ELECTROMAGNETISM

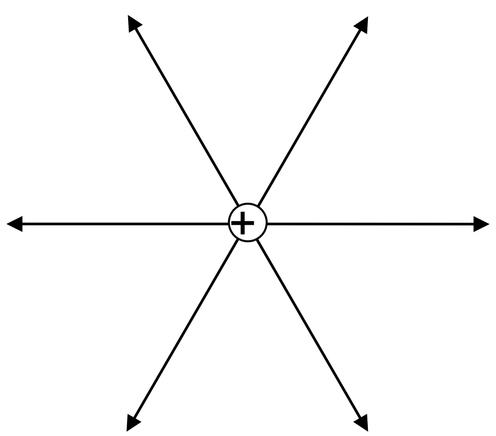
Electromagnetism is primarily concerned with the vector quantities:

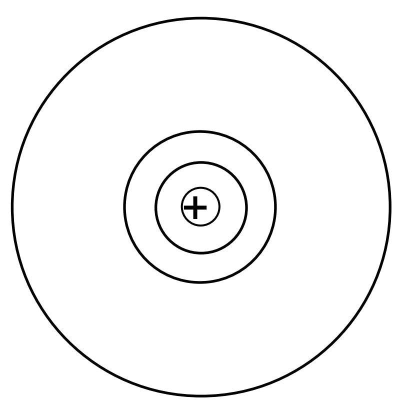
$\mathbf{E}(\mathbf{r},t)$ "The Electric Field" &
$\mathbf{B}(\mathbf{r},t)$ "The Magnetic Field"

They are continuous functions of position $\mathbf{r}$ and time t, defined over a given region, or "field".

Other "vector fields", such as the current density $\mathbf{J}(\mathbf{r},t)$ will be used, as will some "scalar fields" such as the electric charge density $\rho(\mathbf{r})$ and the electrostatic potential $\phi(\mathbf{r})$.

Recall: Vector fields can be visualised using "field lines" or "flux lines", & scalar fields by contour lines / surfaces.





1. Maxwell's Equations

Electromagnetism is governed by 4 differential equations in $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, called Maxwell's equations.

In a vacuum, they are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

where t is time, ε_0 and μ_0 are constants;

$\rho(\mathbf{r})$ is electric charge density
$\mathbf{J}(\mathbf{r},t)$ is electric current density

- note that these are the only "source terms"

$\Rightarrow$ All electromagnetic fields are caused by CHARGES.

We will assume that the charge-carrying particle is much smaller than any length scale of interest.

(POINT CHARGE)

Hence we have "Classical Electrodynamics", and not "Quantum Electrodynamics" (QED).

Maxwell's equations can almost entirely be derived from the "elementary" Laws of electricity & magnetism; ie

Coulomb's Law
Gauss's Law
Biot-Savart Law
Ampère's Law
Faraday's Law

But we first need these in their most general (integral) form.

So to get started, we will need some

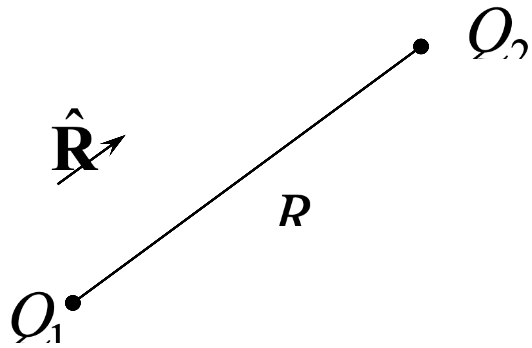
VECTOR CALCULUS

- see "Mathematical operations on field quantities" on moodle

1.1 Electrostatic fields

Coulomb's Law (experimental)

The force between 2 **stationary** point charges is given by

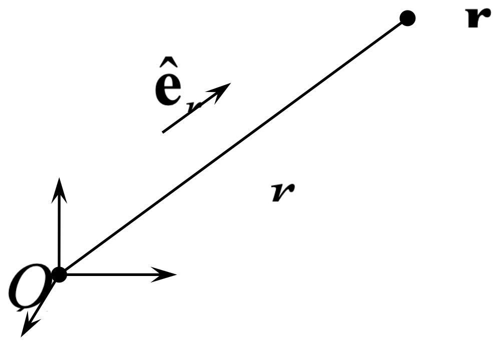


$$\mathbf{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{\mathbf{R}}.$$

ε_0 is the "permittivity of free space"
$\approx 8.854\times 10^{-12}$ $C^2N^{-1}m^{-2}$ or Fm^{-1}.

The **Electric Field** $\mathbf{E}(\mathbf{r})$ is the force experienced by a unit positive charge placed at $\mathbf{r}$.

$\therefore$ from Coulomb's Law, the $\mathbf{E}$-field due to a point charge $Q_1 = Q$ at the origin of coordinates is given by



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r.$$

units Vm^{-1}

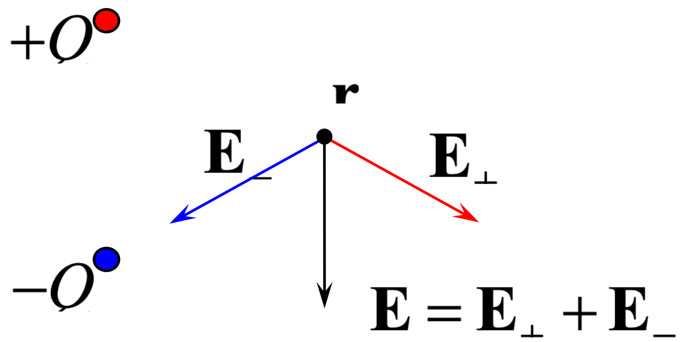
- serves as a definition of $\mathbf{E}$.

Note that the force on a charge Q_2 at $\mathbf{r}$ is

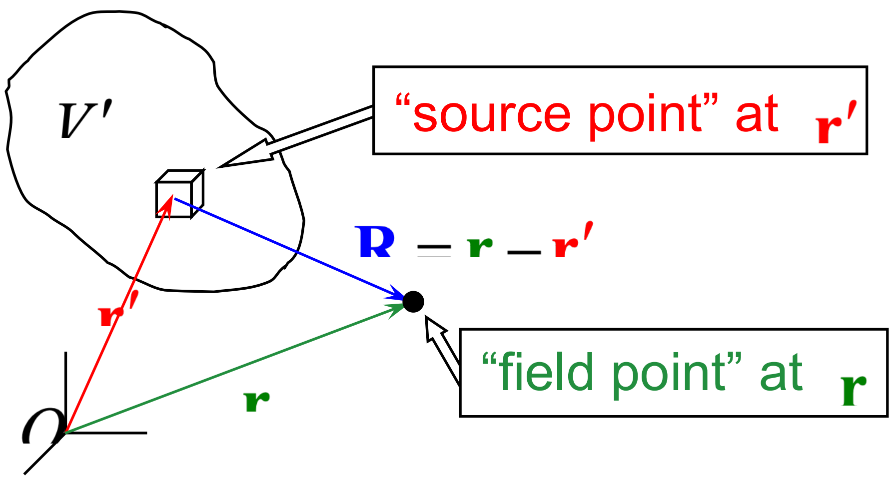
$$\mathbf{F}_e = Q_2 \mathbf{E}.$$

The **Principle of Superposition** states that the $\mathbf{E}$-field due to a distribution of charges is the vector sum of the fields due to individual charges:

2 charges:



Use $\rho(\mathbf{r}')$ to describe a general charge distribution inside volume V':

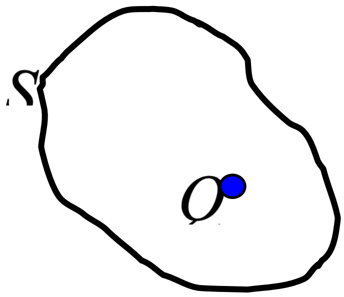


Charge in dV' is $\rho(\mathbf{r}')dV'$ so, by superposition,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{R^2} \hat{\mathbf{R}}\, dV'.$$

Gauss's Law

Gauss's Law, derived from Coulomb's Law, states that if charge Q is **completely** surrounded by a surface S **of any shape**, then



$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}.$$

To show this is true…

With Q at the origin, $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{e}}_r$, so

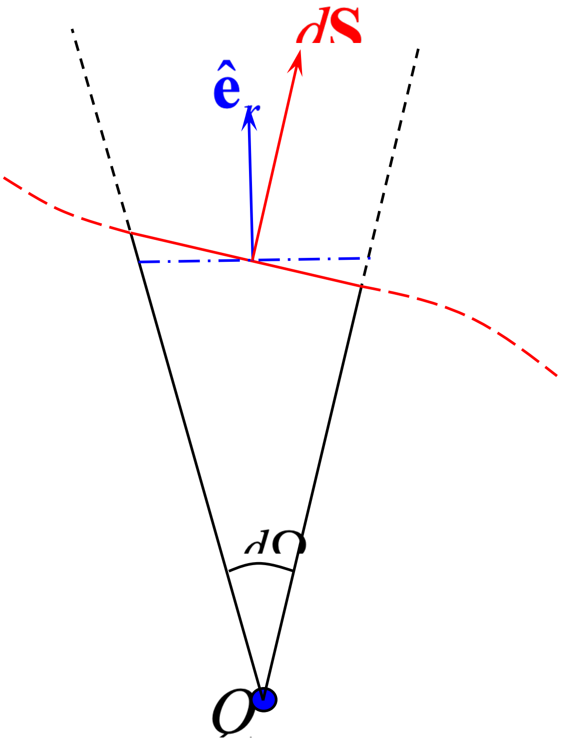
$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0} \oint_S \frac{\hat{\mathbf{e}}_r \cdot d\mathbf{S}}{r^2}.$$

For some shapes (e.g. a sphere, see PS1) this is an easy integral.

For general shapes, use **Solid Angles…**

[Aside on planar and solid angles]

Consider an arbitrary part of a general surface S:



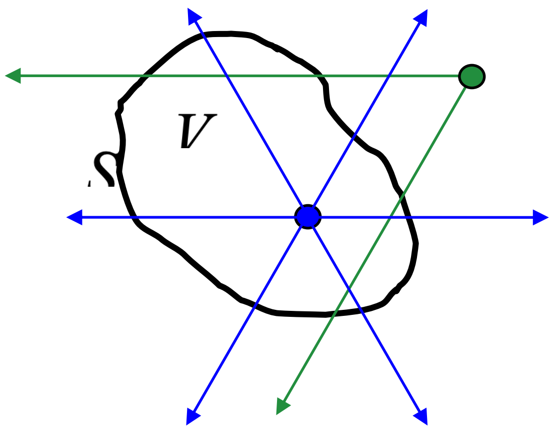
$$\hat{\mathbf{e}}_r \cdot d\mathbf{S} = dS\cos\theta$$
$$= dS',$$
= perpendicular area.

Therefore
$$\frac{\hat{\mathbf{e}}_r \cdot d\mathbf{S}}{r^2} = \frac{dS'}{r^2} = d\Omega,$$
so

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0} \oint_S d\Omega = \frac{Q}{\varepsilon_0}.$$

- shape of S doesn't matter!

This result is not surprising in terms of flux lines:



Also, any charge **outside** S contributes nothing to the flux integral – all its flux lines enter then leave the enclosed volume V.

Finally, extend to a general charge distribution $\rho(\mathbf{r})$ inside V:

Total charge within S $= \int_V \rho(\mathbf{r})\,dV$, so

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r})\,dV.$$

This is **Gauss's Law in integral form.**

Now apply the **Divergence Theorem** to LHS:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V [\nabla \cdot \mathbf{E}]\,dV$$

$$\Rightarrow \quad \int_V [\nabla \cdot \mathbf{E}]\,dV = \int_V \left[\frac{\rho(\mathbf{r})}{\varepsilon_0}\right] dV.$$

We have made no assumption about the size and shape of S and V, or the function $\rho(\mathbf{r})$. These volume integrals can only be equal for all cases if the integrands are equal. i.e.:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

This is **Gauss's Law in differential form.**

Notes:

1. Though derived for static charges, the result is always true. This is our first Maxwell Equation.

2. Divergence measures direct sources and sinks of vector fields. So $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ says

"The direct source of electric field is electric charge"

3. The result is true at all positions $\mathbf{r}$, not just in "V":

Where there is charge, $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Away from charge, $\nabla \cdot \mathbf{E} = 0$, though this **doesn't** mean $\mathbf{E} = \mathbf{0}$.