PH20014/61 - Electric dipole

In the notes, you can find the following expressions for the length of the vectors $\vec{r_+}$ and r_- connecting the respective charges with the point at $\vec{r}$ at which we want to calculate the electric potential $V(\vec{r})$,

$$r_+ = \sqrt{\left(\vec{r} - \frac{\vec{d}}{2}\right) \cdot \left(\vec{r} - \frac{\vec{d}}{2}\right)}, \tag{1}$$

$$r_- = \sqrt{\left(\vec{r} + \frac{\vec{d}}{2}\right) \cdot \left(\vec{r} + \frac{\vec{d}}{2}\right)}. \tag{2}$$

I will now focus on the first expression to explain the calculation in the notes in detail (please go through the other part of the calculation if you find it difficult). We have

$$r_+ = \sqrt{\left(\vec{r} - \frac{\vec{d}}{2}\right) \cdot \left(\vec{r} - \frac{\vec{d}}{2}\right)} = \sqrt{\vec{r} \cdot \vec{r} - \vec{d} \cdot \vec{r} + \frac{\vec{d} \cdot \vec{d}}{4}} = \sqrt{r^2 - dr\cos\theta + \frac{d^2}{4}}, \tag{3}$$

where I use the symbols without the vector arrows to denote the length of the respective vectors. In the next step, we have

$$r_+ = \sqrt{r^2 - dr\cos\theta + \frac{d^2}{4}} = r\sqrt{1 - \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2}. \tag{4}$$

Notice that, for $d \ll r$, the term $\left(\frac{d}{2r}\right)^2$ is the smallest. Therefore, we will forget about it and write

$$r_+ = r\sqrt{1 - \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \approx r\sqrt{1 - \frac{d}{r}\cos\theta}. \tag{5}$$

Now, in turn, we will expand the above expression using $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ (you should be able to derive this either from binomial expansion, Taylor's expansion or using other tricks), so that

$$r_+ = r\sqrt{1 - \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \approx r\sqrt{1 - \frac{d}{r}\cos\theta} \approx r\left(1 - \frac{d}{2r}\cos\theta\right) = r - \frac{d}{2}\cos\theta, \tag{6}$$

which is the result in the notes. Note that, it can be shown by using Taylor's expansion rigorously on the whole square root that what we have done is correct up to first order (linear in $\frac{d}{r}$)). The reason this is the case is because the additional term we ignored, $\left(\frac{d}{2r}\right)^2$, is second order (quadratic) in $\frac{d}{r}$ and hence can only contribute in second order expansion and not in first. Those of you interested in confirming this statement can go through the mathematical details - the only thing needed is a calculation of the first and second derivatives of the square root with respect to $\frac{d}{r}$ and use of Taylor's expansion.

Following similar calculation, we obtain

$$r_- = r\sqrt{1 + \frac{d}{r}\cos\theta + \left(\frac{d}{2r}\right)^2} \approx r + \frac{d}{2}\cos\theta, \tag{7}$$

so that we can write the electric potential as

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r_+} + \frac{-q}{4\pi\epsilon_0 r_-} \approx \frac{q}{4\pi\epsilon_0\left(r - \frac{d}{2}\cos\theta\right)} - \frac{q}{4\pi\epsilon_0\left(r + \frac{d}{2}\cos\theta\right)}. \tag{8}$$

To handle this expression, we will expand again in the powers of $\frac{d}{r}$. For example, for the first term we have

$$\frac{q}{4\pi\epsilon_0\left(r - \frac{d}{2}\cos\theta\right)} = \frac{q}{4\pi\epsilon_0 r}\frac{1}{\left(1 - \frac{d}{2r}\cos\theta\right)} \approx \frac{q}{4\pi\epsilon_0 r}\left(1 + \frac{d}{2r}\cos\theta\right), \tag{9}$$

where this time we used the expansion $\frac{1}{1+x} \approx 1 - x$. Applying this to both terms, we get

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r_+} + \frac{-q}{4\pi\epsilon_0 r_-} \approx \frac{q}{4\pi\epsilon_0\left(r - \frac{d}{2}\cos\theta\right)} - \frac{q}{4\pi\epsilon_0\left(r + \frac{d}{2}\cos\theta\right)} \tag{10}$$

$$\approx \frac{q}{4\pi\epsilon_0 r}\left(1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right) = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2}. \tag{11}$$

The last result is the final one in the lecture notes.