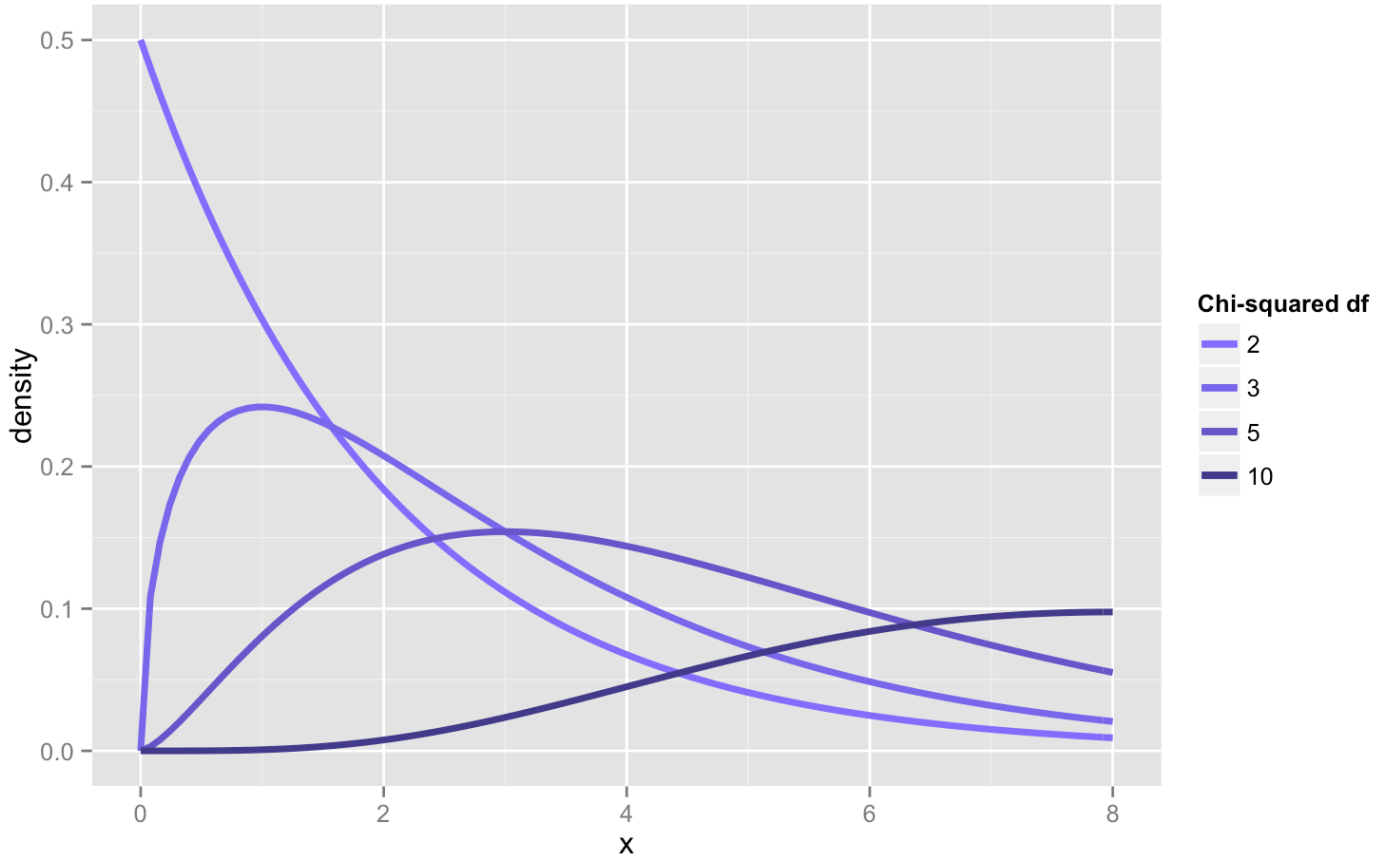
# Class 11:
# SLR outliers and diagnostics

Alison Presmanes Hill

# 3 new distributions…

- If you see squares of normal random variables, think chi-squared.
- If you see the ratio of an independent normal random variable to the square-root of a chi-squared, think Student's t.
- If you see the ratio of two chi-squared random variables (i.e., sample variances from a normal distribution), think F distribution.

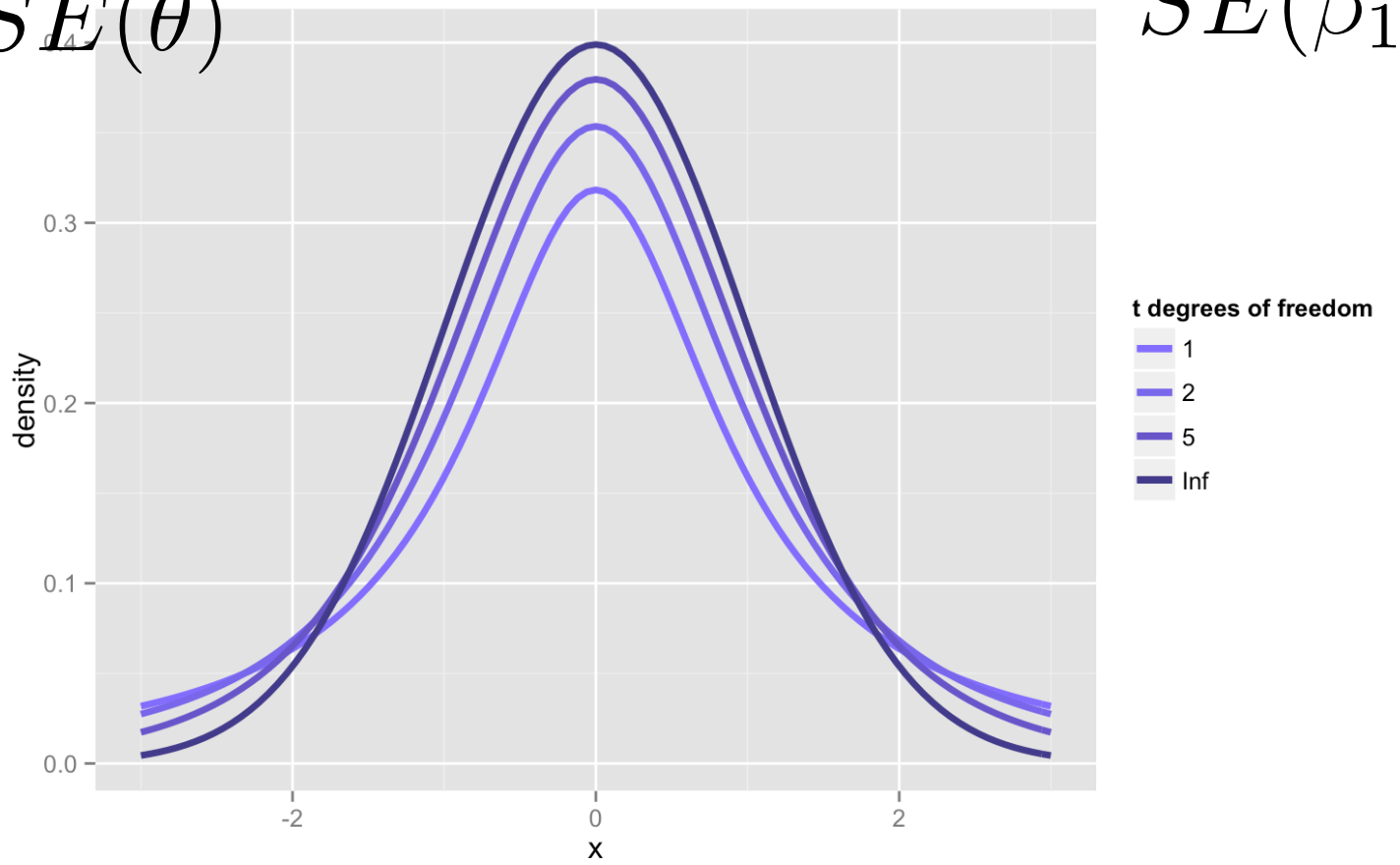
# If you see squares of normal random variables, think chi-squared



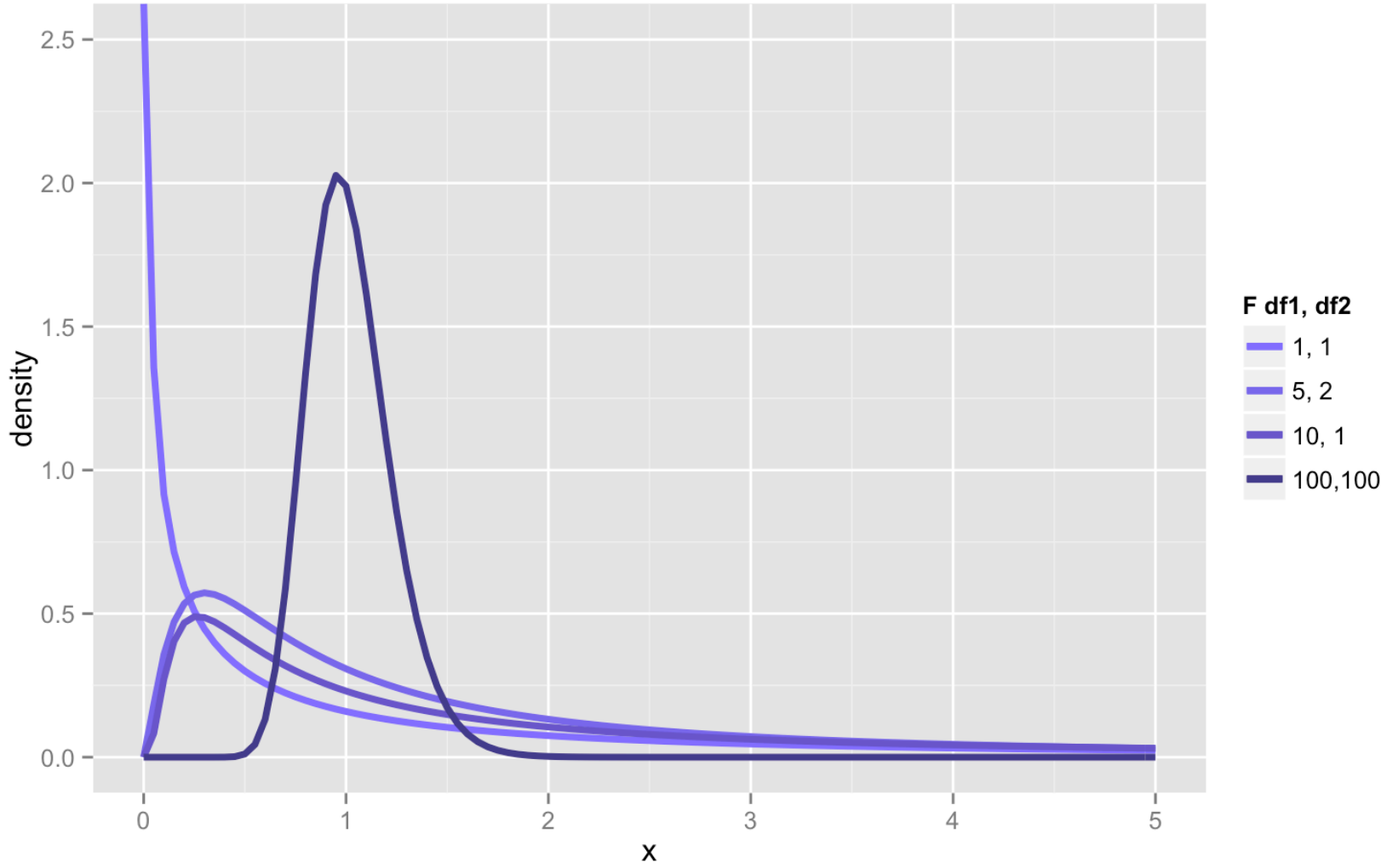
If you see the ratio of an independent normal random variable to the square-root of a chi-squared, think Student's t

$$T = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})}$$

$$\frac{\hat{\beta}_1 - \beta_1}{SE(\hat{\beta}_1)}$$



If you see the ratio of two chi-squared random variables (i.e., sample variances from a normal distribution), think F distribution



# Swiss Fertility and Socioeconomic Indicators (1888)

**Details** (paraphrasing Mosteller and Tukey):

- Switzerland, in 1888, was entering a period known as the *demographic transition*; i.e., its fertility was beginning to fall from the high level typical of underdeveloped countries.
- The data collected are for 47 French-speaking "provinces" at about 1888.
- Here, all variables are scaled to *[0, 100]*.

# 6 swiss variables

- Fertility: (common standardized measure)
- Agriculture: % of males involved in agriculture as an occupation
- Examination: % draftees received highest mark on army exam
- Education: % draftees educated beyond primary school
- Catholic: % Catholic (as opposed to Protestant)
- InfantMortality: Percent of live births who live less than one year

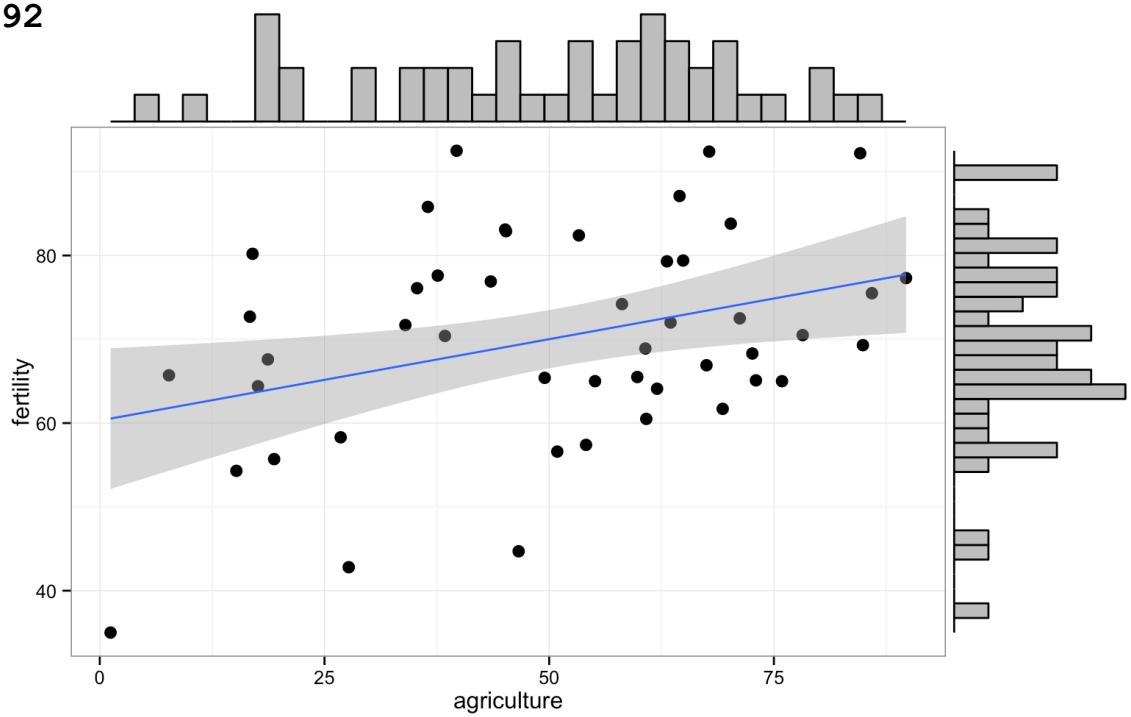
```
> summary(swiss)
 fertility      agriculture     examination
 Min.   :35.00   Min.   : 1.20   Min.   : 3.00
 1st Qu.:64.70   1st Qu.:35.90   1st Qu.:12.00
 Median :70.40   Median :54.10   Median :16.00
 Mean   :70.14   Mean   :50.66   Mean   :16.49
 3rd Qu.:78.45   3rd Qu.:67.65   3rd Qu.:22.00
 Max.   :92.50   Max.   :89.70   Max.   :37.00

  education       catholic       infant.mortality
 Min.   : 1.00   Min.   :  2.150   Min.   :10.80
 1st Qu.: 6.00   1st Qu.:  5.195   1st Qu.:18.15
 Median : 8.00   Median : 15.140   Median :20.00
 Mean   :10.98   Mean   : 41.144   Mean   :19.94
 3rd Qu.:12.00   3rd Qu.: 93.125   3rd Qu.:21.70
 Max.   :53.00   Max.   :100.000   Max.   :26.60
```

```
> with(swiss, cor.test(fertility, agriculture))

        Pearson's product-moment correlation

data:  fertility and agriculture
t = 2.5316, df = 45, p-value = 0.01492
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
 0.07334947 0.58130587
sample estimates:
      cor
0.3530792
```




```
> m_agr <- lm(fertility ~ agriculture, data = swiss) # simple linear regression
> summary(m_agr)

Call:
lm(formula = fertility ~ agriculture, data = swiss)

Residuals:
     Min       1Q   Median       3Q      Max
-25.5374  -7.8685  -0.6362   9.0464  24.4858

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 60.30438    4.25126  14.185   <2e-16 ***
agriculture  0.19420    0.07671   2.532   0.0149 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 11.82 on 45 degrees of freedom
Multiple R-squared:  0.1247,      Adjusted R-squared:  0.1052
F-statistic: 6.409 on 1 and 45 DF,  p-value: 0.01492
```

```
> m_mean <- lm(fertility ~ 1, data = swiss) # intercept ONLY
> m_agr <- lm(fertility ~ agriculture, data = swiss)
> anova(m_mean, m_agr)
Analysis of Variance Table

Model 1: fertility ~ 1
Model 2: fertility ~ agriculture
  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1     46 7178.0
2     45 6283.1  1    894.84 6.4089 0.01492 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
> glance(m_agr)
  r.squared adj.r.squared    sigma statistic   p.value df    logLik
1 0.1246649      0.105213 11.81629  6.408884 0.0149172  2 -181.7337

> m_agr <- lm(fertility ~ agriculture, data = swiss) # simple linear regression
> summary(m_agr)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 60.30438    4.25126  14.185   <2e-16 ***
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```
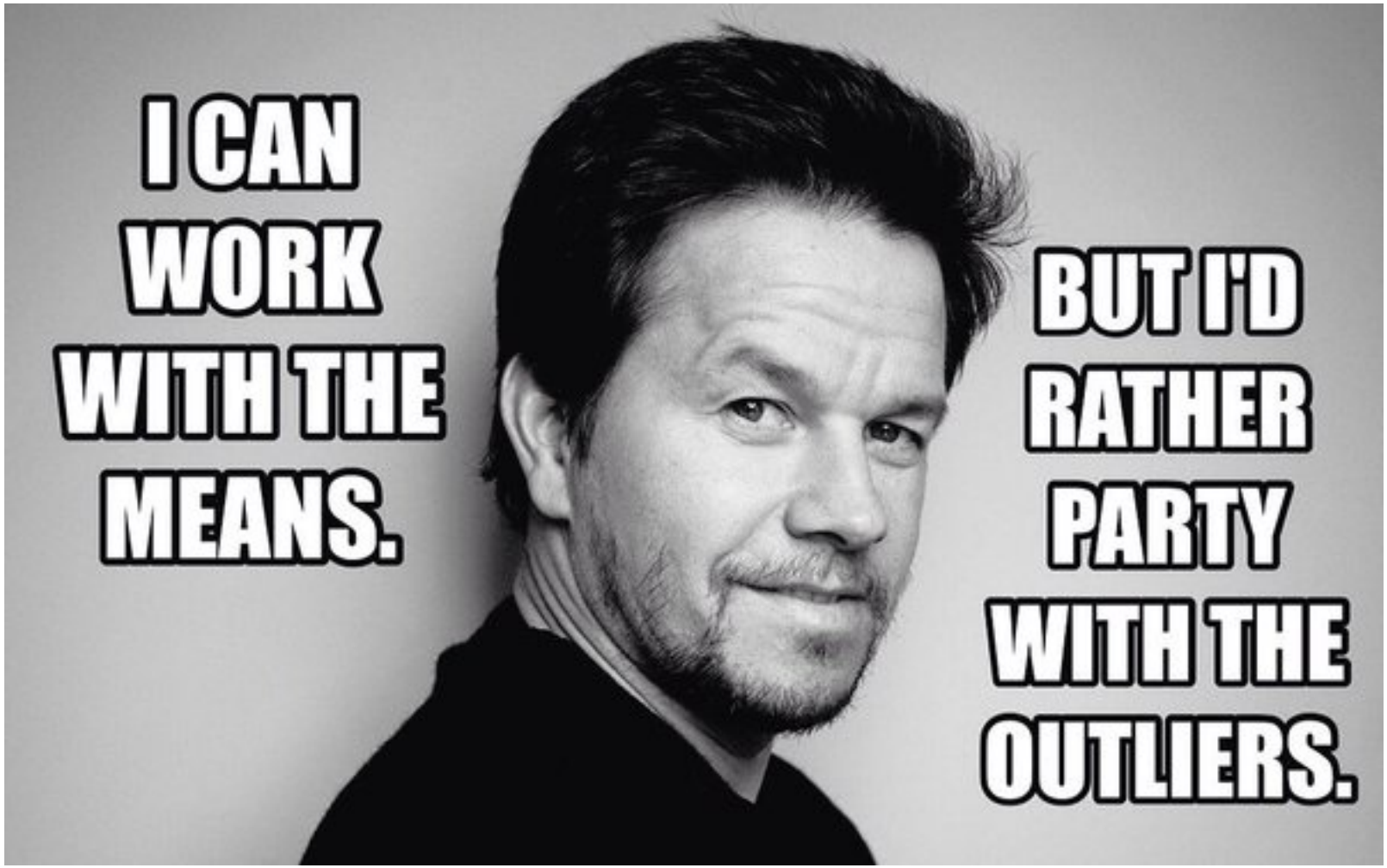

# Sequences of nested models (revisited)

- Full(er) model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

- (Default) reduced model: $Y_i = \beta_0 + \varepsilon_i$


I CAN WORK WITH THE MEANS.
BUT I'D RATHER PARTY WITH THE OUTLIERS.

# Types of influence

- Extremity on the x's (leverage)
- Extremity on y (discrepancy)
- Influence on the regression estimates
  - Global
  - Specific coefficients

```
m_agr <- lm(fertility ~ agriculture, data = swiss) # original model
slr_vars <- augment(m_agr) # broom
```

# Contaminated observation or special snowflake?

- A *contaminated observation* is one that has been damaged in some way. Some examples:
  - Error of execution of the research procedure.
  - Inaccurate measurement of the dependent measure.
  - Data entry error.
  - Error in calculating a measure.
  - Nonattentive or distracted participants.
- The outlier may simply be an *extremely rare case*. For example, a college freshman might be 12 years old and have an 800 SAT in math. Such an individual is extremely rare, but data is valid.


SUP BRO! YEAH I SAW YOU REMOVE MY OUTLIERS WITHOUT JUSTIFICATION.
HILARIOUS.

# Extremity on the x's: leverage

- Standardized measure of how far the observed value for each observation is from the mean value on the set of x's
- Observations with high leverage have the potential to be influential, especially if also extreme on Y

# Leverage

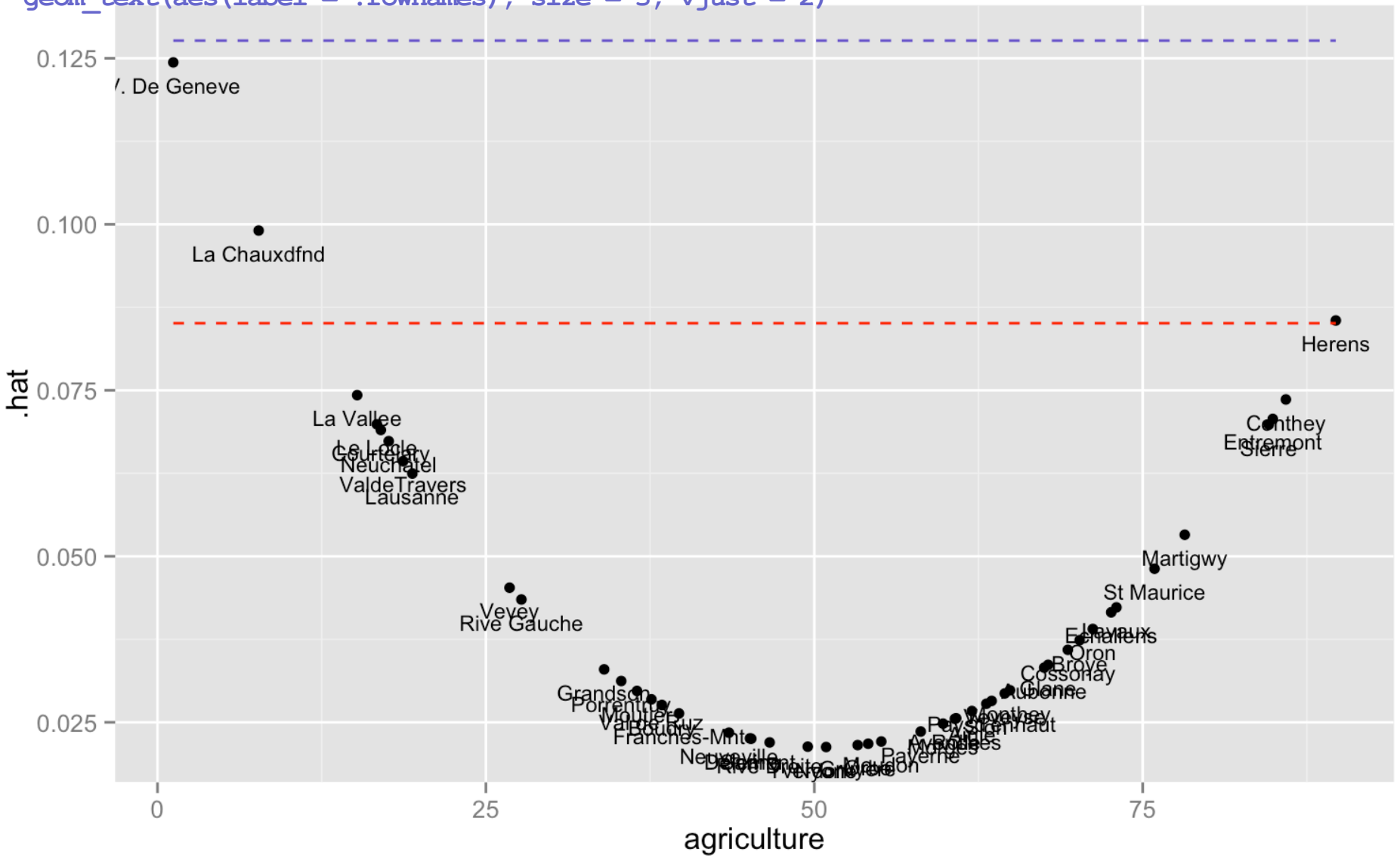
- Measure of how unusual the X value of a point is, relative to the X observations as a whole. Absolute minimum of 1/n. Leverage describes how unusual an observation is in predictor(s) data.
- If $h_{ii}$ is large then the ith observation has considerable impact on the fitted value
- **`broom::.hat`**

$$h_{ii} = \frac{1}{n} + \frac{(x_i - \overline{x})^2}{\sum x^2}$$

# Leverage

```
slr_vars %>%
   filter(.hat > (2*mean(.hat))) %>%
   select(.rownames, fertility, agriculture, .hat)
     .rownames fertility agriculture       .hat
1       Herens      77.3        89.7 0.08551436
2 La Chauxdfnd      65.7         7.7 0.09905898
3 V. De Geneve      35.0         1.2 0.12437740

## small samples
slr_vars %>%
   filter(.hat > (3*mean(.hat))) %>%
   select(.rownames, fertility, agriculture, .hat)
[1] .rownames   fertility   agriculture .hat
<0 rows> (or 0-length row.names)
```

```
## plot leverage versus xs
ggplot(slr_vars, aes(x = agriculture, y = .hat)) +
  geom_point() +
  geom_line(aes(y = 2*mean(.hat)), colour = "red", lty = "dashed") +
  geom_line(aes(y = 3*mean(.hat)), colour = "slateblue", lty = "dashed") +
  geom_text(aes(label = .rownames), size = 3, vjust = 2)
```



# Just the x's

```
slr_vars %>%
  select(.rownames, fertility, agriculture, .hat) %>%
  mutate(mean_dev = agriculture - mean(agriculture)) %>%
  arrange(mean_dev)
```

| | .rownames | fertility | agriculture | .hat | mean_dev |
|---|---|---|---|---|---|
| 1 | V. De Geneve | 35.0 | 1.2 | 0.12437740 | -49.4595745 |
| 2 | La Chauxdfnd | 65.7 | 7.7 | 0.09905898 | -42.9595745 |
| 3 | La Vallee | 54.3 | 15.2 | 0.07427080 | -35.4595745 |
| … | | | | | |
| 45 | Entremont | 69.3 | 84.9 | 0.07068941 | 34.2404255 |
| 46 | Conthey | 75.5 | 85.9 | 0.07361779 | 35.2404255 |
| 47 | Herens | 77.3 | 89.7 | 0.08551436 | 39.0404255 |

# Extremity on the y's: discrepancy

- The discrepancy (or distance$^2$) between each predicted and observed value of $y_i$
- A studentized residual is an observed residual divided by its standard error; two types:
  - Internally studentized (**`rstandard`**): re-normalize the residuals to have unit variance, using a measure of the error variance.
    - Also **`broom::.std.resid`**
  - Externally studentized (**`rstudent`**): re-normalize the residuals to have unit variance, using a leave-one-out measure of the error variance. This is a measure of the size of the residual, standardized by the estimated standard deviation of residuals based on all the data but that observation. Sometimes called jackknifed residuals.

# Internally studentized residuals

- **`rstandard`**
- Also **`broom::augment(.std.resid)`**

$$e'_i = \frac{e_i}{SE_e\sqrt{1-h_i}}$$

# Externally studentized residuals

- **`rstudent`**
- Sadly, not available in **`broom::augment`**

$$e_i^* = \frac{e_i}{SE_{e-i}\sqrt{1-h_i}}$$

# Save externally studentized residuals (ESR)

```
m_agr <- lm(fertility ~ agriculture, data = swiss) # original model
slr_vars <- augment(m1) # broom
slr_vars <- slr_vars %>%
  mutate(.extsr = rstudent(m1)) # add ESR
```

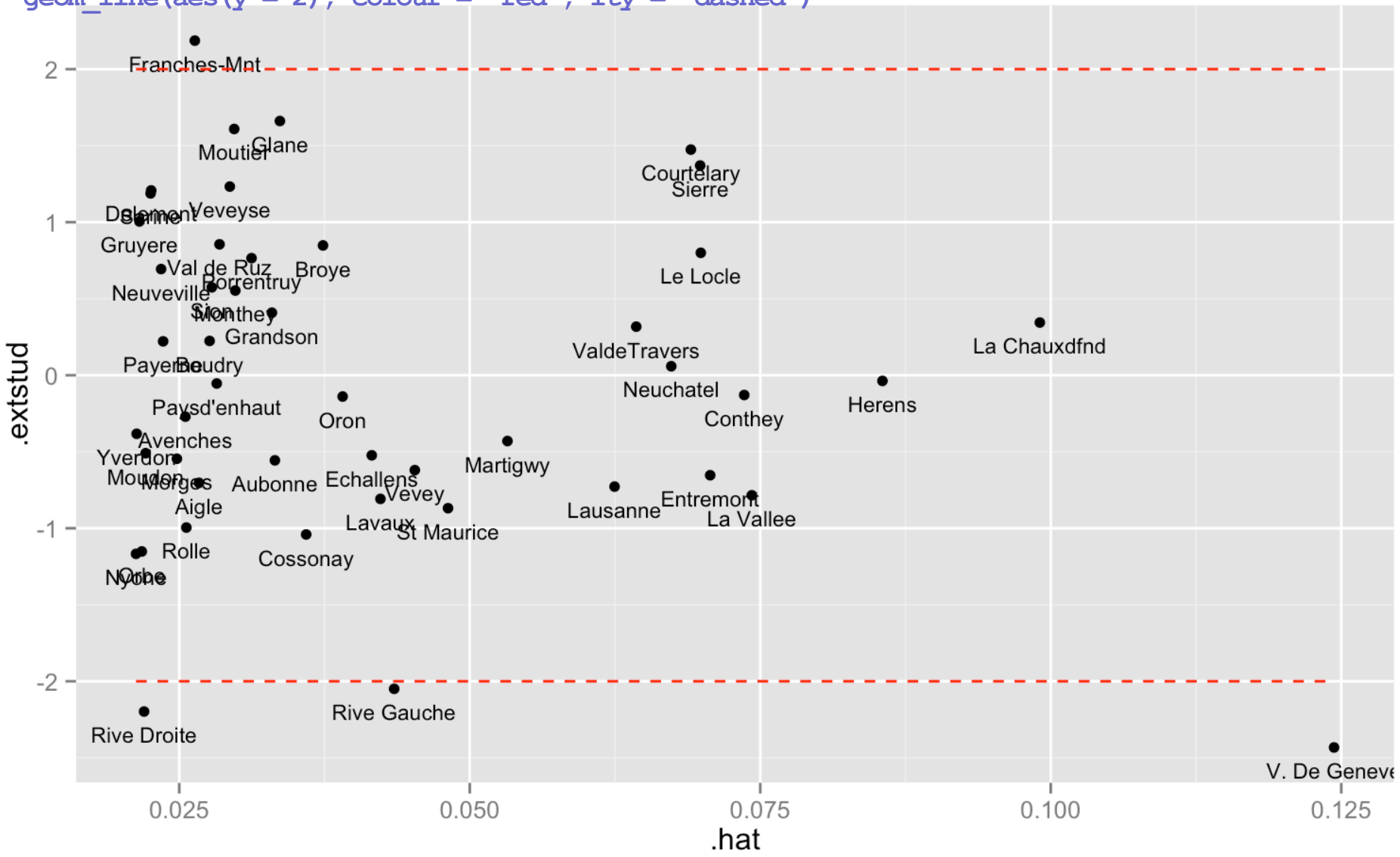
```
> slr_vars %>%
   select(.std.resid, .extsr) %>%
   head()
  .std.resid    .extsr
1  1.4554806 1.4743391
2  1.2015896 1.2076959
3  2.1000488 2.1864890
4  1.5814696 1.6091557
5  0.6977688 0.6937354
6  0.7687025 0.7651537
```

# Expect 5% with `abs(.extsr)` ≥ 2

```
> slr_vars %>%
    filter(abs(.extsr) >= 2) %>%
    select(.rownames, fertility, agriculture, .resid, .std.resid, .extsr)
    .rownames fertility agriculture    .resid .std.resid    .extsr
1 Franches-Mnt      92.5        39.7  24.48582   2.100049  2.186489
2 V. De Geneve      35.0         1.2 -25.53742  -2.309602 -2.432516
3  Rive Droite      44.7        46.6 -24.65418  -2.109762 -2.197709
4  Rive Gauche      42.8        27.7 -22.88376  -1.980169 -2.049364
```

4 out of 47:
8.5% of observations with ESR considered to be relatively large
(expected ≈ 2 or 3 observations)

```
## plot leverage versus ESR
ggplot(slr_vars, aes(x = .hat, y = .extsr)) +
  geom_point() +
  geom_text(aes(label = .rownames), size = 3, vjust = 2) +
  geom_line(aes(y = -2), colour = "red", lty = "dashed") +
  geom_line(aes(y = 2), colour = "red", lty = "dashed")
```



# car::outlierTest()

```
> outlierTest(m1) # library(car)

No Studentized residuals with Bonferonni p < 0.05
Largest |rstudent|:
                rstudent unadjusted p-value Bonferonni p
V. De Geneve -2.432516             0.01913       0.89909
```

ESR follows $t_{n-k-2}$ distribution (k = number of predictors, n = number of observations), $t_{44}$

Alpha = .025/n (two-tailed)

```
> qt(1-.025/47, 44) # 1 - (alpha/2)/n, df = n-k-2
[1] 3.504708
> pt(-2.432516, 44)*2
[1] 0.01912966
```

# Bonferroni adjustment

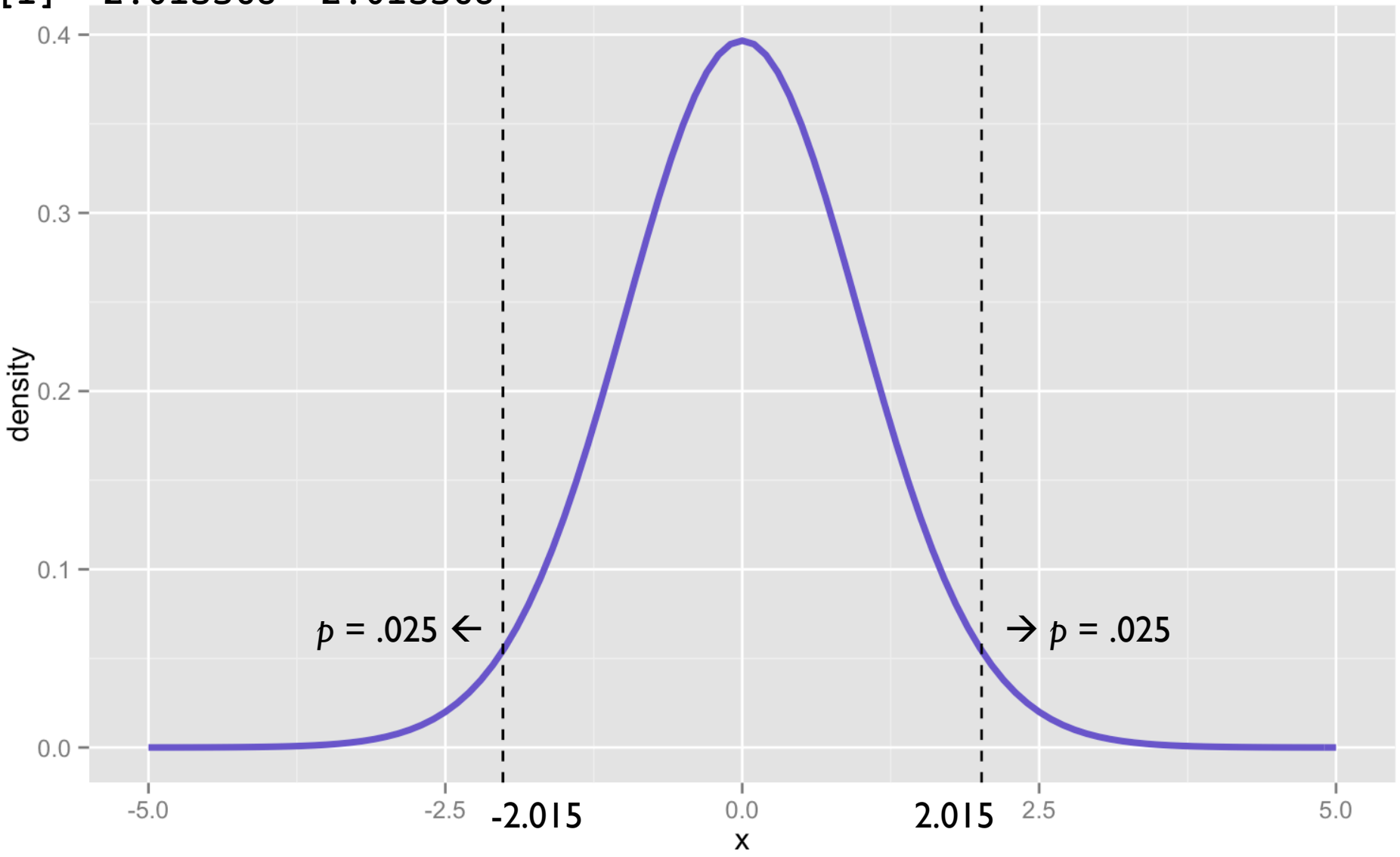
- Problem: if n is large, and we "threshold" at $t_{n-k-2}$ we will get many outliers by chance alone, even if our model is correct
- Solution: adjust the "threshold" to reflect reality- we are screening all observed ESR values and looking for the max, so we should account for multiple testing somehow
- Recall: ESRs are t-distributed variables
- For nominal $\alpha = .05$, the nominal critical t-value value is $t_{n-k-2}$ where k = number of regression coefficients (including intercept), n = number of observations
- Instead, we would set $\alpha = .05/n$ to find ESRs that are in fact outliers

# Bonferroni adjustment

- If I only wanted to know whether one datapoint was more extreme at $\alpha = .05$ (two-tailed), we compare to critical $t_{44}$ =

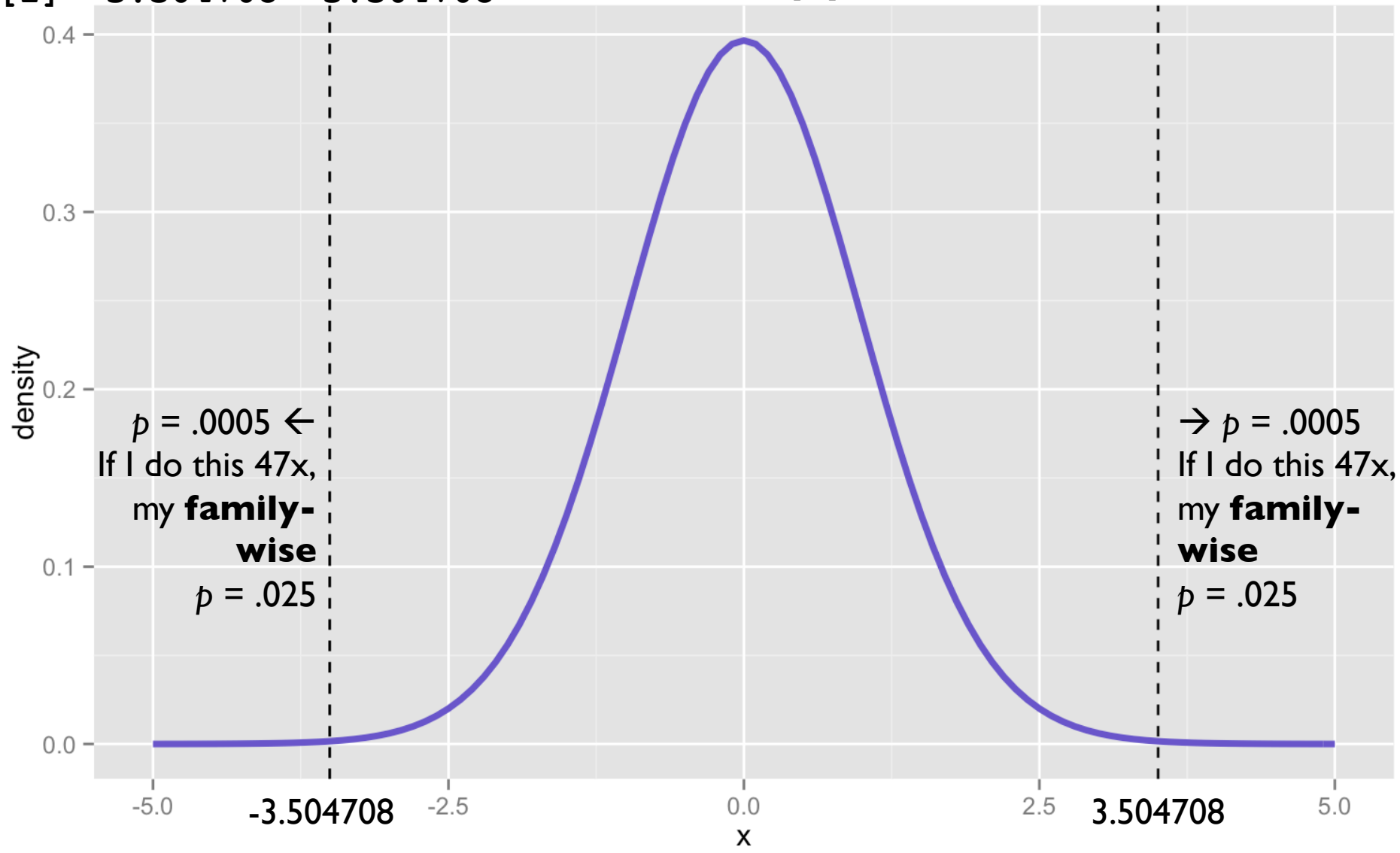
```
> lowert <- qt(.025, 44)
> uppert <- qt(1 - .025, 44)
> c(lowert, uppert)
[1] -2.015368  2.015368
```

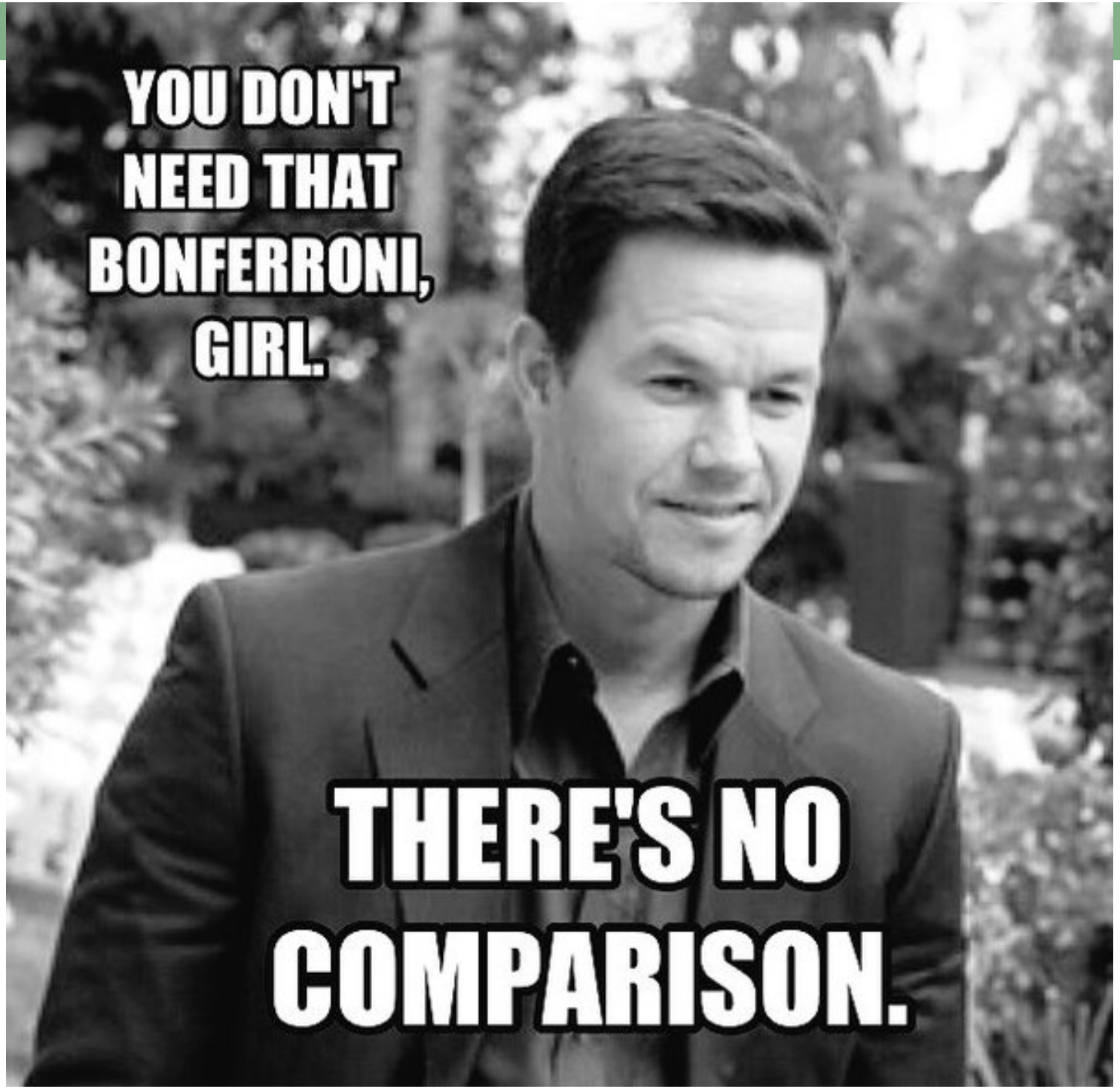
```
> lowert <- qt(.025, 44)
> uppert <- qt(1 - .025, 44)
> c(lowert, uppert)
[1] -2.015368  2.015368
```



```
> lowertstar <- qt(.025/47, 44)
> uppertstar <- qt(1-.025/47, 44)
> c(lowertstar, uppertstar)
[1] -3.504708  3.504708
```

```
> pstar <- 1 - pt(3.504708, 44)
> pstar_fam <- pstar*47*2
> c(pstar, pstar_fam)
[1] 0.0005319156 0.0500000646
```




YOU DON'T NEED THAT BONFERRONI, GIRL.
THERE'S NO COMPARISON.

# Influence on regression estimates

- Global: DFFITS and Cook's D provide information about how each case affects overall regression equation
  - Both are deletion statistics (like the externally studentized residuals)
  - Answer similar questions, but scale differently
- Specific: DFBETAS provide information about how each case affects each individual coefficient estimate (i.e., each beta)
  - Will return to this in multiple regression

# DFFITS

- Compares predicted values of $Y_i$ when (1) observation is included and (2) when observation is excluded
- Number of standard deviations by which yhat would change for each case if it were deleted
- Thus: "difference in fit, standardized"
- Higher absolute values mean greater influence; 0 means no influence
- But: extreme outliers could also change the regression equation, and hence the predicted values, for other observations too…

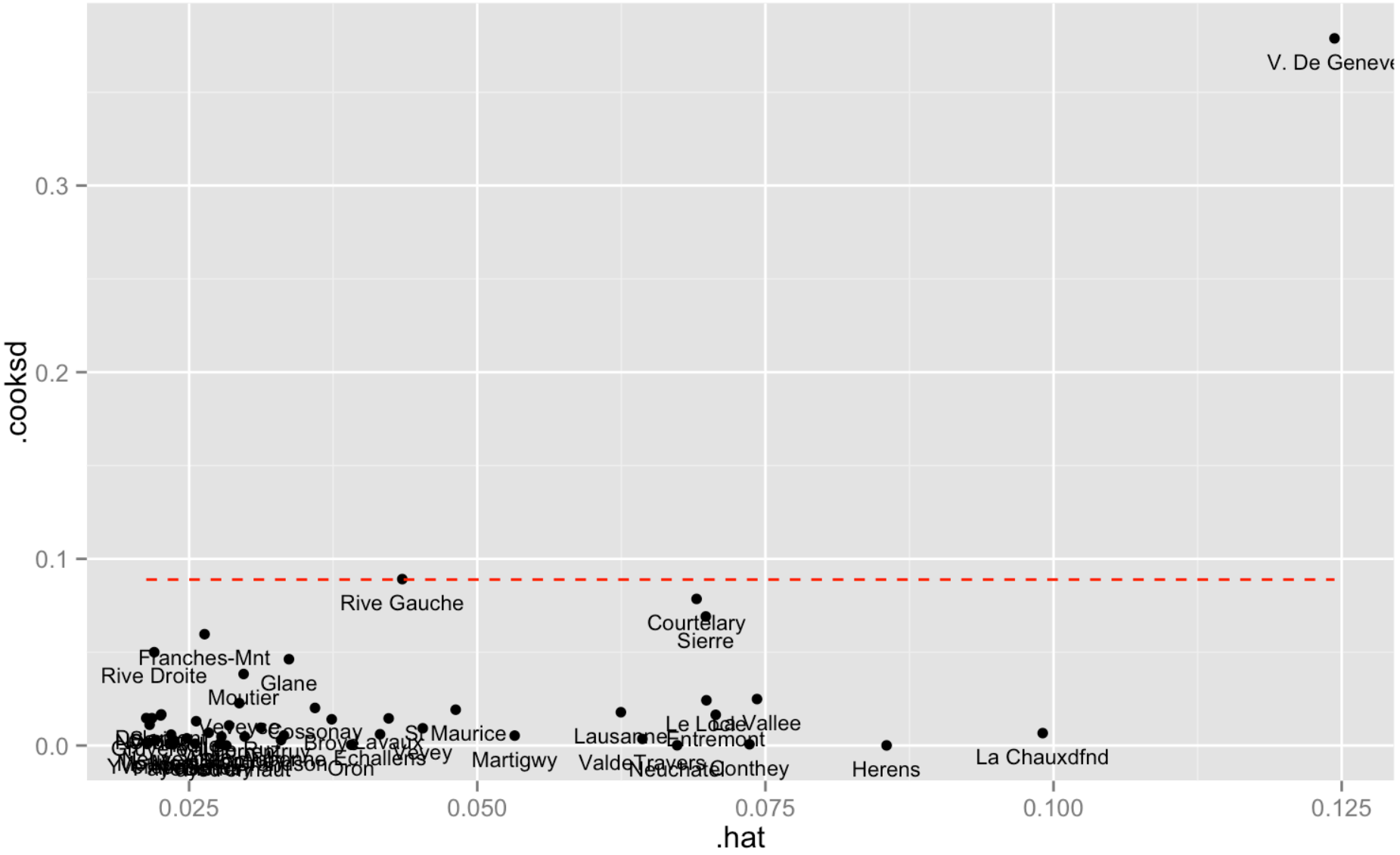
# Cook's distance (Cook's D)

- Influence on regression line, measured by how much the regression line would change if the point were not included in the analysis.
- COOK's distance measures the influence of case i *on all n fitted values* $y_i$ (not just the fitted value for case i as DFFITS)
- Cook's distance refers to how far, on average, *predicted y-values* will move if the observation in question is dropped from the data set
- With > 1 predictor, Cook's distance is presumably more important to you if you are doing predictive modeling, whereas dfbeta is more important in explanatory modeling.

# Cook's distance

```
n <- nrow(slr_vars)
k <- 1 # predictors (not including intercept), number regressors
d <- 4/(n - k - 1)
d
[1] 0.08888889
slr_vars %>%
  filter(.cooksd > d)
     .rownames fertility agriculture  .fitted  .se.fit    .resid       .hat   .sigma    .cooksd .std.resid
1 V. De Geneve      35.0         1.2 60.53742 4.167274 -25.53742 0.12437740 11.21922 0.37885147  -2.309602
2  Rive Gauche      42.8        27.7 65.68376 2.464307 -22.88376 0.04349378 11.41733 0.08914847  -1.980169

## plot leverage versus cooks d
ggplot(slr_vars, aes(x = .hat, y = .cooksd)) +
  geom_point() +
  geom_text(aes(label = .rownames), size = 3, vjust = 2) +
  geom_line(aes(y = d), colour = "red", lty = "dashed")
```

# Cook's D

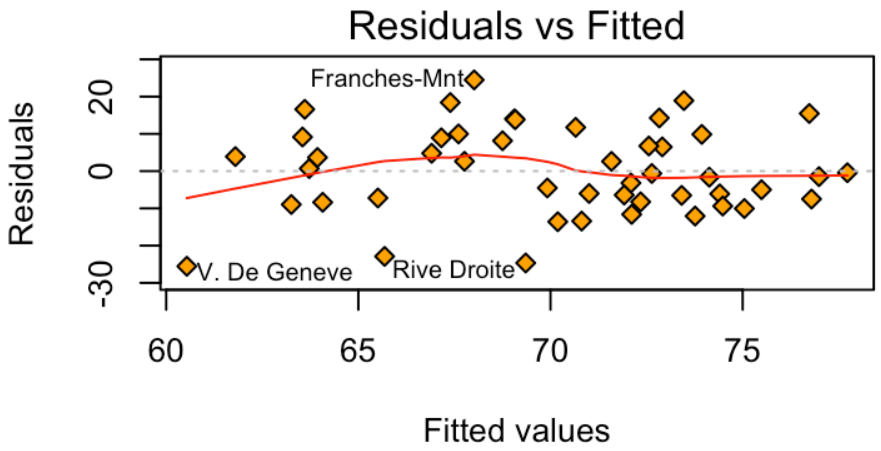

V. De Geneve
0.3
0.2
0.1
0.0
.cooksd
Rive Gauche
Courtelary
Sierre
Franches-Mnt
Rive Droite
Glane
Moutier
Le Locle
Vallee
St Maurice
Lausanne
Entremont
Martigwy
Valde
Travers
Monthey
Herens
La Chauxdfnd
Oron
0.025
0.050
0.075
0.100
0.125
.hat

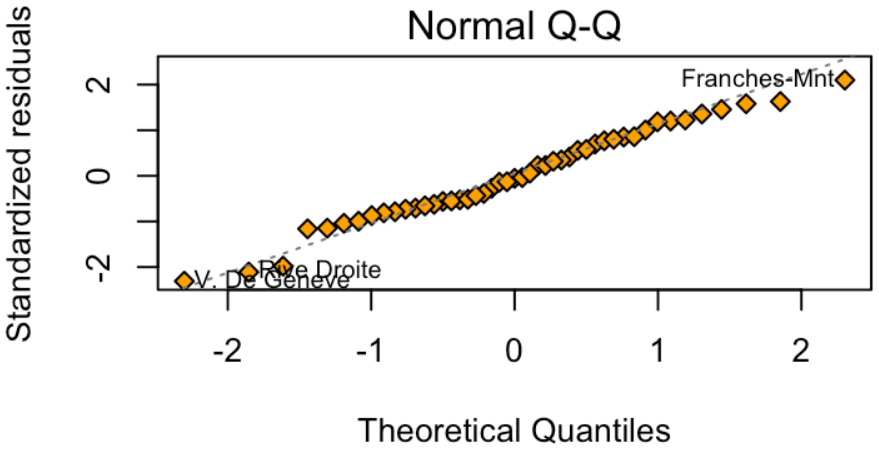
# Specific measures of influence

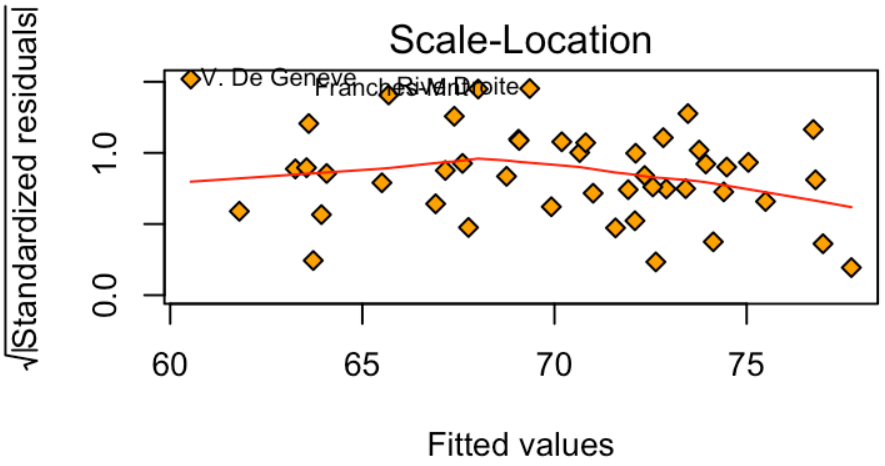
- How each individual regression coefficient is changed by deleting each case from the dataset
- Number of DFBETAS is the number of predictors in your model
- Also for the intercept- generally not interesting one bit!
- Recommendation: I would look at DFBETAS after Cook's to isolate which x('s) might be unduly influencing your overall regression equation
- How to: when we do multiple regression…

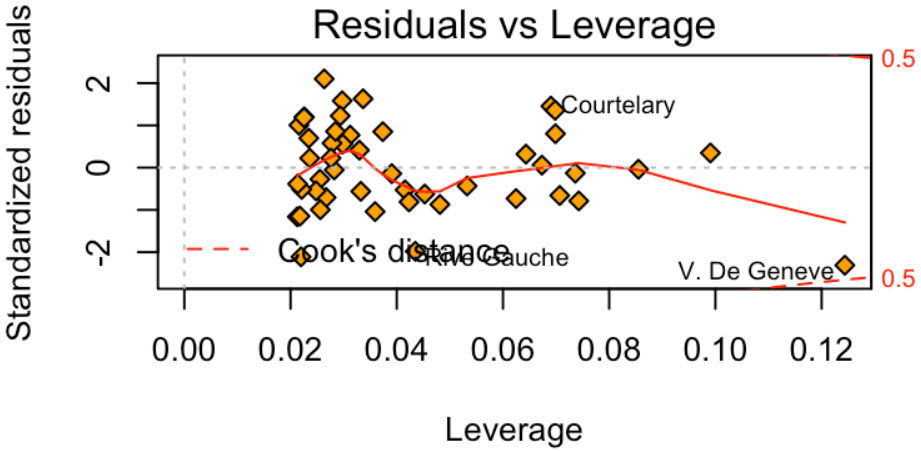
| Influence | Recommended index | R | broom::augment? |
|---|---|---|---|
| Leverage | Hat values | `hatvalues(model)` | `.hat` |
| Discrepancy | Externally studentized residual | `rstudent(model)` | `Sadly no` |
| Global influence | Cook's d | `cooks.distance(model)` | `.cooksd` |
| Specific influence | DFBETAS | `dfbetas(model)` | `Sadly no` |

```
m_agr <- lm(fertility ~ agriculture, data = swiss)
plot(m_agr, bg = "orange", pch = 23)
```



```
m_agr <- lm(fertility ~ agriculture, data = swiss)
library(ggfortify)
autoplot(m_agr) # from ggfortify
```