2. Electromagnetic waves in a vacuum

2.1 The Wave Equation

Maxwell's equations 1-4 are coupled, but may be de-coupled to give separate equations for $\mathbf{E}$ and $\mathbf{B}$:

For $\mathbf{E}$, take curl of equation 2:

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}).$$

Use vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and substitute from MEs for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$:

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

- no $\mathbf{B}$

source terms

From now on we look only for solutions in "source-free" vacuum – we set $\rho = 0$; $\mathbf{J} = \mathbf{0}$, so

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

This is the wave equation for $\mathbf{E}$ in source-free vacuum.

Similarly, for $\mathbf{B}$, take curl of equation 4…

substitute from MEs for $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{E}$…

set $\mathbf{J} = \mathbf{0}$…

to obtain the wave equation for $\mathbf{B}$ in source-free vacuum:

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

- Identical in form to the wave equation for $\mathbf{E}$.

- Implies both will have the same form of solution.

Among these solutions will be waves…

Revision of Waves

- See handout

2.2 Wave solutions for $\mathbf{E}$

Look for plane wave solutions $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$:

Substitute into $\nabla^2\mathbf{E}=\varepsilon_0\mu_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ and see if it works…

LHS:
$$\nabla^2\mathbf{E}=\hat{\mathbf{i}}\underbrace{\nabla^2 E_x}+\hat{\mathbf{j}}\nabla^2 E_y+\hat{\mathbf{k}}\nabla^2 E_z$$

Look at this term…

$$E_x=E_{0x}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=E_{0x}\mathrm{e}^{i(k_x x+k_y y+k_z z-\omega t)}$$ so

$$\begin{aligned}\nabla^2 E_x&=\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\\&=-(k_x^2+k_y^2+k_z^2)E_{0x}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\&=-k^2E_x.\end{aligned}$$

Similarly, $\nabla^2E_y=-k^2E_y$ and $\nabla^2E_z=-k^2E_z$ so overall

$$\begin{aligned}\nabla^2\mathbf{E}&=-k^2(\hat{\mathbf{i}}E_x+\hat{\mathbf{j}}E_y+\hat{\mathbf{k}}E_z)\\&=-k^2\mathbf{E}.\end{aligned}$$

RHS is easier: $\dfrac{\partial^2 \mathbf{E}}{\partial t^2} = -\omega^2 \mathbf{E}.$

Thus the W.E. becomes $-k^2\mathbf{E} = -\varepsilon_0 \mu_0 \omega^2 \mathbf{E}.$

This means that plane waves **are** solutions to the wave equation, provided their wavenumber k and angular frequency ω are related by

$$k^2 = \varepsilon_0 \mu_0 \omega^2 .$$

BUT

For any wave $\dfrac{\omega}{k} = f\lambda = v$, the phase speed.

$\therefore$ only waves in source-free vacuum with phase speed $v = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ are allowed.

$\varepsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}

$\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} so allowed v is

$v = 3 \times 10^8$ ms^{-1} $= c$, the speed of LIGHT in a vacuum

From theory, Maxwell asserted that light is a form of e.m. wave, and that other forms (different ω, k; same c) should exist. Hertz found another in 1887.

2.3 Monochromatic electromagnetic waves

Consider an electric plane wave travelling in the $+z$ direction: $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0 \mathrm{e}^{i(k_z z-\omega t)}$.

- What direction is $\mathbf{E}_0$ in?
- What $\mathbf{B}$ is associated with $\mathbf{E}$?

$$\mathbf{E}(\mathbf{r},t)=\hat{\mathbf{i}}\,E_{0x}\mathrm{e}^{i(k_z z-\omega t)}+\hat{\mathbf{j}}\,E_{0y}\mathrm{e}^{i(k_z z-\omega t)}+\hat{\mathbf{k}}\,E_{0z}\mathrm{e}^{i(k_z z-\omega t)}.$$

Therefore

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\\ &= \quad 0 \quad + \quad 0 \quad + ik_z E_{0z}\mathrm{e}^{i(k_z z-\omega t)}\\ &= ik_z E_z.\end{aligned}$$

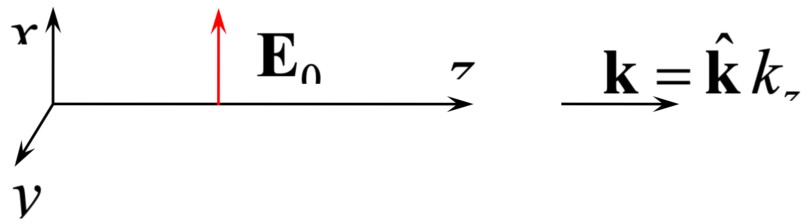
In source-free vacuum, $\rho(\mathbf{r})=0$ so $\nabla\cdot\mathbf{E}=0$.

Clearly $k_z \neq 0$, so we must have $E_z=0$ for this wave.

i.e. The electric field in a plane wave in source-free vacuum is TRANSVERSE.

What is $\mathbf{B}$?

Let $\mathbf{E}_0 = \hat{\mathbf{i}}\, E_0$.



Then our wave solution to Maxwell's equations is $\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z - \omega t)}$. Use the ME $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ to find $\mathbf{B}$…

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 \mathrm{e}^{i(k_z z - \omega t)} & 0 & 0 \end{vmatrix} = \hat{\mathbf{j}}\, i k_z E_0 \mathrm{e}^{i(k_z z - \omega t)} = -\frac{\partial \mathbf{B}}{\partial t}$$

Integrate this with respect to t (integration const $= 0$) to give $\quad \mathbf{B} = -\hat{\mathbf{j}}\left[\dfrac{i k_z}{-i\omega} E_0 \mathrm{e}^{i(k_z z - \omega t)}\right]$.

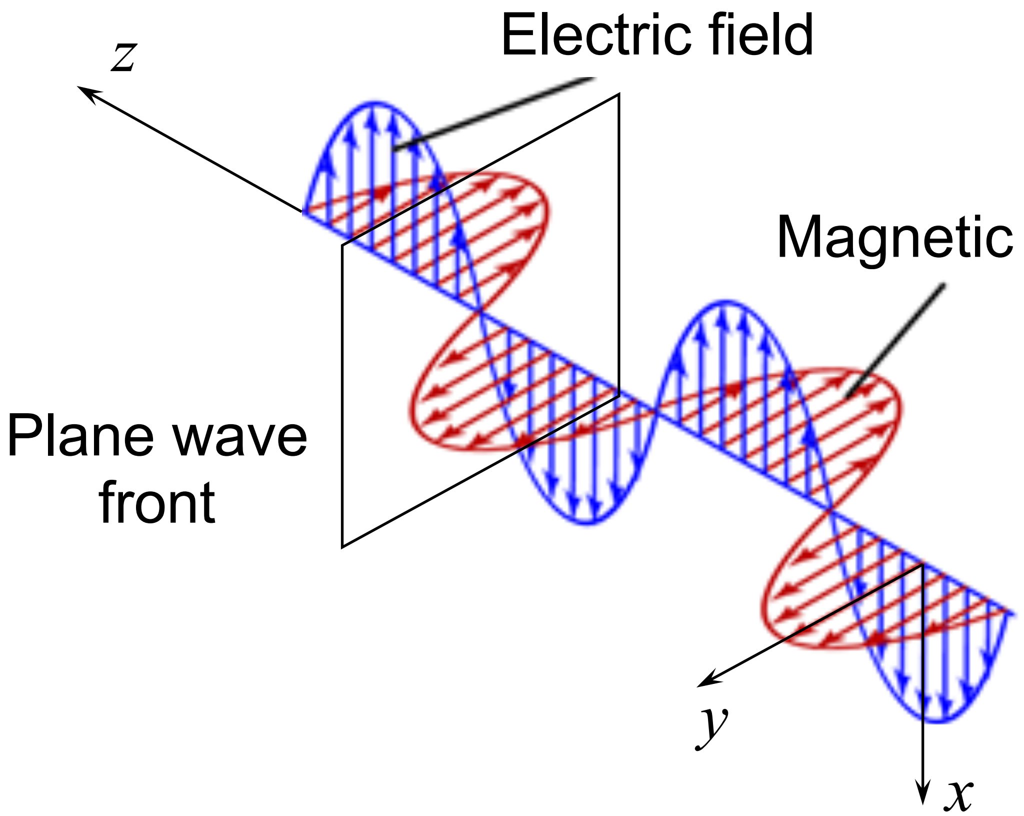
Hence $\mathbf{B} = \hat{\mathbf{j}}\, B_0 \mathrm{e}^{i(k_z z - \omega t)}$ with $B_0 = \dfrac{E_0}{c}$.

$\mathbf{B}$-wave is also transverse, and is perpendicular to $\mathbf{E}$. In source-free vacuum, $\mathbf{E}$ & $\mathbf{B}$ are in phase.	Does NOT imply $\mathbf{B}$ is less important than $\mathbf{E}$.

Summary

In source-free vacuum, we have transverse electromagnetic waves. If we **fix** $\mathbf{k}$ in the z-dir^n, $\mathbf{E}_0$ in the x-dir^n, then

$$\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z - \omega t)} \qquad \mathbf{B}(\mathbf{r},t) = \hat{\mathbf{j}}\, B_0 \mathrm{e}^{i(k_z z - \omega t)}$$



Note that $\mathbf{E}\times\mathbf{B}$ points in the direction of energy flow. This is true for all electromagnetic fields. In this case, $\mathbf{E}\times\mathbf{B} \propto \hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$.

A note on Polarisation

Maxwell's equations require $\mathbf{E}\cdot\mathbf{k} = \mathbf{B}\cdot\mathbf{k} = \mathbf{E}\cdot\mathbf{B} = 0$.

BUT nothing in them prevents $\mathbf{E}$ and $\mathbf{B}$ rotating in the plane perpendicular to the wavevector $\mathbf{k}$.

So far we've assumed that $\mathbf{E}_0$ is unchanging with time. We have therefore assumed we have "linearly polarised" or "plane polarised" waves.

[By convention, the plane of polarisation is that of $\mathbf{E}_0$ and $\mathbf{k}$.]

Other polarisations are possible, even useful, as we shall see later…

$\mathbf{H}$-fields and wave impedance

$\mathbf{H}$-fields are often used to describe magnetic fields in materials. In a vacuum, their relation to $\mathbf{B}$-fields is very simple: $\mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B}$.

In terms of $\mathbf{H}$, our plane-polarised wave moving in the $+z$ direction is

$$\mathbf{E} = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z - \omega t)} \qquad\qquad \mathbf{H} = \hat{\mathbf{j}}\left(\frac{k_z E_0}{\mu_0 \omega}\right)\mathrm{e}^{i(k_z z - \omega t)}$$

Therefore $\dfrac{|\mathbf{E}|}{|\mathbf{H}|} = \dfrac{\mu_0 \omega}{k_z} = \mu_0 c = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}.$

- Same for all waves in source-free vacuum.

The value of $\dfrac{|\mathbf{E}|}{|\mathbf{H}|}$ gives the wave “impedance”.

For all e.m. waves in source-free vacuum, the wave impedance is

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega.$$

2.4 Electromagnetic energy (in a vacuum)

Look at examples:

1. Energy stored in a capacitor $W = \tfrac{1}{2}CV^2$.
e.g. a large parallel-plate capacitor.
$C = \dfrac{\varepsilon_0 A}{d}$; $V = Ed$. [A =plate area; d =spacing]
Therefore $W = \tfrac{1}{2}\varepsilon_0 E^2 \cdot \left(Ad\right)$ so the energy density in this purely electric field is $U = \tfrac{1}{2}\varepsilon_0 E^2 = \tfrac{1}{2}\varepsilon_0 \left|\mathbf{E}\right|^2$.

2. Energy stored in an inductor $W = \tfrac{1}{2}LI^2$.

e.g. a long circular solenoid; radius a, length b.
$L = \mu_0 \pi a^2 b n^2$; $B = \mu_0 n I$. [n =turns per m]
Therefore $W = \tfrac{1}{2}\dfrac{1}{\mu_0}B^2 \cdot \left(\pi a^2 b\right)$ so the energy density in this purely magnetic field is
$U = \tfrac{1}{2}\dfrac{B^2}{\mu_0} = \tfrac{1}{2}\mu_0 H^2 = \tfrac{1}{2}\mu_0 \left|\mathbf{H}\right|^2$.

These specific examples reveal a general result.
The energy density of an e.m. field in vacuum is

$$U = \tfrac{1}{2}\varepsilon_0 \left|\mathbf{E}\right|^2 + \tfrac{1}{2}\mu_0 \left|\mathbf{H}\right|^2$$

= energy density in $\mathbf{E}$-field + e.d. in $\mathbf{H}$-field

The Poynting Vector (source-free vacuum)

The Poynting vector is $\mathbf{N} = \mathbf{E} \times \mathbf{H}$. At any point,

Direction of $\mathbf{N} \rightarrow$ direction of energy flow
Magnitude of $\mathbf{N} \rightarrow$ rate of energy flow across unit area perpendicular to $\mathbf{N}$.

So $\mathbf{N}$ does for energy what $\mathbf{J}$ does for charge.

If this is true, we should expect a continuity equation for energy

$$\nabla \cdot \mathbf{N} = -\frac{\partial U}{\partial t}$$

analogous to the continuity equation for charge

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

We will derive this new continuity equation in s.f.v.
(use $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{J} = \mathbf{0}$)…

Start from $\mathbf{N} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$.

$$\begin{aligned}\nabla\cdot(\mathbf{E}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}) \\ &= \mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right) - \mathbf{E}\cdot\left(\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\right).\end{aligned}$$

Rearranging, $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{B}\cdot\dfrac{\partial\mathbf{B}}{\partial t} - \varepsilon_0\mu_0\mathbf{E}\cdot\dfrac{\partial\mathbf{E}}{\partial t}$.

Now, $\mathbf{B}\cdot\dfrac{\partial\mathbf{B}}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \dfrac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{B}|^2\right)$, so

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\varepsilon_0|\mathbf{E}|^2 + \tfrac{1}{2}\mu_0|\mathbf{H}|^2\right\} = -\frac{\partial U}{\partial t}.$$

The Poynting vector $\mathbf{N} = \mathbf{E}\times\mathbf{H}$ is then, as claimed, the vector which describes energy flow (just as $\mathbf{J}$ describes charge flow).

This is true for all electromagnetic fields.