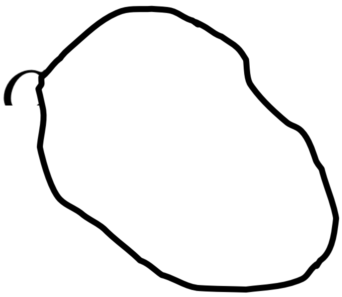
The curl of $\mathbf{E}$. (Electrostatic case)



The work done in moving a charge Q round **any** closed path C in an electrostatic field $\mathbf{E}_S(\mathbf{r})$ is zero.

(Electrostatic fields are "conservative")

Work done $= \oint_C \mathbf{F}_e \cdot d\mathbf{r}$ and from earlier $\mathbf{F}_e = Q\mathbf{E}_S$, so

$$\oint_C \mathbf{E}_S \cdot d\mathbf{r} = 0.$$

Now apply **Stokes' Theorem** to the LHS:

$$\oint_C \mathbf{E}_S \cdot d\mathbf{r} = \int_S \left[\nabla \times \mathbf{E}_S\right] \cdot d\mathbf{S} = 0,$$

where S is **any** surface enclosed by path C.

Since C and S are "arbitrary", we have

$$\nabla \times \mathbf{E}_S = \mathbf{0}.$$

- Not quite a Maxwell Equation – only valid for electrostatic fields.
- Curl measures "rotational" sources; this result implies there aren't any in electrostatics.

Electrostatics via the electrostatic potential

All electrostatic fields $\mathbf{E}_S(\mathbf{r})$ are solutions to our 2 differential equations

$$\nabla \cdot \mathbf{E}_S = \frac{\rho}{\varepsilon_0} \qquad \text{and} \qquad \nabla \times \mathbf{E}_S = \mathbf{0}.$$

Since $\nabla \times \nabla \psi = \mathbf{0}$ for **any** scalar field ψ, we can choose to write $\mathbf{E}_S$ in terms of a potential function:

$$\mathbf{E}_S = -\nabla \phi.$$

(The minus sign is a convention.)

$\phi(\mathbf{r})$ is the **electrostatic potential**.

Then $\nabla \cdot \mathbf{E}_S = \nabla \cdot \left(-\nabla \phi\right) = \dfrac{\rho}{\varepsilon_0}.$

But $\nabla \cdot \left(\nabla \phi\right) = \nabla^2 \phi$, the "Laplacian" of ϕ, so

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}. \qquad \text{The Poisson Equation.}$$

This scalar, 2^{nd} order PDE governs electrostatics.

If we want a solution in a region away from charges, where $\rho(\mathbf{r}) = 0$, then the equation to solve is

$$\nabla^2 \phi = 0. \qquad \text{Laplace's Equation.}$$

Solutions to the Poisson Equation

For a single charge Q at the origin,

$$\mathbf{E}_S(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r,$$

and we now have $\mathbf{E}_S = -\nabla\phi$. In this case we can find $\phi(\mathbf{r})$ by inspection:

Since $$-\nabla\left(\frac{1}{r}\right) = -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = +\frac{\hat{\mathbf{e}}_r}{r^2},$$

it follows that the electrostatic potential due to a single charge at the origin is

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r}.$$

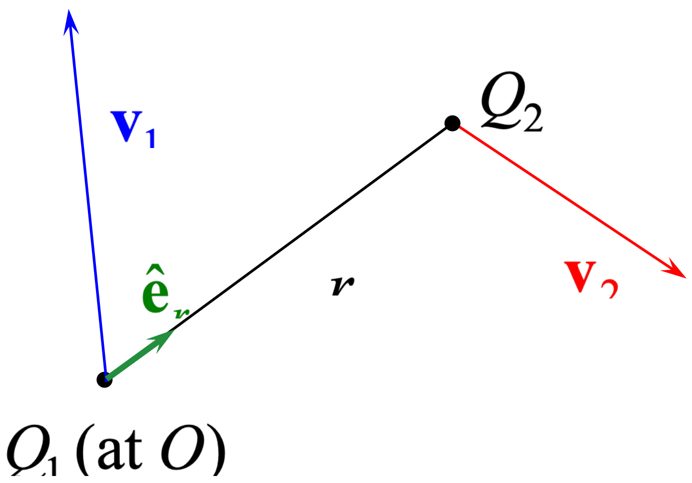
For a general charge distribution $\rho(\mathbf{r}')$ within volume V', superposition of electrostatic potentials yields

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'.$$

1.2 The Magnetic field $\mathbf{B}$

To deduce the origin of the magnetic field, consider a "thought experiment", in which we revisit the 2 charges Q_1 and Q_2 used to write down Coulomb's Law.

This time, let each charge move with **constant** velocity ($\mathbf{v}_1$ and $\mathbf{v}_2$) and re-measure the force between them:



If we could do this experiment, we would find an **additional** force $\mathbf{F}_m$ between the charges.

Total force:

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m.$$

Experimental Force	$\mathbf{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}\hat{\mathbf{e}}_r$	$\mathbf{F}_m = \frac{\mu_0}{4\pi}\frac{Q_1 Q_2}{r^2}\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$ μ_0 = permeability of free space $= 4\pi \times 10^{-7}$ Ns^2C^{-2} or Hm^{-1}
Use Q_2 as a test charge	$\mathbf{F}_e = Q_2\mathbf{E}$ defines $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r.$	$\mathbf{F}_m = Q_2(\mathbf{v}_2 \times \mathbf{B})$ defines $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$

Notes:

1. $\mathbf{B}(\mathbf{r})$ is the **Magnetic Field** due to a single charge Q_1 at the origin, moving at constant velocity $\mathbf{v}_1$. Units are Ns/(Cm) or Tesla T.

2. $\mathbf{B}=\mathbf{0}$ if $\mathbf{v}_1=\mathbf{0}$ or if $\mathbf{v}_1$ is parallel to $\hat{\mathbf{e}}_r$.

3. $\mathbf{B}$ is perpendicular to $\hat{\mathbf{e}}_r$ (and to $\mathbf{v}_1$)

- What does this imply about $\nabla\cdot\mathbf{B}$?

4. The total force on charge Q_2 is

$$\mathbf{F}=Q_2\left(\mathbf{E}+\mathbf{v}_2\times\mathbf{B}\right)$$

THE LORENTZ FORCE

$\mathbf{E}$ and $\mathbf{B}$ and Special Relativity

So far: 2 static charges $\rightarrow \mathbf{F}_e \rightarrow \mathbf{E}$

Giving each a constant velocity $\rightarrow \mathbf{F}_m \rightarrow \mathbf{B}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r) \qquad \xrightarrow{\quad \mathbf{v}_1 \quad}$$

What happens if we move to a new “Inertial Reference Frame” (IRF), moving with Q_1?

Now $\mathbf{v}_1' = \mathbf{0}$ which $\Rightarrow \mathbf{B}' = \mathbf{0}$!!

We just transformed away our new field.

Special Relativity tells us:

“The Laws of Physics are the **same** in all IRFs”

$\therefore$ “Electric Fields” and “Magnetic Fields” must essentially be the same thing, viewed from different frames.