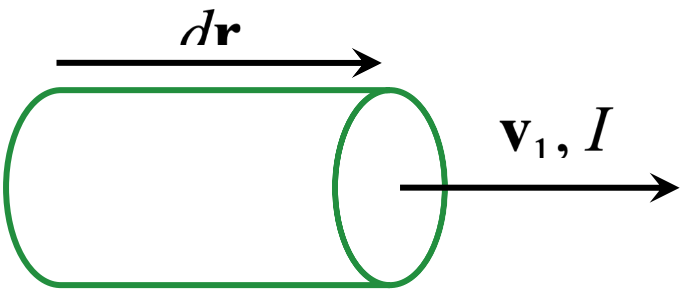
$\mathbf{B}$-field due to a current element

A current element is a small region within which all charges are moving with the same velocity:



For a single charge at the origin

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$$

$Q_1\mathbf{v}_1 =$ charge x distance / time

$=$ current x distance

So, replace $Q_1\mathbf{v}_1$ by $I\,d\mathbf{r}$ to find $\mathbf{B}$ from a current element at the origin:

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\, I\, \frac{d\mathbf{r} \times \hat{\mathbf{e}}_r}{r^2}.$$

This is the Biot-Savart Law (experimental)

- Confirms that $\mathbf{B}$ from our thought experiment **is** the magnetic field
- $\mathbf{v}_1$ = constant, so only Direct Current (DC) so far.

Divergence of $\mathbf{B}$ - informal approach

Relativity argument $\Rightarrow$ $\mathbf{B}$ - fields have no direct "point source" of their own; they arise from moving sources of $\mathbf{E}$ - fields.

Also, the Biot-Savart Law $\Rightarrow$ $\mathbf{B}$ - field lines are perpendicular to $\hat{\mathbf{e}}_r$, so they cannot come directly to or from the current element.

Intuition would therefore lead us to expect

$$\nabla \cdot \mathbf{B} = 0$$

(i.e. "There are no magnetic charges")

This is, in fact, a correct field equation, and is our 2^{nd} "complete" Maxwell Equation.

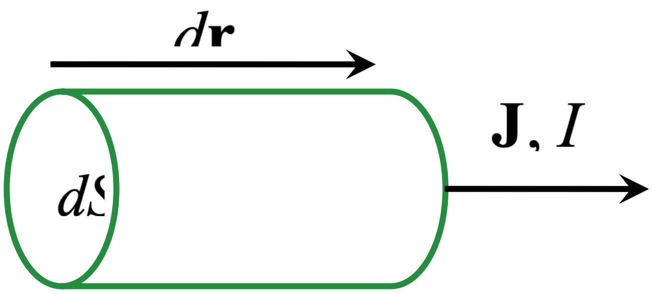
Divergence of $\mathbf{B}$ - more formal approach

Find $\nabla \cdot \mathbf{B}$ for a general current distribution…

Define the **vector current density** $\mathbf{J}(\mathbf{r}')$:

Direction of $\mathbf{J}$ $\rightarrow$ direction of current flow at $\mathbf{r}'$
Magnitude of $\mathbf{J}$ $\rightarrow$ current crossing unit area $\perp \mathbf{J}$

Thus $I = \int_S \mathbf{J} \cdot d\mathbf{S}$ - current is the flux of the vcd.

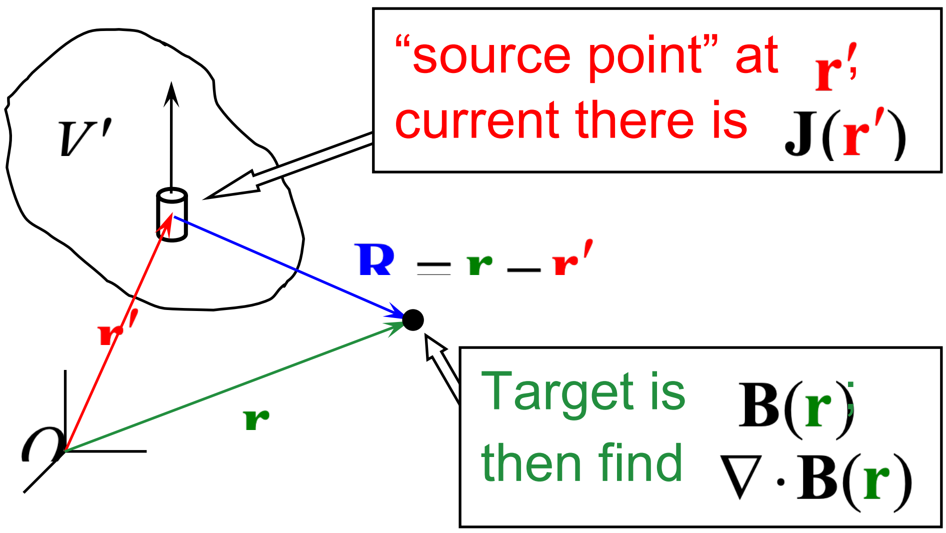


A current element at the origin ($\mathbf{r}' = \mathbf{0}$) gives rise to

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\, dV\, \frac{\mathbf{J} \times \hat{\mathbf{e}}_r}{r^2}.$$

[To see this - $I = |\mathbf{J}|\, dS$ and $\mathbf{J}$ is parallel to $d\mathbf{r}$.
Therefore $I\, d\mathbf{r} = \mathbf{J}\, dS dr = \mathbf{J}\, dV$.]

Now for the general current distribution…



For a single current element

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\, dV'\, \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2}.$$

Then, by the superposition principle,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2}\, dV' .$$

We now find the divergence (w.r.t. $\mathbf{r}$, not $\mathbf{r}'$).

Assume it is OK to swap order of $\nabla\cdot$ and $\int_{V'}$:

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \nabla_{\mathbf{r}}\cdot\left\{\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2}\right\} dV' .$$

Concentrate on the integrand. Use the vector identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$:

$$\nabla_{\mathbf{r}}\cdot\left\{\mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2}\right\} = \frac{\hat{\mathbf{R}}}{R^2}\cdot\left\{\nabla_{\mathbf{r}}\times\mathbf{J}(\mathbf{r}')\right\} - \mathbf{J}(\mathbf{r}')\cdot\left\{\nabla_{\mathbf{r}}\times\frac{\hat{\mathbf{R}}}{R^2}\right\}.$$

This is always zero. Why?

$\nabla_{\mathbf{r}}\times\frac{\hat{\mathbf{R}}}{R^2} = \mathbf{0}$ as $\frac{\hat{\mathbf{R}}}{R^2} = -\nabla_{\mathbf{r}}\frac{1}{R}$ and $\nabla\times\nabla\psi = \mathbf{0} \quad \forall\psi$

$\nabla_{\mathbf{r}}\times\mathbf{J}(\mathbf{r}') = \mathbf{0}$ as $\mathbf{J}$ is not a function of $\mathbf{r}$.

Hence the integrand is 0 and $\nabla\cdot\mathbf{B} = 0$ as expected.

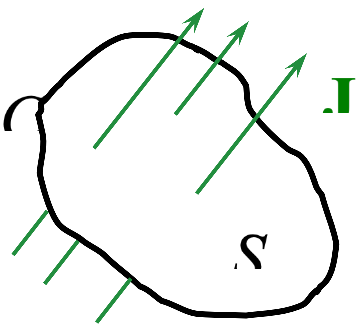
(N.B. true at all $\mathbf{r}$ - even inside V'.)

The curl of $\mathbf{B}$ (DC only)

We start from Ampère's Law, another Law derived from experiments:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \text{(Total DC current passing through } C\text{)}$$

where C is any closed curve bounding a surface S.



i.e. $$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Use Stokes' Theorem on the LHS:

$$\rightarrow \qquad \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Again, since $C, S, \mathbf{J}$ are arbitrary, we must have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

"Current is a rotational source of magnetic field"

Not quite a Maxwell Equation – restricted to DC.

1.3 Time-varying currents and fields

So far, we have for steady currents $\mathbf{J}$ and static charge distributions ρ placed in vacuum:

1 $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ Coulomb's Law / Gauss's Law

2 $\nabla \times \mathbf{E} = \mathbf{0}$ Electrostatic field is conservative

3 $\nabla \cdot \mathbf{B} = 0$ No magnetic monopoles

4 $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampère's Law

Note that $\mathbf{E}$ and $\mathbf{B}$ are still uncoupled.

That is, we still have "electricity" and "magnetism".

1 and 3 are completely correct (we think!)

2 and 4 require modification for time-varying fields.

Faraday's Law

A static $\mathbf{B}$ field induces current in a moving circuit.
(and vice versa)

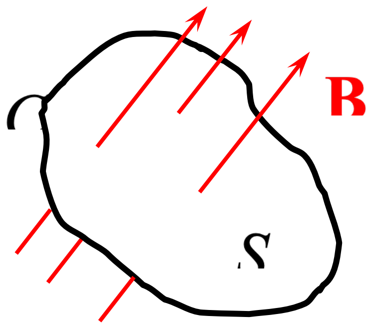
The induced "emf" (actually an electric potential difference) is

$$\varepsilon = -\frac{\partial \Phi_B}{\partial t},$$

Faraday's Law of Induction

where Φ_B = magnetic flux through circuit C

$$= \int_S \mathbf{B} \cdot d\mathbf{S}.$$



ε is an electric potential difference **not** derived from Coulomb's Law – it must be due to an extra force $\mathbf{F}'$

Assume that the induced current consists of a single charge Q moving around C.

Then $Q\varepsilon$ =work$= \oint_C \mathbf{F}' \cdot d\mathbf{r}$, so

$$\varepsilon = \frac{1}{Q} \oint_C \mathbf{F}' \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}.$$

We now have 2 choices:

1. $\mathbf{B}$ is static; circuit moves

2. Circuit is static; $\mathbf{B}$ changes with time

In 1., $\mathbf{F}' = Q(\mathbf{u} \times \mathbf{B})$ $\quad$ $\mathbf{u}$ is extra velocity of Q due to motion of circuit

In 2., $\mathbf{u} = \mathbf{0}$, so force must be due to an extra electric field $\mathbf{F}' = Q\mathbf{E}'$.

Make choice 2. Expression for ε now reads

$$\varepsilon = \oint_C \mathbf{E}' \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}.$$

Use Stokes' Theorem on LHS.
Change order of $\partial / \partial t$ and $\int_S$ on RHS:

$$\int_S \nabla \times \mathbf{E}' \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

Since S is arbitrary, we have

$$\nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}}{\partial t}.$$

If an electrostatic field $\mathbf{E}_S$ is also present, superposition gives the total electric field as

$$\mathbf{E} = \mathbf{E}_S + \mathbf{E}'$$

So

$$\nabla \times \mathbf{E} = \nabla \times \mathbf{E}_S + \nabla \times \mathbf{E}'$$
$$= \quad 0 \qquad - \frac{\partial \mathbf{B}}{\partial t}.$$

Thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

(3rd of Maxwell's Equations)

Notes:

1. A general electric field is **not** conservative. We cannot therefore use electrostatic potentials in the general, time-varying case.

2. A time-varying $\mathbf{B}$ is a rotational source of $\mathbf{E}$.

The Maxwell-Ampère Law

To see that $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ is incomplete, take the divergence of both sides…

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{\text{always } 0 \text{ - a vector identity}} = \mu_0 \nabla \cdot \mathbf{J}$$

$\therefore \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ can only be correct when $\nabla \cdot \mathbf{J} = 0$; i.e. when there are no direct sources of current.

To find a general expression for $\nabla \cdot \mathbf{J}$, consider…

Conservation of Charge

In English:

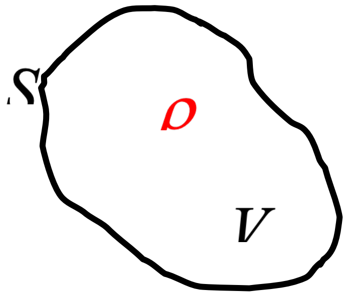
Electric charge is neither created nor destroyed

- Verified experimentally to at least 1 in 10^{20}

In Vector Calculus?

Consider an arbitrary volume V enclosed by surface S, containing charge with density $\rho(\mathbf{r}, t)$.

Conservation of charge implies…



Rate of flow of charge across surface S = Rate of change of charge within V

or

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho \, dV .$$

Use the divergence theorem on LHS

$$\rightarrow \quad \int_V \nabla \cdot \mathbf{J} \, dV = \int_V \left(-\frac{\partial \rho}{\partial t} \right) dV .$$

Since V is arbitrary,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

The “Continuity Equation” – a mathematical expression of charge conservation.

From this, we see that $\nabla \cdot \mathbf{J}$ is **not** always 0 (as required by Ampère's Law), but $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$ IS.

To correct the Law, substitute for ρ from Gauss's Law:

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \\ &= \nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) \\ &= \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \end{aligned}$$

So to make Ampère's Law consistent with charge conservation, we can simply replace $\mathbf{J}$ with $\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$; then the divergence of each side will be 0

This yields the "Maxwell-Ampère Law"

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

(our 4th Maxwell Equation)

The extra term $\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ has same dimensions as $\mathbf{J}$ - called the "vacuum displacement current density".

This was Maxwell's sole contribution to the Laws of Electricity and Magnetism, but it is a crucial one; it leads to the notion of **electromagnetism** and to **electromagnetic waves**.

[Aside: Why did Faraday miss the displacement current?

Integrating the MA Law over an arbitrary surface S enclosed by path C yields (see PSQ):

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S}$$

which shows that a time-varying $\mathbf{E}$-flux must induce a magnetic field.

BUT $\varepsilon_0 \mu_0$ is a tiny number, so $\mathbf{E}$-flux must vary extremely rapidly to induce a detectable $\mathbf{B}$.

Happens in e.g. electromagnetic waves, which Faraday didn't know about.

Effect verified by Hertz in 1887, long after Maxwell predicted it.]

We now have

MAXWELL'S EQUATIONS IN A VACUUM

1 $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$ Gauss's Law

2 $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's Law

3 $$\nabla \cdot \mathbf{B} = 0$$ Absence of magnetic monopoles

4 $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$ Maxwell-Ampère Law

KNOW THEM

All electromagnetic fields in a vacuum, on all length scales down to those governed by quantum mechanics, are solutions to these equations.

We will concentrate on wave solutions…