rate of change of a particle's angular momentum about tthe origin O is equal to the net applied torque about O.

(3.27) is the rotational analog of the equation $\dot{P} = F$ for linear momentum. This is often described as the rotational form of Newton's second law. In many one particle problems, one can choose the origin O so that the net torque is zero. In this case, the particle's angular momentum about O is constant.
A crucial property of the gravitational pull GmM/R^2 of the sun is that the force is central. i.e, directed along the line joining the two centres.

This mean that F is parallel to the position vector r measured from the sun, and hence $r \times F = O$. This simplifies the analysis of planetary motion.

3.1.3 Angular Momentum for a several particles

For system of N- particles, $\alpha = 1, 2...N$ each with angular momentum

$$l_\alpha = r_\alpha \times P$$

with all positions(r_α) measured from its origin. We define the total angular momentum

$$L = \sum_{\alpha=1}^{N} l_\alpha \quad = \sum_{\alpha=1}^{N} r_\alpha \times P_\alpha$$

$$\dot{L} = \sum_{\alpha} \dot{l}_\alpha \quad = \sum_{\alpha} r_\alpha \times F_\alpha$$

Net force on particle α; $F_\alpha = \sum_{\beta \neq \alpha} F_{\alpha\beta} + F_\alpha^{ext}$

$$\dot{L} = \sum_\alpha \sum_{\alpha \neq \beta} r_\alpha \times F_{\alpha\beta} + \sum_\alpha r_\alpha \times F_\alpha^{ext}$$

$$\sum_\alpha \sum_{\beta \neq \alpha} r_\alpha \times F_{\alpha\beta} = \sum_\alpha \sum_{\beta > \alpha} (r_\alpha \times F_{\alpha\beta} + r_\beta \times F_{\beta\alpha})$$

If $F_{\alpha\beta} = -F_{\beta\alpha} : 3^{rd} \quad law$

$$\sum_\alpha \sum_{\beta > \alpha} (r_\alpha - r_\beta) \times F_{\alpha\beta}$$

Examine the vector $r_\alpha - r_\beta = r_{\alpha\beta}$ $F_{\alpha\beta}$ is called central. $r_{\alpha\beta}$ and $F_{\alpha\beta}$ are parallel and therefore cross product = O.

$$\dot{L} = F^{ext} = \tau^{ext}$$

The validity of this particle depends on our assumption that all internal forces $F_{\alpha\beta}$ are central and satisfy the third law.

3.1.4 Angular momentum about the centre of mass

$$\dot{L} = \tau^{ext}$$

This required after both L and τ^{ext} be measured about an origin O fixed in the same reference inertial frame. This also holds if both are measured about the centre of mass–even if the CM is being accelerated and so is not fixed in an inertial frame.

$$\frac{d}{dt} L(about \quad CM) = \tau^{ext}(about \quad CM)$$

If $\tau^{ext}(about \quad CM) = 0$, then $L(about \quad CM)$ is conserved.
Total momentum and angular are both conserved in gravitational force, electrostatic force. Some forces do not obey this properties.
E.g. The Lorentz force on two moving particles with electri charge Q; crossing point T

$$F_{ij} = qV_i \times B_j = qVB\sin\theta$$
$$V_i = Vel = qVB\sin\{0, \pi/2\}$$
$$B = magnetic \quad field \quad = 0, qvB$$

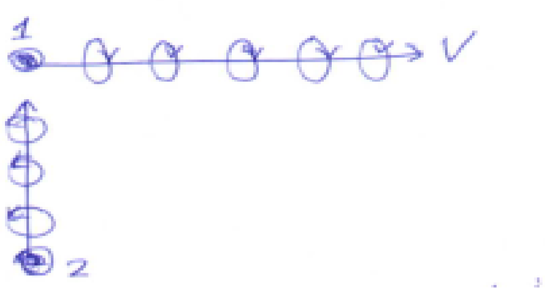


Figure 9

Force on particle 1 from particle 2 = $q_2 VB \sin 0 = 0$. $\theta = 0$.
Force on particle 2 from particle 1 = $q_1 VB \sin \pi/2 = 0$. $\theta = 0$.
The force is non-zero because $\theta = \pi/2$. q_1 is moving at $\pi/2$ to q_2
Electromagnetic field itself carries angular momentum which restores the conservation law.

4 Introduction to Dynamics

4.1 Introduction and Review

Dynamics is the science of how things move. A complete solution to the motion of a system means that we know the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, this means we want to know its position vector $r(t)$ as a function of time. If there are many particles, the motion is described by a set of functions $r_i(t)$, where i labels which particle we are talking about. So generally speaking, solving for the motion means being able to predict where a particle will be at any given instant of time. Of course, knowing the function $r_i(t)$ means we can take its derivative and obtain the velocity $v_i(t) = \frac{dr_i}{dt}$ at any time as well. The complete motion for a system is not given to us outright, but rather is encoded in a set of differential equations, called the equations of motion. An example of an equation of motion is

$$m\frac{d^2x}{dt^2} = -mg \tag{4.1}$$

with the solution

$$x(t) = x_0 + v_0 t - \frac{1}{2}gt^2 \tag{4.2}$$

where x_0 and v_0 are constants corresponding to the initial boundary conditions on the position and velocity: $x(0) = x_0$, $v(0) = v_0$. This particular solution describes the vertical motion of a particle of mass m moving near the earth's surface.

In this class, we shall discuss a general framework by which the equations of motion may be obtained, and methods for solving them. That "general framework" is Lagrangian Dynamics, which itself is really nothing more than an elegant restatement of Isaac Newton's Laws of Motion.

4.1.1 Newton's Laws of Motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens. To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's Principia in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

I. A body remains in uniform motion unless acted on by a force.

II. Force equals rate of change of momentum: $F = \frac{dp}{dt}$.

III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $r(t) = r_0 + v_0 t$. Now r(t) is measured in some coordinate system, and if instead we choose to measure r(t) in a different coordinate system whose origin R moves according to the function R(t), then in this new "frame of reference" the position of our particle will be

$$\begin{aligned} r'(t) &= r(t) - R(t) \\ &= r_0 + v_0 t - R(t) \end{aligned} \tag{4.3}$$

If the acceleration $\dfrac{d^2R}{dt^2}$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does not move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called inertial frames in which Newton's Laws hold. A transformation from one frame K to another frame K' which moves at constant velocity V relative to K is called a Galilean transformation. The equations of motion of classical mechanics are invariant (do not change) under Galilean transformations.

At first, the issue of inertial and non-inertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we are in an inertial frame. The earth's surface, where most physics experiments are done, is not an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum p, defined as the product $p = mv$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force F depends on the coordinates (or possibly velocities) themselves. This is known as a force law. Examples of force laws include:

Constant force: $F = -mg$

Hooke's Law: $F = -kx$

Gravitation: $F = -\dfrac{GMm}{r^2}\hat{r}$

Lorentz force: $F = qE + q\dfrac{v}{c} \times B$

Fluid friction: $F = -\gamma v$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $F = ma$, where $a = dv/dt = d^2r/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of central forces which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $F_1 = -F_2$ and hence the accelerations are parallel to each other, with

$$\frac{a_1}{a_2} = -\frac{m_2}{m_1}$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the total momentum $P = p_1 + p_2$ is a constant, a result known as the conservation of momentum.

4.1.2 Aside: Inertial vs Gravitational Mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force F_{ij} exerted by a particle i by another particle j is

$$F_{ij} = -GMm\frac{r_i - r_j}{|r_i - r_j|^3} \tag{4.4}$$

where G, the Cavendish constant (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11} N \cdot m^2/kg^2$$

.

Notice Newton's Third Law in action: $F_{ij} + F_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_e$, with $r_i - r_j \simeq R_e\hat{r}$ and obtain

$$F = -mg\hat{r} \equiv -mg \tag{4.5}$$

where $\hat{r}$ is a radial unit vector pointing from the earth's center and $g = GM_e/R_e^2 \simeq 9.8m/s^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $a = -g$, i.e. objects accelerate as they fall to earth. However, it is not a priori clear why the inertial mass which enters into the definition of momentum should be the same as the gravitational mass which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$a = -\frac{m'}{m}g \tag{4.6}$$

and unless the ratio $\dfrac{m'}{m}$ were the same number for all objects, then bodies would fall with different accelerations. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, i.e. $m' = m$.

4.1.3 Examples of Motion in One Dimension

To gain some experience with solving equations of motion in a physical setting, we consider some physically relevant examples of one-dimensional motion.

4.1.4 Uniform Force

With $F = -mg$, appropriate for a particle falling under the influence of a uniform gravitational field, we have $m\dfrac{d^2x}{dt^2} = -mg$, or $\ddot{x} = -g$

With $v = \dot{x}$, we solve $dv/dt \equiv -g$:

$$\int_{v(0)}^{v(t)} dv = \int_0^t ds(-g) \tag{4.7}$$

$$v(t) - v(0) = -gt \tag{4.8}$$

Note that there is a constant of integration, v(0), which enters our solution.

$$\int_{x(0)}^{x(t)} dx = \int_0^t dsv(s) \tag{4.9}$$

$$x(t) = x(0) + \int_0^t ds\{v(0) - gs\} \tag{4.10}$$

$$= x(0) + v(0)t - \frac{1}{2}gt^2 \tag{4.11}$$

Note that a second constant of integration, x(0), has appeared.

4.1.5 Uniform Force with Linear Frictional Damping

In this case,

$$m\frac{dv}{dt} = -mg - \gamma v \tag{4.12}$$

which may be rewritten as

$$\frac{dv}{v + mg/\gamma} = -\frac{\gamma}{m}dt \tag{4.13}$$

$$d\ln(v + mg/\gamma) = -(\gamma/m)dt \tag{4.14}$$

Integrating then gives

$$\ln\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) = -\frac{\gamma t}{m} \tag{4.15}$$

$$v(t) = -\frac{mg}{\gamma} = (v(0) + \frac{mg}{\gamma})e^{-\frac{\gamma t}{m}} \tag{4.16}$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, v(0). One can further integrate to obtain the motion

$$x(t) = x(0) + \frac{m}{\gamma}(v(0) + \frac{mg}{\gamma})(1 - e^{-\frac{\gamma t}{m}}) - \frac{mg}{\gamma}t \tag{4.17}$$

The solution to the second order ODE $m\ddot{x} = -mg - \gamma\dot{x}$ thus entails two constants of integration: v(0) and x(0). Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$ where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau = m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$. Where -e is the electron charge i.e. ($e > 0$) and E is the electric field. The terminal velocity is the obtained from

$$v_\infty = \frac{eE}{\gamma} = \frac{e\tau E}{m} \tag{4.18}$$

The current density is a product:

$$j = n \cdot (-e) \cdot (-v_\infty) \tag{4.19}$$

$$= \frac{me^2\tau}{m}E \tag{4.20}$$

The ration j/E is called the conductivity of the metal,σ. According to our theory, $\sigma = \frac{me^2\tau}{m}$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibration ("photons"). In high purity copper at low temperatures ($T \lesssim 4K$), the scattering time τ is about a nanosecond ($\tau \approx 10^{-9}s$).

4.1.6 Uniform Force with Quadratic Frictional Damping

At higher velocities, the frictional damping is proportional to the square of the velocity. The frictional force is then $F_f = -ev^2 sgn(v)$ where sgn(v) is the sign of v: $sgn(v) = +1$ if $v > 0$ and $sgn(v) = -1$ if $v < 0$. (Note one can also write $sgn(v) = v/|v|$ where $|v|$ is the absolute value.) Why all this trouble with sgn(v)? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be oppositely directed with respect to the velocity v. We will assume that $v < 0$ hence $F_f = +ev^2$

Notice that there is a terminal velocity since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$ where $v_\infty = \sqrt{mg/c}$. One can write the equation of the motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty}^2 (v^2 - v_\infty^2) \tag{4.21}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left[\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty}\right] \tag{4.22}$$

we obtain

$$\frac{dv}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \tag{4.23}$$

$$= \frac{1}{2v_\infty} d\ln\left(\frac{v - v_\infty}{v + v_\infty}\right) \tag{4.24}$$

$$= \frac{g}{v_\infty^2} dt \tag{4.25}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\frac{1}{2v_\infty}\ln\left(\frac{v - v_\infty}{v + v_\infty}\right) = \frac{gt}{v_\infty^2} \tag{4.26}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \tag{4.27}$$

Recall that the hyperbolic tangent function tanh(x) is given by

$$tanh(x) = \frac{sinh(x)}{cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{4.28}$$

Again as $t \to \infty$ one has $v(t) \to -v_\infty$ i.e. $v(\infty) = -v_\infty$.

Advanced Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity $v(v > 0)$. During a time $\triangle\ t$, all the fluid molecules inside the volume $\triangle\ v = S \cdot v \triangle\ t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $u - v$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\triangle\ u = 2(v - u)$. The total momentum change is obtained by multiplying $\triangle\ u$ by the total mass $M = \rho \triangle\ V$, where ρ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\triangle\ P = 2(v - u) \cdot \rho S v \triangle\ t \tag{4.29}$$

and the force on the fluid is

$$F = \frac{\triangle\ P}{\triangle\ t} = 2S\rho v(v - u) \tag{4.30}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u \rangle = 0$, we obtain the result $\langle F = 2S\rho v^2 \rangle$, where $\langle ... \rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_f = -F\langle F \rangle = -cv^2$ where $c = 2S\rho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu\rho S$, where μ is a dimensionless constant which depends on the shape of the moving object.

4.1.7 Crossed Electric and Magnetic Fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular E and B fields. We'll throw in gravity for good measure. We take $E = E\hat{x}$, $B = B\hat{z}$, and $g = -g\hat{z}$. The equation of motion is Newton's 2nd Law again:

$$m\ddot{x} = mg + qE + \frac{q}{c}\dot{r} \times B \tag{4.31}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$m\ddot{x} = qE + \frac{qB}{c}\dot{y} \tag{4.32}$$

$$m\ddot{y} = -\frac{qB}{c}\dot{x} \tag{4.33}$$

$$m\ddot{z} = -mg \tag{4.34}$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0) - \frac{1}{2}gt^2 \tag{4.35}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$

$$\dot{v_x} = \omega_c(v_y + u_D)$$

$$\dot{v_y} = -\omega_c v_x$$

where $\omega_c = qB/mc$ is the cyclotron frequency and $u_D = cE/B$ is the drift speed for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$v_x(t) = v_x(0)\cos(\omega_c t) + (v_y(0) + u_D)\sin(\omega_c t) \tag{4.36}$$

$$v_y(t) = -u_D + (v_y(0) + u_D)\cos(\omega_c t) - v_x(0)\sin(\omega_c t) \tag{4.37}$$

Integrating again, the full motion is given by:

$$x(t) = x(0) + A\sin\delta + A\sin(\omega_c t - \delta) \tag{4.38}$$

$$y(r) = y(0) - u_D t - A\cos\delta + A\cos(\omega_c t - \delta) \tag{4.39}$$

where

$$A = \frac{1}{\omega_c}\sqrt{\dot{x}^2(0)} + (\dot{y}(0) + u_D)^2 \tag{4.40}$$

$$\delta = \tan^{-1}\left(\frac{y(0) \dot{+} u_D}{\dot{x}(0)}\right) \tag{4.41}$$

Thus, in the full solution of the motion there are six constants of integration:

$$x(0), y(0), z(0), A, \delta, \dot{z}(0)$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

5 Newtonian Mechanics

Classical mechanics is an ambitious theory. Its purpose is to predict the future and reconstruct the past, to determine the history of every particle in the Universe.

In this course, we will cover the basics of classical mechanics as formulated by Galileo and Newton. Starting from a few simple axioms, Newton constructed a mathematical framework which is powerful enough to explain a broad range of phenomena, from the orbits of the planets, to the motion of the tides, to the scattering of elementary particles. Before it can be applied to any specific problem, the framework needs just a single input: a force. With this in place, it is merely a matter of turning a mathematical handle to reveal what happens next.

We start this course by exploring the framework of Newtonian mechanics, understanding the axioms and what they have to tell us about the way the Universe works. We then move on to look at a number of forces that are at play in the world. Nature is kind and the list is surprisingly short. Moreover, many of forces that arise have special properties, from which we will see new concepts emerging such as energy and conservation principles. Finally, for each of these forces, we turn the mathematical handle. We turn this handle many many times. In doing so, we will see how classical mechanics is able to explain large swathes of what we see around us.

Despite its wild success, Newtonian mechanics is not the last word in theoretical physics. It struggles in extremes: the realm of the very small, the very heavy or the very fast. We finish these lectures with an introduction to special relativity, the theory which replaces Newtonian mechanics when the speed of particles is comparable to the speed of light. We will see how our common sense ideas of space and time are replaced by something more intricate and more beautiful, with surprising consequences. Time goes slow for those on the move; lengths get smaller; mass is merely another form of energy.

Ultimately, the framework of classical mechanics falls short of its ambitious goal to tell the story of every particle in the Universe. Yet it provides the basis for all that follows. Some of the Newtonian ideas do not survive to later, more sophisticated, theories of physics. Even the seemingly primary idea of force will fall by the wayside. Instead other concepts that we will meet along the way, most notably energy, step to the fore. But all subsequent theories are built on the Newtonian foundation.

Moreover, developments in the past 300 years have confirmed what is perhaps the most important legacy of Newton: the laws of Nature are written in the language of mathematics. This is one of the great insights of human civilisation. It has ushered in scientific, industrial and technological revolutions. It has given us a new way to look at the Universe. And, most crucially of all, it means that the power to predict the future lies in hands of mathematicians rather than, say, gypsy astrologers. In this course, we take the first steps towards grasping this power.

5.1 Newton's Laws of Motion

Classical mechanics is all about the motion of particles. We start with a definition.

Definition: A particle is an object of insignificant size. This means that if you want to say what a particle looks like at a given time, the only information you have to specify is its position.

During this course, we will treat electrons, tennis balls, falling cats and planets as particles. In all of these cases, this means that we only care about the position of the object and our analysis will not, for example, be able to say anything about the look on the cat's face as it falls. However, it's not immediately obvious that we can meaningfully assign a single position to a complicated object such as a spinning, mewing cat. Should we describe its position as the end of its tail or the tip of its nose? We will not provide an immediate answer to this question, but we will return to it in Section 5 where we will show that any object can be treated as a point-like particle if we look at the motion of its centre of mass.

To describe the position of a particle we need a reference frame. This is a choice of origin, together with a set of axes which, for now, we pick to be Cartesian. With respect to this frame, the position of a particle is specified by a vector x, which we denote using bold font. Since the particle moves, the position depends on time, resulting in a *trajectory* of the particle described by

$$X = X(t)$$

In these notes we will also use both the notation $X(t)$ and $r(t)$ to describe the trajectory of a particle.

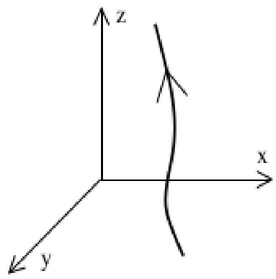


Figure 10

The *velocity* of a particle is defined to be

$$V \equiv \frac{dx(t)}{dt}$$

Throughout these notes, we will often denote differentiation with respect to time by a "dot" above the variable. So we will also write $V = \dot{X}$. The acceleration of the particle is defined to be

$$a \equiv \ddot{X} = \frac{d^2X(t)}{dt^2}$$

A Comment on Vector Differentiation

The derivative of a vector is defined by differentiating each of the components. So, if $X = (x1, x2, x3)$ then

$$\frac{dX}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right)$$

Geometrically, the derivative of a path $X(t)$ lies tangent to the path (a fact which you will see in the Vector Calculus course).

In this course, we will be working with vector differential equations. These should be viewed as three, coupled differential equations – one for each component. We will frequently come across situations where we need to differentiate vector dot-products and cross-products. The meaning of these is easy to see if we use the chain rule on each component. For example, given two vector functions of time, $\mathbf{f}(t)$ and $\mathbf{g}(t)$, we have

$$\frac{d}{dt}(\mathbf{f} \cdot \mathbf{g}) = \frac{d\mathbf{f}}{dt} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{d\mathbf{g}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dt} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dt}$$

As usual, it doesn't matter what order we write the terms in the dot product, but we have to be more careful with the cross product because, for example, $d\mathbf{f}/dt \times \mathbf{g} =$ **-** $\mathbf{g} \times d\mathbf{f}/dt.$

5.1.1 Newton's Laws

Newtonian mechanics is a framework which allows us to determine the trajectory $\mathbf{x}(t)$ of a particle in any given situation. This framework is usually presented as three axioms known as Newton's laws of motion. They look something like:

- **N1:** Left alone, a particle moves with constant velocity.
- **N2:** The acceleration (or, more precisely, the rate of change of momentum) of a particle is proportional to the force acting upon it.
- **N3:** Every action has an equal and opposite reaction.

While it is worthy to try to construct axioms on which the laws of physics rest, the trite, minimalistic attempt above falls somewhat short. For example, on first glance, it appears that the first law is nothing more than a special case of the second law. (If the force vanishes, the acceleration vanishes which is the same thing as saying that the velocity is constant). But the truth is somewhat more subtle. In what follows we will take a closer look at what really underlies Newtonian mechanics.

5.2 Inertial Frames and Newton's First Law

Placed in the historical context, it is understandable that Newton wished to stress the first law. It is a rebuttal to the Aristotelian idea that, left alone, an object will naturally come to rest. Instead, as Galileo had previously realised, the natural state of an object is to travel with constant speed. This is the essence of the law of inertia.

However, these days we're not bound to any Aristotelian dogma. Do we really need the first law? The answer is yes, but it has a somewhat different meaning.

We've already introduced the idea of a frame of reference: a Cartesian coordinate system in which you measure the position of the particle. But for most reference frames you can think of, Newton's first law is obviously incorrect. For example, suppose the coordinate system that I'm measuring from is rotating. Then, everything will appear to be spinning around me. If I measure a particle's trajectory in my coordinates as $\mathbf{X}(t)$, then I certainly won't find that $d^2\mathbf{X}/dt^2 = 0$, even if I leave the particle alone. In rotating frames, particles do not travel at constant velocity.

We see that if we want Newton's first law to fly at all, we must be more careful about the kind of reference frames we're talking about. We define an inertial reference frame to be one in which particles do indeed travel at constant velocity when the force acting on it vanishes. In other words, in an inertial frame

$$\ddot{\mathbf{X}} = 0 \quad \text{when} \quad \mathbf{F} = 0$$

The true content of Newton's first law can then be better stated as

- **N1 Revisited:** Inertial frames exist.

These inertial frames provide the setting for all that follows. For example, the second law — which we shall discuss shortly — should be formulated in inertial frames.

One way to ensure that you are in an inertial frame is to insist that you are left alone yourself: fly out into deep space, far from the effects of gravity and other influences, turn off your engines and sit there. This is an inertial frame. However, for most purposes it will suffice to treat axes of the room you're sitting in as an inertial frame. Of course, these axes are stationary with respect to the Earth and the Earth is rotating, both about its own axis and about the Sun. This means that the Earth does not quite provide an inertial frame.

5.2.1 Galilean Relativity

Inertial frames are not unique. Given one inertial frame, S, in which a particle has coordinates $X(t)$, we can always construct another inertial frame S' in which the particle has coordinates $X'(t)$ by any combination of the following transformations,

- Translations: $\mathbf{X}' = \mathbf{X} + \mathbf{a}$, for constant $\mathbf{a}$.
- Rotations: $\mathbf{X}' = R\mathbf{X}$, for a 3×3 matrix R obeying $R^T R = 1$. (This also allows for reflections if det $R = -1$, although our interest will primarily be on continuous transformations).
- Boosts: $\mathbf{X}' = \mathbf{X} + \mathbf{V}t$, for constant velocity $\mathbf{V}$.

It is simple to prove that all of these transformations map one inertial frame to another. Suppose that a particle moves with constant velocity with respect to frame S, so that $d^2X/dt^2 = 0$. Then, for each of the transformations above, we also have $d^2X'/dt^2 = 0$ which tells us that the particle also moves at constant velocity in S'. Or, in other words, if S is an inertial frame then so too is S'. The three transformations generate a group known as the Galilean group.

The three transformations above are not quite the unique transformations that map between inertial frames. But, for most purposes, they are the only interesting ones! The others are transformations of the form $\mathbf{X}' = \lambda\mathbf{X}$ for some $\lambda \in \mathbf{R}$. This is just a trivial rescaling of the coordinates. For example, we may choose to measure distances in S in units of meters and distances in S in units of parsecs.

We have already mentioned that Newton's second law is to be formulated in an inertial frame. But, importantly, it doesn't matter which inertial frame. In fact, this is true for all laws of physics: they are the same in any inertial frame. This is known as the *principle of relativity*. The three types of transformation laws that make up the Galilean group map from one inertial frame to another. Combined with the principle of relativity, each is telling us something important about the Universe

- Translations: There is no special point in the Universe.
- Rotations: There is no special direction in the Universe.
- Boosts: There is no special velocity in the Universe

The first two are fairly unsurprising: position is relative; direction is relative. The third perhaps needs more explanation. Firstly, it is telling us that there is no such thing as "absolutely stationary". You can only be stationary *with respect* to something else. Although this is true (and continues to hold in subsequent laws of physics) it is not true that there is no special speed in the Universe. The speed of light is special.

So position, direction and velocity are relative. But acceleration is not. You do not have to accelerate relative to something else. It makes perfect sense to simply say that you are accelerating or you are not accelerating. In fact, this brings us back to Newton's first law: if you are not accelerating, you are sitting in an inertial frame.

The principle of relativity is usually associated to Einstein, but in fact dates back at least as far as Galileo. In his book, *"Dialogue Concerning the Two Chief World Systems"*, Galileo has the character Salviati talk about the relativity of boosts,

Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.

Galileo Galilei, 1632

Absolute Time

There is one last issue that we have left implicit in the discussion above: the choice of time coordinate t. If observers in two inertial frames, S and S', fix the units – seconds, minutes, hours – in which to measure the duration time then the only remaining choice they can make is when to start the clock. In other words, the time variable in S and S' differ only by

$$t' = t + t_o$$

This is sometimes included among the transformations that make up the Galilean group.

The existence of a uniform time, measured equally in all inertial reference frames, is referred to as *absolute time*. It is something that we will have to revisit when we discuss special relativity. As with the other Galilean transformations, the ability to shift the origin of time is reflected in an important property of the laws of physics. The fundamental laws don't care when you start the clock. All evidence suggests that the laws of physics are the same today as they were yesterday. They are time translationally invariant.

Cosmology

Notably, the Universe itself breaks several of the Galilean transformations. There was a very special time in the Universe, around 13.7 billion years ago. This is the time of the Big Bang (which, loosely translated, means "we don't know what happened here").

Similarly, there is one inertial frame in which the background Universe is stationary. The "background" here refers to the sea of photons at a temperature of $2.7K$ which fills the Universe, known as the Cosmic Microwave Background Radiation. This is the afterglow of the fireball that filled all of space when the Universe was much younger. Different inertial frames are moving relative to this background and measure the radiation differently: the radiation looks more blue in the direction that you're travelling, redder in the direction that you've come from. There is an inertial frame in which this background radiation is uniform, meaning that it is the same colour in all directions.

To the best of our knowledge however, the Universe defines neither a special point, nor a special direction. It is, to very good approximation, homogeneous and isotropic.

However, it's worth stressing that this discussion of cosmology in no way invalidates the principle of relativity. All laws of physics are the same regardless of which inertial frame you are in. Overwhelming evidence suggests that the laws of physics are the same in far flung reaches of the Universe. They were the same in first few microseconds after the Big Bang as they are now.

5.3 Newton's Second Law

The second law is the meat of the Newtonian framework. It is the famous "$F = ma$", which tells us how a particle's motion is affected when subjected to a force F. The correct form of the second law is

$$\frac{d}{dt}(m\dot{\mathbf{X}}) = \mathbf{F}(\mathbf{X}, \dot{\mathbf{X}}) \tag{5.1}$$

This is usually referred to as the *equation of motion*. The quantity in brackets is called the *momentum*,

$$\mathbf{P} = m\dot{X}$$

Here m is the mass of the particle or, more precisely, the *inertial mass*. It is a measure of the reluctance of the particle to change its motion when subjected to a given force F. In most situations, the mass of the particle does not change with time. In this case, we can write the second law in the more familiar form,

$$m\ddot{X} = \mathbf{F}(\mathbf{X}, \dot{\mathbf{X}}) \tag{5.2}$$

Newton's second law doesn't actually tell us anything until someone else tells us what the force F is in any given situation. We will describe several examples in the next section. In general, the force can depend on the position $\mathbf{X}$ and the velocity $\dot{\mathbf{X}}$ of the particle, but does not depend on any higher derivatives. We could also, in principle, consider forces which include an explicit time dependence, $\mathbf{F}(\mathbf{X}, \dot{\mathbf{X}}, t)$, although we won't do so in these lectures. Finally, if more than one (independent) force is acting on the particle, then we simply take their sum on the right-hand side of (5.2).

The single most important fact about Newton's equation is that it is a *second order* differential equation. This means that we will have a unique solution only if we specify two initial conditions. These are usually taken to be the position $\mathbf{X}(t_o)$ and the velocity $\dot{\mathbf{X}}(t_o)$ at some initial time t_o. However, exactly what boundary conditions you must choose in order to figure out the trajectory depends on the problem you are trying to solve. It is not unusual, for example, to have to specify the position at an initial time t_o and final time t_f to determine the trajectory.

The fact that the equation of motion is second order is a deep statement about the Universe. It carries over, in essence, to all other laws of physics, from quantum mechanics to general relativity to particle physics. Indeed, the fact that all initial conditions must come in pairs — two for each ”degree of freedom“ in the problem — has important ramifications for later formulations of both classical and quantum mechanics.

For now, the fact that the equations of motion are second order means the following: if you are given a snapshot of some situation and asked ”what happens next“? then there is no way of knowing the answer. It's not enough just to know the positions of the particles at some point of time; you need to know their velocities too. However, once both of these are specified, the future evolution of the system is fully determined for all time.

5.4 Looking Forwards: The Validity of Newtonian Mechanics

Although Newton's laws of motion provide an excellent approximation to many phenomena, when pushed to extreme situation they are found wanting. Broadly speaking, there are three directions in which Newtonian physics needs replacing with a different framework: they are

- When particles travel at speeds close to the speed light, $c \approx 3 \times 10^8 ms^{-1}$, the Newtonian concept of absolute time breaks down and Newton's laws need modification. The resulting theory is called special relativity. As we will see, although the relationship between space and time is dramatically altered in special relativity, much of the framework of Newtonian mechanics survives unscathed.
- On very small scales, much more radical change is needed. Here the whole frame- work of classical mechanics breaks down so that even the most basic concepts, such as the trajectory of a particle, become ill-defined. The new framework that holds on these small scales is called quantum mechanics. Nonetheless, there are quantities which carry over from the classical world to the quantum, in particular energy and momentum.
- When we try to describe the forces at play between particles, we need to introduce a new concept: the *field*. This is a function of both space and time. Familiar examples are the electric and magnetic fields of electromagnetism. We won't have too much to say about fields in this course. For now, we mention only that the equations which govern the dynamics of fields are always second order differential equations, similar in spirit to Newton's equations. Because of this similarity, field theories are again referred to as ”classical“.

Eventually, the ideas of special relativity, quantum mechanics and field theories are combined into *quantum field theory*. Here even the concept of particle gets subsumed into the concept of a field. This is currently the best framework we have to describe the world around us. But we're getting ahead of ourselves. Let's firstly return to our Newtonian world....

6 Forces

In this section, we describe a number of different forces that arise in Newtonian mechanics. Throughout, we will restrict attention to the motion of a single particle. We start by describing the key idea of energy conservation, followed by a description of some common and important forces.

6.1 Potentials in One Dimension

Let's start by considering a particle moving on a line, so its position is determined by a single function $x(t)$. For now, suppose that the force on the particle depends only on the position, not the velocity: $F = F(x)$. We define the $potential V(x)$ (also called the *potential energy*) by the equation

$$F(x) = -\frac{dV}{dx} \tag{6.1}$$

The potential is only defined up to an additive constant. We can always invert (6.1) by integrating both sides. The integration constant is now determined by the choice of lower limit of the integral,

$$V(x) = -\int_{x_o}^{x} dx' F(x')$$

Here x' is just a dummy variable. (Do not confuse the prime with differentiation! In this course we will only take derivatives of position x with respect to time and always denote them with a dot over the variable). With this definition, we can write the equation of motion as

$$m\ddot{x} = -\frac{dV}{dx} \tag{6.2}$$

For any force in one-dimension which depends only on the position, there exists a conserved quantity called the energy,

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

The fact that this is *conserved* means that $\dot{E} = 0$ for *any* trajectory of the particle which obeys the equation of motion. While $V(x)$ is called the potential energy, $T = \frac{1}{2}m\dot{x}^2$ is called the *kinetic energy*. Motion satisfying (6.2) is called *conservative*. It is not hard to prove that E is conserved. We need only differentiate to get

$$\dot{E} = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = 0$$

where the last equality holds courtesy of the equation of motion (6.2).

In any dynamical system, conserved quantities of this kind are very precious. We will spend some time in this course fishing them out of the equations and showing how they help us simplify various problems.

An Example: A Uniform Gravitational Field

In a uniform gravitational field, a particle is subjected to a constant force, $F = -mg$ where $g \approx 9.8ms^{-2}$ is the acceleration due to gravity near the surface of the Earth. The minus sign arises because the force is downwards while we have chosen to measure position in an upwards direction which we call z. The potential energy is

$$V = mgz$$

Notice that we have chosen to have $V = 0$ at $z = 0$. There is nothing that forces us to do this; we could easily add an extra constant to the potential to shift the zero to some other height.

The equation of motion for uniform acceleration is

$$\ddot{z} = -g$$

Which can be trivially integrated to give the velocity at time t,

$$\dot{z} = u - gt \tag{6.3}$$

where u is the initial velocity at time $t = 0$. (Note that z is measured in the upwards direction, so the particle is moving up if $\dot{z} > 0$ and down if $\dot{z} < 0$). Integrating once more gives the position

$$z = z_o + ut - \frac{1}{2}gt^2 \tag{6.4}$$

where z_o is the initial height at time $t = 0$. Many high schools teach that (6.3) and (6.4) — the so-called "suvat" equations — are key equations of mechanics. They are not. They are merely the integration of Newton's second law for constant acceleration. Do not learn them; learn how to derive them.

Another Simple Example: The Harmonic Oscillator

The harmonic oscillator is, by far, the most important dynamical system in all of theoretical physics. The good news is that it's very easy. (In fact, the reason that it's so important is precisely because it's easy!). The potential energy of the harmonic oscillator is defined to be

$$V(x) = \frac{1}{2}kx^2$$

The harmonic oscillator is a good model for, among other things, a particle attached to the end of a spring. The force resulting from the energy V is given by $F = -kx$ which, in the context of the spring, is called *Hooke's law*. The equation of motion is

$$m\ddot{x} = -kx$$

which has the general solution
$x(t) = Acos(\omega t) + Bsin(\omega t)$ with $\omega = \sqrt{\frac{k}{m}}$

Here A and B are two integration constants and ω is called the *angular frequency*. We see that all trajectories are qualitatively the same: they just bounce backwards and forwards around the origin. The coefficients A and B determine the amplitude of the oscillations, together with the phase at which you start the cycle. The time taken to complete a full cycle is called the *period*

$$T = \frac{2\pi}{\omega} \tag{6.5}$$

The period is independent of the amplitude. (Note that, annoyingly, the kinetic energy is also often denoted by T as well. Do not confuse this with the period. It should hopefully be clear from the context).

If we want to determine the integration constants A and B for a given trajectory, we need some initial conditions. For example, if we're given the position and velocity at time $t = 0$, then it's simple to check that $A = x(0)$ and $B\omega = \dot{x}(0)$.

6.1.1 Moving in a Potential

Let's go back to the general case of a potential $V(x)$ in one dimension. Although the equation of motion is a second order differential equation, the existence of a conserved energy magically allows us to turn this into a first order differential equation,

$$E = \frac{1}{2}m\dot{x}^2 + V(x) \Rightarrow \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

This gives us our first hint of the importance of conserved quantities in helping solve a problem. Of course, to go from a second order equation to a first order equation, we must have chosen an integration constant. In this case, that is the energy E itself. Given a first order equation, we can always write down a formal solution for the dynamics simply by integrating,

$$t - t_o = \pm\int_{x_o}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}} \tag{6.6}$$

As before, x' is a dummy variable. If we can do the integral, we've solved the problem. If we can't do the integral, you sometimes hear that the problem has been "reduced to quadrature". This rather old-fashioned phrase really means "I can't do the integral". But, it is often the case that having a solution in this form allows some of its properties to become manifest. And, if nothing else, one can always just evaluate the integral numerically (i.e. on your laptop) if need be.

Getting a Feel for the Solutions

Given the potential energy $V_(x)$, it is often very simple to figure out the qualitative nature of any trajectory simply by looking at the form of $V_(x)$. This allows us to answer some questions with very little work. For example, we may want to know whether the particle is trapped within some region of space or can escape to infinity.

Let's illustrate this with an example. Consider the cubic potential

$$V(x) = m(x^3 - 3x) \tag{6.7}$$

If we were to substitute this into the general form (6.6), we'd get a fearsome looking integral which hasn't been solved since Victorian times.

Even without solving the integral, we can make progress. The potential is plotted in Figure 11. Let's start with the particle sitting stationary at some position x_o. This means that the energy is

$$E = V(x_o)$$

and this must remain constant during the subsequent motion. What happens next depends only on x_o. We can identify the following possibilities

- $x_o = \pm 1$: These are the local maximum and minimum. If we drop the particle at these points, it stays there for all time.

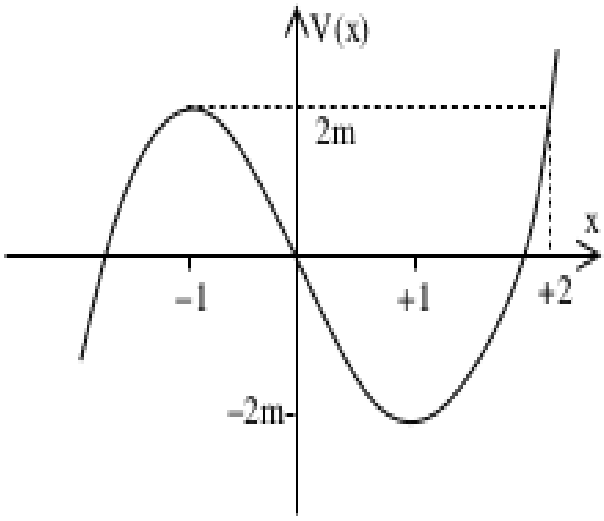


Figure 11: The cubic potential

- $x_o \in (-1, +2)$: Here the particle is trapped in the dip. It oscillates backwards and forwards between the two points with potential energy $V(x_o)$. The particle can't climb to the right because it doesn't have the energy. In principle, it could live off to the left where the potential energy is negative, but to get there it would have to first climb the small bump at $x = -1$ and it doesn't have the energy to do so. (There is an assumption here which is implicit throughout all of classical mechanics: the trajectory of the particle $x(t)$ is a continuous function).

- $x_o > 2$: When released, the particle falls into the dip, climbs out the other side, before falling into the void $x \to -\infty$

- $x_o < -1$: The particle just falls off to the left.

- $x_o = +2$: This is a special value, since $E = 2m$ which is the same as the potential energy at the local maximum $x = -1$. The particle falls into the dip and starts to climb up towards $x = -1$. It can never stop before it reaches $x = -1$ for at its stopping point it would have only potential energy $V < 2m$. But, similarly, it cannot arrive at $x = -1$ with any excess kinetic energy. The only option is that the particle moves towards $x = -1$ at an ever decreasing speed, only reaching the maximum at time $t \to \infty$. To see that this is indeed the case, we can consider the motion of the particle when it is close to the maximum. We write $x \approx -1 + \epsilon$with $\epsilon \ll 1$. Then, dropping the ϵ^3 term, the potential is

  $$V(x = -1 + \epsilon) \approx 2m - 3m\epsilon^2 + ...$$

  and, using (6.6), the time taken to reach $x = -1 + \epsilon$ is

  $$t - t_o = -\int_{\epsilon o}^{\epsilon} \frac{d\epsilon'}{\sqrt{6}\epsilon'} = -\frac{1}{\sqrt{6}} \log(\frac{\epsilon}{\epsilon_o})$$

The logarithm on the right-hand side gives a divergence as $\epsilon \to 0$. This tells us that it indeed takes infinite time to reach the top as promised.

One can easily play a similar game to that above if the starting speed is not zero. In general, one finds that the particle is trapped in the dip $x \in [-1, +1]$ if its energy lies in the interval $E \in [-2m, 2m]$.

6.1.2 Equilibrium: Why (Almost) Everything is a Harmonic Oscillator

A particle placed at an *equilibrium* point will stay there for all time. In our last example with a cubic potential (6.7), we saw two equilibrium points: $x = \pm 1$. In general, if we want $\dot{x} = 0$ for all time, then clearly we must have $\ddot{x} = 0$, which, from the form of Newton's equation (6.2), tells us that we can identify the equilibrium points with the critical points of the potential,

$$\frac{dV}{dx} = 0$$

What happens to a particle that is close to an equilibrium point, x_o? In this case, we can Taylor expand the potential energy about $x = x_o$. Because, by definition, the first derivative vanishes, we have

$$V(x) \approx V(x_o) + \frac{1}{2}V''(x_o)(x - x_o)^2 + ... \tag{6.8}$$

To continue, we need to know about the sign of $V''(x_o)$:

- $V''(x_o) > 0$: In this case, the equilibrium point is a minimum of the potential and the potential energy is that of a harmonic oscillator. From our previous discussion, We know that the particle oscillates backwards and forwards around x_o with frequency

  $$\omega = \sqrt{\frac{V''(x_o)}{m}}$$

  Such equilibrium points are called *stable*. This analysis shows that if the amplitude of the oscillations is small enough (so that we may ignore the $(x - x_o)^3$ terms in the Taylor expansion) then all systems oscillating around a stable fixed point look like a harmonic oscillator.

- $V''(x_o) < 0$: In this case, the equilibrium point is a maximum of the potential. The equation of motion again reads

  $$m\ddot{x} = -V''(x_o)(x - x_o)$$

  But with $V'' < 0$, we have $\ddot{x} > 0$ when $x - x_o > 0$. This means that if we displace the system a little bit away from the equilibrium point, then the acceleration pushes it further away. The general solution is

  $$x - x_o = Ae^{\alpha t} + Be^{-\alpha t} with \alpha = \sqrt{\frac{-V''(x_o)}{m}}$$

  Any solution with the integration constant $A \neq 0$ will rapidly move away from the fixed point. Since our whole analysis started from a Taylor expansion (6.8), neglecting terms of order $(x - x_o)^3$ and higher, our approximation will quickly break down. We say that such equilibrium points are *unstable*.

  Notice that there are solutions around unstable fixed points with $A = 0$ and $B \neq 0$ which move back towards the maximum at late times. These finely tuned solutions arise in the kind of situation that we described for the cubic potential where you drop the particle at a very special point (in the case of the cubic potential, this point was $x = 2$) so that it just reaches the top of a hill in infinite time. Clearly these solutions are not generic: they require very special initial conditions.

- Finally, we could have $V''(x_o) = 0$. In this case, there is nothing we can say about the dynamics of the system without Taylor expanding the potential further.

Another Example: The Pendulum

Consider a particle of mass m attached to the end of a light rod of length l. This counts as a one-dimensional system because we need specify only a single coordinate to say what the system looks like a a given time. The best coordinate to choose is θ, the angle that the rod makes with the vertical. The equation of motion is

$$\ddot{\theta} = -\frac{g}{l} sin\theta \tag{6.9}$$

The energy is

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl\cos\theta$$

(Note: Since θ is an angular variable rather than a linear variable, the kinetic energy is a little different.)

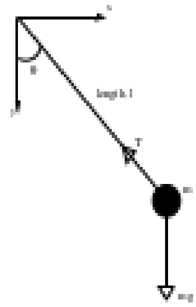


Figure 12

There are two qualitatively different motions of the pendulum. If $E > mgl$, then the kinetic energy can never be zero. This means that the pendulum is making complete circles. In contrast, if $E < mgl$, the pendulum completes only part of the circle before it comes to a stop and swings back the other way. If the highest point of the swing is θ_o , then the energy is

$$E = -mgl\cos\theta_o$$

We can determine the period T of the pendulum using (6.6). It's actually best to calculate the period by taking 4 times the time the pendulum takes to go from $\theta = 0$ to $\theta = \theta_o$. We have

$$T = 4\int_0^{T/4} dt = 4\int_0^{\theta_o} \frac{d\theta}{\sqrt{2E/ml^2 + (2g/l)\cos\theta}}$$

$$= 4\sqrt{\frac{l}{g}} \int_o^{\theta_o} \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\theta_o}} \tag{6.10}$$

We see that the period is proportional to $\sqrt{l/g}$ multiplied by some dimensionless number given by (4 times) the integral. For what it's worth, this integral turns out to be, once again, an elliptic integral.

For small oscillations, we can write $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ and the pendulum becomes a harmonic oscillator with angular frequency $\omega = \sqrt{g/l}$. If we replace the $\cos\theta' s$ in (6.10) by their Taylor expansion, we have

$$T = 4\sqrt{\frac{l}{g}} \int_o^{\theta_o} \frac{d\theta}{\sqrt{\theta_o^2 - \theta^2}} = 4\sqrt{\frac{l}{g}} \int_o^{1} \frac{dx}{\sqrt{1 - x^2}} = 2\pi\sqrt{\frac{l}{g}}$$

This agrees with our result (3.5) for the harmonic oscillator.

6.2 Potentials in Three Dimensions

Let's now consider a particle moving in three dimensional $\mathbf{R}^3$. Here things are more interesting. Firstly, it is possible to have energy conservation even if the force depends on the velocity. We will see how this can happen in Section 3.4. Conversely, forces which only depend on the position do not necessarily conserve energy: we need an extra condition. For now, we restrict attention to forces of the form $\mathbf{F} = \mathbf{F}(x)$. We have the following result:

Claim: There exists a conserved energy if and only if the force can be written in the form

$$\mathbf{F} = -\nabla V \tag{6.11}$$

or some potential function $V(x)$. This means that the components of the force must be of the form $F_i = -\partial V/\partial x^i$. The conserved energy is then given by

$$E = \frac{1}{2} m\dot{X} \cdot \dot{X} + V(X) \tag{6.12}$$

Proof: The proof that E is conserved if $\mathbf{F}$ takes the form (6.11) is exactly the same as in the one-dimensional case, together with liberal use of the chain rule. We have

$$\frac{dE}{dt} = m\dot{X} \cdot \ddot{X} + \frac{\partial V}{\partial x^i}\frac{\partial x^i}{\partial t}$$

using summation convention

$$= \dot{X} \cdot (m\ddot{X} + \nabla V) = 0$$

where the last equality follows from the equation of motion which is $m\ddot{X} = -\nabla V$.

To go the other way, we must prove that if there exists a conserved energy E taking the form (6.12) then the force is necessarily given by (6.11). To do this, we need the concept of work. If a force F acts on a particle and succeeds in moving it from $X(t_1)$ to $X(t_2)$ along a trajectory C, then the work done by the force is defined to be

$$W = \int_c \mathbf{F} \cdot dX$$

This is a line integral (of the kind you've met in the Vector Calculus course). The scalar product means that we take the component of the force along the direction of the trajectory at each point. We can make this clearer by writing

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{dX}{dt} dt$$

The integrand, which is the rate of doing work, is called the *power*, $P = \mathbf{F} \cdot \dot{X}$. Using Newton's second law, we can replace $\mathbf{F} = m\ddot{X}$ to get

$$W = m\int_{t_1}^{t_2} \ddot{X} \cdot \dot{X} dt = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt}(\dot{X} \cdot \dot{X}) dt = T(t_2) - T(t_1)$$

where

$$T \equiv \frac{1}{2}\dot{X} \cdot \dot{X} m$$

is the kinetic energy.(In all advanced courses of theoretical physics, kinetic energy is always denoted T)

So the total work done is proportional to the change in kinetic energy. If we want to have a conserved energy of the form (6.12), then the change in kinetic energy must be equal to the change in potential energy. This means we must be able to write

$$W = \int_c \mathbf{F} \cdot dX = V(X(t_1)) - V(X(t_2)) \tag{6.13}$$

In particular, this result tells us that the work done must be independent of the trajectory C; it can depend only on the end points $X(t_1)$ and $X(t_2)$. But a simple result (which you will prove in your Vector Calculus course) says that (6.13) holds only for forces of the form

$$\mathbf{F} = -\nabla V$$

as required.

Forces in three dimensions which take the form $\mathbf{F} = -\nabla V$ are called *conservative*. forces in $\mathbf{R}^3$ are conservative if and only if $\nabla \times \mathbf{F} = 0$.

6.2.1 Central Forces

A particularly important class of potentials are those which depend only on the distance to a fixed point, which we take to be the origin

$$V(X) = V(r)$$

where $r = |x|$. The resulting force also depends only on the distance to the origin and, moreover, always points in the direction of the origin,

$$\mathbf{F}(r) = -\nabla V = -\frac{dV}{dr}\hat{X} \tag{6.14}$$

Such forces are called *central*. In these lectures, we'll also use the notation $\hat{\mathbf{r}} = \hat{\mathbf{X}}$ to denote the unit vector pointing radially from the origin to the position of the particle. (In other courses, you may see this same vector denoted as $\mathbf{e}_r$).

In the vector calculus course, you will spend some time computing quantities such as ∇V in spherical polar coordinates. But, even without such practice, it is a simple matter to show that the force (6.14) is indeed aligned with the direction to the origin. If $X = (x_1, x_2, x_3)$ then the radial distance is $r^2 = x_1^2 + x_2^2 + x_3^2$, from which we can compute $\partial/\partial x_i = x_i/r$ for $i = 1, 2, 3$. Then, using the chain rule, we have

$$\begin{aligned}\nabla V &= \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right)\\ &= \left(\frac{dV}{dr}\frac{\partial r}{\partial x_1}, \frac{dV}{dr}\frac{\partial r}{\partial x_2}, \frac{dV}{dr}\frac{\partial r}{\partial x_3}\right)\\ &= \frac{dV}{dr}\left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right) = \frac{dV}{dr}\hat{\mathbf{X}}\end{aligned}$$

6.2.2 Angular Momentum

We will devote all of Section 8 to the study of motion in central forces. For now, we will just mention what is important about central forces: they have an extra conserved quantity. This is a vector $\mathbf{L}$ called *angular momentum*,

$$\mathbf{L} = m\mathbf{X} \times \dot{\mathbf{X}}$$

Notice that, in contrast to the momentum $\mathbf{P} = m\dot{\mathbf{X}}$, the angular momentum $\mathbf{L}$ depends on the choice of origin. It is a perpendicular to both the position and the momentum.

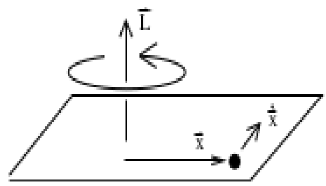


Figure 13

Let's look at what happens to angular momentum in the presence of a general force $\mathbf{F}$. When we take the time derivative, we get two terms. But one of these contains $\dot{\mathbf{X}} \times \dot{\mathbf{X}}$. We're left with

$$\frac{d\mathbf{L}}{dt} = m\mathbf{X} \times \ddot{\mathbf{X}} = \mathbf{XF}$$

The quantity $\tau = \mathbf{X} \times \mathbf{F}$ is called the *torque*. This gives us an equation for the change of angular momentum that is very similar to Newton's second law for the change of momentum,

$$\frac{d\mathbf{L}}{dt} = \tau$$

Now we can see why central forces are special. When the force **F** lies in the same direction as the position **X**of the particle, we have $\mathbf{X} \times \mathbf{F} = 0$. This means that the torque vanishes and angular momentum is conserved

$$\frac{d\mathbf{L}}{dt} = 0$$

We'll make good use of this result in Section 8 where we'll see a number of important examples of central forces.

6.3 Gravity

To the best of our knowledge, there are four fundamental forces in Nature. They are

- Gravity
- Electromagnetism
- Strong Nuclear Force
- Weak Nuclear Force

The two nuclear forces operate only on small scales, comparable, as the name suggests, to the size of the nucleus ($r_o \approx 10^{-15}m$). We can't really give an honest description of these forces without invoking quantum mechanics and, for this reason, we won't discuss them in this course. (A very rough, and slightly dishonest, classical description of the strong nuclear force can be given by the potential $V(r) \sim e^{-r/r_o}/r$). In this section we discuss the force of gravity; in the next, electromagnetism.

Gravity is a conservative force. Consider a particle of mass M fixed at the origin. A particle of mass m moving in its presence experiences a potential energy

$$V(r) = -\frac{GMm}{r} \tag{6.15}$$

Here G is Newton's constant. It determines the strength of the gravitational force and is given by

$$G \approx 6.67 \times 10^{-11} m^3 Kg^{-1} s^{-2}$$

The force on the particle is given by

$$\mathbf{F} = -\nabla V = -\frac{GMm}{r^2}\hat{\mathbf{r}} \tag{6.16}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of the particle. This is Newton's famous inverse-square law for gravity. The force points towards the origin. We will devote much of Section 8 to studying the motion of a particle under the inverse-square force.

6.3.1 The Gravitational Field

The quantity V in (6.15) is the potential energy of a particle of mass m in the presence of mass M . It is common to define the gravitational field of the mass M to be

$$\Phi(r) = -\frac{GM}{r}$$

Φ is sometimes called the Newtonian gravitational field to distinguish it from a more sophisticated object later introduced by Einstein. It is also sometimes called the *gravitational potential*. It is a property of the mass M alone. The potential energy of the mass m is then given by $V = m\Phi$.

The gravitational field due to many particles is simply the sum of the field due to each individual particle. If we fix particles with masses M_i at positions $\mathbf{r}_i$, then the total gravitational field is

$$\Phi(r) = -G\sum_i \frac{M_i}{|\mathbf{r}-\mathbf{r}_i|}$$

The gravitational force that a moving particle of mass m experiences in this field is

$$\mathbf{F} = -Gm\sum_i \frac{M_i}{|\mathbf{r}-\mathbf{r}_i|^3}(\mathbf{r}-\mathbf{r}_i)$$

The Gravitational Field of a Planet

The fact that contributions to the Newtonian gravitational potential add in a simple linear fashion has an important consequence: the external gravitational field of a spherically symmetric object of mass M – such as a star or planet – is the same as that of a point mass M positioned at the origin.

The proof of this statement is an example of the volume integral. We let the planet have density $\rho(r)$ and radius R. Summing over the contribution from all points x inside the planet, the gravitational field is given by

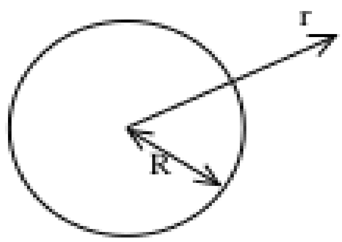


Figure 14

$$\Phi(\mathbf{r}) = -\int_{|X|\le R} d^3X \frac{G\rho(\mathbf{X})}{\mathbf{r}-\mathbf{X}}$$

It's best to work in spherical polar coordinates and to choose the polar direction, $\theta = 0$, to lie in the direction of $\mathbf{r}$. Then $\mathbf{r}\times\mathbf{X} = rx\cos\theta$. We can use this to write an expression for the denominator: $|\mathbf{r}-\mathbf{X}^2| = r^2+x^2-2rx\cos\theta$. The gravitational field then becomes

$$\Phi(r) = -G\int_0^R dx \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\rho(x)x^2\sin\theta}{\sqrt{r^2+x^2-2rx\cos\theta}}$$

$$= -2\pi G\int_0^R dx \int_0^\pi d\theta \frac{\rho(x)x^2\sin\theta}{\sqrt{r^2+x^2-2rx\cos\theta}}$$

$$= -2\pi G\int_0^R dx\rho(x)x^2\frac{1}{rx}\left[\sqrt{r^2+x^2-2rx\cos\theta}\right]_{\theta=0}^{\theta=\pi}$$

$$= -\frac{2\pi G}{r}\int_0^R dx\rho(x)x(|r+x|-|r-x|)$$

So far this calculation has been done for any point $\mathbf{r}$, whether inside or outside the planet. At this point, we restrict attention to points external to the planet. This means that $|r+x| = r+x$ and $|r-x| = r-x$ and we have

$$\Phi(\mathbf{r}) = -\frac{4\pi G}{r}\int_0^R dx\rho(x)x^2 = -\frac{GM}{r}$$

This is the result that we wanted to prove: the gravitational field is the same as that of a point mass M at the origin.

6.3.2 Escape Velocity

Suppose that you're trapped on the the surface of a planet of radius R. (This should be easy). Let's firstly ask what gravitational potential energy you feel. Assuming you can only rise a distance $z \ll R$ from the planet's surface, we can Taylor expand the potential energy,

$$V(R+z) = -\frac{GMm}{R+z} = -\frac{GMm}{R}\left(1 - \frac{z}{R} + \frac{z^2}{R^2} + ...\right)$$

If we're only interested in small changes in $z \ll R$, we need focus only on the second term, giving

$$V(z) \approx constant + \frac{GMm}{R^2}z + ...$$

This is the familiar potential energy that gives rise to constant acceleration. We usually write $g = GM/R^2$. For the Earth, $g \approx 9.8ms^{-2}$.

Now let's be more ambitious. Suppose we want to escape our parochial, planet-bound existence. So we decide to jump. How fast do we have to jump if we wish to truly be free? This, it turns out, is the same kind of question that we discussed previously in the context of particles moving in one dimension and can be determined very easily using gravitational energy $V = -GMm/r$. If you jump directly upwards (i.e. radially) with velocity v, your total energy as you leave the surface is

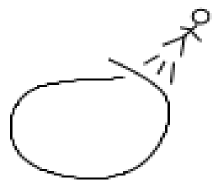


Figure 15

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

For any energy $E < 0$, you will eventually come to a halt at position $r = -GMm/E$, before falling back. If you want to escape the gravitational attraction of the planet for ever, you will need energy $E \geq 0$. At the minimum value of $E = 0$, the associated velocity

$$v_{escape} = \sqrt{\frac{2GM}{R}} \tag{6.17}$$

This is the *escape velocity*.

6.3.3 Earth's satellite

Satellites can be launched from the Earth's surface to circle the Earth. They are kept in their orbit by the gravitational attraction of the Earth.

Consider a satellite of mass m which just circles the Earth of mass M close to its surface in an orbit of radius R

$$\frac{mV^2}{R} = \frac{GMm}{R^2} = Mg$$

$$V^2 = Rg \quad or \quad V = \sqrt{Rg}; \quad orbital \quad velocity$$

NB; Escape velocity $=\sqrt{2Rg}$

$$= \sqrt{2} \times orbital \quad velocity$$

$$\therefore V_{es} = \sqrt{2} \times orbital \quad velocity$$

Black Holes and the Schwarzchild Radius

Let's do something a little dodgy. We'll take the formula above and apply it to light. The reason that this is dodgy is because, the laws of Newtonian physics need modifying for particles close to the speed of light where the effects of special relativity are important. Nonetheless, let's forget this for now and plough ahead regardless.

Light travels at speed $c \approx 3 \times 10^8 ms^{-1}$. Suppose that the escape velocity from the surface of a star is greater than or equal to the speed of light. From (6.17), this would happen if the radius of the star satisfies

$$R \leq R_s = \frac{2GM}{c^2}$$

What do we see if this is the case? Well, nothing! The star is so dense that light can't escape from it. It's what we call a black hole.

Although the derivation above is not trustworthy, by some fortunate coincidence it turns out that the answer is correct. The distance $R_s = 2GM/c^2$ is called the *Schwarzchild radius*. If a star is so dense that it lies within its own Schwarzchild radius, then it will form a black hole. (To demonstrate this properly, you really need to work with the theory of general relativity).

For what it's worth, the Schwarzchild radius of the Earth is around $1cm$. The Schwarzchild radius of the Sun is about $3km$. You'll be pleased to hear that, because both objects are much larger than their Schwarzchild radii, neither is in danger of forming a black hole any time soon.

6.3.4 Inertial vs Gravitational Mass

We have seen two formulae which involve mass, both due to Newton. These are the second law and the inverse-square law for gravity. Yet the meaning of mass in these two equations is very different. The mass appearing in the second law represents the reluctance of a particle to accelerate under any force. In contrast, the mass appearing in the inverse-square law tells us the strength of a particular force, namely gravity. Since these are very different concepts, we should really distinguish between the two different masses. The second law involves the *inertial mass*, m_I

$$m_I \ddot{\mathbf{X}} = \mathbf{F}$$

while Newton's law of gravity involves the *gravitational mass*, m_G

$$\mathbf{F} = -\frac{GM_G m_G}{r^2}\hat{\mathbf{r}}$$

It is then an experimental fact that

$$m_I = m_G \tag{6.18}$$

Much experimental effort has gone into determining the accuracy of (6.18), most notably by the Hungarian physicist Eötvösh at the turn of the (previous) century. We now know that the inertial and gravitational masses are equal to within about one part in 10^{13} . Currently, the best experiments to study this equivalence, as well as searches for deviations from Newton's laws at short distances, are being undertaken by a group at the University of Washington in Seattle who go by the name Eöt-Wash. A theoretical understanding of the result (6.18) came only with the development of the general theory of relativity.

6.4 Electromagnetism

Throughout the Universe, at each point in space, there exist two vectors, E and B. These are known as the *electric* and *magnetic* fields. Their role – at least for the purposes of this course – is to guide any particle that carries electric charge.

The force experienced by a particle with electric charge q is called the $Lorentz force$,

$$\mathbf{F} = q\Big(\mathbf{E(X)} + \dot{\mathbf{X}} \times \mathbf{B(X)}\Big) \tag{6.19}$$

Here we have used the notation **E(X)** and **B(X)** to stress that the electric and magnetic fields are functions of space. Both their magnitude and direction can vary from point to point.

The electric force is parallel to the electric field. By convention, particles with positive charge q are accelerated in the direction of the electric field; those with negative electric charge are accelerated in the opposite direction. Due to a quirk of history, the electron is taken to have a negative charge given by

$$q_{electron} \approx -1.6 \times 10^{-19} Coulombs.$$

As far as fundamental physics is concerned, a much better choice is to simply say that the electron has charge 1. All other charges can then be measured relative to this.

The magnetic force looks rather different. It is a velocity dependent force, with magnitude proportional to the speed of the particle, but with direction perpendicular to that of the particle. We shall see its effect in simple situations shortly.

In principle, both **E** and **B** can change in time. However, here we will consider only situations where they are static. In this case, the electric field is always of the form

$$\mathbf{E} = -\nabla\phi$$

For some function $\phi(X)$ called the *electric potential* (or *scalar potential* or even just the *potential* as if we didn't already have enough things with that name).

For time independent fields, something special happens: energy is conserved.

Claim: The conserved energy is

$$E = \frac{1}{2} m\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} + q\phi(\mathbf{X})$$

Proof:

$$\dot{\mathbf{E}} = m\dot{\mathbf{X}} \cdot \ddot{\mathbf{X}} + q\nabla\phi \cdot \dot{\mathbf{X}} = \dot{\mathbf{X}} \cdot (\dot{\mathbf{F}} + q\nabla\phi) = q\dot{\mathbf{X}} \cdot (\dot{\mathbf{X}} \times \mathbf{B}) = 0$$

where the last equality occurs because $\dot{\mathbf{X}} \times \mathbf{B}$ is necessarily perpendicular to $\dot{\mathbf{X}}$. Notice that this gives an example of something we promised earlier: a velocity dependent force which conserves energy. The key part of the derivation is that the velocity dependent force is perpendicular to the trajectory of the particle. This ensures that the force does no work.

6.4.1 The Electric Field of a Point Charge

Charged objects do not only respond to electric fields; they also produce electric fields. A particle of charge Q sitting at the origin will set up an electric field given by

$$\mathbf{E} = -\nabla\left(\frac{Q}{4\pi\epsilon_o r}\right) = \frac{Q}{4\pi\epsilon o}\frac{\hat{\mathbf{r}}}{r^2} \tag{6.20}$$

where $r^2 = \mathbf{X} \cdot \mathbf{X}$ The quantity ϵ_o has the grand name *Permittivity of Free Space* and is a constant given by

$$\epsilon_o \approx 8.85 \times 10^{-12} m^{-3} Kg^{-1} s^2 C^2$$

This quantity should be thought of as characterising the strength of the electric interaction.

The force between two particles with charges Q and q is given by $\mathbf{F} = q\mathbf{E}$ with $\mathbf{E}$ given by (6.2). In other words,

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_o}\frac{\hat{\mathbf{r}}}{r^2}$$

This is known as the *Coulomb force*. It is a remarkable fact that, mathematically, the force looks identical to the Newtonian gravitational force (6.16): both have the characteristic inverse-square form. We will study motion in this potential in detail in Section 8, with particular focus on the Coulomb force in 8.4.

Although the forces of Newton and Coulomb look the same, there is one important difference. Gravity is always attractive because mass $m > 0$. In contrast, the electrostatic Coulomb force can be attractive or repulsive because charges q come with both signs. Further differences between gravity and electromagnetism come when you ask what happens when sources (mass or charge) move; but that's a story that will be told in different courses.

6.4.2 Circles in a Constant Magnetic Field

Motion in a constant electric field is simple: the particle undergoes constant acceleration in the direction of $\mathbf{E}$. But what about motion in a constant magnetic field $\mathbf{B}$? The equation of motion is

$$m\ddot{\mathbf{X}} = q\dot{\mathbf{X}} \times \mathbf{B}$$

Let's pick the magnetic field to lie in the z-direction and write

$$\mathbf{B} = (0, 0, B)$$

We can now write the Lorentz force law (6.19) in components. It reads

$$m\ddot{x} = qB\dot{y} \tag{6.21}$$

$$m\ddot{y} = -qB\dot{x} \tag{6.22}$$

$$m\ddot{z} = 0$$

The last equation is easily solved and the particle just travels at constant velocity in the z direction. The first two equations are more interesting. There are a number of ways to solve them, but a particularly elegant way is to construct the complex variable $\xi x + iy$. Then adding (6.21) to i times (6.22) gives

$$m\ddot{\xi} = -iqB\dot{\xi}$$

which can be integrated to give

$$\xi = \alpha e^{-i\omega t} + \beta$$

where α and β are integration constants and ω is given by

$$\omega = \frac{qB}{m}$$

If we choose our initial conditions to be that the particle starts life at $t = 0$ at the origin with velocity $-v$ in the y-direction, then α and ω are fixed to be

$$\xi = \frac{v}{\omega}\left(e^{-i\omega t} - 1\right)$$

Translating this back into x and y coordinates, we have
$x = \frac{v}{\omega}(\cos\omega t - 1)$ and $y = -\frac{v}{\omega}(\sin\omega t)$

The end result is that the particle undergoes circles in the plane with angular frequency ω, known as the *cyclotron frequency*. The time to undergo a full circle is fixed: $T = 2\pi/\omega$. In contrast, the size of the circle is v/ω and arises as an integration constant. Circles of arbitrary sizes are allowed; the only price that you pay is that you have to go faster.

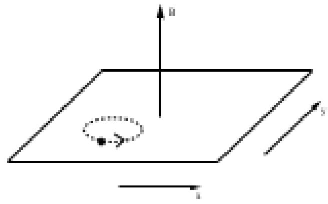


Figure 16

A Comment on Solving Vector Differential Equations

The Lorentz force equation (6.19) gives a good example of a vector differential equation. The straightforward way to view these is always in components: they are three, coupled, second order differential equations for x, y and z. This is what we did above when understanding the motion of a particle in a magnetic field.

However, one can also attack these kinds of questions without reverting to components. Let's see how this would work in the case of Larmor circles. We start with the vector equation

$$m\ddot{\mathbf{X}} = q\dot{\mathbf{X}} \times \mathbf{B} \tag{6.23}$$

To begin, we take the dot product with **B**. Since the right-hand side vanishes, we're left with

$$\ddot{\mathbf{X}} \cdot \mathbf{B} = 0$$

This tells us that the particle travels with constant velocity in the direction of **B**. This is simply a rewriting of our previous result $\ddot{z} = 0$. For simplicity, let's just assume that the particle doesn't move in the **B** direction, remaining at the origin. This tells us that the particle moves in a plane with equation

$$\mathbf{X} \cdot \mathbf{B} = 0 \tag{6.24}$$

However, we're not yet done. We started with (6.23) which was three equations. Taking the dot product always reduces us to a single equation. So there must still be two further equations lurking in (6.23) that we haven't yet taken into account. To find them, the systematic thing to do would be to take the cross product with B. However, in the present case, it turns out that the simplest way forwards is to simply integrate (6.23) once, to get

$$m\dot{\mathbf{X}} = q\mathbf{X} \times \mathbf{B} + \mathbf{c}$$

with **c** a constant of integration. We can now substitute this back into the right-hand side of (6.23) to find

$$\begin{aligned} m^2\ddot{\mathbf{X}} &= \mathbf{d} + q^2(\mathbf{X} \times \mathbf{B}) \times \mathbf{B} \\ &= \mathbf{d} + q^2((\mathbf{X} \cdot \mathbf{B})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{X}) \\ &= -q^2B^2\Big(\mathbf{X} - \mathbf{d}/q^2B^2\Big) \end{aligned}$$

where the integration constant now sits in $\mathbf{d} = q\mathbf{c} \times \mathbf{B}$ which, by construction, is perpendicular to B. In the last line, we've used the equation (6.24). (Note that if we'd considered a situation in which the particle was moving with constant velocity in the **B** direction, we'd have to work a little harder at this point). The resulting vector equation looks like three harmonic oscillators, displaced by the vector $\mathbf{d}/q^2B^2$, oscillating with frequency $\omega = qB/m$. However, because of the constraint (6.24), the motion is necessarily only in the two directions perpendicular to $\mathbf{B}$. The end result is

$$\mathbf{X} = \frac{\mathbf{d}}{q^2B^2} + \alpha_1 \cos\omega t + \alpha_2 \sin\omega t$$

with $\alpha_i = 1, 2$ integration constants satisfying $\alpha_1 \cdot \mathbf{B} = 0\cdot$ This is the same result we found previously.

Admittedly, in this particular example, working with components was somewhat easier than manipulating the vector equations directly. But this won't always be the case — for some problems you'll make more progress by playing the kind of games that we've described here.

6.4.3 An Aside: Maxwell's Equations

In the Lorentz force law, the only hint that the electric and magnetic fields are related is that they both affect a particle in a manner that is proportional to the electric charge. The connection between them becomes much clearer when things depend on time. A time dependent electric field gives rise to a magnetic field and vice versa. The dynamics of the electric and magnetic fields are governed by Maxwell's equations. In the absence of electric charges, these equations are given by

$$\begin{array}{ccc} \nabla \cdot \mathbf{E} = 0 & , & \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & , & \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \end{array}$$

with c the speed of light.

For now, it's worth making one small comment. When we showed that energy is conserved, we needed both the electric and magnetic field to be time independent. What happens when they change with time? In this case, energy is still conserved, but we have to worry about the energy stored in the fields themselves.

6.5 Friction

Friction is a messy, dirty business. While energy is always conserved on a fundamental level, it doesn't appear to be conserved in most things that you do every day. If you slide along the floor in your socks you don't keep going for ever. At a microscopic level, your kinetic energy is transferred to the atoms in the floor where it manifests itself as heat. But if we only want to know how far our socks will slide, the details of all these atomic processes are of little interest. Instead, we try to summarise everything in a single, macroscopic force that we call *friction*.

6.5.1 Dry Friction

Dry friction occurs when two solid objects are in contact. Think of a heavy box being pushed along the floor, or some idiot sliding in his socks. Experimentally, one finds that the complicated dynamics involved in friction is usually summarised by the force

$$F = \mu R$$

where R is the reaction force, normal to the floor, and μ is a constant called the *coefficient of friction*. Usually $\mu \approx 0.3$, although it depends on the kind

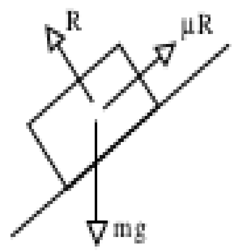


Figure 17

of materials that are in contact. Moreover, the coefficient is usually, more or less, independent of the velocity. We won't have much to say about dry friction in this course. In fact, we've already said it all.

6.5.2 Fluid Drag

Drag occurs when an object moves through a fluid — either liquid or gas. The resistive force is opposite to the direction of the velocity and, typically, falls into one of two categories

- Linear Drag:

$$\mathbf{F} = -\gamma \mathbf{V}$$

  where the coefficient of friction, γ, is a constant. This form of drag holds for objects moving slowly through very viscous fluids. For a spherical object of radius L, there is a formula due to Stokes which gives $\gamma = 6\pi\eta L$ where η is the viscosity of the fluid.

- Quadratic Drag:

$$\mathbf{F} = -\gamma|\mathbf{V}|\mathbf{V}$$

  Again, γ is called the coefficient of friction. For quadratic friction, γ is usually proportional to the surface area of the object, i.e. $\gamma \sim L^2$. (This is in contrast to the coefficient for linear friction where Stokes' formula gives $\gamma \sim L$). Quadratic drag holds for fast moving objects in less viscous fluids. This includes objects falling in air such as, for example, the various farmyard animals dropped by Galileo from the leaning tower.

Quadratic drag arises because the object is banging into molecules in the fluid, knocking them out the way. There is an intuitive way to see this. The force is proportional to the change of momentum that occurs in each collision. That gives one factor of v. But the force is also proportional to the number of collisions. That gives the second factor of v, resulting in a force that scales as v^2.

One can ask where the cross-over happens between linear and quadratic friction. Naively, the linear drag must always dominate at low velocities simply because $x \gg x^2$ when $x \ll 1$. More quantitatively, the type of drag is determined by a dimensionless number called the *Reynolds number*,

$$R = \frac{\rho v L}{\eta} \tag{6.25}$$

where ρ is the density of the fluid while η is the viscosity. For $R \ll 1$, linear drag dominates; for $R \gg 1$, quadratic friction dominates.

What is Viscosity?

Above, we've mentioned the viscosity of the fluid, η, without really defining it. For completeness, below is how to measure viscosity.

Place a fluid between two plates, a distance d apart. Keeping the lower plate still, move the top plate at a constant speed v. This sets up a velocity gradient in the fluid. But, the fluid pushes back. To keep the upper plate moving at constant speed, you will have to push with a force per unit area which is proportional to the velocity gradient,

$$\frac{F}{A} = \eta \frac{v}{d}$$

The coefficient of proportionality, η, is defined to be the *(dynamic) viscosity*.

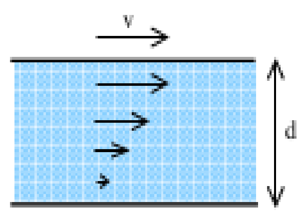


Figure 18

6.5.3 An Example: The Damped Harmonic Oscillator

We start with our favourite system: the harmonic oscillator, now with a damping term. This was already discussed in your Differential Equations course and we include it here only for completeness. The equation of motion is

$$m\ddot{x} = -kx - \gamma\dot{x}$$

Divide through by m to get

$$\ddot{x} = -\omega_o^2 x - 2\alpha\dot{x}$$

where $\omega_o^2 = k/m$ is the frequency of the undamped harmonic oscillator and $\alpha = \gamma/2m$. We can look for solutions of the form

$$x = e^{i\beta t}$$

Remember that x is real, so we're using a trick here. We rely on the fact that the equation of motion is linear so that if we can find a solution of this form, we can take the real and imaginary parts and this will also be a solution. Substituting this ansatz into the equation of motion, we find a quadratic equation for β. Solving this, gives the general solution

$$x = Ae^{i\omega_+ t} + Be^{i\omega_- t}$$

with $\omega_\pm = i\alpha \pm \sqrt{\omega_o^2 - \alpha^2}$. We identify three different regimes,

- Underdamped: $\omega_o^2 > \alpha^2$. Here the solution takes the form,

  $$x = e^{-\alpha t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right)$$

  where $\Omega = \sqrt{\omega^2 - \alpha^2}$. Here the system oscillates with a frequency $\Omega < \omega_o$, while the amplitude of the oscillations decays exponentially.

- Overdamped: $\omega_o^2 < \alpha^2$. The roots $\omega_\pm$ are now purely imaginary and the general solution takes the form,

  $$x = e^{-\alpha t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right)$$

  Now there are no oscillations. Both terms decay exponentially. If you like, the amplitude decays away before the system is able to undergo even a single oscillation.

- Critical Damping: $\omega_o^2 = \alpha^2$. Now the two roots $\omega_\pm$ coincide. With a double root of this form, the most general solution takes the form,

  $$x = (A + Bt)e^{-\alpha t}$$

  Again, there are no oscillations, but the system does achieve some mild linear growth for times $t < 1/\alpha$, after which it decays away.

6.5.4 Terminal Velocity with Quadratic Friction

You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, a horse splashes.

J.B.S. Haldane, *On Being the Right Size*

Let's look at a particle of mass m moving in a constant gravitational field, subject to quadratic friction. We'll measure the height z to be in the upwards direction, meaning that if $v = dz/dt > 0$, the particle is going up. We'll look at the cases where the particle goes up and goes down separately.

6.5.5 Coming Down

Suppose that we drop the particle from some height. The equation of motion is given by

$$m\frac{dv}{dt} = -mg + \gamma v^2$$

It's worth commenting on the minus signs on the right-hand side. Gravity acts downwards, so comes with a minus sign. Since the particle is falling down, friction is acting upwards so comes with a plus sign. Dividing through by m, we have

$$\frac{dv}{dt} = -g + \frac{\gamma v^2}{m} \tag{6.26}$$

Integrating this equation once gives

$$t = -\int_0^v \frac{dv'}{g - \gamma v'^2/m}$$

which can be easily solved by the substitution $v = \sqrt{mg/\gamma}\tanh x$ to get

$$t = -\sqrt{\frac{m}{\gamma g}}\tanh^{-1}\left(\sqrt{\frac{\gamma}{mg}}v\right)$$

Inverting this gives us the speed as a function of time

$$v = -\sqrt{\frac{mg}{\gamma}}\tanh\left(\sqrt{\frac{\gamma g}{m}}t\right)$$

We now see the effect of friction. As time increases, the velocity does not increase without bound. Instead, the particle reaches a maximum speed,

$$v \to -\sqrt{\frac{mg}{\gamma}} \tag{6.27}$$

as

$$t \to \infty$$

This is the *terminal velocity*. The sign is negative because the particle is falling downwards. Notice that if all we wanted was the terminal velocity, then we don't need to go through the whole calculation above. We can simply look for solutions of (6.26) with constant speed, so $dv/dt = 0$. This obviously gives us (6.27) as a solution. The advantage of going through the full calculation is that we learn how the velocity approaches its terminal value.

We can now see the origin of the quote we started with. The point is that if we compare objects of equal density, the masses scale as the volume, meaning $m \sim L^3$ where L is the linear size of the object. In contrast, the coefficient of friction usually scales as surface area, $\gamma \sim L^2$. This means that the terminal velocity depends on size. For objects of equal density, we expect the terminal velocity to scale as $v \sim \sqrt{L}$. I have no idea if this is genuinely a big enough effect to make horses splash. (Haldane was a biologist, so he should know what it takes to make an animal splash. But in his essay he assumed linear drag rather than quadratic, so maybe not).

6.5.6 Going Up

Now let's think about throwing a particle upwards. Since both gravity and friction are now acting downwards, we get a flip of a minus sign in the equation of motion. It is now

$$\frac{dv}{dt} = -g - \frac{\gamma v^2}{m} \tag{6.28}$$

Suppose that we throw the object up with initial speed u and we want to figure out the maximum height, h, that it reaches. We could follow our earlier calculation and integrate (6.28) to determine $v = v(t)$. But since we aren't asking about time, it's much better to instead consider velocity as a function of distance: $v = v(z)$. We write

$$\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz} = -g - \frac{\gamma v^2}{m}$$

which can be rewritten as

$$\frac{1}{2}\frac{d(v^2)}{dz} = -g - \frac{\gamma v^2}{m}$$

Now we can integrate this equation to get velocity as a function of distance. Writing $y = v^2$, we have

$$\int_{u^2}^{0} \frac{dy}{g + \gamma y/m} = -2\int_0^h dz \Rightarrow \frac{m}{\gamma}\left[\log\left(g + \frac{\gamma y}{m}\right)\right]_{y=u^2}^{y=0} = -2h$$

which we can rearrange to get the final answer,

$$h = \frac{m}{2\gamma}\log\left(1 + \frac{\gamma u^2}{mg}\right)$$

It's worth looking at what happens when the effect of friction is small. Naively, it looks like we're in trouble here because as $\gamma \to 0$, the term in front gets very large. But surely the height shouldn't go to infinity just because the friction is small. The resolution to this is that the log is also getting small in this limit. Expanding the log, we have

$$h = \frac{u^2}{2g}\left(1 - \frac{\gamma u^2}{2mg} + ...\right)$$

Here the leading term is indeed the answer we would get in the absence of friction; the subleading terms tell us how much the friction, γ, lowers the attained height.

6.5.7 Linear Drag and Ohm's Law

Consider an electron moving in a conductor. As we've seen, a constant electric field causes the electron to accelerate. A fairly good model for the physics of a conductor, known as the Drude model, treats the electron as a classical particle with linear damping. The resulting equation of motion is

$$m\ddot{x} = -eE - \gamma v$$

As in the previous example, we can figure out the terminal velocity by setting $\ddot{x} = 0$, to get

$$v = -\frac{eE}{\gamma}$$

In a conductor, the velocity of the electron v gives the current density, j,

$$j = -env$$

where n is the density of electrons. This then gives us a relationship between the current density and the electric field

$$j = \sigma E$$

The quantity $\sigma = e^2 n/\gamma$ is called the *conductivity*. This equation is Ohm's law. However, it's probably not yet in the form you know and love. If the wire has length L and cross-sectional area A, then the current I is defined as $I = jA$. Meanwhile, the voltage dropped across the wire is $V = EL$. With this in hand, we can rewrite Ohm's law as

$$V = IR$$

where the resistance is given by $R = L/\sigma A$

6.5.8 A 3d Example: A Projectile with Linear Drag

All our examples so far have been effectively one-dimensional. Here we give a three dimensional example which provides another illustration of how to treat vector differential equations and, specifically, how to work with vector constants on integration. We will consider a projectile, moving under gravity, experiencing linear drag. (Think of a projectile moving very slowly in a viscous liquid). At time $t = 0$, we throw the object with velocity **u**. What is its subsequent motion? The equation of motion is

$$m\frac{d\mathbf{V}}{dt} = m\mathbf{g} - \gamma\mathbf{V} \tag{6.29}$$

We can solve this by introducing the integrating factor $e^{\gamma t/m}$ to write the equation as

$$\frac{d}{dt}\left(e^{\gamma t/m}\mathbf{V}\right) = e^{\gamma t/m}\mathbf{g}$$

We now integrate, but have to introduce a vector integration constant – let's call it **c** – for our troubles. We have

$$\mathbf{V} = \frac{m}{\gamma}\mathbf{g} + \mathbf{c}e^{-\gamma t/m}$$

We specified above that at time $t = 0$, the velocity is **V** = **u**, so we can use this information to determine the integration constant **c**. We get

$$\mathbf{V} = \frac{m}{\gamma}\mathbf{g} + \left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m}$$

Now we integrate $\mathbf{V} = d\mathbf{x}/dt$ a second time to determine x as a function of time. We get another integration constant, **b**,

$$\mathbf{X} = \frac{m}{\gamma}\mathbf{g}t - \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)e^{-\gamma t/m} + \mathbf{b}$$

To determine this second integration constant, we need some further information about the initial conditions. Lets say that $\mathbf{X} = 0$ at $t = 0$. Then we have

$$\mathbf{X} = \frac{m}{\gamma}\mathbf{g}t + \frac{m}{\gamma}\left(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\right)\left(1 - e^{-\gamma t/m}\right)$$

We can now look at this in components to get a better idea of what's going on. We'll write $\mathbf{X} = (x, y, z)$ and we'll send the projectile off with initial velocity $\mathbf{u} = (u\cos\theta, 0, u\sin\theta)$. With gravity acting downwards, so $\mathbf{g} = (0, 0, -g)$, our vector equation becomes three equations. One is trivial: $y = 0$. The other two are

$$x = \frac{m}{y}u\cos\theta\left(1 - e^{-\gamma t/m}\right)$$

$$z = -\frac{mgt}{\gamma} + \frac{m}{\gamma}\left(u\sin\theta + \frac{mg}{\gamma}\right)\left(1 - e^{-\gamma t/m}\right)$$

Notice that the time scale m/γ is important. For $t \gg m/\gamma$, the horizontal position is essentially constant. By this time, the particle is dropping more or less vertically.

Finally, we can revisit the question that we asked in the last example: what happens when friction is small? Again, there are a couple of terms that look as if they are going to become singular in this limit. But that sounds very unphysical. To resolve this, we should ask what γ is small relative to. In the present case, the answer lies in the exponential terms. To say that γ is small, really means $\gamma \ll m/t$ or, in other words, it means that we are looking at short times, $t \ll m/\gamma$. Then we can expand the exponential. Reverting to the vector form of the equation, we find

$$\begin{aligned}\mathbf{X} &= \frac{m}{\gamma}\mathbf{g}t + \frac{m}{\gamma}\Big(\mathbf{u} - \frac{m}{\gamma}\mathbf{g}\Big)\Big(1 - 1 + \frac{\gamma t}{m} - \frac{1}{2}\Big(\frac{\gamma t}{m}\Big)^2 + ...\Big) \\ &= \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2 + O\Big(\frac{\gamma t}{m}\Big)\end{aligned}$$

So we see that, on small time scales, we indeed recover the usual story of a projectile without friction. The friction only becomes relevant when $t \sim m/\gamma$.

7 Conservative and Non-conservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{r}^2$. The work done on the particle during its mechanical evolution is

$$W^{(A-B)} = \int_{t_A}^{t_B} dt F \cdot v \tag{7.1}$$

where $v = \dot{r}$. This is the most general expression for the work done. If the force F depends only on the particle's position r, we may write $dr = vdt$, and then

$$W^{(A-B)} = \int_{r_A}^{r_B} dr F(r) \tag{7.2}$$

Consider now the force

$$F(r) = K_1 y\hat{x} + K_2 x\hat{y} \tag{7.3}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in the figure below:

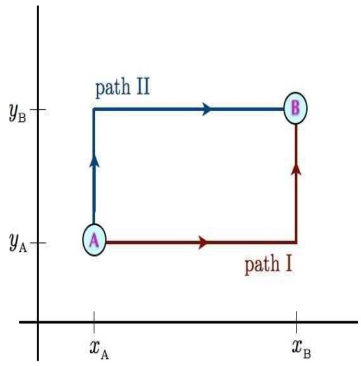


Figure 19: Two Paths Joining A and B

$$W^I = K_1 \int_{x_A}^{x_B} dx y_A + K_2 \int_{y_A}^{y_B} dy x_B = K_1 y_A(x_B - x_A) + K_2 x_B(y_B - y_A)$$

$$W^{II} = K_1 \int_{x_A}^{x_B} dx y_B + K_2 \int_{y_A}^{y_B} dy x_A = K_1 y_B(x_B - x_A) + K_2 x_A(y_B - y_A)$$

Note that in general, $W^I \neq W^{II}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^I - W^{II} = (K_2 - K_1)(x_B - x_A)(y_B - y_A) \tag{7.4}$$

Thus, we see that if $K_1 - K_2$, the work is the same for the two paths. In fact, if $K_1 - K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for any path, and not just piece-wise linear paths. The reason for this is Stokes theorem:

$$\oint_{\partial C} dl \cdot F = \int_C dS\hat{n} \cdot \nabla \times F \tag{7.5}$$

Here, C is a connected region in three-dimensional space, ∂C is mathematical notation for the boundary of C, which is a closed path, dS is the scalar differential area element, $\hat{n}$ is the unit normal to that differential area element, and $\nabla \times F$ is the curl of F:

$$\nabla \times F = det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial_x} & \dfrac{\partial}{\partial_y} & \dfrac{\partial}{\partial_z} \\ F_x & F_y & F_z \end{pmatrix}$$

$$= \Big(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\Big)\hat{x} + \Big(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\Big)\hat{y} + \Big(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\Big)\hat{z} \tag{7.6}$$

For the force under consideration, $F(r) = K_1 y\hat{x} + K_2 x\hat{y}$, the curl is

$$\nabla \times F = (K_2 - K_1)\hat{z} \tag{7.7}$$

which is a constant. The RHS (7.5) is then simply proportional to the area enclosed by C. When we compute the work difference in (7.4), we evaluate the integral $\oint_C dl \cdot F$ along path $\gamma_{II}^{-1} \circ \gamma_I$ which is to say path I followed by the inverse of path II. In this case, $\hat{n} = \hat{z}$ and the integral of $\hat{n} \cdot \nabla \times F$ over the rectangle C is given by the RHS of (7.4).

When $\nabla \times F = 0$ everywhere in space, we can always write $F = -\nabla U$, where $U(r)$ is the potential energy. Such forces are called conservative forces because the total kinetic energy of the system $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done

$$W^{(A-B)} = \int_{r_A}^{r_B} dr F(r)$$

$$= -\int_{r_A}^{r_B} dr \cdot \nabla U$$

$$= U(r_A) - U(r_B) \tag{7.8}$$

The work-energy theorem then gives

$$T^B - T^A = U(r_A) - U(r_B) \tag{7.9}$$

which says

$$E^B = T^B + U(r_B = T^A + U(r_A = E^A \tag{7.10}$$

Thus total energy $E = T + V$ is conserved

Example: Integrating $F = -\nabla U$
If $\nabla \times F = 0$, we can compute U(r) by integrating. *viz*

$$U(r) = U(0) - \int_0^r dr' \cdot F(r') \tag{7.11}$$

The integral does not depend on the path chosen connecting 0 and r. For example, we can take

$$U(x,y,z) = U(0,0,0) - \int_{0,0,0}^{x,0,0} dx' F_x(x',0,0) - \int_{x,0,0}^{x,y,0} dy' F_y(x,y',0) - \int_{z,y,0}^{x,y,z} dz' F_z(x,y,z') \tag{7.12}$$

The constant $U(0,0,0)$ us arbitrary and impossible to determine from F alone.

As an example, consider the force

$$F(r) = -ky\hat{x} - kx\hat{y} - 4bz^3\hat{z} \tag{7.13}$$

where k and b are constants. We have

$$(\nabla \times F)_x = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0 \tag{7.14}$$

$$(\nabla \times F)_y = \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0 \tag{7.15}$$

$$(\nabla \times F)_y = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \tag{7.16}$$

so $\nabla \times F = 0$ and F must be expressible as $F = -\nabla U$. Integrating using (7.12) we have

$$U(x,y,z) = U(0,0,0) + \int_{0,0,0}^{x,0,0} dx' k \cdot 0 + \int_{x,0,0}^{x,y,0} dy' kxy' + \int_{z,y,0}^{x,y,z} dz' 4bz'^3 \tag{7.17}$$

$$= U(0,0,0) + kxy + bz^4 \tag{7.18}$$

Another approach is to integrate the partial differential equation $\nabla U = -F$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{x}$-component,

$$\frac{\partial U}{\partial x} = ky$$

Integrating, we obtain

$$U(x,y,z) = kxy + f(y,z)$$

where $f(y,z)$ is at this point an arbitrary function of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$ Next we have

$$\frac{\partial U}{\partial y} = kx \Rightarrow U(x,y,z) = kxy + g(x,z)$$

Finally the z component integrate to give

$$\frac{\partial U}{\partial z} = 4bz^3 \Rightarrow U(x,y,z) = bz^4 + h(x,y)$$

we now equate the first two expressions:

$$kxy + f(y, z) = kxy + g(x, z)$$

Subtracting kxy from each side, we obtain the equation $f(y, z) = g(x, z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y, z) = g(x, z) = q(z)$$

where q(z) is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x, y) = kxy + q(z)$$

The only possible solution is $h(x, y) = C + kxy$ and $q(z) = C + bz^4$ where C is a constant. Therefore,

$$U(x, y, z) = C + kxy + bz^4$$

Note that it would be very wrong to integrate $\partial U/\partial x = ky$ and obtain $U(x, y, z) = kxy + C'$ where C' is a constant. As we have seen, the constant of integration we obtain upon integratinf this first order PDE is in fact a *function* of y and z. The fact that $f(y, z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y, z)$ is a solution to the PDE $\partial f/\partial x = ky$, for any arbitrary function $f(y, z)$.

7.1 Conservative Forces in Many Particle Systems

$$T = \sum_i \frac{1}{2} m_i \dot{r_i}^2$$

$$U = \sum_i V(r_i) + \sum_{i<j} v(|r_i - r_j|) \tag{7.19}$$

Here, $V(r)$ is the external (or one-body) potential, and $v(r - r')$ is the interparticle potential which we assume to be central depending only on the distance between any pair of particles. The equations of motion are

$$m_i \ddot{r_i} = F_i^{ext} + F_i^{int} \tag{7.20}$$

with

$$F_i^{ext} = -\frac{\partial V(r_i)}{\partial r_i} \tag{7.21}$$

$$F_i^{int} = -\sum_j \frac{\partial v(|r_i - r_j|)}{\partial r_i} r_i \equiv \sum_j F_{ij}^{int} \tag{7.22}$$

Here, F_{ij}^{int} us the force exerted on particle i by particle j:

$$F_{ij}^{int} = -\sum \frac{\partial v(r_i - r_j)}{\partial r_i} = -\frac{r_i - r_j}{|r_i - r_j|} v'(|r_i - r_j|) \tag{7.23}$$

Note that $F_{ij}^{int} = -F_{ji}^{int}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $r_{ij} \equiv r_i - r_j$ in which case we may write the interparticle force as

$$F_{ij}^{int} = -\hat{r_{ij}} v'(r_{ij}) \tag{7.24}$$

7.2 Linear and Angular Momentum

Consider now the total momentum of the system , $P = \sum_i p_i$, its rate of change is

$$\frac{dP}{dt} = \sum_i \dot{p}_i = \sum_i F_i^{ext} + \sum_{i\neq j} F_{ij}^{int} = F_{tot}^{ext} \tag{7.25}$$

Since the sum over all internal forces cancels as a result of Newton's Third Law, we can write

$$P = \sum_i m_i \dot{r}_i = M\ddot{R}$$

$$M = \sum_i m_i \quad (total\ mass) \tag{7.26}$$

$$R = \frac{\sum_i m_i r_i}{\sum_i m_i} \quad (center\ of\ mass)$$

Next we consider the total angular momentum,

$$L = \sum_i r_i \times p_i = \sum_i m_i r_i \times \dot{r}_i$$

The rate if change of L is then

$$\frac{dL}{dt} = \sum_i \{m_i \dot{r}_i \times \dot{r}_i + m_i r_i \times \ddot{r}_i\} \tag{7.27}$$

$$= sum_i \times F_i^{ext} + \sum_{i\neq j} r_i \times F_{ij}^{int} \tag{7.28}$$

$$= \sum_i r_i \times F_i^{ext} + \frac{1}{2}\sum_{i\neq j}(r_i - r_j) \times F_{ij}^{int}$$

Finally, it useful to establish the result

$$T = \frac{1}{2}\sum_i m_i \dot{r}_i{}^2 = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\sum_i m_i(\dot{r}_i - \dot{R})^2$$

Which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center of mass motion, and the kinetic energy of the particles relative to the center of mass. Recall the "work-energy theorem" for conservative systems

$$0 = \int_{initial}^{final} dE = \int_{initial}^{final} dT + \int_{initial}^{final} dU$$

$$= T^B - T^A - \sum_i \int dr_i \cdot F_i \tag{7.29}$$

Which is to say

$$\Delta T = T^B - T^A = \sum_i \int dr_i \cdot F_i = -\Delta U$$

In other words, the total energy $E = T + V$ is conserved

$$E = \sum \frac{1}{2} m_i \dot{r}_i{}^2 + \Sigma V(r_i) + \sum v(|r_i - r_j|) \tag{7.30}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz*

$$\sum m_i \phi(r_i) \longrightarrow \int d^3r \rho(\mathbf{r})\phi(\mathbf{r})$$

where $\rho(\mathbf{r})$ is the mass density and $\phi(\mathbf{r})$ is any function.

8 Central Forces

In this section we will study the three-dimensional motion of a particle in a central force potential. Such a system obeys the equation of motion

$$m\ddot{X} = -\nabla V(r) \tag{8.1}$$

where the potential depends only on $r = |x|$. Since both gravitational and electrostatic forces are of this form, solutions to this equation contain some of the most important results in classical physics.

Our first line of attack in solving (8.1) is to use angular momentum. Recall that this is defined as

$$\mathbf{L} = m\mathbf{X} \times \dot{\mathbf{X}}$$

We already saw in previous discussion that angular momentum is conserved in a central potential. The proof is straightforward:

$$\frac{d\mathbf{L}}{dt} = mX \times \ddot{\mathbf{X}} = -\mathbf{X} \times \nabla V = 0$$

where the final equality follows because ∇V is parallel to x.

The conservation of angular momentum has an important consequence: all motion takes place in a plane. This follows because $\mathbf{L}$ is a fixed, unchanging vector which, by construction, obeys

$$\mathbf{L} \cdot \mathbf{X} = 0$$

So the position of the particle always lies in a plane perpendicular to $\mathbf{L}$. By the same argument, $\mathbf{L} \cdot \dot{\mathbf{X}}$ so the velocity of the particle also lies in the same plane. In this way the three-dimensional dynamics is reduced to dynamics on a plane.

8.1 Polar Coordinates in the Plane

We've learned that the motion lies in a plane. It will turn out to be much easier if we work with polar coordinates on the plane rather than Cartesian coordinates. For this reason, we take a brief detour to explain some relevant aspects of polar coordinates.

To start, we rotate our coordinate system so that the angular momentum points in the z-direction and all motion takes place in the (x, y) plane. We then define the usual polar coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

Our goal is to express both the velocity and acceleration in polar coordinates. We introduce two unit vectors, $\hat{r}$ and $\hat{\theta}$ in the direction of increasing r and θ respectively as shown in the diagram. Written in Cartesian form, these vectors are

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad , \quad \hat{\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

These vectors form an orthornormal basis at every point on the plane. But the basis itself depends on which angle θ we sit at.

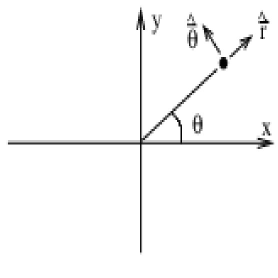


Figure 20

Moving in the radial direction doesn't change the basis, but moving in the angular direction we have

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\theta} \quad , \quad \frac{d\hat{\theta}}{d\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = \hat{\mathbf{r}}$$

This means that if the particle moves in a way such that θ changes with time, then the basis vectors themselves will also change with time. Let's see what this means for the velocity expressed in these polar coordinates. The position of a particle is written as the simple, if somewhat ugly, equation

$$\mathbf{X} = r\hat{\mathbf{r}}$$

From this we can compute the velocity, remembering that both r and the basis vector $\hat{\mathbf{r}}$ can change with time. We get

$$\begin{aligned} \dot{\mathbf{X}} &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \end{aligned} \tag{8.2}$$

The second term in the above expression arises because the basis vectors change with time and is proportional to the *angular velocity*, $\dot{\theta}$. (Strictly speaking, this is the angular speed. In the next section, we will introduce a vector quantity which is the angular velocity).

Differentiating once more gives us the expression for acceleration in polar coordinates,

$$\begin{aligned} \ddot{\mathbf{X}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\dot{\theta} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \end{aligned} \tag{8.3}$$

An Example: Circular Motion

Let's look at an example that we're already all familiar with. A particle moving in a circle has $\dot{r} = 0$. If the particle travels with constant angular velocity $\dot{\theta} = \omega$ then the velocity in the plane is

$$\dot{\mathbf{X}} = r\omega\hat{\theta}$$

so the speed in the plane in $v = |\dot{X}| = r\omega$. Similarly, the acceleration in the plane is

$$\ddot{X} = -r\omega^2\hat{\mathbf{r}}$$

The magnitude of the acceleration is $a = |\ddot{X}| = r\omega^2 = v^2/r$. From Newton's second law, if we want a particle to travel in a circle, we need to supply a force $F = mv^2/r$ towards the origin. This is known as a *centripetal force*.

8.2 Back to Central Forces

We've already seen that the three-dimensional motion in a central force potential actually takes place in a plane. Let's write the equation of motion (8.1) using the plane polar coordinates that we've just introduced. Since $V = V(r)$, the force itself can be written using

$$\nabla V = \frac{dV}{dr}\hat{\mathbf{r}}$$

and, from (8.3) the equation of motion becomes

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -\frac{dV}{dr}\hat{\mathbf{r}} \tag{8.4}$$

The $\hat{\theta}$ component of this is particularly simple. It is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \Rightarrow \frac{1}{r}\frac{d}{dt}\Big(r^2\dot{\theta}\Big) = 0$$

It looks as if we've found a new conserved quantity since we've learnt that

$$l = r^2\dot{\theta} \tag{8.5}$$

does not change with time. However, we shouldn't get too excited. This is something that we already know. To see this, let's look again at the angular momentum **L**. We already used the fact that the direction of **L** is conserved when restricting motion to the plane. But what about the magnitude of **L**? Using (8.2), we write

$$\mathbf{L} = m\mathbf{X} \times \dot{X} = mr\hat{r} \times \Big(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\Big) = mr^2\dot{\theta}\Big(\hat{\mathbf{r}} \times \hat{\theta}\Big)$$

Since $\hat{r}$ and $\hat{\theta}$ are orthogonal, unit vectors, $\hat{\mathbf{r}} \times \hat{\theta}$ is also a unit vector. The magnitude of the angular momentum vector is therefore

$$|\mathbf{L}| = ml$$

and l, given in (8.5), is identified as the angular momentum per unit mass, although we will often be lazy and refer to l simply as the angular momentum.

Let's now look at the $\hat{\mathbf{r}}$ component of the equation of motion (8.4). It is

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}$$

Using the fact that $l = r^2\dot{\theta}$ is conserved, we can write this as

$$m\ddot{r} = -\frac{dV}{dr} + \frac{ml^2}{r^3} \tag{8.6}$$

It's worth pausing to reflect on what's happened here. We started in (8.1) with a complicated, three dimensional problem. We used the direction of the angular momentum to reduce it to a two dimensional problem, and the magnitude of the angular momentum to reduce it to a one dimensional problem. This was all possible because angular momentum is conserved.

This should give you some idea of how important conserved quantities are when it comes to solving anything. Roughly speaking, this is also why it's not usually possible to solve the N -body problem with $N \geq 3$. For the $N = 2$ mutually interacting particles, we can use the symmetry of translational invariance to solve the problem. But for $N \geq 3.$, we don't have any more conserved quantities to come to our rescue.

Returning to our main storyline, we can write (8.6) in the suggestive form

$$m\ddot{r} = -\frac{dV_{eff}}{dr} \tag{8.7}$$

where V_{eff} is called the *effective potential* and is given by

$$V_{eff}(r) = V(r) + \frac{ml^2}{2r^2} \tag{8.8}$$

The extra term, $ml^2/2r^2$ is called the *angular momentum barrier* (also known as the centrifugal barrier). It stops the particle getting too close to the origin, since there is must pay a heavy price in "effective energy".

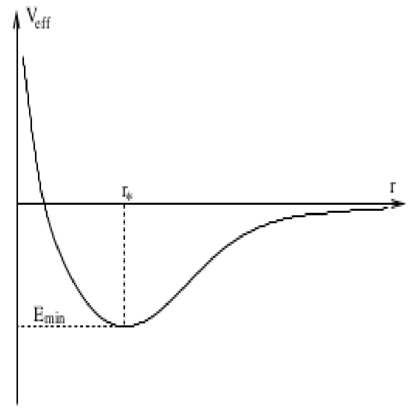


Figure 21: The effective potential arising from the inverse square force law.

8.2.1 The Effective Potential: Getting a Feel for Orbits

Let's just check that the effective potential can indeed be thought of as part of the energy of the full system. Using (8.2), we can write the energy of the full three dimensional problem as

$$
\begin{aligned}
E &= \frac{1}{2}m\dot{X}\cdot\dot{X} + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} + V(r) \\
&= \frac{1}{2}m\dot{r}^2 + V_{eff}(r)
\end{aligned}
$$

This tells us that the energy E of the three dimensional system does indeed coincide with the energy of the effective one dimensional system that we've reduced to. The effective potential energy is the real potential energy, together with a contribution from the angular kinetic energy.

We already had a discussion how we can understand qualitative aspects of one dimensional motion simply by plotting the potential energy. Let's play the same game here. Starting with the most useful example of a central potential: $V(r) = -k/r$, corresponding to an attractive inverse square law for $k > 0$. The effective potential is

$$V_{eff} = -\frac{k}{r} + \frac{ml^2}{2r^2}$$

and is drawn in the figure.

The minimum of the effective potential occurs at $r_* = ml^2/k$ and takes the value $V_{eff}(r_*) = -k^2/2ml^2$. The possible forms of the motion can be characterised by their energy E.

- $E = E_{min} = -k^2/2ml^2$: Here the particle sits at the bottom of the well r_* and stays there for all time. However, remember that the particle also has angular velocity, given by $\dot{\theta} = l/r_*^2$. So although the particle has fixed radial position, it is moving in the angular direction. In other words, the trajectory of the particle is a circular orbit about the origin.

Notice that the radial position of the minimum depends on the angular momentum l. The higher the angular momentum, the further away the minimum. If there is no angular momentum, and $l = 0$, then $V_{eff} = V$ and the potential has no minimum. This is telling us the obvious fact that there is no way that r can be constant unless the particle is moving in the θ direction. In a similar vein, notice that there is a relationship between the angular velocity $\dot{\theta}$ and the size of the orbit, r_* , which we get by eliminating l: it is $\dot{\theta}^2 = k/mr_*^3$.

- $E_{min} < E < 0$: Here the 1d system sits in the dip, oscillating backwards and forwards between two points. Of course, since $l \neq 0$, the particle also has angular velocity in the plane. This describes an orbit in which the radial distance r depends on time. Although it is not yet obvious, we will soon show that for $V = -k/r$, this orbit is an ellipse.

  The smallest value of r that the particle reaches is called the *periapsis*. The furthest distance is called the *apoapsis*. Together, these two points are referred to as the *apsides*. In the case of motion around the Sun, the periapsis is called the *perihelion* and the apoapsis the *aphelion*.

- $E > 0$. Now the particle can sit above the horizontal axis. It comes in from infinity, reaches some minimum distance r, then rolls back out to infinity. We will see later that, for the $V = -k/r$ potential, this trajectory is hyperbola.

8.2.2 The Stability of Circular Orbits

Consider a general potential $V(r)$. We can ask: when do circular orbits exist? And when are they stable?

The first question is quite easy. Circular orbits exist whenever there exists a solution with $l \neq 0$ and $\dot{r} = 0$ for all time. The latter condition means that $\ddot{r} = 0$ which, in turn, requires

$$V'_{eff}(r_*) = 0$$

In other words, circular orbits correspond to critical points, r_*, of V_{eff}. The orbit is stable if small perturbations return us back to the critical point. Stability requires that we sit at the minimum of the effective potential. This usually translates to the requirement that

$$V''_{eff}(r_*) > 0$$

If this condition holds, small radial deviations from the circular orbit will oscillate about r_* with simple harmonic motion.

Although the criterion for circular orbits is most elegantly expressed in terms of the effective potential, sometimes it's necessary to go back to our original potential $V(r)$. In this language, circular orbits exist at points r_* obeying

$$V'(r_*) = \frac{ml^2}{r_*^3}$$

These orbits are stable if

$$V''(r_*) + \frac{3ml^2}{r_*^4} = V''(r_*) + \frac{3}{r_*}V'(r_*) > 0 \tag{8.9}$$

We can even go right back to basics and express this in terms of the force. $F(r) = -V'(r)$. A circular orbit is stable if

$$F'(r_*) + \frac{3}{r_*}F(r_*) < 0$$

An Example

Consider a central potential which takes the form

$$V(r) = -\frac{k}{r^n} n \geq 1$$

For what powers of n are the circular orbits stable? By our criterion (5.9), stability requires

$$V'' + \frac{3}{r}V' = -\Big(n(n+1) - 3n\Big)\frac{k}{r^{n+2}} > 0$$

which holds only for $n < 2$. We can easily see this pictorially in the figures where we've plotted the effective potential for $n = 1$ and $n = 3$.

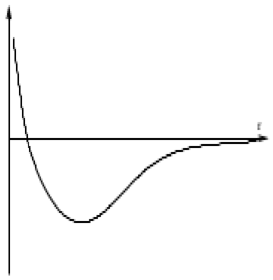


Figure 22: V_{eff} for $V = -1/r$

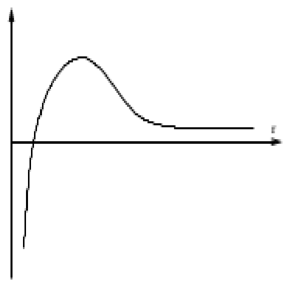


Figure 23: V_{eff} for $V = -1/r^3$

Curiously, in a Universe with d spatial dimensions, the law of gravity would be $F \sim 1/r^{d-1}$ corresponding to a potential energy $V \sim -1/r^{d-2}$. We see that circular planetary orbits are only stable in $d < 4$ spatial dimensions. Fortunately, this includes our Universe. We should all be feeling very lucky right now.

8.3 The Orbit Equation

Let's return to the case of general V_{eff} . If we want to understand how the radial position $r(t)$ changes with time, then the problem is essentially solved. Since the energy E is conserved, we have

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

which we can view as a first order differential equation for dr/dt. Integrating then gives

$$t = \pm\sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - V_{eff}(r)}}$$

However, except for a few very special choices of $V_{eff}(r)$, the integral is kind of a pain. What's more, often trying to figure out $r(t)$ is not necessarily the information that we're looking for. It's better to take a more global approach, and try to learn something about the whole trajectory of the particle, rather than its position at any given time. Mathematically, this means that we'll try to understand something about the shape of the orbit by computing $r(\theta)$.

In fact, to proceed, we'll also need a little trick. It's trivial, but it turns out to make the resulting equations much simpler. We introduce the new coordinate

$$u = \frac{1}{r}$$

Firstly, we can rewrite the radial velocity as

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{l}{r^2} = -l\frac{du}{d\theta}$$

Meanwhile, the acceleration is

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\Big(-l\frac{du}{d\theta}\Big) = -l\frac{d^2u}{d\theta^2}\dot{\theta} = -l^2\frac{d^2u}{d\theta^2}\frac{1}{r^2} = -l^2u^2\frac{d^2u}{d\theta^2} \tag{8.10}$$

The equation of motion for the radial position, which we first derived back in (5.6), is

$$m\ddot{r} - \frac{ml^2}{r^3} = F(r)$$

where, we've reverted to expressing the right-hand side in terms of the force $F(r) = -dV/dr$. Using (8.10), and doing a little bit of algebra (basically dividing by ml^2u^2), we get the second order differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}F(1/u) \tag{8.11}$$

This is the *orbit equation*.

8.3.1 The Kepler Problem

The *Kepler problem* is the name given to understanding planetary orbits about a star. It is named after the astronomer Johannes Kepler.

The inverse-square force law of gravitation is described by the central potential

$$V(r) = -\frac{km}{r} \tag{8.12}$$

where $k = GM$. However, the results that we will now derive will equally well apply to motion of a charged particle in a Coulomb potential if we instead use $k = -qQ/4\pi\epsilon_o m$.

For the potential (8.12), the orbit equation (8.11) becomes very easy to solve. It is just

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{l^2}$$

But this is just the equation for a harmonic oscillator, albeit with its centre displaced by k/l^2 . We can write the most general solution as

$$u = A\cos(\theta - \theta_o) + \frac{k}{l^2} \tag{8.13}$$

with A and θ_o integration constants. (You might be tempted instead to write $u = A\cos\theta + B\sin\theta + k/l^2$ with A and B as integration constants. This is equivalent to our result above but, as we will now see, it's much more useful to use θ_o as the second integration constant).

At the point where the orbit is closest to the origin (the periapsis), u is largest. From our solution, we have $u_{max} = A + k/l^2$. We will choose to orient our polar coordinates so that the periapsis occurs at $\theta = 0$. this choice means that set $\theta_o = 0$. In terms of our original variable $r = 1/u$, we have the final expression for the orbit

$$r = \frac{r_o}{e\cos\theta + 1} \tag{8.14}$$

where

$$r_o = \frac{l^2}{k}$$

and

$$e = \frac{Al^2}{k}$$

Notice that r_o is fixed by the angular momentum, while the choice of e is now effectively the integration constant in the problem.

(8.14) describes a *conic section*. The integration constant e is called the *eccentricity* and it determines the shape of the orbit.

Ellipses: $e < 1$

For $e < 1$, the radial position of the particle is bounded in the interval

$$\frac{r_o}{r} \in [1-e, 1+e]$$

We can convert (8.14) back to Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$, writing

$$r = r_o - er\cos\theta \Rightarrow x^2 + y^2 = (r_o - ex)^2$$

Multiplying out the square, collecting terms, and rearranging allow us to write this equation as

$$\frac{(x-x_c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

with

$$x_c = -\frac{er_o}{1-e^2} \quad and \quad a^2 = \frac{r_o^2}{(1-e^2)^2} \quad and \quad b^2 = \frac{r_o^2}{1-e^2} < a^2 \tag{8.15}$$

This is the formula for an ellipse, with its centre shifted to $x = x_c$. The orbit is drawn in the figure. The two semi-axes of the ellipse have lengths a and b. The centre of attraction of the

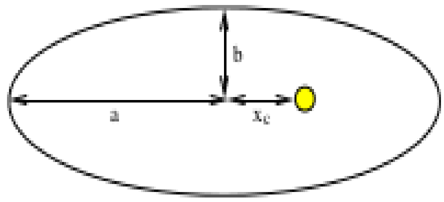


Figure 24: The elliptical orbit with the origin at a focus

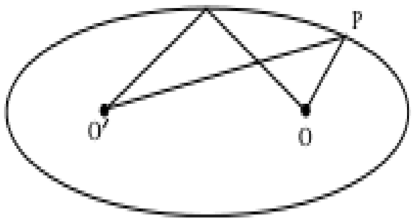


Figure 25: The distance from between the two foci and a point on the orbit is constant

gravitational force (for example, the sun) sits at $r = 0$. This is marked by the yellow disc in the figure. Notice that it is not the centre of the ellipse: the two points differ by a distance

$$|x_c| = \frac{r_o e}{1 - e^2} = ea$$

The origin where the star sits has special geometric significance: it is called the *focus* of the ellipse. In fact, it is one of two foci: the other, shown as O' in Figure above, sits at equal distance from the centre along the major axis. A rather nice geometric property of the ellipse is that the distance OPO' shown in the second figure is the same for all points P on the orbit. (You can easily prove this with some messy algebra).

In the Solar System, nearly all planets have $e < 0.1$. This means that the difference between the major and minor axes of their orbits is less than 1% and the orbits are very nearly circular. The only exception is Mercury, the closest planet to the Sun, which has $e \approx 0.2$. For very eccentric orbits, we need to look at comets. The most famous, Halley's comet, has $e \approx 0.97$, a fact which most scientists hold responsible for the Chas and Dave lyric "Halley's comet don't come round every year, the next time it comes into view will be the year 2062". However, according to astronomers, it will be the year 2061.

Hyperbolae: $e > 1$

For $e > 1$, there are two values of θ for which $r \to \infty$. They are $\cos\theta = -1/e$. Repeating the algebraic steps that lead to the ellipse equation, we instead find that the orbit is described by

$$\frac{1}{a^2}\left(x - \frac{r_o e}{e^2 - 1}\right)^2 - \frac{y^2}{b^2} = 1$$

with $a^2 = r_o^2/(e^2-1)^2$ and $b^2 = r_o^2/(e^2-1)$. This is the equation for a hyperbola. It is plotted in the figure, where the dashed lines are the asymptotes. They meet at the point $x = r_o e/(e^2-1)$. Again, the centre of the gravitational attraction sits at the origin denoted by the yellow disc. Notice that the orbit goes off to $r \to \infty$ when $\cos\theta = -1/e$. Since the righthand side is negative, this must occur for some angle $\theta > \pi/2$. This is one way to see why the orbit sits in the left-hand quadrant as shown.

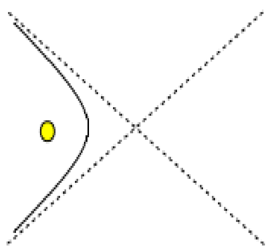


Figure 26: A hyperbola

Parabolae: $e = 1$

Finally, in the special case of $e = 1$, the algebra is particularly simple. The orbit is described by the equation for a parabola,

$$y^2 = r_o^2 - 2r_o x$$

The Energy of the Orbit Revisited

The energy of a given orbit is

$$= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} - \frac{km}{r}$$

$$= \frac{1}{2}m\left(\frac{dr}{d\theta}\right)^2 \dot{\theta}^2 + \frac{ml^2}{2r^2} - \frac{km}{r}$$

$$= \frac{1}{2}m\left(\frac{dr}{d\theta}\right)^2 \frac{l^2}{r^4} + \frac{ml^2}{2r^2} - \frac{km}{r}$$

We can substitute in our solution (5.14) for the orbit to get

$$\frac{dr}{d\theta} = \frac{r_o e \sin\theta}{(1 + e\cos\theta)^2}$$

After a couple of lines of algebra, we find that all the θ dependence vanishes in the energy (as it must since the energy is a constant of the motion). We are left with the pleasingly simple result

$$E = \frac{mk^2}{2l^2}(e^2 - 1) \tag{8.16}$$

We can now compare this with the three cases

- $e < 1 \Rightarrow E < 0$: These are the trapped, or bounded, orbits that we now know are ellipses.
- $e > 1 \Rightarrow E > 0$: These are the unbounded orbits that we now know are hyperbolae.
- $e = 1 \Rightarrow E = -mk^2/2t^2$: This coincides with the minimum of the effective potential V_{eff} which we previously understood corresponds to a circular orbit.

A Repulsive Force

In the analysis above, we implicitly assumed that the force is attractive, so $k > 0$. This, in turn, ensures that $r_o = l^2/k > 0$. For a repulsive interaction, we choose to write the solution (8.14) as

$$r = \frac{|r_o|}{e\cos\theta - 1} \tag{8.17}$$

where $|r_o| = l^2/|k|$ and $e = Al^2/|k|$. Note that with this choice of convention, $e > 0$. Since we must have $r > 0$, we only find solutions in the case $e > 1$. This is nice: we wouldn't expect to find bound orbits between two particles which repel each other. For $e > 1$, the unbounded hyperbolic orbits look like those shown in the figure. Notice that the orbits go off to $r \to \infty$ when $\cos\theta = 1/e$ which, since $e > 0$, must occur at an angle $\theta < \pi/2$. This is the reason that the orbit sits in the right-hand quadrant.

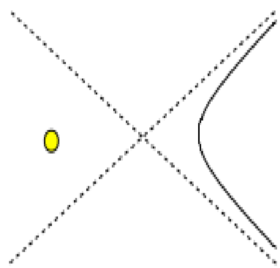


Figure 27

8.3.2 Kepler's Laws of Planetary Motion

In 1605, Kepler published three laws which are obeyed by the motion of all planets in the Solar System. These laws were the culmination of decades of careful, painstaking observations of the night sky, firstly by Tycho Brahe and later by Kepler himself. They are:

- **K1:** Each planet moves in an ellipse, with the Sun at one focus.
- **K2:** The line between the planet and the Sun sweeps out equal areas in equal times.
- **K3:** The period of the orbit is proportional to the $radius^{3/2}$.

Now that we understand orbits, let's see how Kepler's laws can be derived from Newton's inverse-square law of gravity.

We'll start with Kepler's second law. This is nothing more than the conservation of angular momentum. From the figure, we see that in time δt, the area swept out is

$$\delta A = \frac{1}{2}r^2\delta\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}$$

which we know is constant. This means that Kepler's second law would hold for *any* central force.

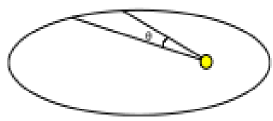


Figure 28

What about Kepler's third law? This time, we do need the inverse-square law itself. However, if we assume that the gravitational force takes the form $F = -GMm/r^2$, then Kepler's third law follows simply by dimensional analysis. The only parameter in the game is GM which has dimensions

$$[GM] = L^3T^{-2}$$

So if we want to write down a formula relating the period of an orbit, T , with some average radius of the orbit R (no matter how we define such a thing), the formula must take the form

$$T^2 \sim \frac{R^3}{GM}$$

For circular orbits, $\dot{\theta}^2 \sim 1/r^3$. For a general elliptical orbit, we can be more precise. The area of an ellipse is

$$A = \pi ab = \pi a^2\sqrt{1-e^2} = \frac{\pi r_o^2}{(1-e^2)^{3/2}}$$

Since area is swept out at a constant rate, $dA/dt = l/2$, the time for a single period is

$$T = \frac{2A}{l} = \frac{2\pi r_o^2}{l(1-e^2)^{3/2}} = \frac{2\pi}{\sqrt{GM}}\left(\frac{r_o}{1-e^2}\right)^{3/2}$$

The quantity in brackets indeed has the dimension of a length. But what length is it? In fact, it has a nice interpretation. Recall that the periapsis of the orbit occurs at $r_{min} = r_o/(1+e)$ and the apoapsis at $r_{max} = r_o/(1-e)$. It is then natural to define the average radius of the orbit to be $R = \frac{1}{2}(r_{min} + r_{max}) = r_o/(1-e^2)$ We have

$$T = \frac{2\pi}{\sqrt{GM}}R^{3/2}$$

The fact that the inverse-square law implies Kepler's third law was likely known to several of Newton's contemporaries, including Hooke, Wren and Halley. However, the proof that the inverse-square law also gives rise to Kepler's first law – a proof which we have spent much of this section deriving – was Newton's alone. This is one of the highlights of Newton's famous Principia.

8.3.3 Orbital Precession

For extremely massive objects, Newton's theory of gravity needs replacing. Its successor is Einstein's theory of general relativity which describes how gravity can be understood as the bending of space and time.

However, for certain problems, the full structure of general relativity reduces to something more familiar. It can be shown that for planets orbiting a star, much of the effect of the curvature of spacetime can be captured in a simple correction to the Newtonian force law, with the force now arising from the potential

$$V(r) = -\frac{GMm}{r}\left(1 + \frac{3GM}{c^2 r}\right)$$

where c is the speed of light. For $r \gg GM/c^2$, this extra term is negligible and we return to the Newtonian result. Here we will see the effect of keeping this extra term.

We again define $k = GM$. After a little bit of algebra, the orbit equation (8.11) can be shown to be

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{6k^2}{c^2l^2}\right)u = \frac{k}{l^2}$$

The solution to this equation is very similar to that of the Kepler problem (8.13). It is

$$u(\theta) = A\cos\left(\sqrt{1 - \frac{6k^2}{c^2l^2}}\,\theta\right) + \frac{k}{l^2 - 6k^2/c^2}$$

where we have once again chosen our polar coordinates so that the integration constant is $\theta_o = 0$.

This equation again describes an ellipse. But now the ellipse *precesses*, meaning that the periapsis (the point of closest approach to the origin) does not sit at the same angle on each orbit. This is simple to see. A periapsis occurs whenever the cos term is 1. This first happens at $\theta_o = 0$. But the next time round, it happens at

$$\theta = 2\pi\left(1 - \frac{6k^2}{c^2l^2}\right)^{-1/2} \approx 2\pi\left(1 + \frac{3k^2}{c^2l^2}\right)$$

We learn that the orbit does not close up. Instead the periapsis advances by an angle of $6\pi G^2M^2/c^2l^2$ each turn.

The general relativistic prediction of the perihelion advance of Mercury – the closest planet to the sun – was one of the first successes of Einstein's theory.

8.4 Scattering: Throwing Stuff at Other Stuff

In the past century, physicists have developed a foolproof and powerful method to understand everything and anything: you take the object that you're interested in and you throw something at it. Ideally, you throw something at it really hard. This technique was pioneered by Rutherford who used it to understand the structure of the atom. It was used by Franklin, Crick and Watson to understand the structure of DNA. And, more recently, it was used at the LHC to demonstrate the existence of the Higgs boson. In short, throwing stuff at other stuff is the single most important experimental method available to science. Because of this, it is given a respectable sounding name: it is called *scattering*.

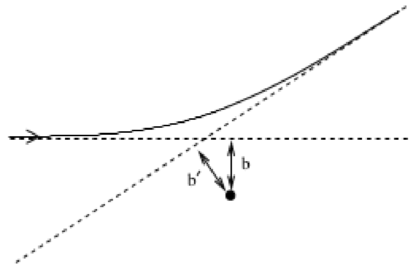


Figure 29

Before we turn to any specific problem, there are a few aspects that apply equally well to particles scattering off any central potential $V(r)$. We will only need to assume $V(r) \to 0$ as $r \to \infty$. We do our experiment and throw the particle from a large distance which we will take to be $r \to \infty$. We want to throw the particle towards the origin, but our aim is not always spot on. If the interaction is repulsive, we expect the particle to be deflected and its trajectory will be something like that shown in the figure. (However, much of what we're about to say will hold whether the force is attractive or repulsive).

Firstly, by energy conservation, the speed of the particle at the end of its trajectory must be the same as the initial speed. (This is true since at $r \to \infty$ at both the beginning and end and there is no contribution from the potential energy). Let's call this initial/final speed v.

But, in a central potential, we also have conservation of angular momentum, $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$. We can get an expression for $l = |\vec{L}|/m$ as follows: draw a straight line tangent to the initial velocity. The closest this line gets to the origin is distance b, known as the *impact parameter*. The modulus of the angular momentum is then

$$l = bv \tag{8.18}$$

If this equation isn't immediately obvious mathematically, the following words may convince you. Suppose that there was no force acting on the particle at all. In this case, the particle would indeed follow the straight line shown in the figure. When it's closest to the origin, its velocity $\dot{\mathbf{r}}$ is perpendicular to its position $\mathbf{r}$ and is its angular momentum is obviously $l = bv$. But angular momentum is conserved for a free particle, so this must also be its initial angular momentum. But, if this is the case, it is also the angular momentum of the particle moving in the potential $V(r)$ because there too the angular momentum is conserved and can't change from its initial value.

At the end of the trajectory, by the same kind of argument, the angular momentum l is $l = b'v$ where b' is the shortest distance from the origin to the exit asymptote as shown in the figure. But since the angular momentum is conserved, we must have

$$b = b'$$

8.4.1 Rutherford Scattering

It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.

Ernest Rutherford

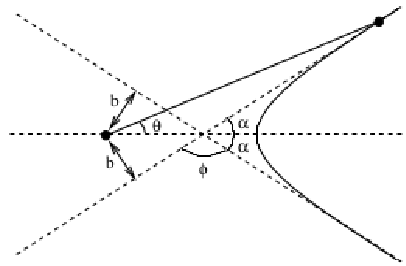


Figure 30

Here we'll look at the granddaddy of all scattering experiments. We take a particle of charge q and mass m and throw it at a fixed particle of charge Q. We'll ignore the gravitational interaction and focus just on the repulsive Coulomb force. The potential is

$$V = \frac{qQ}{4\pi\epsilon_o r}$$

This is mathematically identical to the gravitational force, so we can happily take all the results from the last section and replace $k = -qQ/4\pi\epsilon_o m$ in our previous equations.

Using our knowledge that $b' = b$, we can draw another scatting event as shown. Here θ is the position of the particle. We will denote the total angle through which the particle is deflected as ϕ. However, in the short term the angle α, shown in the figure, will prove more useful. This is related to ϕ simply by

$$\phi = \pi - 2\alpha \tag{8.19}$$

Our goal is to understand how the scattering angle ϕ depends on the impact parameter b and the initial velocity v. Using the expression (8.17) for the orbit that we derived earlier, we know that the particle asymptotes to $4 \to \infty$ when the angle is at $\theta = \alpha$. This tells us that

$$\cos\alpha = \frac{1}{e}$$

As we mentioned previously, $e > 1$ which ensures that $\alpha < \pi/2$ as shown in the figure.

There are a number of ways to proceed from here. Probably the easiest is if we use the expression for energy. When the particle started its journey, it had $E = \frac{1}{2}mv^2$ (where *v* is the initial velocity). We can equate this with (8.16) to get

$$E = \frac{1}{2}mv^2 = \frac{mk^2}{2l^2}(e^2 - 1) = \frac{mk^2}{2l^2}\tan^2\alpha$$

Finally, we replace $l = bv$ to get an the expression we wanted, relating the scattering angle ϕ to the impact parameter *b*,

$$\phi = 2\tan^{-1}\left(\frac{|k|}{bv^2}\right) \tag{8.20}$$

The result that we've derived here is for a potential with all the charge ***Q*** sitting at the origin. We now know that this is a fairly good approximation to the nucleus of the atom. But, in 1909, when Rutherford, Geiger and Marsden, first did this experiment, firing alpha particles (Helium nuclei) at a thin film of gold, the standard lore was that the charge of the nucleus was smeared throughout the atom in the so-called "plum pudding model ". In that case, the deflection of the particle at high velocities would be negligible. But, from (8.20), we see that, regardless of the initial velocity *v*, if you fire a particle directly at the nucleus, so that $b = 0$, the particle will always be deflected by a full $\phi = 180°$. This was the result that so surprised Rutherford.

9 Systems of Particles

So far, we've only considered the motion of a single particle. If our goal is to understand everything in the Universe, this is a little limiting. In this section, we take a small step forwards: we will describe the dynamics of N , interacting particles.

The first thing that we do is put a label $i = 1, ..., N$ on everything. The i^{th} particle has mass m_i, position X_i and momentum $P_i = m_i \dot{X}_i$. (A word of warning: do not confuse the label i on the vectors with index notation for vectors!) Newton's second law should now be written for each particle,

$$\dot{\mathbf{P}}_i = \mathbf{F}_i$$

where $\mathbf{F}_i$ is the force acting on the i^{th} particle. The novelty is that the force $\mathbf{F}_i$ can be split into two parts: an external force $\mathbf{F}_i^{ext}$ (for example, if the whole system sits in a i gravitational field) and a force due to the presence of the other particles. We write

$$\mathbf{F}_i = \mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{F}_{ij}$$

where $\mathbf{F}_{ij}$ is the force on particle i due to particle j. At this stage, we get to provide a more precise definition of Newton's third law. Recall the slogan: every reaction has an equal and opposite reaction. In equations this means,

- **N3 Revisited:** $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

9.1 Centre of Mass Motion

The total mass of the system is

$$M = \sum_{i=1}^{N} m_i$$

We define the *centre of mass* to be

$$\mathbf{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{X}_i$$

The total momentum of the system, **P**, can then be written entirely in terms of the centre of mass motion,

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{P}_i = M\dot{\mathbf{R}}$$

We can now look at how the centre of mass moves. We have

$$\dot{\mathbf{P}} = \sum_i \dot{\mathbf{P}}_i = \sum_i \left(\mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{F}_{ij} \right) = \sum_i \mathbf{F}_i^{ext} + \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji})$$

But Newton's third law tells us that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and the last term vanishes, leaving

$$\dot{\mathbf{P}} = \sum_i \mathbf{F}_i^{ext} \tag{9.1}$$

This is an important formula. It tells us if you just want to know the motion of the centre of mass of a system of particles, then only the external forces count. If you throw a wriggling, squealing cat then its internal forces $\mathbf{F}_{ij}$ can change its orientation, but they can do nothing to change the path of its centre of mass. That is dictated by gravity alone. (Actually, this statement is only true for conservative forces. The shape of the cat could change friction coefficients which would, in turn, change the external forces).

It's hard to overstate the importance of (9.1). Without it, the whole Newtonian framework for mechanics would come crashing down. After all, nothing that we really describe is truly a point particle. Certainly not a planet or a cat, but even something as simple as an electron has an internal spin. Yet none of these details matter because everything, regardless of the details, any object acts as a point particle if we just focus on the position of its centre of mass.

9.1.1 Conservation of Momentum

There is a trivial consequence to (9.1). If there is no net external force on the system, so $\sum_i \mathbf{F}_i^{ext} = 0$, then the total momentum of the system is conserved: $\dot{\mathbf{P}} = 0$.

9.1.2 Angular Momentum

The total angular momentum of the system about the origin is defined as

$$\mathbf{L} = \sum_i \mathbf{X}_i \times \mathbf{P}_i$$

Recall that when we take the time derivative of angular momentum, we get $d/dt(\mathbf{X}_i \times \mathbf{P}_i) = \dot{\mathbf{X}}_i \times \mathbf{P}_i + \mathbf{X}_i \times \dot{\mathbf{P}}_i = \mathbf{X}_i \times \dot{\mathbf{P}}_i$ because $\mathbf{P}_i$ is parallel to $\dot{\mathbf{X}}_i$. Using this, the change in the total angular momentum is

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{X}_i \times \dot{\mathbf{P}}_i = \sum_i \mathbf{X}_i \times \left(\mathbf{F}_i^{ext} + \sum_{j\neq i} \mathbf{F}_{ij}\right) = \tau + \sum_i \sum_{j\neq i} \mathbf{X}_i \times \mathbf{F}_{ij}$$

where $\tau \equiv \sum_i \mathbf{X}_i \times \mathbf{F}_i^{ext}$ is the *total external torque*. The second term above still involves the internal forces. What are we going to do about it? Since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, we can write it as

$$\sum_i \sum_{i\neq j} \mathbf{X}_i \times \mathbf{F}_{ij} = \sum_{i<j} (\mathbf{X}_i - \mathbf{X}_j) \times \mathbf{F}_{ij}$$

This would vanish if the force between the i^{th} and j^{th} particle is parallel to the line $(\mathbf{X}_i - \mathbf{X}_j)$ joining the two particles. This is indeed true for both gravitational and Coulomb forces and this requirement is sometimes elevated to a strong form of Newton's third law:

- **N3 Revisited Again:** $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and is parallel to $(\mathbf{X}_i - \mathbf{X}_j)$. In situations where this strong form of Newton's third law holds, the change in total angular momentum is again due only to external forces,

$$\frac{d\mathbf{L}}{dt} = \tau \qquad (9.2)$$

9.1.3 Energy

The total kinetic energy of the system of particles is

$$T = \frac{1}{2} \sum_i m_i \dot{\mathbf{X}}_i \cdot \dot{\mathbf{X}}_i$$

We can decompose the position of each particle as

$$\mathbf{X}_i = \mathbf{R} + \mathbf{y}_i$$

where $\mathbf{y}_i$ is the position of the particle i relative to the centre of mass. In particular, since $\sum_i m_i \mathbf{X}_i = M\mathbf{R}$, the $\mathbf{y}_i$ must obey the constraint $\sum_i m_i \dot{\mathbf{y}}_i = 0$. The kinetic energy can then be written as

$$T = \frac{1}{2} \sum_i m_i \left(\dot{\mathbf{R}} + \dot{\mathbf{y}}_i\right)^2$$

$$= \frac{1}{2}\sum_i m_i \dot{\mathbf{R}}^2 + \dot{\mathbf{R}} \cdot \sum_i m_i \dot{\mathbf{y}}_i + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i{}^2$$

$$= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\mathbf{y}}_i{}^2 \tag{9.3}$$

This tells us that the kinetic energy splits up into the kinetic energy of the centre of mass, together with the kinetic energy of the particles moving around the centre of mass.

We can repeat the analysis that lead to the construction of the potential energy. When the i^{th} particle moves along a trajectory C_i, the difference in kinetic energies is given by

$$T(t_2) - T(t_1) = \sum_i \int_{C_i} \mathbf{F}_i^{ext} \cdot d\mathbf{X}_i + \sum_i \sum_{j \neq i} \int_{C_i} \mathbf{F}_{ij} \cdot d\mathbf{X}_i$$

If we want to define a potential energy, we require that both external and internal forces are conservative. We usually do this by asking that

- Conservative External Forces: $\mathbf{F}_i^{ext} = -\nabla_i V_i(\mathbf{X}_i)$
- Conservative Internal Forces: $\mathbf{F}_{ij} = -\nabla_i V_{ij}(|\mathbf{X}_i - \mathbf{X}_j|)$

Note that, for once, we are not using the summation convention here. We are also working with the definition $\nabla_i \equiv \partial/\partial\mathbf{X}_i$. In particular, internal forces of this kind obey the stronger version of Newton's third law if we take the potentials to further obey $V_{ij} = V_{ji}$. With these assumptions, we can define a conserved energy given by

$$E = T + \sum_i V_i(\mathbf{X}_i) + \sum_{i<j} V_{ij}(|\mathbf{X}_i - \mathbf{X}_j)|$$

9.1.4 Work-Energy Theorem

Consider a system of many particles, with positions r_i and velocities $\dot{r}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum \frac{1}{2} m_i \dot{r}_i{}^2 \tag{9.4}$$

Now let us consider how the kinetic energy of the system changes in time. Assuming each m_i is independent, we have

$$\frac{dT_i}{dt} = m_i \dot{r}_i \cdot \ddot{r}_i \tag{9.5}$$

Here we have used the relation

$$\frac{d}{dt}A^2 = 2A \cdot \frac{dA}{dt}$$

We now invoke Newton's 2nd Law, $m_i \ddot{r}_i = F_i$ to write (6.5) as

$$\dot{T}_i = F_i \cdot \dot{r}_i$$

. We integrate this equation from time t_A to t_B:

$$T_i^B - T_i^A = \int_{t_A}^{t_B} dt \frac{dT_i}{dt}$$

$$= \int_{t_A}^{t_B} dt F_i \cdot \dot{r}_i \equiv \sum W_i^{(A-B)} \tag{9.6}$$

Where $W_i^{(A-B)}$ is the total work done on particle i during its motion from state A to state B. Clearly, the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{(A-B)} = \sum W_i^{(A-B)}$. (6.6) is is known as the work-energy theorem. It says that

In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done:

$$T^B - T^A = W^{(A-B)}$$

.

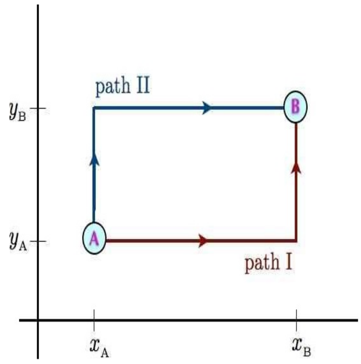


Figure 31: Two Paths Joining A and B

9.1.5 In Praise of Conservation Laws

We have introduced three quantities that, under the right circumstances, are conserved: momentum, angular momentum and energy. There is a beautiful theorem, due to Emmy Noether, which relates these conserved quantities to symmetries of space and time.

- Conservation of momentum follows from the translational invariance of space. In our formulation, we saw that momentum is conserved if the total external force vanishes. But without an external force pushing the particles one way or another, any point in space is just as good as any other. This is the deep reason for momentum conservation.
- Conservation of angular momentum follows from the rotational invariance of space. Again, there are hints of this already in what we have seen since a vanishing external torque can be guaranteed if the background force is central, and therefore rotational symmetric.
- Conservation of energy follows from invariance under time translations. This means that it doesn't matter when you do an experiment, the laws of physics remain unchanged. We can see one aspect of this in our discussion of potential energy, where it was important that there was no explicit time dependence. (This is not to say that the potential energy doesn't change with time. But it only changes because the position of the particle changes, not because the potential function itself is changing).

9.1.6 Why the Two Body Problem is Really a One Body Problem

Solving the dynamics of N mutually interacting particles is hard. Here "hard" means that no one knows how to do it unless the forces between the particles are of a very special type (e.g. harmonic oscillators).

However, when there are no external forces present, the case of two particles actually reduces to the kind of one particle problem.

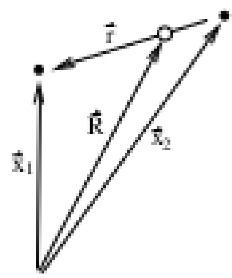


Figure 32: The particles are the black dots; the centre of mass is the white dot.

We have already defined the centre of mass to be,

$$M\mathbf{R} = m_1\mathbf{X}_1 + m_2\mathbf{X}_2$$

We'll also define the relative separation,

$$\mathbf{r} = \mathbf{X}_1 - \mathbf{X}_2$$

Then we can write

$$\mathbf{X}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r} \quad and \quad \mathbf{X}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}$$

We assume that there are no external forces at work on the system, so $\mathbf{F}_i^{ext} = 0$ which ensures that the centre of mass travels with constant velocity: $\ddot{\mathbf{R}} = 0$. Meanwhile, the relative motion is governed by

$$\ddot{\mathbf{r}} = \ddot{\mathbf{X}}_1 - \ddot{\mathbf{X}}_2 = \frac{1}{m_1}\mathbf{F}_{12} - \frac{1}{m_2}\mathbf{F}_{21} = \frac{m_1 + m_2}{m_1 m_2}\mathbf{F}_{12}$$

where, in the last step, we've used Newton's third law $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The equation of motion for the relative position can then be written as

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$$

where μ is the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

But this is really nice. It means that we've already solved the problem of two mutually interacting particles because their centre of mass motion is trivial, while their relative separation reduces to the kind of problem that we've already seen. In particular, if they interact through a central force of the kind $\mathbf{F}_{12} = -\nabla V(\mathbf{r})$ — which is true for both gravitational and electrostatic forces — then we simply need to adopt the methods of Section 6, with m in (6.1) replaced by μ.

In the limit when one of the particles involved is very heavy, say $m_2 \gg m_1$, then $\mu \approx m_1$ and the heavy object remains essentially fixed, with the lighter object orbiting around it. For example, the centre of mass of the Earth and Sun is very close to the centre of the Sun. Even for the Earth and moon, the centre of mass is 1000 miles below the surface of the Earth.

9.2 Collisions

This subject is strictly speaking off-syllabus but, nonetheless, there's a couple of interesting things to say. Of particular interest are elastic collisions, in which both kinetic energy and momentum are conserved. As we have seen, such collisions will result from any conservative inter-particle force between the two particles.

Consider the situation of a particle travelling with velocity $\mathbf{v}$, colliding with a second, stationary particle. After the collision, the two particles have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. Even without knowing anything else about the interaction, there is a pleasing, simple result that we can derive. Conservation of energy tells us

$$\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}\mathbf{v}_1^2 + \frac{1}{2}\mathbf{v}_2^2$$

while the conservation of momentum reads

$$m\mathbf{v} = m\mathbf{v}_1 + m\mathbf{v}_2 \tag{9.7}$$

Squaring this second equation, and comparing to the first, we learn that the cross-term on the right-hand side must vanish. This tells us that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \tag{9.8}$$

In other words, either one of the particles is stationary, or the two particles scatter at right-angles.

Although the conservation of energy and momentum gives us some information about the collision, it is not enough to uniquely determine the final outcome. It's easy to see why: we have six unknowns in the two velocities $\mathbf{v}_1$ and $\mathbf{v}_1$, but just four equations in (9.7) and (9.8).

10 One-Dimensional Conservative Systems

10.1 A Dynamical System

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \tag{10.1}$$

A system is conservative if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \frac{1}{2}m\dot{x}^2 + U(x) \tag{10.2}$$

is then conserved. This may be verified explicitly:

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + U(x)\right] \\ &= \Big[m\ddot{x} + U'(x)\Big]\dot{x} = 0 \end{aligned} \tag{10.3}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\Big(E - U(x)\Big)} \tag{10.4}$$

Note that the constant **E** is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(\mathbf{E})$ which satisfy

$$\mathbf{E} = \mathbf{U}(x) \quad \Rightarrow \quad x(\mathbf{E}) = \mathbf{U}^{-1}(\mathbf{E}) \tag{10.5}$$

where U^{-1} is the inverse function, are called **turning points**. When the total energy is **E**, the motion of the system is bounded by the turning points, and confined to the region(s)$U(x) \leq E$. We can integrate equation (10.4) to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}} \tag{10.6}$$

This to be inverted to obtain the function $x(t)$. Note that there are now two constants of integration, E and x_0. Since

$$E = E_0 = \frac{1}{2}mv_0^2 + U(x_0) \tag{10.7}$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two independent constants of integration. For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m}\int_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} \tag{10.8}$$

10.1.1 Example: Harmonic Oscillator

In the case of the harmonic oscillator, we have $U(x) = \frac{1}{2}kx^2$, hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2E - kx^2}} \tag{10.9}$$

The turning points are $x_\pm(E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the solution, let us substitute

$$x = \sqrt{\frac{2E}{k}} \sin\theta \tag{10.10}$$

We then find

$$dt = \sqrt{\frac{m}{k}} d\theta \tag{10.11}$$

with solution

$$\theta(t) = \theta_0 + \omega t \tag{10.12}$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the complete motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \sin(\omega t + \theta_0) \tag{10.13}$$

Note the two constants of integration E and θ_0

10.2 One-Dimensional Mechanics as a Dynamical System

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, viz.

$$\frac{dx}{dt} = v \tag{10.14}$$

$$\frac{dv}{dt} = \frac{1}{m}F(x) \tag{10.15}$$

This may be written in matrix-vector form as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}F(x) \end{pmatrix} \tag{10.16}$$

This is an example of a dynamical system described by the general form

$$\frac{d\varphi}{dt} = \mathbf{V}(\varphi) \tag{10.17}$$

where $\varphi = (\varphi_1, ...\varphi_N)$ is an N-dimensional vector in phase space. For the model of equation 9.16, we evidently have $N = 2$ The object $\mathbf{V}(\varphi)$ is called a **vector field**. It is itself a vector existing at every point in phase space R^N. Each of the components of $\mathbf{V}(\varphi)$ is a function (in general) of all the components of φ

$$V_j = V_j(\varphi_1, ...\varphi_N) \quad (j = 1, ..., N)) \tag{10.18}$$

Solutions to the equation $\dot{\varphi} = \mathbf{V}(\varphi)$ are called integral curves. Each such integral curve $\varphi(t)$ is uniquely determined by N constants of integration, which may be taken to be the initial value $\varphi(0)$.

The collection of all integral curves is known as the phase portrait of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}(E - U(x))}$ The phase curves are thus curves of constant energy.

10.2.1 Sketching Phase Curves

To plot the phase curves,

(i) Sketch the potential U(x).

(ii) Below this plot, sketch $v(x; E) = \pm\sqrt{\frac{2}{m}(E - U(x))}$

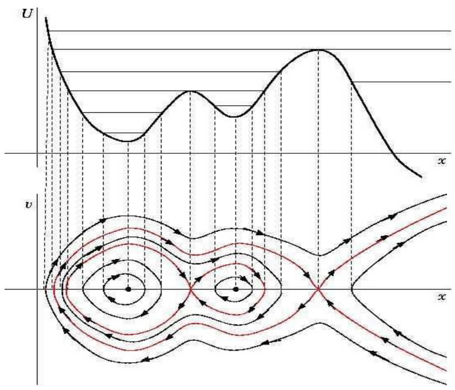


Figure 33: A potential U(x) and the corresponding phase portraits. Separatrices are showed in red.

(iii) When E lies at a local extremum of U(x), the system is at a fixed point.

(a) For E slightly above E_{min}, the phase curves are ellipses.

(b) For E slightly below E_{max}, the phase curves are (locally) hyperbolae.

(c) For $E = E_{min}$ the phase curve is called a separatrix.

(iv) When$E > U(\infty)$ or $E > U(-\infty)$, the motion is unbounded.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for$v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area A in phase space enclosed by a bounded phase curve is

$$A(E) = \oint_E v dx = \sqrt{\frac{8}{m}} \int_{x_-((E))}^{x_+(E)} dx' \sqrt{E - U(x')} \tag{10.19}$$

Thus, the period is proportional to the rate of change of A(E) with E:

$$T = m\frac{\partial A}{\partial E} \tag{10.20}$$

10.3 Fixed Points and their Vicinity

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*)(x - x^*) + \frac{1}{2}U''(x^*)(x - x^*)^2 + \frac{1}{6}U'''(x^*)(x - x^*)^3 + ... \tag{10.21}$$

$U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \frac{1}{2}k(\delta x)^2 \tag{10.22}$$

where $U_0 = U(x^*)$ and $k = U''(x^*) > 0$. The solutions to the motion in this potential are:

$$U''(x^*) > 0: \quad \delta x(t) = \delta x_0 \cos(\omega t) + \frac{\delta v_0}{\omega}\sin(\omega t) \tag{10.23}$$

$$U''(x^*) < 0: \quad \delta x(t) = \delta x_0 \cosh(\gamma t) + \frac{\delta v_0}{\gamma}\sinh(\gamma t) \tag{10.24}$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \frac{1}{2}m(\delta v_0)^2 + \frac{1}{2}k(\delta x_0)^2 \tag{10.25}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm\gamma\delta x_0$. Let's take $\delta v_0 = -\gamma\delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). i.e. the initial velocity is negative if we are to the right of the UFP $\delta x_0 > 0$ and positive if we are to the left of the UFP $\delta x_0 < 0$. The motion of the system is then

$$\delta x(t) = \delta x_0 \exp(\gamma t) \tag{10.26}$$

The particle gets closer and closer to the unstable fixed point at$\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point$\delta x < \delta x_0$ is

$$t = \gamma^{-1}\ln\left(\frac{\delta x_0}{\delta x}\right) \tag{10.27}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, **the period of motion along a separatrix is infinite**.

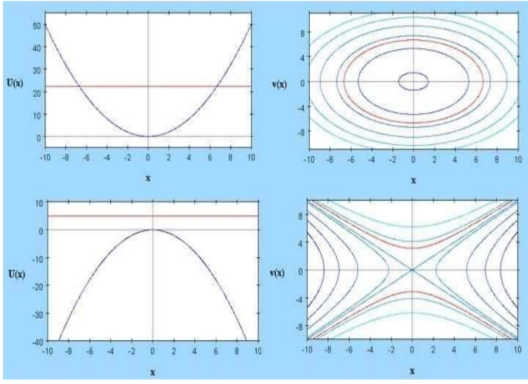


Figure 34: Phase curves in the vicinity of centers and saddles.

10.3.1 Linearized Dynamics in the Vicinity of a Fixed Point

Linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{m}U''(x^*) & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} + ..., \tag{10.28}$$

This is a linear equation, which we can solve completely.

Consider the general linear equation $\dot{\varphi} = A\varphi$, where A is a fixed real matrix. Now whenever we have a problem involving matrices, we should start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$A\psi_\alpha = \lambda_\alpha \psi_\alpha \tag{10.29}$$

Here ψ_α is the α^{th} *right eigenvector*[3] of A. The eigenvalues are roots of the characteristic equation $P(\lambda) = 0$, where $P(\lambda) = det(\lambda\cdot \parallel - A)$. Let's expand $\varphi(t)$ in terms of the right eigenvectors of A:

$$\varphi(t) = \sum_\alpha C_\alpha(t)\psi_\alpha \tag{10.30}$$

Assuming, for the purposes of this discussion, that A is nondegenerate, and its eigenvectors span R^N, the dynamical system can be written as a set of decoupled first order ODEs for the coefficients$C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha C_\alpha, \tag{10.31}$$

with solutions

$$C_\alpha(t) = C_\alpha(0)\exp(\lambda_{\alpha t}) \tag{10.32}$$

If $Re(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $Re(\lambda_\alpha) > 0$,$C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $lm(\lambda_\alpha)$. For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D, \tag{10.33}$$

where $T = Tr(A)$ and $D = det(A)$. Then eigenvalues are then

$$\lambda_\pm = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4D}. \tag{10.34}$$

We'll study the general case in Physics 110B. For now, we focus on our conservative mechanical system of equation 9.28. The trace and determinant of the above matrix are $T = 0$ and $D =$

[3] If A is symmetric, the right and left eigenvectors are the same. If A is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same.

$\frac{1}{m}U''(x^*)$. Thus, there are only two (generic) possibilities: centers, when $U''(x^*) > 0$ and saddles, when $U''(x^*) < 0$.

10.4 Examples of Conservative One-Dimensional Systems

10.4.1 Harmonic Oscillator

Recall again the harmonic oscillator, discussed in lecture 3. The potential energy is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m\frac{d^2x}{dx^2} = -\frac{dU}{dx} = -kx, \tag{10.35}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} \tag{10.36}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known

$$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \tag{10.37}$$

$$v(t) = v_0\cos(\omega t) - \omega x_0 \sin(\omega t) \tag{10.38}$$

The phase curves are ellipses:

$$\omega_0 x^2(t) + \omega_0^{-1} v^2(t) = C \tag{10.39}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in Fig. 35. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{10.40}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{max} = \sqrt{\frac{2E}{k}}, \quad v_{max} = \sqrt{\frac{2E}{m}} \tag{10.41}$$

The area of the elliptical phase curves is thus

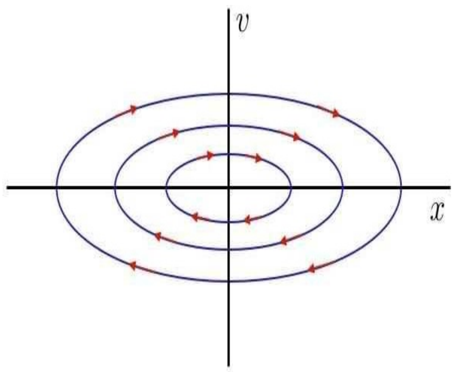


Figure 35: Phase curves for the harmonic oscillator

$$A(E) = \pi x_{max} v_{max} = \frac{2\pi E}{\sqrt{mk}} \tag{10.42}$$

The period of motion is therefore

$$T(E) = m\frac{\partial A}{\partial E} = 2\pi\sqrt{\frac{m}{k}} \tag{10.43}$$

which is independent of E.

10.4.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length *l*. The potential is $U(\theta) = -mgl\cos\theta$, hence

$$ml^2\ddot{\theta} = -\frac{dU}{d\theta} = -mgl\sin\theta \tag{10.44}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix}\omega \\ -\omega_0^2\sin\theta\end{pmatrix}, \tag{10.45}$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/l}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + U(\theta) \tag{10.46}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$'

$$\frac{2E}{ml^2} = \dot{\theta}^2 - 2\omega_0^2\cos\theta = -2\omega_0^2\cos\theta_0 \tag{10.47}$$

The period for motion of amplitude θ_0 is then

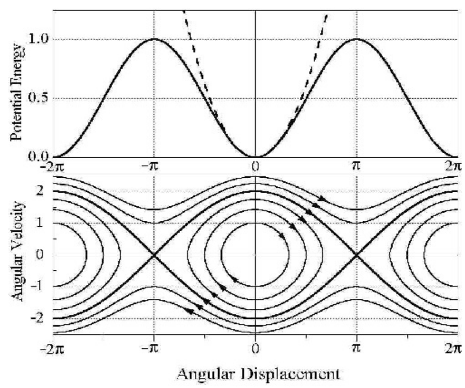


Figure 36: Phase curves for the simple pendulum. The separatrix divides phase space into regions of rotation and libration.

$$T(\theta_0) = \frac{\sqrt{8}}{\omega_0}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}K(\sin^2\frac{1}{2}\theta_0), \tag{10.48}$$

where K(z) is the complete elliptic integral of the first kind. Expanding K(z), we have

$$T(\theta_0) = \frac{2\pi}{\omega_0}\left[1 + \frac{1}{4}\sin^2(\frac{1}{2}\theta_0) + \frac{9}{64}\sin^4(\frac{1}{2}\theta_0) + ...\right] \tag{10.49}$$

For$\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\theta$. As $\theta_0 \to \pi/2$, the period diverges logarithmically.

The phase curves for the pendulum are shown in Fig. 36. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called **librations**. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are **rotations**. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a **separatrix**.

11 Lagrangian and Hamiltonian Mechanics

In the first part of this course we were introduced to the basic ideas of Newtonian mechanics via concrete examples, such as motion of a particle in a gravitational potential, the simple harmonic oscillator, charge moving in electric and magnetic field, etc.

In this section, we will develop a more abstract viewpoint in which one thinks of the dynamics of a system described by an arbitrary number of generalised coordinates, but in which the dynamics can be nonetheless encapsulated in a single scalar function: The Lagrangian, named after the French Mathematician Joseph Louis Lagrange or the Hamiltonian named after Irish Mathematician Willian Hamilton.

The abstract view point is enormously powerful and underpins quantum mechanics and modern nonlinear dynamics. It may or may not be more efficient than elementary approaches for solving simple problems, but in order to feel comfortable with the formulation, it is very instructive to do some elementary problems using abstract methods. Thus, we will be revisiting such examples as the harmonic oscillator and the pendulum.

Therefore in summary, the formulation in mechanics are equivalent to Newtonian mechanics.

Generalized Coordinates

Suppose we have a system of N particles each moving in 3-space and interacting through arbitrary (finite) forces, then the dynamics of the total system is described by the motion of a point $q \equiv \{q_i/i = 1, 2, ...3N\} = \{x_1, y_1, z_1, x_2, y_2, z_2, ...x_N, Y_N, Z_N\}$ in a 3N - dimensional generalized configurational space for the radius of the particle, since a radius vector is specified by the three coordinates. Therefore, for N particles, there N radii vectors.

Definition

The number S of possible independent quantities which must be specified in order to define uniquely the position of any system is called the ***number of degrees of freedom.*** Thus for N - particles in space, $S = 3N$ of generalized coordinates q_i. No particular metric is assumed-e.g. we could equally as well use spherical polar coordinates. $q_i = \{r_1, \theta_1, \phi_1, r_2, \theta_2, \phi_2, ..., r_N, \theta_N, \phi_N\}$ or more general curvillinear coordinate system.

In other systems, the generalized "coordinates" need not even be spatial coordinates -e.g. they could be the charges flowing in and electrical circuit.

Thus the conventional vector-like notation for the array of generalized coordinates should not be confused with the notation for the position vector in 3-space. Vectors are entities independent of which coordinates are used to represent them, whereas the set of generalized coordinates changes if we change variables.
For instance, consider the position vector of a particle in cartesian coordinates $x, y, z,$, cylindrical coordinates $P(x, y, \theta)$, $P(r, \phi, z)$, and spherical polar coordinate $S(r, \phi, \theta)$ in the figure below
The vector

$$\begin{aligned} r &= xe_x + ye_y + ze_z \\ &= re_r(\theta, \phi) \end{aligned}$$

A generalized coordinate for

$$r_c \equiv \{x, y, z\}$$

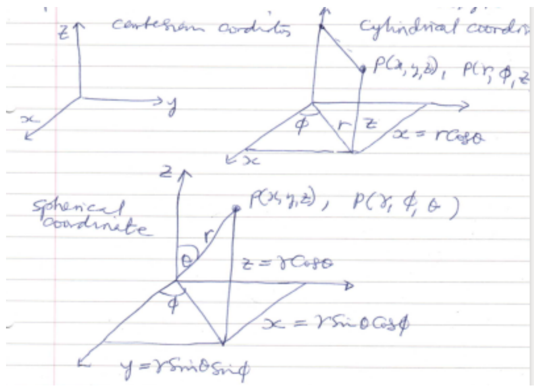


Figure 37

and

$$r_{sph} \equiv \{r, \phi, \theta)$$

are distinct entities. They are points in two different (though related) configuration spaces describing the particles. The generalized coordinates of a system are the independent parameters $q_1, q_2, q_3, ..., q_i$ which completely specify the the configuration of the system i.e. the position of all its particles with respect to a frame of reference.

11.0.1 Constraints

Constraints are applied to rigid body, the motion of the particles are such that the distances between the particles (with reference to the center of mass, m) remains constant. Example: The beads of an abacus are constrained to move in one direction, billiard balls constrained to move within a plane, gas molecules within a container are constrained by the walls of the container. Constraints are meant to limit the motion of the system.

Types of Constraints

1. **Holonomic Constraint (Integral)**
   The condition of a constraint can be expressed in equation connecting the coordinates of the particles and time having the form

   $$f(q_1, q_2, q_3, ...t) = 0$$

   . The simplest example of holonomic constraint is the rigid body, where the constraints are expressed by equations of the form $(q_i - q_j)^2 - c_{ij=0}$

Illustration

Two particles tied together with a rod of length so that

$$(x_1 - x_2)^2 + (y_1 - y_2)^2) + (z_1 - z_2)^2 = L^2$$

. A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint. Thus, the N-particles are moving on the surface of a sphere

$$(x_i)^2 + (y_i)^2 + (z_i)^2 = R^2$$

$$\forall_i = 1, 2, ...N$$

A ball in a squash court $x \geq 0, \quad z \geq 0$

2. **Nonholonomic Constraint** Constraints not expressible in the fashion

$$f(q_1, q_2, q_3, ...t) = 0 \tag{11.1}$$

i.e. Nonholonomic

$$f(q_1, q_2, q_3, ...t) \neq 0$$

.

Example:

$$x^2 + y^2 + z^2 - R^2 = 0$$

$$q_1 = \frac{x}{R}, \quad q_2 = \frac{y}{R}, \quad , q_3 = \frac{z}{R},$$

$$q_1^2 + q_2^2 + q_3^2 - 1 = 0 \quad (Reversegivestheaboveequation)$$

The walls of a gas container constitutes a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geq 0$$

where a = radius of the sphere, which is not in the form of (11.1) Thus, in a gravitational field a particle placed on top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicit dependent on time (scleronomous). Constraints introduce two types of difficulties in the solution of mechanics problems.

- The coordinates are no longer all independent since they are connected to the equations of constraint, hence the equations of motion are not all independent.
- The forces of constraint imposing constraints on the system is simply another method of saying that there are forces present in the problem that cannot be specified directly but are known rather in terms of their effect on the motion of the system.

In the case of holonomic constraints, the first difficulty is solved by the introduction of generalized coordinates. A system of N particles free from constraints, has $3N$ independent coordinates or degree of freedom. If there exists holonomic constraints, expressed in (11.1), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N - k$ independent coordinates and the system is said to have $3N - k$ degrees of freedom.

This elimination of the dependent coordinates can be expressed in another way by the introduction of new $3N - k$ independent variables $q_1, q_2, ...q_{3N-k}$ in terms of which the old coordinates $r_1, r_2, ...r_N$ are expressed by equations of the form

$$r = r_1(q_1, q_2, ...q_{3N-k}, t) \tag{11.2}$$

$$r_N = r_N(q_1, q_2, ...q_{3N-k}, t)$$

containing the constraint in them implicitly. These are transformation equations from the set of r_i variables to the q_i. The parametric representation of the r variables. It is always assumed that one can transform back from (q_i) to (r_i) set.

Usually the generalized coordinates (q_i) unlike the cartesian coordinates, will not divide int oconvenient groups of three that can be associated together to form vectors.

If the constraint is nonholonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates.

An often quoted example of nonholonomic constraint is an object rolling on a rough suface without slipping.
The coordinates used to describe the system involve angular coordinates to specify the orientation of the body plus a set of coordinates describing the location of the point of contact on the surface. The constraint of "rolling" connects these two sets of coordinates; They are not independent. A change in the position of the point of constraint inevitably means a change in its orientation. Yet we cannot reduce the number of coordinates, for the "rolling" condition is not expressible as an equation between the coordinates. Rather, it s condition on the velocities (i.e. the point of contact is stationary), a differential condition that can be given in an integral form only after the problem is solved.

example:

$$x^2 + y^2 + z^2 - R^2 = 0$$

$$q_1 = \frac{x}{R}, q_2 = \frac{y}{R}, q_3 = \frac{z}{R}$$

$$q_1^2 + q_2^2 + q_3^2 - 1 = 0$$

Illustration

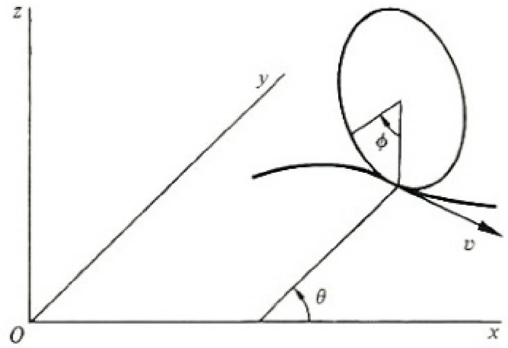


Figure 38: Vertical disk rolling on a horizontal plane.

The coordinates used to describe the motion might be the $x - y$ coordinates of the center of the disk, an angle Φ of rotation about axis of the disk, θ the angle between the axis of the disk and say the x-axis.
Because of the constraint, the velociy is proportional to Φ

$$v = a\dot{\Phi} \quad a = radiusofthedisk$$

$$\dot{x} = v \sin\theta$$

$$\dot{y} = -v \cos\theta$$

Substituting the velocity writing the two set of differential equations

$$\frac{dx}{dt} = a \sin\theta \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = a \cos\theta \frac{d\phi}{dt}$$

$$\Rightarrow \quad dx - a \sin\theta d\phi$$

$$dy + a \cos\theta d\phi \tag{11.3}$$

The two eequations cannot be found or integrated without in fact solving the problem i.e. we cannot find and integrating factor

$$f(x, y, \theta, \phi)$$

that will turn either of the equation into perfect differential. Therefore, the constraint cannot be reduced to the form of (11.1) and are therefore nonholonomic.

Physically we can see that there can be no direct functional relation between ϕ and the other coordinates x, y, and θ, because at any pooint on the disk path can be made to roll around in a circle tangent to the path nd of arbitrary radius. At the end, x, y, and θ would have returned to their original values, but Φ has changed by and amount depending on the radius of the circle.

Nonintegrable differential constraint of the form of (11.3) are fo course not the only type of nonholonomic constraints. The constraint conditions may involve higher-order derivatives, or in the form of inequalities.

Consider the set of all conceivable paths through configuration space as shown in figure 39 below:

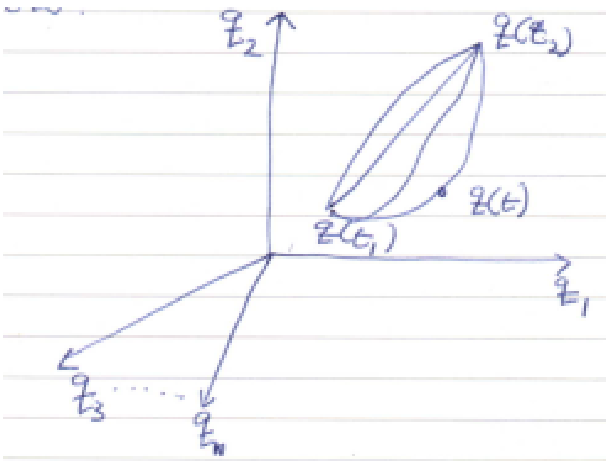


Figure 39

Each one path may be parameterized by the time,

$$t: \; q = q(t)$$

by differentiating

$$f_j(q) = 0$$

with respect to time, we find the set of constraints on the generalized velocities

$$\dot{q}_i \equiv \frac{dq_i}{dt}$$

in a differential notation

$$\sum_{i=1}^{n} \frac{\partial f_j(q)}{\partial q_i} dq_i \equiv \frac{\partial f_j(q)}{\partial q_i} dq_i = 0 \tag{11.4}$$

where

$$\frac{\partial f}{\partial q} \equiv \left[\frac{\partial f}{\partial q_i}, \cdots \frac{\partial f}{\partial q_n}\right]$$

In short-hand notation

$$a\frac{\partial f}{\partial b} \equiv \sum_{i\leq 1}^{n} a_i \frac{\partial f}{\partial b_i} \tag{11.5}$$

where a_i and b_i are arbitrary configuration space variables.

Lagrangian equations of motion are especially appropriate for dealing with problem in which the motion of a particle is subject to constraints.

There are situations where there is an infinite number of generalized coordinates. For instance, consider a scalar field (such as the instantaneous amplitude of a wave), $\psi(r,t)$. Here ψ is a generalized coordinate of the system and the position vector r replaces the index i. Since r is a continuous variable, it ranges over and infinite number of values.

11.1 Lagrangian and Hamiltonian Formulation

What are the pros/cons of each approach? What questions are more naturally solved in each? For example, I believe Fermat's Principle of Least Time is something that's very naturally explained in Lagrangian mechanics ("minimize the time it takes to get between these two points"), but more difficult to explain in Newtonian mechanics since it requires knowing your endpoint.
What's the overall difference in layman's terms? From what I've read so far, it sounds like Newtonian mechanics takes a more local "cause-and-effect"/"apply a force, get a reaction" view, while Lagrangian mechanics takes a more global "minimize this quantity" view. Or, to put it more axiomatically, Newtonian mechanics starts with Newton's three laws of motion, while Lagrangian mechanics starts with the Principle of Least Action.

How do the approaches differ mathematically/when you're trying to solve a problem? Kind of similar to above, I'm guessing that Newtonian solutions start with drawing a bunch of force vectors, while Lagrangian solutions start with defining some function (calculating the Lagrangian...?) you want to minimize, but I really have no idea.

In Newtonian mechanics you have to use mainly rectangular co ordinate system and consider all the constraint forces. Lagrange's scheme avoids the considerations of the constraint forces deftly and you can use any set of "generalized coordinates" like angle, radial distance etc. consistent with the constraint relations. The number of those generalized coordinates are the same with the number of degrees of freedom of the system. In all dynamical systems we arbitrarily choose some generalized co ordinates consistent with the constraints of the system. In Newtonian mechanics, the difference between the kinetic and potential energy of the system gives you the so called Lagrangian. Then we have n number of differential equations.

The main advantage of Lagrangian mechanics is that we don't have to consider the forces of constraints and given the total kinetic and potential energies of the system we can choose some generalized coordinates and blindly calculate the equation of motions totally analytically unlike Newtonian case where one has to consider the constraints and the geometrical nature of the system.

11.1.1 Introduction to Calculus of Variation

The calculus of variations involves finding an extremum (maximum or minimum) of a quantity that is expressible as an integral. Let's look at a couple of examples: The shortest path between two points, Fermat's principle (light follows a path that is an extremum). What is the shortest path between two points in a plane? You certainly know the answer—a straight line—but you probably have not seen a proof of this—the calculus of variations provides such a proof. Consider two points in the $x - y$ plane, as shown in the figure below:

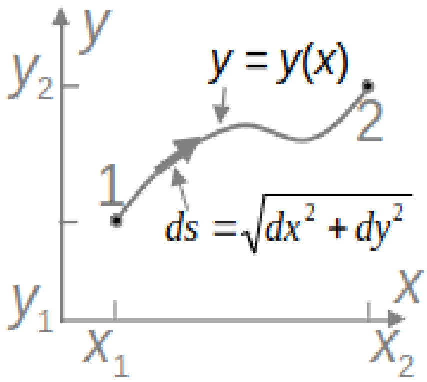


Figure 40

An arbitrary path joining the points follows the general curve y = y(x), and an element of length along the path is

$$ds = \sqrt{dx^2 + dy^2}$$

We can rewrite this as

$$ds = \sqrt{1 + y'x^2 dx}$$

which is valid because $dy = \dfrac{dy}{dx}dx$. Thus length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'x^2 dx}$$

Variational Principles

In our usual minimizing or maximizing of a function $f(x)$, we would take the derivative and find its zeroes (i.e. the values of x for which the slope of the function is zero). These points of zero slope may be minima, maxima, or points of inflection, but in each case we can say that the function is stationary at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope). In analogy with this familiar approach, we want to be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called **calculus of variations**. The methods we will develop are called **variational methods**, and a principle like Fermat's Principle are called **variational principles**. These principles are common, and of great importance, in many areas of physics (such as quantum mechanics and general relativity).

General formulation of calculus of variation in a single independent variable and fixed end points. Find the path $y = y(x)$ between two fixed points (x_1, y_1) and (x_2, y_2) such that the line integral

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x) dx \tag{11.6}$$

where

$$\dot{y} = \frac{dy}{dx}$$

x play the role of parameter, is and extremum i.e. a minimum or maximum.

The problem is an extremum one so it must be put in a form that enables us to use the apparatus of differential calculus for obtaining and extremum value. This is done by labelling all the possible

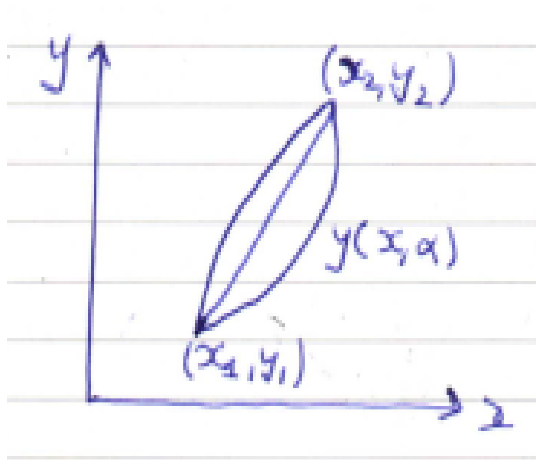


Figure 41

curve $y(x)$ in question with different values of a parameter α such that for some value $\alpha = 0$, say the curve will coincide with the path given an extremum for the integral.

The quantity y will then be a function of both x and the independent parameter α. If α is small then we can represent α by its Taylor series.

$$y(x, \alpha) = y(x, 0) + \alpha n(x) \tag{11.7}$$

where

$$n(x) = \left.\frac{\partial y}{\partial \alpha}\right|_{\alpha=0}$$

it is assumed to be continuously differentiable in the open interval (x_1, x_2) and such that

$$n(x_1) = n(x_2) = 0$$

then

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), \dot{y}(x, \alpha), x)dx \tag{11.8}$$

and the condition for obtaining and extremum is the familiar one

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{11.9}$$

thus differentiating (11.8) we obtain

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial \dot{y}}{\partial \alpha}\right] dx \tag{11.10}$$

Now let us consider the second term of (11.10)

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial \dot{y}}{\partial \alpha} dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \cdot \frac{\partial^2 y}{\partial x \partial \alpha}$$

since

$$\dot{y} = \frac{\partial y}{\partial x}$$

Now let us integrate the L.H.S by parts.

$$\int \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} dx = \left.\frac{\partial f}{\partial \dot{y}} \frac{\partial y}{\partial \alpha}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) \frac{\partial y}{\partial \alpha} dx \tag{11.11}$$

$$i.e. \quad \int udv = uv - \int vdu$$

The condition on all the varied curves are that they pass through the points (x_1, y_1) and (x_2, y_2) and hence

$$\frac{\partial y}{\partial \alpha} = n(x)$$

must vanish at x_1 and x_2. This means the first term on the R.H.S of (11.11) must varnish. Therefore

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} = -\int_{x_1}^{x_2} \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\frac{\partial y}{\partial \alpha} dx$$

$$\Rightarrow \quad \int_{x_1}^{x_2} \Big[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\frac{\partial y}{\partial \alpha}\Big] dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \Big[\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\Big]\frac{\partial y}{\partial \alpha} dx$$

Multiply through by $d\alpha$

$$\Big(\frac{\partial J}{\partial \alpha}\Big)_{d\alpha} = \int \Big[\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\Big]\Big(\frac{\partial y}{\partial \alpha}\Big) d\alpha dx = 0$$

$$i.e. \quad \Big(\frac{\partial J}{\partial \alpha}\Big)_{d\alpha} = 0$$

The quantity $\delta y = \Big(\frac{\partial y}{\partial \alpha}\Big) d\alpha$ is known as the variation of y. It is an arbitrary quantity and therefore not equal to zero generally.

$$\int_{x_1}^{x_2} \Big[\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\Big]\partial y \partial x = 0$$

Hence

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) = 0 \tag{11.12}$$

Thus J is an extremum only for curve $y(x)$ such that

$$f(y, \dot{y}, x)$$

satisfy the differential equation from (11.12). This differential is known as the **Euler-Lagrange Equation**

11.1.2 Particular forms of the Euler-Lagrange Equation

As indicated above, the function f is generally functions of y, $\dot{y}$, and x however, in particular cases, any one of these quantities may be absent form the expression of f. That is f may not explicitly depend on any one fo them. We can therefore distinguish for different cases.

1. **Case I**:
   f depends on all the quantities: y, $\dot{y}$, and x. The curve $y = y(x)$ is given by the full differential equation of (11.12)

2. **Case II**:
   f does not explicitly depend of $\dot{y}$, then
$$\frac{\partial f}{\partial \dot{y}} = 0$$
   and
$$\frac{\partial f}{\partial y} = 0$$
   Trivial case and the differential equation in (11.12) possesses no definite solution.

3. **Case III**:
   f does not explicitly depend on y: then
$$\frac{\partial f}{\partial y} = 0$$
   and the differential equation (11.12) assumes the form
$$\frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) = 0$$
   or integrating
$$\frac{\partial f}{\partial \dot{y}} = c \quad constant \tag{11.13}$$

4. **Case IV**:
   f does not depend explicitly depend on x. The differential equation (11.12) then possesses a first integral. Consider the quantity
$$\frac{d}{dx}\Big[f - \dot{y}\frac{df}{d\dot{y}}\Big] = \frac{df}{dx} - \frac{d\dot{y}}{dx}\frac{df}{d\dot{y}} - \dot{y}\frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) \tag{11.14}$$
   Also
$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial x}$$
   substitute into the (11.14): we obtain the following
$$\begin{aligned}
\frac{d}{dx}\Big[f - \dot{y}\frac{\partial f}{\partial \dot{y}}\Big] &= \frac{df}{dx} - \frac{s\dot{y}}{dx}\frac{\partial f}{\partial \dot{y}} - \dot{y}\frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) \\
&= \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{y}}{\partial x}\frac{\partial f}{\partial \dot{y}} - \dot{y}\frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) \\
&= \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} - \dot{y}\frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big) \\
&= \dot{y}\Big[\frac{\partial f}{\partial x} - \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)\Big] \equiv 0 \\
\Rightarrow \quad & \frac{d}{dx}\Big[f - \dot{y}\frac{\partial f}{\partial \dot{y}}\Big] = 0 \\
f - \dot{y}\frac{\partial f}{\partial \dot{y}} &= c \quad constant
\end{aligned} \tag{11.15}$$
   In other words, whenever the function f does not explicitly depend on x, the differential equation reduces to the first order differential equation
$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = c \quad \text{.....}\textbf{\textit{BELTRAMI IDENTITY}}$$
   Its use greatly simplifies the problem of finding the curve $y = y(x)$

11.1.3 Lagrangian equation

Lagrangian equations of motion are especially appropriate for dealing with problem in which the motion of a particle is subject to constraints.
According to Lagragian generalized force F_{qr} on a particle may be expressed as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_r}\right) - \frac{\partial T}{\partial q_r} = F_{qr} \tag{11.16}$$

where T = kinetic energy of the particle q_r is any one of the coordinates appearing in T

$$F_{qr} = F_x \frac{\partial x}{\partial q_r} + F_y \frac{\partial y}{\partial q_r} + F_z \frac{\partial z}{\partial q_r}$$

(11.16) is generally applied to conservative systems in which

$$F_i = -\nabla_i V \quad ; \quad V = potevtial$$

$\therefore$ The generalized force

$$F_{qj} = -\frac{\partial V}{\partial q_j} \tag{11.17}$$

From (11.16), we can write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0 \tag{11.18}$$

V is dependent only on position and therefore without lost of generality. We can write the following:

$$\frac{d}{dt}\frac{\partial (T - V)}{\partial \dot{q}_j} - \frac{\partial (T - V)}{\partial q_j} = 0 \tag{11.19}$$

We define:

$$L = T - V;$$

The Lagrangian equation becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q_j} = 0 \tag{11.20}$$

Lagrangian equation of motion; other definitions:

- Generaized momentum, P_j

  $P_j = \frac{\partial L}{\partial \dot{q}_j}$

- Generalized force
  $F_j = \frac{\partial L}{\partial q_j}$

- $\dot{P}_j = F_j$

11.2 Geodesics

A geodesic is a curved between two points whose length (calculated using the given metric) is stationary against infinitesimal variations about that path. Thus the task of finding geodesics fits within the class of variational problem we have discussed, and we can use the Euler-Lagrange equations to find them. Perhaps the best know result on geodesics is the fact that the shortest path two points in an Euclidean space (one where $g_{ij} = 0$ for $i \neq j$ and $g_{ij} = 1$ for $i = j$) is a straight line. Another well-known result is that the shortest path between two points on the surface of a sphere is a great circle.

Geodesics are not necessary purely geometrical objects, but can have physical interpretations. For instance, suppose we want to find the shape of and elastic string stretched over a slippery surface. The string will adjust its shape to minimize its elastic energy. Since the elastic potential energy is a monotonically increasing function of the length of the string, the string will settle onto a geodesic on the surface.

Geodesics also play and important role in General Relativity, because the world line of a photon is a geodesic in 4-dimensional space time with the metric tensor obeying Einstein's equations. If the metric is sufficiently distorted, it can happen that there is not one but several geodesics between two points, a fact which explains the phenomenon of gravitational lensing (multiple images of a distant galaxy behind a closer massive object).
In general, curves that give the shortest distance between two points on a surface are called the ***the geodesics*** of the surface.

Example 1:

The geodesic problem in a plane.
Finding the curve joining two points in a plane along which the distance between the two points is minimum. This is a 2-D

$$ds^2 = dx^2 + dy^2$$
$$ds = \sqrt{dx^2 + dy2}$$
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$
$$ds = \sqrt{1 + \dot{y}^2}dx$$

The entire length

$$L = \int_{x_1}^{x_2} ds$$
$$= \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2}dx$$

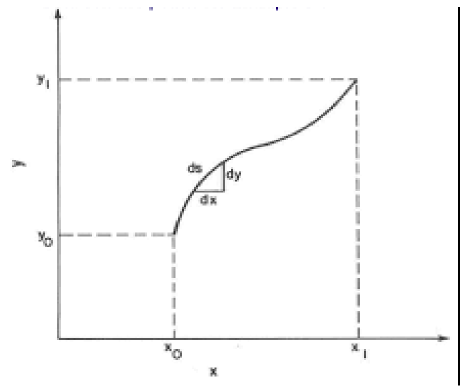


Figure 42: A Diagram showing the shortest distance between two points

set $f = \sqrt{1 + \dot{y}^2}$,

$$\therefore \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

Thus, this is the line integral that must be minimized. For it to be minimum, $f = \sqrt{1 + \dot{y}^2}$ must satisfy the differential equation (11.12) or since f does not contain y explicitly, the differential equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

or integrating we have

$$\frac{\partial f}{\partial \dot{y}} = c$$

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c\,; \quad squaring\ both\ sides$$

$$\dot{y}^2 = c^2(1+\dot{y}^2)$$

$$\dot{y}^2(1-c^2) = c^2$$

$$\dot{y}^2 = \frac{c^2}{1-c^2}$$

$$\dot{y} = \frac{c}{\sqrt{1-c^2}}$$

$$\frac{dy}{dx} = \frac{c}{\sqrt{1-c^2}} = a; \quad constant$$

$$\Rightarrow \quad \frac{dy}{dx} = a$$

$$dy = \int a dx$$

$$y = ax + b;$$

which is a straight line equation. The constraints a and b being determined by the condition that the curve passes through the two points. The required curve being a straight line.

Strictly speaking, it has only been proved to be an extremum path but for this problem, it is a minimum path.

In general, curves that gives the shortest distance between two points in a given space are called ***GEODESICS*** of the sphere.

Example 2:

The minimum space of Revolution.

$$A = \pi r^2$$

$$dA = 2\pi r dr$$

Find the curve $y(x)$ for which the surface area is minimum. The curve of a strip of surface

$$ds = \sqrt{1+\dot{y}^2}dx; \quad r = variablealongx - axis$$

$$dA = 2\pi x ds = 2\pi x\sqrt{1+\dot{y}^2}dx$$

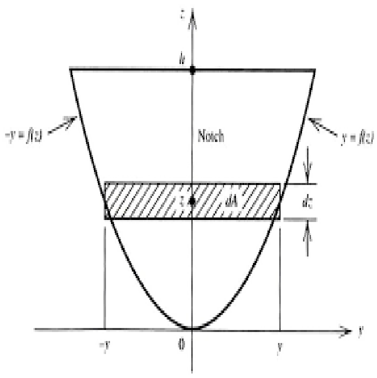


Figure 43

The total area is

$$A = 2\pi \int_{x_1}^{x_2} x\sqrt{1+\dot{y}^2}dx$$

The extremum of this integral is given by the differential equation (11.12).

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\Big(\frac{\partial f}{\partial \dot{y}}\Big)$$

$$f = x\sqrt{1+\dot{y}^2}$$

f does not explicitly depend on y.

$$\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\therefore \frac{d}{dx}\Big(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\Big) = 0 \quad or \quad \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = c \quad after \quad integration$$

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = 0$$

$$x^2\dot{y}^2 = c^2(1+\dot{y}^2)$$

$$(x^2 - c^2)\dot{y}^2 = c^2$$

$$\dot{y}^2 = \frac{c^2}{x^2 - c^2}$$

$$\dot{y} = \frac{dy}{dx} = \frac{c^2}{x^2 - c^2}dx$$

$$y + b = c\int \frac{dx}{x^2 - c^2}$$

set

$$x = c\cosh\theta$$

$$dx = c\sinh\theta d\theta$$

$$y + b = c\int \frac{c\sinh\theta d\theta}{c\sqrt{\cosh^2\theta - 1}} = \int \frac{c\sinh\theta d\theta}{\sqrt{\sinh^2\theta}}$$

$$= \int \frac{c\sinh\theta d\theta}{\sinh\theta} = c\int d\theta = c\theta$$

$$y + b = c\theta$$

NB: hyperbolic function $\cosh^2\theta - \sinh^2\theta = 1$

$$\Rightarrow \quad \frac{y+b}{c} = \cosh^{-1}\frac{x}{c} \quad or \quad \frac{x}{c} = \cosh\left(\frac{y+b}{c}\right)$$

$$x = c\cosh\left(\frac{y+b}{c}\right) \tag{11.21}$$

This is the equation of catenary. A catenary is the shape assumed by a heavy flexible uniform chain or cable freely suspended between two fixed points.

11.3 The Brachistochrone Problem

Two greek words: Brachistos – shortest and chronos – time. The brachistochrone problem is a least-time variational problem which was first solved in 1696 by Johann Bernoulli(1667-1748). The problem can be stated as follows:

A particle (bead) of mass m sliding down under the influence of gravity on a frictionless wire that connects the origin to a given point B. The question posed by the brachistochrone problem is to determine the shape y(x) of the wire for which the frictionless descent of the particle under gravity takes the shortest amount of time.

$$\Delta t = \frac{ds}{v} = \frac{\sqrt{1+\dot{y}^2}}{v}dx$$

If v is the speed along the element arc ds, then the time required to traverse that arc length is $\dfrac{ds}{v}$ and the problem is to find a minimum of the integral $t_{A\to B}$

$$t_{AB} = \int_A^B \frac{\sqrt{1+\dot{y}^2}}{v}dx$$

Now if y is measured down in the initial point of release, the conservation theorem for the energy of the particle can be written as

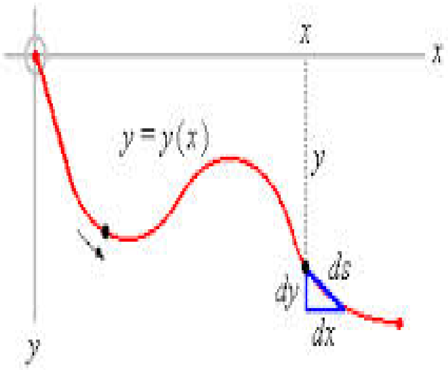


Figure 44

$$\frac{1}{2}mv^2 = mgh$$

loss in potential energy is gain in kinetic energy

$$v = \sqrt{2gy}$$

$$\therefore t_{AB} = \int_A^B \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}dx$$

$$= \int_A^B \sqrt{\frac{1+\dot{y}^2}{2gy}}$$

$$= \frac{1}{\sqrt{2g}}\int_A^B \sqrt{\frac{1+\dot{y}^2}{y}}dx$$

For the minimum time path, let us set

$$f = \sqrt{\frac{1+\dot{y}^2}{y}}$$

and must satisfy the differential equation (11.12) or

$$\frac{\partial f}{\partial \dot{y}} = \frac{1}{\sqrt{y}}\frac{1}{2}\left(1+\dot{y}^2\right)^{-1/2}2\dot{y} = \frac{\dot{y}}{\sqrt{y(1+\dot{y}^2)}}$$

since f is not explicitly dependent on x, the differential equation (11.15)

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = c$$

Note: Beltrami Identity is used

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{y}} - \frac{\dot{y}^2}{\sqrt{y(1+\dot{y}^2)}} = c$$

$$\frac{1+\dot{y}^2-\dot{y}^2}{\sqrt{y(1+\dot{y}^2)}} = c$$

$$\frac{1}{\sqrt{y(1+\dot{y}^2)}} = c$$

squaring both sides we obtain

$$y(1+\dot{y}^2) = \frac{1}{c^2} = a$$

$$1+\dot{y}^2 = \frac{a}{y} \quad \Rightarrow \quad \dot{y}^2 = \frac{a}{y} - 1$$

$$\frac{dy}{dx} = \dot{y} = \sqrt{\frac{a-y}{y}}$$

$$\int dx = \int \sqrt{\frac{y}{a-y}} dy$$

$$x + b = \int \sqrt{\frac{y}{a-y}} dy \quad b = cosntant\ of\ integration$$

Let us set

$$y = a \sin^2\theta$$

$$dy = 2a \sin\theta \cos\theta d\theta$$

substitute into the equation:

$$x + b = \int \sqrt{\frac{a\sin^2\theta}{a(1-\sin^2\theta)}} \cdot 2a \sin\theta\cos\theta d\theta$$

$$= 2a \int \sqrt{\frac{\sin^2\theta}{(1-\sin^2\theta)}} \cdot \sin\theta\cos\theta d\theta$$

$$= 2a \int \frac{\sin\theta}{\cos\theta} \cdot \sin\theta\cos\theta$$

$$= 2a \int \sin^2\theta d\theta$$

$$= a \int (1-\cos 2\theta) d\theta$$

Note the following identities:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin^2\theta + \cos^2\theta = 1, \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

we can write

$$= a(\theta - \frac{1}{2}\sin 2\theta)$$

$$x + b = \frac{a}{2}(\rho - \sin\rho) \quad \rho = 2\theta$$

$$\Rightarrow \quad x = \frac{a}{2}(\rho - \sin\rho) - b$$

Now $y = a\sin^2\theta$

$$= \frac{a}{2}(1 - \cos^2\theta)$$

$$= \frac{a}{2}(1 - \cos 2\theta)$$

$$= \frac{a}{2}(1 - \cos\rho)$$

hence we have

$$x = \frac{a}{2}(\rho - \sin\rho) - b$$

$$y = \frac{a}{2}(1 - \cos\rho)$$

For x and y to varnish at the point $(0,0)$ we must have $b = 0$ then we finally have

$$x = \frac{a}{2}(\rho - \sin\rho)$$

These are the parametric equations of the cycloid.
A cycloid is the locus of a point on the rim of a disk or ring rolling without slipping on a horizontal plane.

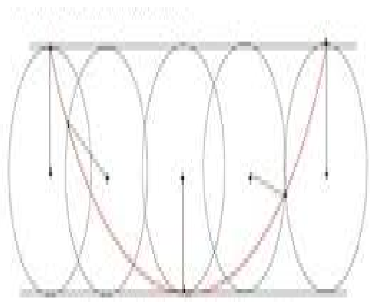
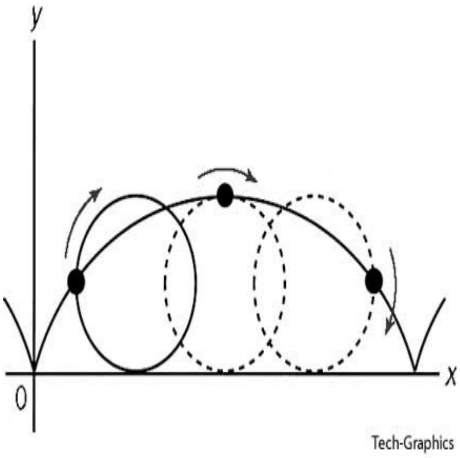
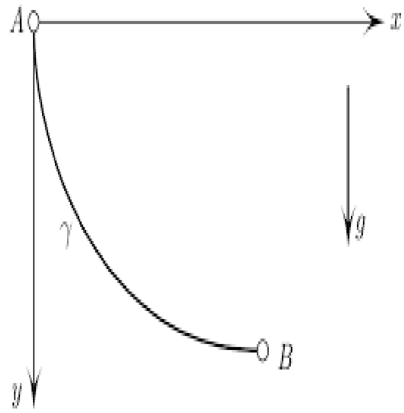


Figure 45: The path of the Brachistochrone is shown in red



(a)



(b)

11.3.1 Principle of Least Action: Fermats Principle in Geometric Optics

A ray of light in a plan medium with variable refractive index $n(x, y)$ will follow the path which requires the shortest traveling time or the path taken between two points by a ray of light is the path that can be traversed in the least time.

Problem:

Given the refractive index of a plane medium, find the curve along which a ray of light will travel in the shortest possible time between any two fixed points.

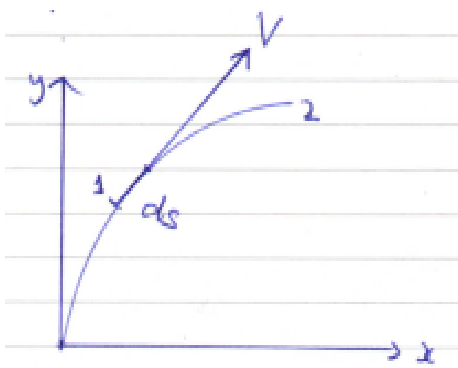


Figure 47

$$time \quad t_{1-2} = \int_1^2 \frac{ds}{v}$$

$$= \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{v} dx$$

refractive index $\eta(x, y) = c/v$ or

$$\frac{1}{v} = \frac{\eta(x, y)}{c}$$

$$t_{1-2} = \int_1^2 \sqrt{1+\dot{y}^2} \cdot \frac{\eta(x, y)}{c} dx$$

$$= \frac{1}{c}\int_1^2 \eta(x,y)\sqrt{(1+\dot{y}^2)}dx$$

The minimum time-path will be the one for which

$$f = \eta(x,y)\sqrt{(1+\dot{y}^2)}$$

will satisfy the differential equation (7.12) Numerous curves can be obtained here. Let us consider the case when

$$\eta(x,y) = \frac{1}{y}$$

$$\therefore f = \frac{\sqrt{1+\dot{y}^2}}{y}$$

Since the function is not explicitly dependent on x, we must use the Beltrami identity.

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = c$$

$$\Rightarrow \frac{\sqrt{1+\dot{y}^2}}{y} - \frac{\dot{y}^2}{y\sqrt{1+\dot{y}^2}} = c$$

$$1+\dot{y}^2 - \dot{y}^2 = cy\sqrt{1+\dot{y}^2}$$

$$1 = cy\sqrt{1+\dot{y}^2}$$

$$1+\dot{y}^2 = \frac{1}{c^2y^2} = \frac{a^2}{y^2} \qquad a^2 = \frac{1}{c^2}$$

$$\dot{y}^2 = \frac{a^2}{y^2} - 1$$

$$= \frac{a^2-y^2}{y^2}$$

$$\frac{dy}{dx} = \dot{y} = \sqrt{\frac{a^2-y^2}{y^2}} = \frac{\sqrt{a^2-y^2}}{y}$$

$$dx = \int \frac{y}{\sqrt{a^2-y^2}}dy$$

$$x - b = \int \frac{y}{\sqrt{a^2-y^2}}dy$$

set $y = a\cos\theta$; $\qquad dy = -a\sin\theta d\theta$

$$x - b = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\cos^2\theta}}(-a\sin\theta d\theta)$$

$$= -a\int \cos\theta d\theta$$

$$x - b = -a\sin\theta$$

$$y = a\cos\theta$$

square both sides and add them

$$(x-b)^2 + y^2 = a^2\sin^2\theta + a^2\cos^2\theta$$

$$= a^2$$

$$(x-b)^2 + y^2 = a^2$$

This is a circle with radius a and center $(b, 0)$.

11.4 Generalized Coordinates

Seek generalization of coordinates. A radius vector is needed to specify the position of a particle in space
A radius vector is specified by three cordinates
For N-particles, there N-radius vectors and hence 3N-cordinates If there are M constraint equations that limit the motion of particle by for instance relating some of coordinates, then the number of independent coordinates is limited to **3N-M**. One then describes the system as having **3N-M degrees of freedom**.

Important note:

If $s = 3N - M$ coordinates are required to describe a system, it is NOT necessary these s coordinates be rectangular or curvilinear coordinates. One can choose any combination of independent parameters as long as they completely specify the system. Note further that these coordinates (parameters) need not even have the dimension of length (e.g. q in our previous example). We use the term generalized coordinates to describe any set of coordinates that completely specify the state of a system. Generalized coordinates will be noted:$q_1, q_2, ..., q_n$. A set of generalized coordinates whose number equals the number s of degrees of freedom of the system, and not restricted by the constraints is called a proper set of generalized coordinates. In some cases, it may be useful/convenient to use generalized coordinates whose number exceeds the number of degrees of freedom, and to explicitly use constraints through Lagrange multipliers. Useful e.g. if one wishes to calculate forces due to constraints. The choice of a set of generalized coordinates is obviously not unique. They are in general (infinitely) many possibilities. In addition to generalized coordinates, we shall also consider time derivatives of the generalized coordinates called generalized velocities.

Notation:

$$q_1, q_2, ..., q_s \quad or \quad q_i \quad i = 1, ..., s$$
$$\dot{q_1}, \dot{q_2}, ..., \dot{q_s} \quad or \quad \dot{q_i} \quad i = 1, ..., s$$

Transformation:

The "normal" coordinates can be expressed as functions of the generalized coordinates - and vice-versa

$$x_{a,i} = x_{a,i}(q_1, q_2, ..., q_s, t) \quad \alpha = 1, 2, ..., n \quad i = 1, 2, 3$$
$$= x_{a,i}(q_j, t) \quad j = 1, 2, ..., s$$

Rectangular components of the velocities depend on the generalized coordinates, the generalized velocities, and the time.

$$\dot{x_{a,i}} = \dot{x_{a,i}}(q_j, \dot{q_j}, t)$$

Inverse transformations are noted:

$$q_j = q_j(x_{a,i}, t)$$
$$\dot{q_j} = \dot{q_j}(x_{a,i}, \dot{x_{a,i}}, t)$$

There are m=3n-s equations of constraint

$$f_k(x_{a,1}, t) = 0 \quad k = 1, 2, ..., m$$

Example:

Question: Find a suitable set of generalized coordinates for a point particle moving on the surface of a hemisphere of radius R whose center is at the origin.

Solution: Motion on a spherical surface implies:

$$x^2 + y^2 + z^2 - R^2 = 0 \quad z \geq 0$$

Choose cosines as generalized coordinates

$$q_1 = \frac{x}{R}$$

$$q_2 = \frac{y}{R}$$

$$q_3 = \frac{z}{R}$$

$$q_1^2 + q_2^2 + q_3^2 = 1$$

q_1, q_2, q_3 do not constitute a proper set of generalized of coordinates because they are not independent. One may however choose e.g. q_1, q_2, and the constraint equation

$$x^2 + y^2 + z^2 = R^2$$

11.4.1 Lagrange Equations In Generalized Coordinates

Of all possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian for the system.

Remarks

Lagrangian defined as the difference between kinetic and potential energies.

- Energy is a scalar quantity (at least in Galilean relativity).
- Lagrangian is a scalar function.
- Implies the lagrangian must be invariant with respect to coordinate transformations.
- Certain transformations that change the Lagrangian but leave the Eqs of motion unchanged are allowed.

E.G. if L is replaced by $L + d/dt\ f(q_i, t)$, for a function with continuous 2nd partial derivatives. (Fixed end points)

The choice of reference for U is also irrelevant, one can add a constant to L.

The choice of specific coordinates is therefore immaterial

$$L = T(\dot{x_{\alpha,i}}) - U(x_{\alpha,i})$$

$$= T(q_j, \dot{q}_j, t) - U(q_j, t)$$
$$= L(q_j, \dot{q}_j, t)$$

Hamilton's principle becomes

$$\delta \int_1^2 L(q_j, \dot{q}_j, t) = 0$$

$$x \to t$$

$$y_i(x) \to q_i(t)$$

$$y_i{}'(x) \to \dot{q}_i(t)$$

$$f[y_i(x), y_i{}'(x); x] \to L(q_i(t), \dot{q}_i(t))$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad i = 1, 2, ..., s$$

"s" equations "m" constraint equations.

Applicability:

1. Force derivable from one/many potential.
2. Constraint Eqs connect coordinates, may be fct(t)

Holonomic constraints

$$f_k(x_{a,1}, t) = 0 \quad k = 1, 2, ..., m$$

Scleronomic constraints

Independent of time.

Rheonomic

Dependent on time.

Example:

Projectile in 2D
Consider the motion of a projectile under gravity in two dimensions. Find equations of motion in Cartesian and polar coordinates.

$$T = \frac{1}{2}mv^2$$

$$U = mgy \quad U = 0 \; at \; y = 0$$

$$L = T - U = \frac{1}{2}mv^2 - mgy$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$0 - \frac{d}{dt}m\dot{x} = 0$$

$$\ddot{x} = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$-mg - \frac{d}{dt} m\dot{y} = 0$$
$$\ddot{y} = -g$$

In polar coordinates

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$U = mgr\sin\theta \quad U = 0 \; at \; \theta = 0$$
$$L = T - U = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$
$$mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$$
$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$$
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$
$$-mge\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$
$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

Example:

Motion in a cone
A particle of mass "m" is constrained to move on the inside surface of a smooth cone of hal-angle a. The particle is subject to a gravitational force. Determine a set of generalized coordinates and determine the constraints. Find Lagrange Equations of motion.
Constraint: $z - r\cot\alpha = 0$ 2 degrees of freedom only! and 2 generalized coordinates. Choose to eliminate " z " $z = r\cot\alpha$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$$
$$= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha$$
$$= \dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2$$
$$U = mgz = mgr\cot\alpha$$
$$L = T - U = \frac{1}{2} m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2) - mgr\cot\alpha$$

L is independent of θ. hence

$$\frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$
$$\frac{\partial L}{\partial \dot{\theta}} = constant = mr^2\dot{\theta}$$
$$mr^2\dot{\theta} = mr^2\omega \quad is\; the\; angular\; momentum\; relative\; to\; the\; axis\; of\; cone$$

For r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$
$$\ddot{r} - r\dot{\theta}^2\sin^2\alpha + g\sin\alpha\cos\alpha = 0$$

Lagrange's Equations with underdetermined multipliers

Constraints that can be expressed as algebraic equations among the coordinates are holonomic constraints. If a system is subject to such equations, one can always find a set of generalized coordinates in terms of which Equations of motion are independent of these constraints. Constraints which depend on the velocities have the form

$$f(x_{a,i}, \dot{x_{a,i}}, t) = 0$$

Consider

$$\sum_i A_i \dot{x}_i + B = 0 \quad i = 1, 2, 3$$

Generally non-integrable unless

$$A_i = \frac{f}{x_i}, \quad B_i = \frac{f}{t} = 0 \quad f = f(x, t)$$

One thus has:

$$\sum_i \frac{\partial f}{\partial x_i}\dot{x}_i + \frac{\partial f}{\partial t} = 0$$

Or

$$\frac{df}{dt} = 0$$

which yields

$$f(x_i, t) - constant = 0$$

So the constraints are actually holonomic. We therefore conclude that if constraints can be expressed as

$$\sum_i \frac{f_k}{q_i} dq_i + \frac{f}{t} dt = 0$$

Constraints Eqs given in differential form can be integrated in Lagrange Equations using undetermined multipliers. For:

$$\sum_i \frac{f_k}{q_i} dq_i = 0$$

One gets:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0$$

Forces of Constraint

The underdetermined multipliers are the forces of constraint:

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}$$

11.5 Equivalence of Lagrange's and Newton's Equation

Lagrange and Newton formulations of mechanic are equivalent
Different view point, same equations of motion.
Explicit demonstration

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, 3$$

$$\frac{\partial (T-U)}{\partial x_i} - \frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{x}_i} = 0 \quad i = 1, 2, 3$$

$$-\frac{\partial U}{\partial x_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}$$

$$\frac{U}{x_i} = F_i$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(\sum_{j=1}^{3} \frac{1}{2} m \dot{x}_j^2\right) = \frac{d}{dt}(m\dot{x}_i) = \dot{p}_i$$

$$F_i = \dot{p}_i \quad i = 1, 2, 3$$

$$x_i = x_i(q_i, t)$$

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$

Generalized momentum

$$p_j = \frac{\partial T}{\partial \dot{q}_j}$$

Generalized force defined through virtual work δW

$$W = \sum_i F_i \delta x_i$$

$$\delta W = \sum_{ij} F_i \frac{x_i}{q_j} q_j$$

$$\delta W = \sum_j Q_j \delta q_j$$

$$Q_j = \sum_i F_i \frac{x_i}{q_j}$$

For a conservative system:

$$Q_j = -\frac{\partial U}{\partial q_j}$$

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\Big(\sum_i \frac{1}{2} m \dot{x}_i{}^2\Big)$$

$$p_j = \sum_i m \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_i}$$

remember

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$

$$p_j = \sum_i m \dot{x}_i \frac{\partial x_i}{\partial q_j}$$

$$\dot{p}_j = \sum_i \left(m \ddot{x}_i \frac{\partial x_i}{\partial q_j} + m \dot{x}_i \frac{d}{dt} \frac{\partial x_i}{\partial q_{j1}} \right)$$

$$\frac{d}{dt} \frac{\partial x_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 x_i}{\partial q_j \partial t}$$

$$\dot{p}_j = \sum_i m \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_{i,k} m \dot{x}_i \frac{\partial^2 x_i}{\partial q_k \partial q_j} \dot{q}_k + \sum_i \frac{\partial^2 x_i}{\partial q_j \partial t}$$

$$\dot{p}_j = Q_j + \frac{\partial T}{\partial q_j}$$

$$p_j = \frac{\partial T}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}$$

Because U does not depend on $\dot{q}_j$, one has

$$\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right) - \frac{\partial (T-U)}{\partial q_j} = 0$$

and with $L = T - U$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\right) = 0$$

The Atwood's machine

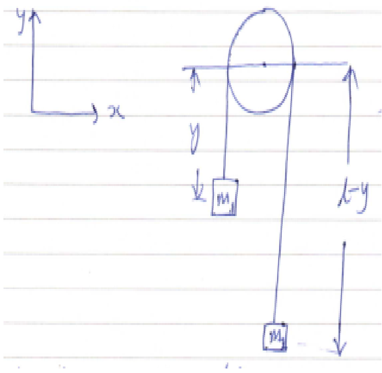


Figure 48

l: length of the whole thread.
For this system, there is only one independent coordinate. The position of m_2 is determined by the constraint that the length l of the robe between m_1 and m_2 is constant. That is, l is not extensible.
To derive the equation of motion:
Potential energy:

$$P.E = -m_1gy - m_2g(l-y)$$

The negative sign is due to the reference chosen or considered

$$\frac{dy}{dt} = \dot{y}; \quad velocity \quad of \quad the \quad system$$

$$K.E : T = \frac{1}{2}(m_1+m_2)\dot{y}^2$$

$$L = T - V$$

$$= \frac{1}{2}(m_1+m_2)\dot{y}^2 + m_2g(l-y) + m_1gy$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0; \quad Lagragian \quad equation \quad of \quad motion$$

$$\frac{\partial L}{\partial \dot{y}} = (m_1+m_2)\dot{y}$$

$$\frac{\partial L}{\partial y} = m_1g - m_2g = (m_1-m_2)g$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}(m_1+m_2)\dot{y} = \ddot{y}(m_1+m_2)$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \ddot{y}(m_1+m_2) - (m_1-m_2)g = 0$$

$$\therefore \quad \ddot{y}(m_1+m_2) = (m_1-m_2)g$$

$$\Rightarrow \ddot{y} = \frac{(m_1-m_2)g}{m_1+m_2}$$

11.5.1 Canonical momenta and conservation

Assume the Lagragian L does not depend explicit on the coordinate q_i. Such coordinates are called cyclic. The Euler-Lagragian equation for the cyclic coordinate q_i becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q_i} = 0$$

$$\Rightarrow = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad or \quad \frac{\partial L}{\partial \dot{q}} = \int 0dt = P_i$$

$$\frac{\partial L}{\partial \dot{q}} \equiv P_i = constant.$$

P_i is called **canonical momentum conjugate** to P_i

$$L = T - V$$

$$= \frac{1}{2}m\sum_j \dot{x}_j^2 - V(x)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}_i = P_i = ordinary \quad momentum \quad or \quad linear \quad Momentum$$

So we have found the law of conservation of momentum P_i if the potential V does not depend on the coordinate x_i, i.e; if the system is translationally invariant in the i- direction. Note that if V does not depend on x_i this implies that there are no net forces in the i- direction.

We may in a similar way demonstrate conservation of total momentum for a system of n particles if the potential energy does not depend on the center of mass coordinate.
But the concept of cononical momentum is much more general and powerful than this and can be used to derive a whole host of other conservation laws.
One such important law is the angular momentum.

Angular Momentum

For a particle in 2D system

$$L = T - V$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

θ is a cyclic coordinate and the canonical momentum P_θ is therefore conserved.

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}(\frac{1}{2}mr^2\dot{\theta}^2) = mr^2\dot{\theta}$$

$$\dot{\theta} = \omega(angular \quad velocity)$$

V_θ in the angular direction is

$$V_\theta = r\omega$$

$$P_\theta = r(mr\omega) = r(mV_\theta)$$

$$L_z = (r \times \vec{P})_z = rmV_\theta$$

So the canonical momentum conjugate to the angle θ is the angular momentum, which is conserved if the system is rotationally symmetric. i.e. The Lagragian does not depend on θ.

11.5.2 Energy conservation: The Hamiltonian

When we have conservation forces, the potential enrgy depends only on positions and not time, and total energy is conserved.
We have derived conservation of linear and angular momentum in Lagragian mechanics so we can do the same here but its more general than we already know.
Now, let us take the total derivative of the Lagragian with respect to

$$L = L(q_i, \dot{q}_i, t)$$

and using chain rule E-L equations, we get

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \cdot \frac{\partial q_i}{\partial t} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t} \quad chain \quad rule$$

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i} \cdot \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t}$$

According to E-L,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial q}; \quad E - L \quad equation$$

$$= \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}_i}{\partial t} + \frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial t} = \frac{d}{\partial t}\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial t}; \quad chain \quad rule \quad was \quad applied$$

$$\Leftrightarrow 0 = \frac{d}{dt}\left(\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) + \frac{\partial L}{\partial t} \quad factorized \quad \frac{d}{dt}$$

$$0 = \frac{dH}{dt} + \frac{\partial L}{\partial t} \quad where \quad H = \sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

We define:

$$H = \sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \quad = \sum P_i \dot{q} - L; \quad the \quad Hamiltonian.$$

So we find that if the Lagragian does not depend explicitly on time, the Hamiltonian or energy function is conserved.

To see this, consider a system of particles in cartesian coordinates, described by the Lagragian

$$L = T - V$$

$$= \frac{1}{2}\sum m_i \dot{q}_i^2 - V(q); \quad \frac{\partial L}{\partial \dot{q}} = (\sum m_i \dot{q})$$

The Hamiltonian for the system

$$H = \sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

$$= \sum (m_i \dot{q})\dot{q}_i - L$$

$$H = \sum (m_i \dot{q})\dot{q}_i - \left[\frac{1}{2}\sum m_i \dot{q}_i^2 - V(q)\right]$$

$$= \sum m_i\dot{q}_i^2 - \frac{1}{2}\sum m_i\dot{q}_i^2 + V(q)$$

$$\frac{1}{2}\sum m_i\dot{q}_i^2 + V(q)$$

$$H = T + V$$

So we find that the Hamiltonian is equal to the total energy, so Conservation of the Hamiltonian is the same as energy conservation in this particular(most common) case.

11.5.3 Hamiltonian equation of motion

The Hamiltonian fuction H of a holonomic system with s degrees of freedom is a fuction of the generalized coordinate and momenta of the system as well as time.

$$H(P, q, t) = \sum_i P_i\dot{q}_i - L(q_i, \dot{q}, t) \tag{11.22}$$

Let us consider the L.H.S of (7.22) as

$$H = H(P, q, t)$$

$$dH = \sum \frac{\partial H}{\partial q_i}dq_i + \sum \frac{\partial H}{\partial P_i}dP_i + \sum \frac{\partial H}{\partial t}dt \tag{11.23}$$

Similarly, let us consider the R.H.S of (7.22)

$$dH = \sum \dot{q}dP_i + \sum P_i d\dot{q} - \sum \frac{\partial L}{\partial q_i}dq_i - \sum \frac{\partial L}{\partial \dot{q}}d\dot{q} - \sum \frac{\partial L}{\partial t}dt$$

Recall that: $P_i = \frac{\partial L}{\partial \dot{q}}$. Lets substitute.

$$dH = \sum \dot{q}dP_i + \sum P_i d\dot{q}_i - \sum \frac{\partial L}{\partial q_i}dq_i - \sum P_i d\dot{q}_i - \sum \frac{\partial L}{\partial t}dt$$

$$= \sum \dot{q}dP_i - \sum \frac{\partial L}{\partial q_i}dq_i - \sum \frac{\partial L}{\partial t}dt \tag{11.24}$$

Now comparing (7.22) and (7.23) according to similar terms:

1. dP_i:
   $\Rightarrow \dot{q} = \frac{\partial H}{\partial P_i}$ or $\sum \dot{q}dP_i = \sum \frac{\partial H}{\partial P_i}dP_i$

2. dq_i:
   $-\sum \frac{\partial L}{\partial q_i}dq_i = \sum \frac{\partial H}{\partial q_i}dq_i$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i} = \quad Force$$

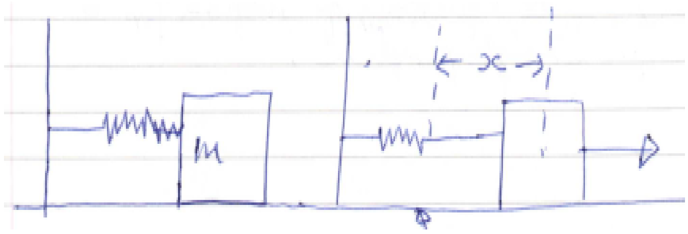
$$\dot{P}_i = F_i = -\frac{\partial H}{\partial q_i}$$

3. dt:

$$\sum \frac{\partial H}{\partial t} dt = -\frac{\partial L}{\partial t} dt$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Example: Harmonic oscillator.



$$T = \frac{1}{2} m \dot{x}^2 = \frac{P_x^2}{2m} \qquad P = m\dot{x}$$

$$V = \frac{1}{2} k x^2$$

$$\begin{aligned} H &= T + V \\ &= \frac{P_x^2}{2m} + \frac{1}{2} k x^2 \end{aligned}$$

(a)

$$\dot{q} = \frac{\partial H}{\partial P_i}$$

$$\dot{q} = \dot{x} = \frac{\partial}{\partial P_x} \left(\frac{P_x^2}{2m} + \frac{1}{2} k x^2 \right) = \frac{P_x}{m}$$

(b)

$$\dot{P}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{P}_x = -\frac{\partial}{\partial x} \left(\frac{1}{2} P_x^2 + \frac{1}{2} k x^2 \right) = -kx$$

11.5.4 Functions and Functionals

A function is a mathematical object which takes a real (or complex) variable, or several such variables, and returns a real (or complex number)
A Functional is a methematical object which takes an entire function and returns a number. In the present case;

$$T[y(x)] = \int_{x1}^{x2} dx L(y, \dot{y}, x),$$

where the function $L(y, \dot{y}, x)$ is given by

$$L(y, \dot{y}, x) = \frac{1}{V(x)} \sqrt{1 + \dot{y}^2}$$

Note,

$$\delta T = \int_{x1}^{x2} dx \Big[\frac{\partial L}{\partial y} - \frac{d}{dx}\Big(\frac{\partial L}{\partial \dot{y}}\Big)\Big]\delta y$$

We say that the functional derivative of T with respect to $y(x)$ is

$$\frac{\delta T}{\delta y(x)} = \Big[\frac{\partial L}{\partial y} - \frac{d}{dx}\Big(\frac{\partial L}{\partial \dot{y}}\Big)\Big]_x$$

11.5.5 Canonical Equations of Motion- Hamiltonian Dynamics

Whenever the potential energy is velocity independent:

$$p_j = \frac{\partial L}{\partial \dot{x_j}}$$

Result extended to define the Generalized Momenta:

$$p_j = \frac{\partial L}{\partial \dot{q_j}}$$

Given Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q_j}} = 0$$

One also finds:

$$\dot{p_j} = \frac{\partial L}{\partial q_j}$$

The Hamiltonian may then be considered a function of the generalized coordinates, q_j, and momenta p_j:

$$H = \sum_j p_j \dot{q_j} - L$$

whereas the Lagrangian is considered a function of the generalized coordinates, q_i, and their time derivative

$$H(q_k, p_k, t) = \sum_j p_j \dot{q_j} - L(q_k, p_k, t)$$

To " convert " from the Lagrange formulation to the Hamiltonian formulation, we consider:

$$dH = \sum_j \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k\right) + \frac{\partial H}{\partial t} dt$$

But given $H = \sum_j p_j \dot{q_j} - L$, one can also write:

$$dH = \sum_j (p_k d\dot{q_k} + \dot{q_k} dp_k) - dL$$

$$= \sum_j \left(p_k d\dot{q_k} + \dot{q_k} dp_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q_k}} d\dot{q_k}\right) - \frac{\partial L}{\partial t} dt$$

$$dH = \sum_j (\dot{q_k} dp_k - \dot{p_k} dq_k) - \frac{\partial L}{\partial t} dt$$

That must also equal:

$$dH = \sum_j \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k\right) + \frac{\partial H}{\partial t} dt$$

We then conclude the **Hamilton Equations**:

$$\frac{\partial H}{\partial p_k} = \dot{q_k} \quad ; \quad \frac{\partial H}{\partial q_k} = -\dot{p_k} \quad and \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$

Let us now rewrite:

$$dH = \sum_j \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt$$

and calculate:

$$\frac{dH}{dt} = \sum_j (-\dot{p_k}\dot{q_k} + \dot{p_k}\dot{q_k}) + \frac{\partial H}{\partial t}$$

We then finally conclude:

$$\frac{dH}{dt} = \frac{H}{t}$$

If $\frac{H}{t} = 0$, H is constant of motion. If additionally $H = U + T = E$, then E is a conserved quantity.

Some remarks

- The Hamiltonian formulation requires, in general, more work than the Lagrange formulation to derive the equations of motion.
- The Hamiltonian formulation simplifies the solution of problems whenever cyclic variables are encountered.
- Cyclic variables are generalized coordinates that do not appear explicitly in the Hamiltonian.
- The Hamiltonian formulation forms the basis to powerful extensions of classical mechanics to other fields e.g. Beam physics, statistical mechanics, etc.
- The generalized coordinates and momenta are said to be **canonically conjugates** – because of the symmetric nature of Hamilton's equations.
  If q_k is cyclic, i.e. does not appear in the Hamiltonian, then

$$\dot{q_k} = \frac{\partial L}{\partial q_k} = \frac{\partial H}{\partial q_k} = 0$$

  and p_k is then a constant of motion $p_k = \alpha_k$

A coordinate cyclic in H is also cyclic in L. Note: if qk is cyclic, its time derivative "q-dot" appears explicitly in L. No reduction of the number of degrees of freedom in the Lagrange formulation: still "s" 2^{nd} order equations of motion. Reduction by 2 of the number of equations to be solved in the Hamiltonian formulation – since 2 become trivial.

$$\dot{q_k} = \frac{\partial H}{\partial \alpha_k} = \omega_k \qquad where\ \omega_k is\ possibly\ a\ function\ of\ t$$

One must get the simple (trivial) solution:

$$q_k(t) = \int \omega_k dt$$

The solution for a cyclic variable is thus reduced to a simple integral as above.
The simplest solution to a system would occur if one could choose the generalized coordinates in a way they are ALL cyclic. One would then have "s" equations of the form:

$$q_k(t) = \int \omega_k dt$$

Such a choice is possible by applying appropriate transformations – this is known as Hamilton-Jacobi Theory.

Some remarks on the Calculus of Variation:

Hamilton's Principle:

$$\delta \int_{t_1}^{t_2} L(q_k, \dot{q_k}, t)dt = 0$$

Evaluated to:

$$\int_{t_1}^{t_2} \left(\dot{q_k} dp_k - \frac{\partial L}{\partial q_k} dq_k + \frac{\partial L}{\partial \dot{q_k}} d\dot{q_k} \right) dt = 0$$

where ∂q_k and $\partial \dot{q_k}$ are **not** !independent

$$\partial \dot{q_k} = \delta \left(\frac{dq_k}{dt} \right) = \frac{d}{dt} \delta q_k$$

The above integral becomes after integration by parts:

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q_k}} \right) \delta q_k dt = 0$$

which gives rise to Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q_k}} = 0$$

Alternatively, Hamilton's Principle can be written as:

$$\delta \int_{t_1}^{t_2} \left(\sum_k p_k \dot{q_k} - H \right) dt = 0$$

which evaluates to:

$$\int_1^2 \sum_j \left(p_k \delta q_k + \dot{q_k} \delta p_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \right) dt = 0$$

Consider

$$\int_1^2 \sum_j p_k \dot{q_k} dt = \int_1^2 \sum_j p_k \frac{d}{dt} \delta q_k$$

Integrate by parts:

$$\int_1^2 \sum_j p_k \delta \dot{q_k} dt = - \int_1^2 \sum_j \dot{p_k} \delta q_k dt$$

The variation may be written:

$$\int_1^2 \sum_j \left[\left(\dot{q_k} - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p_k} + \frac{\partial H}{\partial q_k} \right) \delta q_k \right] dt = 0$$

12 Lagragian Revisited

12.1 Lagragian and Hamiltonian Formulation

What are the pros/cons of each approach? What questions are more naturally solved in each? For example, I believe Fermat's Principle of Least Time is something that's very naturally explained in Lagrangian mechanics ("minimize the time it takes to get between these two points"), but more difficult to explain in Newtonian mechanics since it requires knowing your endpoint.
What's the overall difference in layman's terms? From what I've read so far, it sounds like Newtonian mechanics takes a more local "cause-and-effect"/"apply a force, get a reaction" view, while Lagrangian mechanics takes a more global "minimize this quantity" view. Or, to put it more axiomatically, Newtonian mechanics starts with Newton's three laws of motion, while Lagrangian mechanics starts with the Principle of Least Action.
How do the approaches differ mathematically/when you're trying to solve a problem? Kind of similar to above, I'm guessing that Newtonian solutions start with drawing a bunch of force vectors, while Lagrangian solutions start with defining some function (calculating the Lagrangian...?) you want to minimize, but I really have no idea.
In Newtonian mechanics you have to use mainly rectangular co ordinate system and consider all the constraint forces. Lagrange's scheme avoids the considerations of the constraint forces deftly and you can use any set of "generalized coordinates" like angle, radial distance etc. consistent with the constraint relations. The number of those generalized coordinates are the same with the number of degrees of freedom of the system.
In all dynamical systems we arbitrarily choose some generalized co ordinates consistent with the constraints of the system. In Newtonian mechanics, the difference between the kinetic and potential energy of the system gives you the so called Lagrangian. Then we have n number of differential equations.

12.2 Advantages of Lagragian Mechanics

The main advantage of Lagrangian mechanics is that we don't have to consider the forces of constraints and given the total kinetic and potential energies of the system we can choose some generalized coordinates and blindly calculate the equation of motions totally analytically unlike Newtonian case where one has to consider the constraints and the geometrical nature of the system.

12.3 Calculus of variation

- he calculus of variations involves finding an extremum (maximum or minimum) of a quantity that is expressible as an integral. Let's look at a couple of examples:
- The shortest path between two points
- Fermat's principle (light follows a path that is an extremum)
- What is the shortest path between two points in a plane? You certainly know the answer—a straight line—but you probably have not seen a proof of this—the calculus of variations provides such a proof.
- Consider two points in the x-y plane, as shown in the figure.
- An arbitrary path joining the points follows the general curve $y = y(x)$, and an element of length along the path is

$$ds = \sqrt{dx^2 + dy^2} \tag{12.1}$$

We can rewrite this (pay attention, because you will be doing this a lot in the next few chapters) as

$$ds = \sqrt{1 + y'(x)^2}dx, \tag{12.2}$$

which is valid because $dy = \frac{dy}{dx}dx = y'(x)dx$.

Thus, the length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2 dx} \tag{12.3}$$

12.4 Shortest Path Between 2 Points

Note that we have converted the problem from an integral along a path, to an integral over x:

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2 dx} \tag{12.4}$$

We have thus succeeded in writing the problem down, but we need some additional mathematical machinery to find the path for which L is an extremum (a minimum in this case).

12.5 Fermat's Principle:

- A similar but somewhat more interesting problem is to find the path light will take through a medium that has some index of refraction $n \neq 1$. You may recall that light travels more slowly through such a medium, and we define the index of refraction as $n = c/v$. where c is the speed of light in vacuum, and v is the speed of light in the medium. The total travel time is then

$$\tau = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 nds = \frac{1}{c}\int_{x_1}^{x_2} n(x,y)\sqrt{1 + y'(x)^2}dx \tag{12.5}$$

Here we are allowing the index of refraction to vary arbitrarily vs. x and y.

12.6 Variational Principles

- Obviously, both problems on the previous slide are similar, and such cases arise in many other situations.
- In our usual minimizing or maximizing of a function f(x), we would take the derivative and find its zeroes (i.e. the values of x for which the slope of the function is zero). These points of zero slope may be minima, maxima, or points of inflection, but in each case we can say that the function is stationary at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope).
- In analogy with this familiar approach, we want to be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called calculus of variations.
- The methods we will develop are called variational methods, and a principle like Fermat's Principle are called variational principles.
- These principles are common, and of great importance, in many areas of physics (such as quantum mechanics and general relativity).

12.7 Euler-Lagrange Equations

We are now going to discuss a variational method due to Euler and Lagrange, which seeks to find an extremum (to be definite, let's consider this a minimum) for an as yet unknown curve joining two points x 1 and x 2 , satisfying the integral relation

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx \tag{12.6}$$

The function f is a function of three variables, but because the path of integration is $y = y(x)$, the integrand can be reduced to a function of just one variable, x.
To start, let's consider two curves joining points 1 and 2, the "right" curve $y(x)$, and a "wrong" curve $Y(x)$ that is a small displacement from the "right" curve, as shown in the figure.
We will write the difference between these curves as some function $\eta(x)$

$$Y(x) = y(x) + \eta(x); \quad \eta(x_1) = \eta(x_1) = 0 \tag{12.7}$$

12.7.1 Euler-Lagrange Equation-2

There are infinitely many functions $h(x)$, that can be "wrong," but we require that they each be longer that the "right" path. To quantify how close the "wrong" path can be to the "right" one, let's write $Y = y + a\eta$, so that

$$S(a) = \int_{x_1}^{x^2} f[Y, Y'(x), x]dx \tag{12.8}$$

$$= \int_{x_1}^{x^2} f[y + a\eta, y' + a\eta', x]dx \tag{12.9}$$

This is going to allow us to characterize the shortest path as the one for which the derivative $dS/da = 0$ when $a = 0$. To differentiate the above equation with respect to a , we need to evaluate the partial derivative dS/da via the chain rule

$$\frac{df(y + a\eta, y' + a\eta', x}{da} = \eta\frac{df}{dy} + \eta'\frac{df}{dy'} \tag{12.10}$$

so $dS/da = 0$ gives

$$\frac{ds}{da} = \int_{x_1}^{x^2} \frac{df}{da}dx = \int_{x_1}^{x^2} \left(\eta\frac{df}{dy} + \eta'\frac{df}{dy'}\right) = 0 \tag{12.11}$$

12.7.2 Euler-Lagrange Equation-3

We can handle the second term in the previous equation by integration by parts:

$$\int_{x_1}^{x_2} \eta'\frac{df}{dy'}dx = \left[\eta(x)\frac{df}{dy'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{df}{dy'}\right)dx \tag{12.12}$$

but the first term of this relation (the end-point term) is zero because h (x) is zero at the endpoints. Our modified equation is then

$$\frac{dS}{da} = \int_{x_1}^{x_2} \eta(x)\left(\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'}\right)dx \tag{12.13}$$

This leads us to the Euler-Lagrange equation

$$\frac{df}{dy} - \frac{d}{dx}\frac{df}{dy'} = 0 \tag{12.14}$$

We come to this conclusion because the modified equation has to be zero for any $\eta(x)$. See the text for more discussion of this point.

Geodesics: Shortest Path Between Two Points

We earlier showed that the problem of the shortest path between two points can be expressed as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2} dx \tag{12.15}$$

The integrand contains our function

$$f(y, y', x) = \sqrt{1 + y'(x)^2} \tag{12.16}$$

The two partial derivatives in the Euler-Lagrange equation area

$$\frac{df}{dy} = 0 \qquad \text{and} \qquad \frac{df}{dy'} = \frac{y'}{\sqrt{1 + y'^2}} \tag{12.17}$$

Thus, the Euler-Lagrange equation gives us

$$\frac{d}{dx}\frac{df}{dy'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0 \tag{12.18}$$

This says that

$$\frac{y'}{\sqrt{1 + y'2}} = C, \qquad \text{or} \qquad y'^2 = C^2(1 + y'^2) \tag{12.19}$$

A little rearrangement gives the final result: $y'^2 = constant$ (call it m^2), so $y(x) = mx + b$. In other words, a straight line is the shortest path.

12.8 The Brachistochrone

- This is a famous problem in the history of the calculus of variations.

Statement of the problem:
Given two points 1 and 2, with 1 higher above the ground, in what shape could we build a track for a frictionless roller-coaster so that a car released from point 1 would reach point 2 in the shortest possible time? See the figure, which takes point 1 as the origin, with y positive downward.

- Solution:

The time to travel from point 1 to 2 is $\tau = \int_1^2 \frac{ds}{v}$ where $v = \sqrt{2gy}$ from kinetic energy considerations.
Since this depends on y, we will take y as the independent variable, hence

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1} dy \tag{12.20}$$

Our integral now becomes

$$\tau = \frac{1}{2g} \int_0^{y_2} \frac{x'(2)}{+} 1\sqrt{y} dy \tag{12.21}$$

From the Euler-Lagrange equation:

$$\frac{df}{dx} = \frac{d}{dy}\frac{df}{dx'} \tag{12.22}$$

Since

$$f = \frac{\sqrt{x'(2)}}{\sqrt{y}}, \quad clearly \quad \frac{df}{dx} = 0, \quad and \quad so \quad \frac{df}{dx'} = constant \tag{12.23}$$

Evaluating this derivative and squaring it for convenience, we have

$$\frac{x'^2}{y(x'^2+1)} = constant = \frac{1}{2a} \tag{12.24}$$

where the constant is renamed $1/2a$ for future convenience.
Solving for x'we have

$$x^{'} = \sqrt{\frac{y}{2a-y}} \tag{12.25}$$

Finally, to get x we integrate

$$x = \int \frac{y}{2a-y} dy \tag{12.26}$$

It is not obvious, but this can be solved by the substitution $y = a(1 - cos\theta)$, which gives

$$x = a\int(1 - cos\theta)d\theta = a(\theta - sin\theta) + const. \tag{12.27}$$

The two equations that give the path are then:

$$x = a(\theta - sin\theta) \qquad\qquad y = a(1 - cos\theta) \qquad\qquad in \quad terms \quad of \quad \theta \tag{12.28}$$

This curve is called a cycloid, and is a very special curve indeed. As you will show in the homework, it is the curve traced out by a wheel rolling (upside down) along the x axis.
Another remarkable thing is that the time it takes for a cart to travel this path from $2 \rightarrow 3$ is the same, no matter where 2 is placed, from 1 to 3! Thus, oscillations of the cart along that path are exactly isochronous (period perfectly independent of amplitude).

12.9 Generalized Coordinates and constraints

Generalized Coordinates

In analytical mechanics, specifically the study of the rigid body dynamics of multibody systems, the term generalized coordinates refers to the parameters that describe the configuration of the system relative to some reference configuration. These parameters must uniquely define the configuration of the system relative to the reference configuration. This is done assuming that this can be done with a single chart. The generalized velocities are the time derivatives of the generalized coordinates of the system.
An example of a generalized coordinate is the angle that locates a point moving on a circle. The adjective "generalized" distinguishes these parameters from the traditional use of the term coordinate to refer to Cartesian coordinates: for example, describing the location of the point on the circle using x and y coordinates.
Although there may be many choices for generalized coordinates for a physical system, parameters which are convenient are usually selected for the specification of the configuration of the system and which make the solution of its equations of motion easier. If these parameters are independent of one another, the number of independent generalized coordinates is defined by the number of degrees of freedom of the system

Constraints and degrees of freedom

Generalized coordinates are usually selected to provide the minimum number of independent coordinates that define the configuration of a system, which simplifies the formulation of Lagrange's equations of motion. However, it can also occur that a useful set of generalized coordinates may be dependent, which means that they are related by one or more constraint equations.

Holonomic Constraints

For a system of N particles in 3D real coordinate space, the position vector of each particle can be written as a 3-tuple in Cartesian coordinates:

$$r_1 = (x_1, y_1, z_1) \quad , r_2 = (x_2, y_2, z_2), ..., r_N = (x_N, y_N, z_N)$$

$$\mathbf{r}_1 = (x_1, y_1, z_1)\,, \quad \mathbf{r}_2 = (x_2, y_2, z_2)\,, \ldots\,, \mathbf{r}_N = (x_N, y_N, z_N)\,.$$

$$\mathbf{r}_1 = (x_1, y_1, z_1)\,, \quad \mathbf{r}_2 = (x_2, y_2, z_2)\,, \ldots\,, \mathbf{r}_N = (x_N, y_N, z_N)\,.$$

particle k

$$f(rk, t) = 0 f(\mathbf{r}_k, t) = 0 f(\mathbf{r}_k, t) = 0$$

which connects all the 3 spatial coordinates of that particle together, so they are not independent. The constraint may change with time, so time t will appear explicitly in the constraint equations. At any instant of time, when t is a constant, any one coordinate will be determined from the other coordinates, e.g. if xk and zk are given, then so is yk. One constraint equation counts as one constraint. If there are C constraints, each has an equation, so there will be C constraint equations. There is not necessarily one constraint equation for each particle, and if there are no constraints on the system then there are no constraint equations. Any of the position vectors can be denoted rk where $k = 1, 2, ..., N$ labels the particles. A holonomic constraint is a constraint equation of the form for particle

So far, the configuration of the system is defined by $3N$ quantities, but C coordinates can be eliminated, one coordinate from each constraint equation. The number of independent coordinates is $n = 3N - C$. (In D dimensions, the original configuration would need ND coordinates, and the reduction by constraints means $n = ND - C$). It is ideal to use the minimum number of coordinates needed to define the configuration of the entire system, while taking advantage of the constraints on the system. These quantities are known as generalized coordinates in this context, denoted $qj(t)$. It is convenient to collect them into an n-tuple

$$q(t) = (q1(t), q2(t), ..., qn(t))\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t))\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t))$$

which is a point in the configuration space of the system. They are all independent of one other, and each is a function of time. Geometrically they can be lengths along straight lines, or arc lengths along curves, or angles; not necessarily Cartesian coordinates or other standard orthogonal coordinates. There is one for each degree of freedom, so the number of generalized coordinates equals the number of degrees of freedom, n. A degree of freedom corresponds to one quantity that changes the configuration of the system, for example the angle of a pendulum, or the arc length traversed by a bead along a wire. If it is possible to find from the constraints as many independent variables as there are degrees of freedom, these can be used as generalized coordinates. The position vector rk of particle k is a function of all the n generalized coordinates and time
and time

$$rk = rk(q(t), t), \mathbf{r}_k = \mathbf{r}_k(\mathbf{q}(t), t)\,, \mathbf{r}_k = \mathbf{r}_k(\mathbf{q}(t), t)\,,$$

and the generalized coordinates can be thought of as parameters associated with the constraint. The corresponding time derivatives of q are the generalized velocities,

$$q = \dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt} = (\dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t))\dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt} = (\dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t))$$

(each dot over a quantity indicates one time derivative). The velocity vector vk is the total derivative of rk with respect to time

$$vk = \mathbf{v}_k = \dot{\mathbf{r}}_k = \frac{d\mathbf{r}_k}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}\,. \mathbf{v}_k = \dot{\mathbf{r}}_k = \frac{d\mathbf{r}_k}{dt} = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}\,.$$

and so generally depends on the generalized velocities and coordinates. Since we are free to specify the initial values of the generalized coordinates and velocities separately, the generalized coordinates and velocities can be treated as independent variables. The generalized coordinates qj and velocities dqj/dt are treated as independent variables.

Non Holonomic Constraints

A mechanical system can involve constraints on both the generalized coordinates and their derivatives. Constraints of this type are known as non-holonomic. First-order non-holonomic constraints have the form

$$g(g, q.t) = 0, g(\mathbf{q}, \dot{\mathbf{q}}, t) = 0\,, g(\mathbf{q}, \dot{\mathbf{q}}, t) = 0\,,$$

An example of such a constraint is a rolling wheel or knife-edge that constrains the direction of the velocity vector. Non-holonomic constraints can also involve next-order derivatives such as generalized accelerations.

Generalized Momentum

The generalized momentum "canonically conjugate to" the coordinate qi is defined by

$$pi = .p_i = \frac{\partial L}{\partial \dot{q}_i} .p_i = \frac{\partial L}{\partial \dot{q}_i}$$

. If the Lagrangian L does not depend on some coordinate qi, then it follows from the Euler–Lagrange equations that the corresponding generalized momentum will be a conserved quantity, because the time derivative is zero implying the momentum is a constant of the motion;

$$p = 0.\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} = 0\,.\dot{p}_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} = 0\,.$$

12.10 The Hamiltonian Formalism

We'll now move onto the next level in the formalism of classical mechanics, due initially to Hamilton around 1830. While we won't use Hamilton's approach to solve any further complicated problems, we will use it to reveal much more of the structure underlying classical dynamics. If you like, it will help us understands what questions we should ask.

Hamiltonian's Equations

Recall that in the Lagrangian formulation, we have the function $L(q_i; \dot{q}_i; t)$ where qi (i = 1,. . . n) are n generalized coordinates. The equations of motion are

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q}_i = 0$$

These are $n2^{nd}$ order differential equations which require 2n initial conditions, say $q_i(t = 0)$ and $q_i(t = 0)$. The basic idea of Hamilton's approach is to try and place qi and q_i on a more symmetric footing. More precisely, we'll work with the n generalised momenta that we introduced in section 2.3.3,

$$P_i = \frac{\delta L}{\delta \dot{q}_i}\; i = 1, \ldots n$$

so $pi = pi(q_j; \dot{q}_j; t)$. This coincides with what we usually call momentum only if we work in Cartesian coordinates (so the kinetic term is $\frac{1}{2}m_i q_i^2$). If we rewrite Lagrange's equations (4.1) using the definition of the momentum (4.2), they become

$$\dot{P}_1 = \frac{\delta L}{\delta q_{\ i}}$$

The plan will be to eliminate q_i in favor of the momenta p_i, and then to place q_i and p_i on equal footing.

An Example: The Pendulum

Consider a simple pendulum. The configuration space is clearly a circle, S^1, parameterized by an angle $\theta \, \epsilon | -\pi, \pi)$. The phase space of the pendulum is a cylinder $R \times S^1$, with the R factor corresponding to the momentum. We draw this by flattening out the cylinder. The two different types of motion are clearly visible in the phase space flows. For small θ and small momentum,

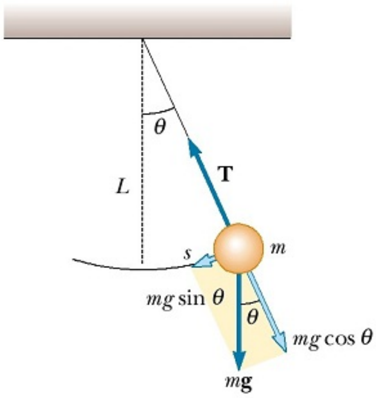


Figure 49: Flows in the phase space of a pendulum

the pendulum oscillates back and forth, motion which appears as an ellipse in phase space. But for large momentum, the pendulum swings all the way around, which appears as lines wrapping around the S^1 of phase space. Separating these two different motions is the special case where the pendulum starts upright, falls, and just makes it back to the upright position. This curve in phase space is called the separatix.

The Legendre Transform

We want to find a function on phase space that will determine the unique evolution of q_i and p_i. This means it should be a function of qi and pi (and not of q_i) but must contain the same information as the Lagrangian $L(q_i; \dot{q}_i, t)$. There is a mathematical trick to do this, known as the Legendre transform. To describe this, consider an arbitrary function f(x, y) so that the total derivative is

$$df = \frac{\delta f}{\delta x} dx + \frac{\delta f}{\delta y} dy$$

Now define a function $g(x, y, u) = ux - f(x, y)$ which depends on three variables, x,y and also u. If we look at the total derivative of g, we have

$$dg = d(ux) - df = udx + xdu - \frac{\delta f}{\delta x} dx - \frac{\delta f}{\delta y} dy$$

At this point u is an independent variable. But suppose we choose it to be a specific function of x and y, defined by

$$u(x, y) = \frac{\delta f}{\delta x}$$

Then the term proportional to dx in (4.5) vanishes and we have

$$dg = xdu - \frac{\delta f}{\delta y} dy$$

Or, in other words, g is to be thought of as a function of u and $y : g = g(u, y)$. If we want an explicit expression for g(u,y), we must first invert (4.6) to get $x = x(u; y)$ and then insert this into the definition of g so that

$$g(u, y) = ux(u, y) - f(x(u, y), y)$$

This is the Legendre transform. It takes us from one function $f(x, y)$ to a different function g(u; y) where $u = \delta f/\delta x$. The key point is that we haven't lost any information. Indeed, we can always recover $f(x, y)$ from $g(u, y)$ by noting that

$$\frac{\delta g}{\delta u}|_y = x(u, y) \text{ and } \frac{\delta g}{\delta u}|_y = \frac{\delta f}{\delta y}$$

which assures us that the inverse Legendre transform $f = (\delta g/\delta u)u - g$ takes us back to the original function. The geometrical meaning of the Legendre transform is captured in the diagram. For fixed y, we draw the two curves f(x, y) and ux. For each slope u, the value of $g(u)$ is the maximal distance between the two curves. To see this, note that extremising this distance means

$$\frac{d}{dx}(ux - f(x)) = 0 \Rightarrow u = \frac{\delta f}{\delta x}$$

Hamiltonian's Equations

The Lagrangian $L(qi, q_i, t)$ is a function of the coordinates q_i, their time derivatives q_i and (possibly) time. We define the Hamiltonian to be the Legendre transform of the Lagrangian with respect to the q_i variables,

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

where q_i is eliminated from the right hand side in favour of p_i by using

$$p_i = \frac{\delta L}{\delta \dot{q}_i} = p_i(q_j, \dot{q}_j, t)$$

and inverting to get $\dot{q}_i = \dot{q}_i(q_j, p_j, t)$. Now look at the variation of H:

$$dH = (dp_i \dot{q}_i + p_i d\dot{q}_i) - \left(\tfrac{\delta L}{\delta q_i} dq_i + \tfrac{\delta L}{\delta \dot{q}_i} d\dot{q}_i + \tfrac{\delta L}{\delta t} dt \right)$$

$$= dp_i \dot{q}_i - \frac{\delta L}{\delta q_i} dq_i - \frac{\delta L}{\delta t} dt$$

but we know that this can be rewritten as

$$dH = \frac{\delta H}{\delta q_i} dq_i + \frac{\delta H}{\delta p_i} dpi + \frac{\delta H}{\delta t} dt$$

So we can equate terms. So far this is repeating the steps of the Legendre transform. The new ingredient that we now add is Lagrange's equation which reads $\dot{p}_i = \delta L/\delta q_i$. We find

$$\dot{p}_i = -\frac{\delta H}{\delta q_i}$$

$$\dot{q}_i = \frac{\delta H}{\delta p_i}$$
$$-\frac{\delta L}{\delta t} = \frac{\delta H}{\delta t}$$

These are Hamilton's equations. We have replaced n $2^n d$ order differential equations by 2n $1^s t$ order differential equations for q_i and p_i. In practice, for solving problems, this isn't particularly helpful. But, as we shall see, conceptually it's very useful!

Example :A Particle in a Potential.

Let's start with a simple example: a particle moving in a potential in 3-dimensional space. The Lagrangian is simply

$$L = \frac{1}{2}m\dot{r}^2 - V(r)$$

We calculate the momentum by taking the derivative with respect to $\dot{r}$

$$P = \frac{\delta L}{\delta \dot{r}} = m\dot{r}$$

which, in this case, coincides with what we usually call momentum. The Hamiltonian is then given by

$$H = p.\dot{r} - L = \frac{1}{2m}p^2 + V(r)$$

where, in the end, we've eliminated $\dot{r}$ in favor of **p** and written the Hamiltonian as a function of **p** and **r**. Hamilton's equations are simply

$$\dot{r} = \frac{\delta H}{\delta p} = \frac{1}{m}p$$

$$\dot{p} = -\frac{\delta H}{\delta r} - \nabla V$$

which are familiar: the first is the definition of momentum in terms of velocity; the second is Newton's equation for this system.

A Particle in an Electromagnetic Field

We have seen that the Lagrangian for a charged particle moving in an electromagnetic field is

$$L = \frac{1}{2}m\dot{r}^2 - e\left(\phi - \frac{1}{c}\dot{r}.A\right)$$

From this we compute the momentum conjugate to the position

$$P = \frac{\delta L}{\delta \dot{r}} = m\dot{r} + \frac{e}{c}A$$

which now differs from what we usually call momentum by the addition of the vector potential A. Inverting, we have

$$\dot{r} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

So we calculate the Hamiltonian to be

$$H(p, r) = p.\dot{r} - L$$

$$= \frac{1}{m}p.\left(p - \frac{e}{c}A\right) - \left[\frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 - e\phi + \frac{e}{cm}\left(p - \frac{e}{c}\right).A\right]$$

$$= \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi$$

Now Hamilton's equations read

$$\dot{r} = \frac{\delta H}{\delta p} = \frac{1}{m}\left(p - \frac{e}{c}A\right)$$

while the $\dot{p}_a = -\delta H/\delta r$ equation is best expressed in terms of components

$$\dot{p}_a = -\frac{\delta H}{\delta r}_a = -e\frac{\delta \phi}{\delta r_a} + \frac{e}{cm}\left(p_b - \frac{e}{c}A_b\right)\frac{\delta A_b}{\delta r_a}$$

An Example of the Example

Let's illustrate the dynamics of a particle moving in a magnetic field by looking at a particular case. Imagine a uniform magnetic field pointing in the *z*-direction: $B = (0, 0, B)$. We can get this from a vector potential $B = \nabla \times A$ with

$$A = (-By, 0, 0)$$

This vector potential isn't unique: we could choose others related by a gauge transform as already described. But this one will do for our purposes. Consider a particle moving in the $(x; y)$-plane. Then the Hamiltonian for this system is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2$$

From which we have four, first order differential equations which are Hamilton's equations

$$\dot{P}x = 0$$

$$\dot{x} = \frac{1}{m}\left(Px + \frac{eB}{c}y\right)$$

$$\dot{p}y = -\frac{eB}{mc}\left(Px + \frac{eB}{c}y\right)$$

$$\dot{y} = \frac{Py}{m}$$

If we add these together in the right way, we find that

$$Py + \frac{eB}{c}x = a = const.$$

and

$$Px = m\dot{x} - \frac{eB}{c}y = b = const.$$

which is easy to solve: we have

$$x = \frac{ac}{eB} + R\sin(\omega(t - t_0))$$

$$y = -\frac{bc}{eB} + R\cos(\omega(t - t_0))$$

with a, b,R and t_0 integration constants. So we see that the particle makes circles in the (x, y)-plane with frequency

$$\omega = \frac{eB}{mc}$$

This is known as the cyclotron frequency.

Some Conservation Laws

We have seen the importance of conservation laws in solving a given problem. The conservation laws are often simple to see in the Hamiltonian formalism. For example,

Claim:

If $\delta H/\delta t = 0$ (i.e. H does not depend on time explicitly) then H itself is a constant of motion.

Proof:

$$\begin{aligned} \frac{dH}{dt} &= \frac{\delta H}{\delta q_i}\dot{q}_i + \frac{\delta H}{\delta p_i}p_i + \frac{\delta H}{\delta t} \\ &= -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i + \frac{\delta H}{\delta t} \\ &= \frac{\delta H}{\delta t} \end{aligned}$$

Claim:

If an ignorable coordinate q doesn't appear in the Lagrangian then, by construction, it also doesn't appear in the Hamiltonian. The conjugate momentum p_q is then conserved.

Proof

$$\dot{P}_q = \frac{\delta H}{\delta q} = 0$$

The Principle of Least Action

Recall that in our previous discussion we saw the principle of least action from the Lagrangian perspective. This followed from defining the action

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt$$

Then we could derive Lagrange's equations by insisting that $\delta S = 0$ for all paths with fixed end points so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. How does this work in the Hamiltonian formalism? It's quite simple! We define the action

$$S = \int_{t_1}^{t_2} (p_i\dot{q}_i - H)dt$$

where, of course, $q_i = q_i(q_i, p_i)$. Now we consider varying qi and pi independently. Notice that this is different from the Lagrangian set-up, where a variation of qi automatically leads to a variation of q_i. But remember that the whole point of the Hamiltonian formalism is that we treat q_i and p_i on equal footing. So we vary both. We have

$$\delta S = \int_{t_1}^{t_2} \delta p_i\dot{q}_i + p_i\delta\dot{q}_i - \frac{\delta H}{\delta p_i}\delta p_i - \frac{\delta H}{\delta q_i}dt$$

$$= \int_{t_1}^{t_2} \left(\left[\dot{q}_i - \frac{\delta H}{\delta p_i}\right]\delta p_i + \left[-\dot{p}_i - \frac{\delta H}{\delta q_i}\right]\delta \right) dt + [p_\delta q_i]_{t_1}^{t_2}$$

Except there's a very slight subtlety with the boundary conditions. We need the last term in (12.38) to vanish, and so require only that

$$\delta q_i(t_i) = \delta q_i(t_2) = 0$$

while δp_i can be free at the end points $t = t_1$ and $t = t_2$. So, despite our best efforts, q_i and p_i are not quite symmetric in this formalism. Note that we could simply impose $\delta p_i(t_1) = \delta p_i(t_2) = 0$ if we really wanted to and the above derivation still holds. It would mean we were being more restrictive on the types of paths we considered. But it does have the advantage that it keeps qi and pi on a symmetric footing. It also means that we have the freedom to add a function to consider actions of the form

$$S = \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q,p) + \frac{dF(q,p)}{dt} \right)$$

so that what sits in the integrand differs from the Lagrangian. For some situations this may be useful.

13 Conic Sections

Numerous functions are encountered in the day-to-day life of a scientist. These functions include polynomials,trigonometry, hyperbolic functions amongst others. However throughout history of science one group of functions, the ***conics***, arise time and time again not only in the development of mathematical theory but also in practical application. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC. However, it was not until the 17th century that Sir Isaac Newton found their application to physics. Conic sections have been known since the ancient Greek era.

In mathematics, a *conic section* is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, and is sufficient interest in its own right that it was sometimes called the fourth type of conic section.

The circle

A circle is "the set of all the points in a plane equidistant from a fixed point(circle)".

We can make the obvious cut, or section, perpendicular to the axis of the cone. This gives us a circle.

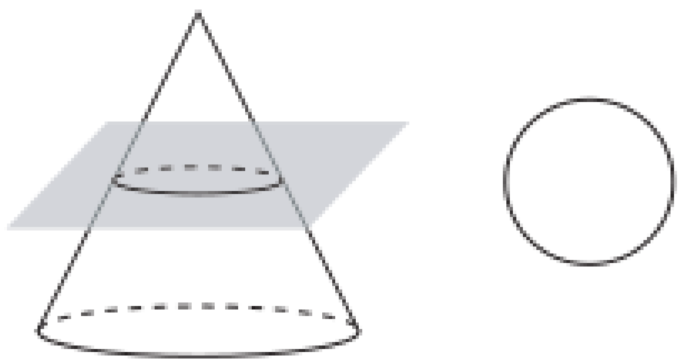


Figure 50: The Circle

Equation of a Circle.

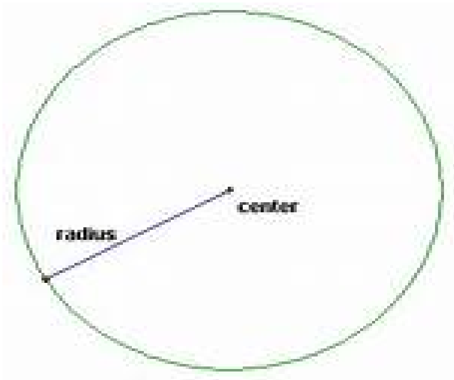


Figure 51

The standard form for a circle with center at the origin is $x^2 + y^2 - r^2$, where r is the radius of the circle. Here the center of the circle is located at the origin $(0, 0)$ and the radius of the circle the center of the circle at a point (α, β) then we use the form:

$$(x - \alpha)^2 + (y - \beta)^2 = r^2$$

The Ellipse

An ellipse informally is an oval or a squished circle. An ellipse is obtained by the plane which is not perpendicular to the core-axis, but cutting the cone in a closed curve. Various ellipses are obtained as the plane continues to rotate.

We can make the cut at an angle to the axis of the cone, so that we still get a closed curve which is no longer a circle. This curve is an ellipse.

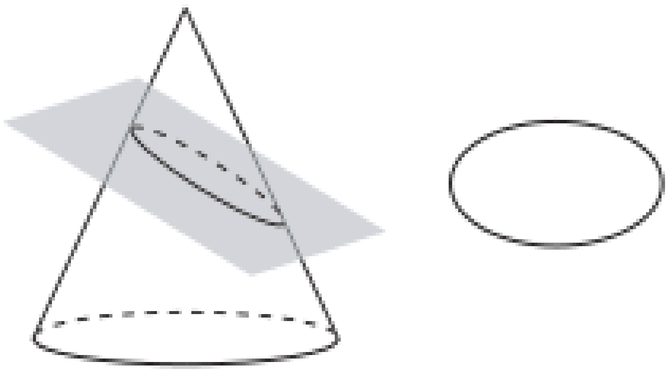


Figure 52: An Ellipse.

Equation of an Ellipse.

The sum of the distances to any point on the ellipse (x, y) from the two foci $(c, 0)$ and $(-c, 0)$ is a constant That will be 2a. If we let d_1 and d_2 bet the distances from the foci to the point then d_1 + $d_2 = 2a$. You can use that definition to derive the equation of an ellipse, but I will give you the short form below. The ellipse is a stretched circle. Begin with the unit circle (circle with a radius of 1) centered at the origin. Stretch the vertex from $x = 1$ to $x = a$ and the point $y = 1$ to $y = b$. What you have done is multiplied every x by a and multiplied every y by b. In translation form, you represent that by X divided by a and y divided by b. So, the equation of the circle changes from $x^2 + y^2 = 1$ to $(\frac{x}{a})^2$ + $(\frac{y}{b})^2 = 1$ and the standard equation for an ellipse centered at the origin.

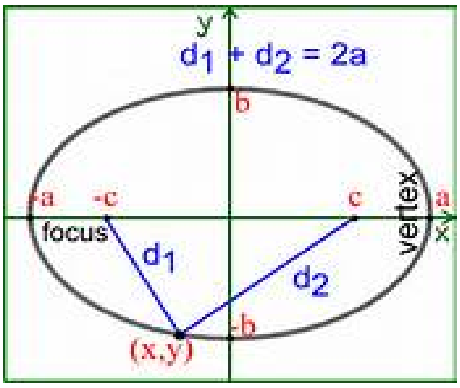


Figure 53

- The center is the starting point at (h, k).
- The major axis contains the foci and the vertices.
- Major axis length = $2a$. This is also the constant the sum of the distances added to be.
- Minor axis length = $2b$.
- Distance between foci = $2c$.
- The foci are within the curve.
- Since the vertices are the farthest away from the center, a is the largest of the three lengths, and the Pythagorean relationship is $a^2 = b^2 + c^2$.

The Parabola

A parabola is "the set of all points in a plane equidistant from a fixed point (focus) and a fixed line (directrix)".

If we make a cut parallel to the generator of the cone, we obtain an open curve. This is a parabola.

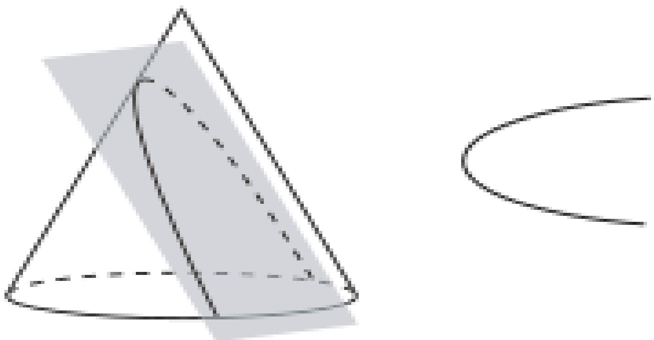


Figure 54: A Parabola.

Equation of a Parabola.

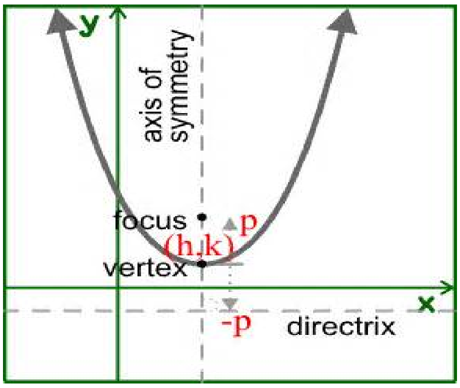


Figure 55

The distances to any point (x, y) on the parabola from the focus $(0, p)$ and the directrix $y = -p$, are equal to each other. This can be used to develop the equation of a parabola. If you take the definition of a parabola and work out the algebra, you can develop the equation of a parabola. This is a short but standard version on how to develop that standard form of $x^2 = 4py$.

- The standing point is the vertex at (h, k)
- There is an axis of symmetry that contains the focus and thr vertex and is perpendicular to the directrix.
- Move p units along the axis of symmetry from the vertex to the focus.
- Move $-p$ units along the axis of symmetry from the vertex to the directrix (which is a line)
- The focus is within the curve.

The parabola has the property that any signal (light, sound, etc) entering the parabola parallel to the axis of symmetry will be reflected through the focus (this is why satellite dishes and those parabolic antennas that the detectives use to eavesdrop on conversations work). Also, any signal originating at the focus will be reflected out parallel to the axis of symmetry (this is why flashlights work).

The Hyperbola

A hyperbola is "the set of all points in a plane such that the difference of the distances from two fixed points (foci) is constant ".

Finally, we make the cut at an even steeper angle. If we imagine that we have a double cone, that is, two cones vertex to vertex, then we obtain the two branches of a hyperbola.

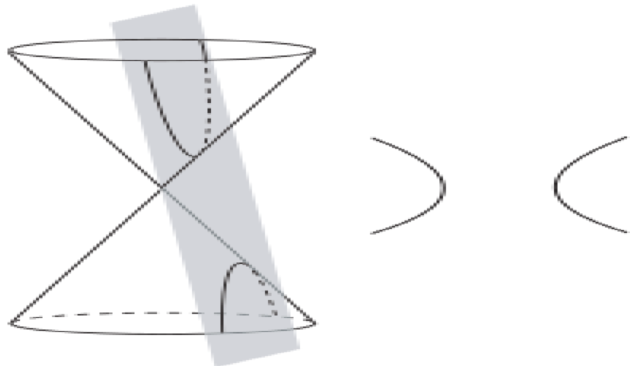


Figure 56: A Hyperbola.

Equation of a Hyperbola.

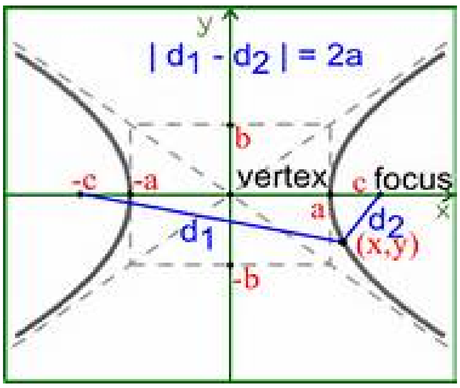


Figure 57

The difference of the distances to any point on the hyperbola (x, y) from the two foci $(c, 0)$ and $(-c, 0)$ is a constant. That constant will be $2a$. If we let d_1 and d_2 be the distances from the foci to the point, them $|d_1 - d_2| = 2a$. The absolute value is around the difference so that it is always positive. You can use the definition to derive the equation of a hyperbola, but below is a short but standard way of deriving it. The only difference in the definition of a hyperbola and that of an ellipse is that the hyperbola is the difference of the distances from the foci that is constant and the ellipse is the sum of the distances from the foci that is constant. Instead of the equation being $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, the equation is $(\frac{x}{a})^2 - (\frac{y}{b}^2) = 1$ The graphs, however , are very different.

- The center is starting point at (h, k).
- The Transverse axis contains the foci and the vertices.
- Transverse axis length = $2a$. This is also the constant that the difference od the distances must be.

- Conjugate axis length = $2b$
- Distance between foci = $2c$.
- the foci within the curve.
- Since the foci are the farthest away from the center, c is the largest of the three lengths, and the Pythagorean relationship is $a^2 + b^2 = c^2$.

General Conics

The conics we have considered above-the ellipse, the parabola and the hyperbol- have all been presented in standard form; their axes are parallel to either the *x* or *y*-axes. However, conics m ay be rotated to any angle with respect to the axes; they clearl remain conics, but what equations do they have? It can be shown that the equation of any conic, can be described by the quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are constants. If not all of A, B, C are zero the graph of this equation is

- an ellipse if $B^2 < 4AC(circle \quad if \quad A = C)$
- a parabola if $B^2 = 4AC$
- a hyperbola if $B^2 > 4AC$

Newton and the Conics

These amongst a few are some of the things that Newton accomplished. He mathematically showed that the orbit of any planet around the Sun should be a conic section with one of its foci located at the Sun, that is if the force of gravity acting between then is inversely proportional to the square of their distances apart. This also fit into Kepler's first law which shows that the orbit of any planet is an ellipse with the sun at one foci. With more observations and experiment ranging from the falling of apples to the moon orbiting around the earth, he asserted that every massive object attracts another with the amount of force being inversely proportional to their distance squared. Another of these real life conics is that the the trajectory of a planet is perfectly elliptical when only the gravitation due to the sun is considered. This however, is not very true since the trajectory deviates slightly away from a perfect ellipse due to the presence of gravitational attraction of other planets; the line connecting the two foci of a planet rotates very slowly rather than being fixed. In the case of Mercury, it rotates by 0.160 degrees per century. However, if you calculate the gravitational effect of other planets, it should rotate by only 0.148 degrees. In 1915, Einstein successfully accounted for this difference by discovering theory of general relativity., In other words, he discovered that the gravitational force is not exactly, but only approximately, inversely proportional to the square of the distance. When a ball is bouncing off, the maths of conics can be used to map its trajectory. A satellite is also designed to be a parabola. If they are designed otherwise, they wont function properly. A signal sent to it will be lost of an odea of where the focus would be is not considered. The solar cook is also not left out. It is designed in a form of parabola in other to receive maximum solar energy for efficiency. The touch light is designed the same way in other for the emerging beam to shine on a wider area. Telescopes also apply the idea of conics.