Energy flow in an electromagnetic plane wave

Take our previous example (monochromatic, s.f.v):

$$\mathbf{E} = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z - \omega t)} \qquad\qquad \mathbf{H} = \hat{\mathbf{j}}\, H_0 \mathrm{e}^{i(k_z z - \omega t)}$$

The wave impedance is $Z_0 = \dfrac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$. Therefore $\sqrt{\varepsilon_0}\,|\mathbf{E}| = \sqrt{\mu_0}\,|\mathbf{H}|$ and, squaring, $\frac{1}{2}\varepsilon_0 |\mathbf{E}|^2 = \frac{1}{2}\mu_0 |\mathbf{H}|^2$.

i.e. the electric and magnetic energy densities are EQUAL in a vacuum plane wave (VPW).

So the total energy density in a VPW is twice that in (say) the electric field: $U_{VPW} = \varepsilon_0 |\mathbf{E}|^2$.

As $|\mathbf{E}| = E_0 \cos(k_z z - \omega t)$, $U_{VPW} = \varepsilon_0 E_0^2 \cos^2(k_z z - \omega t)$

[N.B. Don't confuse $|\mathbf{E}|$ = magnitude of $\mathbf{E}$ at a given $(\mathbf{r}, t)$ and E_0 = constant wave amplitude]

Instantaneous values of energy density vary too quickly to measure; it is more useful to consider the **average energy density.** As the average of $\cos^2\theta$ is $\frac{1}{2}$, we have $\langle U_{VPW} \rangle = \frac{1}{2}\varepsilon_0 E_0^2$.

For our example plane wave,

$$\mathbf{N} = \mathbf{E}\times\mathbf{H} = \hat{\mathbf{i}}\,E_0 \mathrm{e}^{i(k_z z-\omega t)} \times \hat{\mathbf{j}}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \mathrm{e}^{i(k_z z-\omega t)}, \text{ or}$$

$$\mathbf{N} = \hat{\mathbf{k}}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 \cos^2(k_z z-\omega t).$$

As expected, $\mathbf{N}$ points in the direction of wave (& energy) propagation. But the speed of light

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}, \text{ so } \mathbf{N} = \hat{\mathbf{k}}\, c\varepsilon_0 E_0^2 \cos^2(k_z z-\omega t) = \hat{\mathbf{k}}\, c U_{VPW}.$$

(With a bit of thought) this confirms that $|\mathbf{N}|$ =energy crossing unit area per unit time.

The average value of $|\mathbf{N}| = N$ is

$$\langle N\rangle = c\langle U_{VPW}\rangle = \tfrac{1}{2} c\varepsilon_0 E_0^2.$$

We can use $\langle N\rangle$ to find E_0 and H_0 for a given light source.

[Though note that we should be careful not to confuse “beams” and “plane waves”.]

Example.

Consider a 25W continuous-wave laser with a spot size of diameter 1mm. What are the amplitudes of the electric and magnetic fields in the laser light?

25W is the average power. Assume that the laser profile is a $d = $1mm circle cut out of a vacuum plane wave. The spot area is $A = \dfrac{\pi d^2}{4}$.

$\langle N \rangle$ gives us the average power per unit area:

$$\langle N \rangle \approx \frac{25 \cdot 4}{\pi \cdot 10^{-6}} \approx 3.2 \times 10^7 \,\mathrm{Wm^{-2}}.$$

Since $\langle N \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$ we get the amplitude of the electric field as $E_0 \approx 1.5 \times 10^5 \,\mathrm{Vm^{-1}}$.

In a VPW $\mathbf{E}$ and $\mathbf{H}$ are in phase, and $\dfrac{E_0}{H_0} = Z_0 = 377\Omega$. This gives for the amplitude of the magnetic field $H_0 \approx 410 \,\mathrm{Am^{-1}}$.

2.5 Photons and Radiation Pressure

This unit: e.m. waves are classical & continuous in energy. Any $\omega = ck$ is allowed.

Quantum theory: light "wave" consists of "particles"; photons of energy $\mathrm{E} = \hbar\omega$. [ω same as ours]

Special relativity: $\mathrm{E}^2 = p^2c^2 + m^2c^4$.

Here m is the "rest mass" of a particle $= 0$ for photons, so the momentum of a photon is

$$p = \frac{\mathrm{E}}{c} = \hbar\left(\frac{\omega}{c}\right) = \hbar k.$$

[Same as de Broglie result for $m \neq 0$; in 3D $\mathbf{p} = \hbar\mathbf{k}$.]

Through its momentum a photon can exert a RADIATION PRESSURE, P_r.

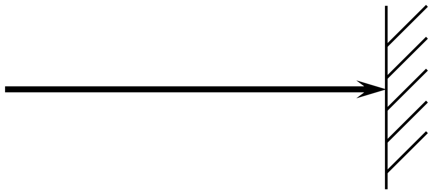
Example. Consider a light beam, cross-sectional area A cut out of a VPW, consisting of identical photons, each with $\mathrm{E} = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, $p = \dfrac{\mathrm{E}}{c}$.

Radiation pressure $P_r = \dfrac{\text{Force}}{\text{Area}} = \dfrac{1}{A}\dfrac{\partial p}{\partial t} = \dfrac{1}{Ac}\dfrac{\partial \mathrm{E}}{\partial t}$.

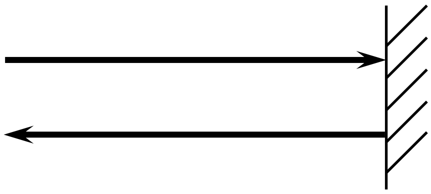
But $\dfrac{1}{A}\dfrac{\partial \mathrm{E}}{\partial t}$ = power per unit area in beam. Treated classically, this is $\langle N \rangle$.

So $P_r = \dfrac{\langle N \rangle}{c} = \langle U_{VPW} \rangle = \frac{1}{2}\varepsilon_0 E_0^2$.

If the beam strikes a surface and is perfectly absorbed, it exerts a radiation pressure $\langle U_{VPW} \rangle$ on the surface.



If it strikes a perfectly reflecting surface, it exerts a pressure $\langle U_{VPW} \rangle - \left(-\langle U_{VPW} \rangle\right) = 2\langle U_{VPW} \rangle$.



2.6 Polarisation of electromagnetic waves

For convenience, consider an e.m. wave travelling in the $+z$ direction:

$$\mathbf{E}(z,t) = \mathbf{E}_0(t)\cos(kz - \omega t) \text{ (and associated } \mathbf{H}\text{)}$$

The "polarisation state" is determined by the behaviour of $\mathbf{E}$ in the transverse (xy) plane as a function of time. (This is NOT fixed by M.Es.)

Write $\mathbf{E}$ (in this section) as the superposition

$$\mathbf{E}(z,t) = \mathbf{E}_x(z,t) + \mathbf{E}_y(z,t),$$

with

$$\mathbf{E}_x(z,t) = \hat{\mathbf{i}} E_{0x}\cos(kz - \omega t) \qquad (1)$$

$$\mathbf{E}_y(z,t) = \hat{\mathbf{j}} E_{0y}\cos(kz - \omega t + \beta) \qquad (2)$$

Note that E_{0x}, E_{0y}, β, k, ω, are all constants. The 1st 3 of these determine the polarisation state.

If $\beta > 0$, wave (2) reaches a given value of $\cos$ a time $\tau = \dfrac{\beta}{\omega}$ LATER than (1), so $\mathbf{E}_y$ LAGS $\mathbf{E}_x$ by β.

1. Plane Polarisation

$\mathbf{E}_0$ is a CONSTANT.

To achieve this, set $\beta = 0$ (or 2π etc); then

$$\mathbf{E}(z,t) = \underbrace{(\hat{\mathbf{i}}E_{0x} + \hat{\mathbf{j}}E_{0y})}_{\mathbf{E}_0}\cos(kz - \omega t).$$

The "plane of polarisation" is the $(\mathbf{E}_0, \mathbf{k})$ plane.

Note that $\beta = \pi$ (or 3π etc) also gives plane polarisation; $\mathbf{E}_0 = (\hat{\mathbf{i}}E_{0x} - \hat{\mathbf{j}}E_{0y})$.

2. Circular Polarisation

$$E_{0x} = E_{0y} = E_0; \qquad \beta = \pm\frac{\pi}{2}.$$

$$\cos(x \pm \tfrac{\pi}{2}) = \cos x \cos\tfrac{\pi}{2} \mp \sin x \sin\tfrac{\pi}{2} = \mp \sin x,$$

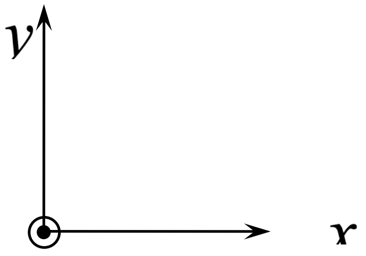
so for $\beta = -\dfrac{\pi}{2}$ we have

$\mathbf{E}_x = \hat{\mathbf{i}}E_0\cos(kz - \omega t)$; $\qquad \mathbf{E}_y = \hat{\mathbf{j}}E_0\sin(kz - \omega t)$ and

$$\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos(kz - \omega t) + \hat{\mathbf{j}}\sin(kz - \omega t)\right].$$

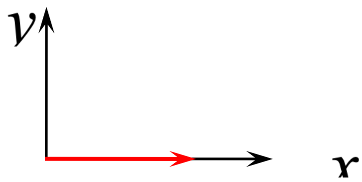
Then $|\mathbf{E}| = E_0$, but the direction of $\mathbf{E}_0$ ROTATES in the transverse plane. The endpoint of $\mathbf{E}_0$ traces out a circle:

Observe the wave at $z = 0$:



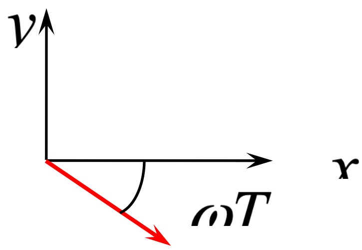
Wave is travelling <u>towards</u> us

At $t = 0$, $\mathbf{E} = E_0 \hat{\mathbf{i}}$



At a later time $t = T$,

$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(\omega T) - \hat{\mathbf{j}} \sin(\omega T) \right]$$



So the $\mathbf{E}$-vector rotates clockwise at angular frequency ω according to our observer.

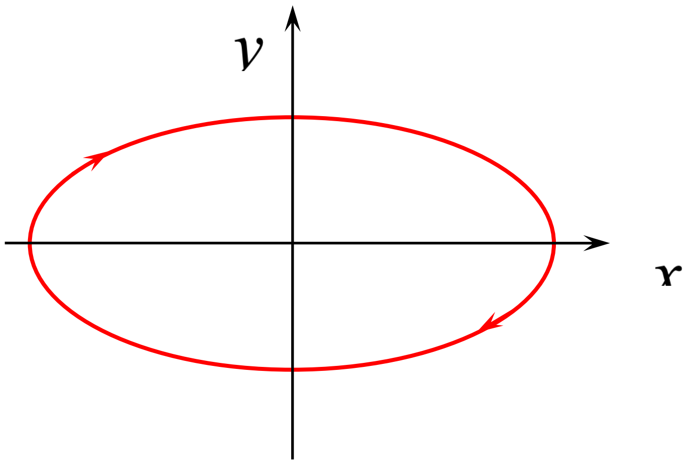
This is a “right-circularly” polarised wave.
[Though beware a confusing array of conventions.]

$\beta = +\dfrac{\pi}{2}$ gives $\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(kz - \omega t) - \hat{\mathbf{j}} \sin(kz - \omega t) \right]$,
a “left circularly” polarised wave, in which $\mathbf{E}$ rotates anti-clockwise at angular frequency ω according to our observer.

3. Elliptical Polarisation

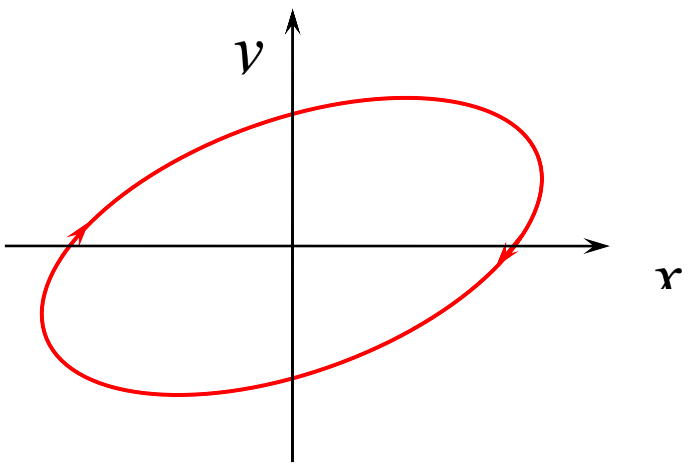
- Tip of $\mathbf{E}$-vector traces out an ellipse in the transverse plane.
  [Includes plane- and circular polarisation.]

e.g. $\beta = -\dfrac{\pi}{2}$, but $E_{0x} \neq E_{0y}$:



$(E_{0x} > E_{0y})$

For most values of β, ellipse will be tilted:



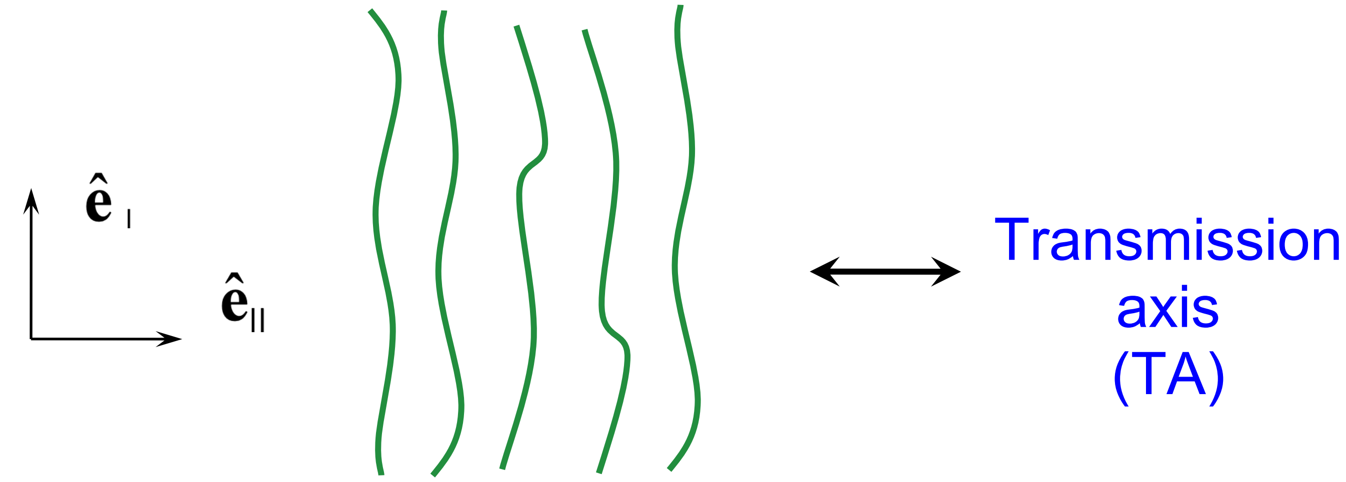
Natural light

- Often called “unpolarised”. “Randomly polarised” would be better; rapidly (and randomly) changing polarisation state.

Polarisers

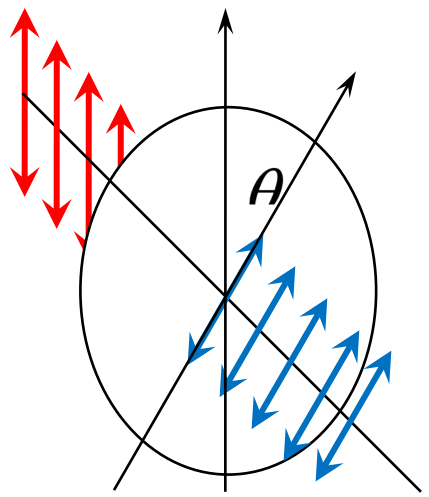
These are optical devices used to manipulate and analyse polarisation states.

Simple example (optical frequencies) – 1 or more sheets of polaroid. Essentially consists of aligned conducting polymer chains…



Only component of $\mathbf{E}_0$ parallel to TA gets through.

Consider, for example, a plane polarised wave, at normal incidence on an "ideal" polaroid sheet. Let θ=angle (in plane of polariser) between $\mathbf{E}_0$ and TA.



$$\mathbf{E}_0 = \underbrace{E_0 \cos\theta}_{\text{Transmitted amplitude}} \hat{\mathbf{e}}_{||} + E_0 \sin\theta \, \hat{\mathbf{e}}_{\perp}$$

Transmitted amplitude

What is the transmitted intensity?

Intensity = av. power per unit area $= \langle N \rangle = \frac{1}{2} c\varepsilon_0 E_0^2$,

so $I(\theta) = \frac{1}{2} c\varepsilon_0 E_0^2 \cos^2\theta.$

Maximum $(\theta = 0)$ intensity is $I(0) = \frac{1}{2} c\varepsilon_0 E_0^2$ so

$$I(\theta) = I(0)\cos^2\theta. \qquad \text{Malus' Law}$$

Malus' Law can be used to analyse the effect of our ideal polariser on other forms of light:

Let I_{inc} = intensity of incident light
I_{tr} = intensity of transmitted light

Circularly polarised light: $\mathbf{E}_0$ has constant magnitude, but is rotating rapidly (at angular frequency ω)

$$\therefore \quad I_{tr} = I_{inc}\left\langle \cos^2\theta \right\rangle = \frac{1}{2} I_{inc}$$

-half the light gets through

Natural light: a similar story. The direction of $\mathbf{E}_0$ varies rapidly and randomly, so again

$$I_{tr} = I_{inc}\left\langle \cos^2\theta \right\rangle = \frac{1}{2} I_{inc}.$$

Birefringence and wave-plates

- are on the syllabus for this part of the unit, BUT

easier of you've studied electromagnetic properties of materials

- coming next…

So, as a piece of self-directed learning, read about

Birefringent crystals (anisotropy of refractive index) & how they can be used to manipulate the polarisation state of electromagnetic waves in (for example) a "half-wave plate".

Chapter 8 of "Optics" by Hecht is an excellent source for this, and other matters relating to polarisation.