Mathematics and Computational Methods for Complex Systems.

Joshua Dare-Cullen

Candidate Number: 207740

Analytical Work.

System information:

$[\dot{A}] = \beta \frac{[B]}{N}[A] + \gamma[B]$

$[\dot{B}] = \beta \frac{[B]}{N}[A] - \gamma[B]$

$N = [A] + [B]$, at all times.

1. Deriving the mean field equation.

$$[\dot{B}] = \beta \frac{[B]}{N}[A] - \gamma[B]$$

$$[\dot{B}] = [B](\beta \frac{[A]}{N} - \gamma)$$

Substitute in: $[A] = N - [B]$

$$[\dot{B}] = [B](\frac{\beta(N - [B])}{N} - \gamma)$$

$$[\dot{B}] = [B](\frac{\beta N - \beta[B]}{N} - \gamma)$$

$$[\dot{B}] = [B](\frac{\beta N}{N} - \frac{\beta[B]}{N} - \gamma)$$

Therefore, the mean field equation is:

$$[\dot{B}] = [B](\beta - \frac{\beta[B]}{N} - \gamma)$$

2. Equilibria and their stability.

The system is now of one dimension. The equilibria of the model, $B*$, can be derived by setting $[\dot{B}]$ = 0.

$$[B](\beta - \frac{\beta[B]}{N} - \gamma) = 0$$

Equilibria point, $[B] = 0$: $(N,0)$

For non-zero equilibria, $B*$:

$$\beta - \frac{\beta[B]}{N} - \gamma = 0$$

$$\beta - \gamma = \frac{\beta B*}{N}$$

$$B* = \frac{N(\beta - \gamma)}{\beta}$$

$$B* = \frac{N\beta}{\beta} - \frac{N\gamma}{\beta}$$

$$B* = N - \frac{N\gamma}{\beta}$$

$$B* = N(1 - \frac{\gamma}{\beta})$$

Results in terms of $R_0 = \frac{\beta}{\gamma}$

$$B* = N(1 - \frac{1}{R_0})$$

In terms of $A*$:

Substitute in : $[B] = N - [A]$

$$N - A^* = N(1 - R_0)$$

$$A^* = N - N - \frac{N}{R0}$$

$$A^* = \frac{N}{R0}$$

Therefore, the equilibria in terms of R_0 and total population:

$$(A^*, B^*)$$

$$A^* = \frac{N}{R0}$$

$$B^* = N(1 - \frac{1}{R_0})$$

The equilibria's with R_0 is such that, if $R_0 > 1$, then the system admits a unique positive equilibrium A^* and B^*. However, if $R_0 < 1$, then the resulting equilibria is not realistic. When $R_0 = 1$ ($\gamma = \beta$) , A asymptotically takes over the population.

Stability of equilibria:

The Jacobian of the mean field equation can provide insight into the stability of the fixed points.

$$F([B], t) = [\dot{B}] = [B](\beta - \frac{\beta[B]}{N} - \gamma)$$

$$[\dot{B}] = [B]\beta - \frac{\beta[B]^2}{N} - [B]\gamma$$

The Jacobian:

$$\frac{dF}{d[B]} = \beta - \frac{2\beta[B]}{N} - \gamma$$

When $[B] = 0$:

$$\frac{dF}{d[B]} = \beta - \frac{2\beta(0)}{N} - \gamma)$$

$$\frac{dF}{d[B]} = \beta - \gamma$$

The Jacobian states if $\frac{dF}{d[B]} \mid_{B=0} < 0$, then $(N{,}0)$ is a stable fixed point.

For the fixed point to be stable: $\beta > \gamma,\ R_0 > 1.$

$\frac{dF}{d[B]} \mid_{B=B*} > 0,\ (N{,}0)$ is unstable.

For the fixed point to be unstable: $\beta < \gamma,\ R_0 < 1.$

When $B* = N(1 - \frac{1}{R_0})$:

$$\frac{dF}{d[B]} = \beta - \frac{2\beta(N(1 - \frac{1}{R_0}))}{N} - \gamma$$

$$\frac{dF}{d[B]} = \beta - 2\beta + \frac{2\beta}{R_0} - \gamma$$

Substitute $\frac{\beta}{\gamma}$ for R_0:

$$\frac{dF}{d[B]} = \beta - 2\beta + \frac{2\beta}{\frac{\beta}{\gamma}} - \gamma$$

$$\frac{dF}{d[B]} = \beta - 2\beta + \frac{2\beta\gamma}{\beta} - \gamma$$

$$\frac{dF}{d[B]} = \beta - 2\beta + 2\gamma - \gamma$$

$$\frac{dF}{d[B]} = \gamma - \beta$$

$\frac{dF}{d[B]} \mid_{B=B*} < 0$, then $B*$ is a stable fixed point.

$$\gamma - \beta < 0$$

For the fixed point to be stable: $\beta > \gamma,\ R_0 > 1.$

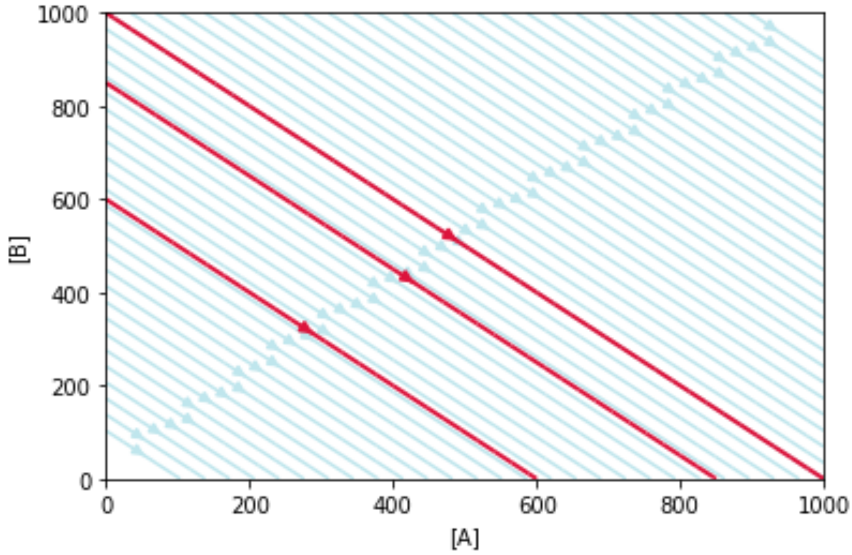
$$\frac{dF}{d[B]} \mid_{B=B*} > 0,\ B* \text{ is unstable.}$$

For the fixed point to be unstable: $\beta < \gamma,\ R_0 < 1.$

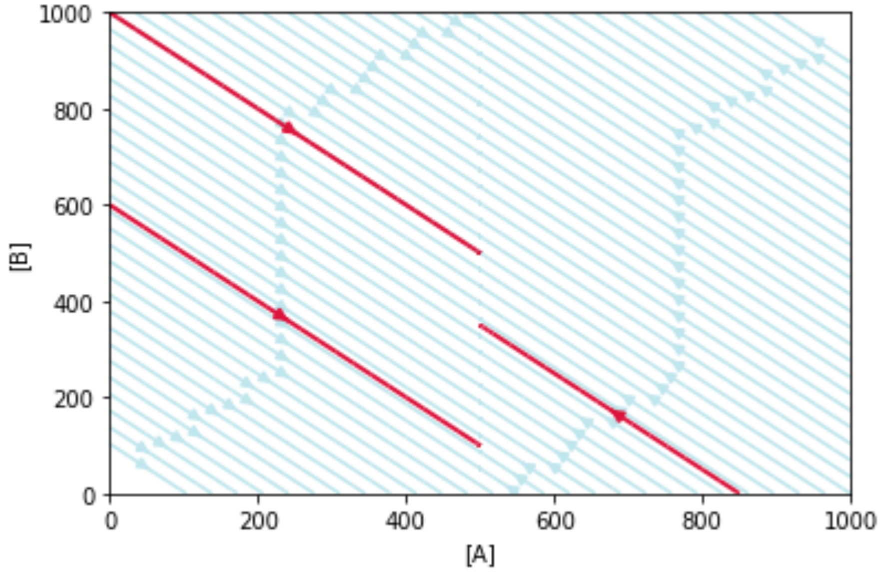
It is of interest to note, when the second derivate is equal to zero ($R_0 = 1, \beta = \gamma$), it could correspond to a potential inflection point. This can be verified by whether the Jacobian changes sign around 0.

Phase Portrait of the system

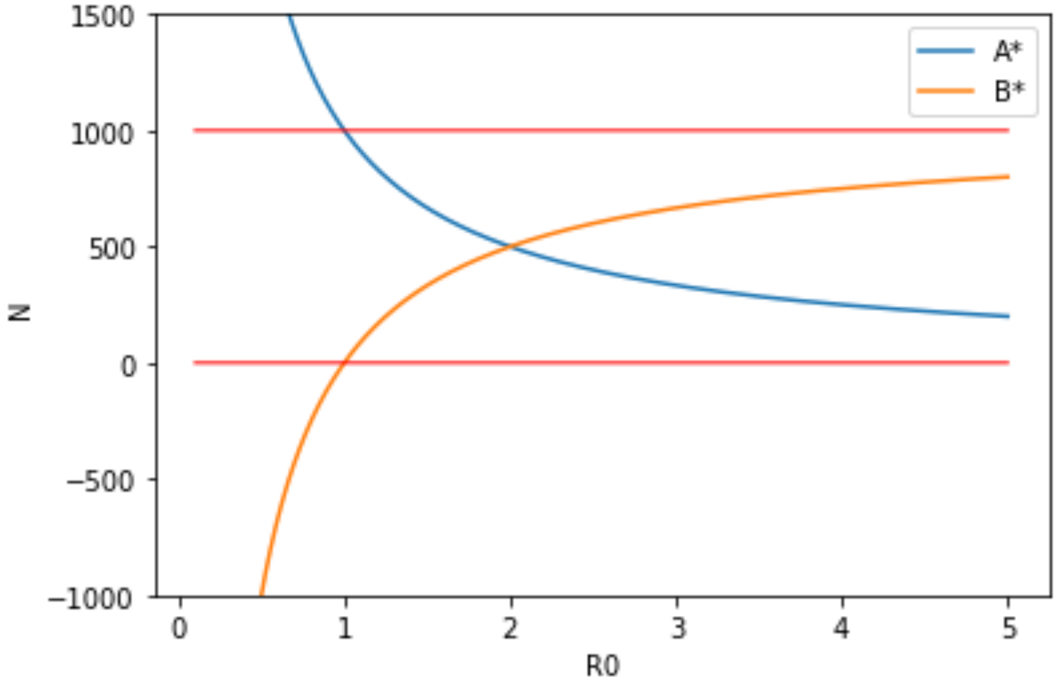
When $R_0 < 1$: exponential decay. $R_0 = 1$ is identical.



When $R_0 > 1$. In this case, $R_0 = 2$ and the equilibria is 500.



3. Bifurcation plot



The bifurcation plot verifies when $R_0 < 1$, the equilibria is unrealistic, denoted by the red lines. An interesting event occurs at $R_0 = 2$, where the equilibria are identical. $R_0 > 2$, B becomes the dominant equilibrium in terms of its share of the total population.

4. Analytical Integration of $[\dot{B}]$

Step 1: Starting from the mean-field equation, factorise the right hand side by $[B]^2$, then write an expression for $\frac{1}{[B]^2}[\dot{B}]$

Mean field: $[\dot{B}] = [B]\beta - \frac{\beta[B]^2}{N} - [B]\gamma$

Factorising by $[B]^2$ gives :

$$[\dot{B}] = [B]^2(\frac{\beta[B]}{[B]^2} - \frac{\beta[B]^2}{N[B]^2} - \frac{[B]\gamma}{[B]^2})$$

$$[\dot{B}] = [B]^2(\frac{\beta}{[B]} - \frac{\beta}{N} - \frac{\gamma}{[B]})$$

$$[\dot{B}] = [B]^2(\frac{\beta}{[B]} - \frac{\beta}{N} - \frac{\gamma}{[B]})$$

In terms of $\frac{1}{[B]^2}[\dot{B}]$:

$$\frac{1}{[B]^2}[\dot{B}] = \frac{\beta - \gamma}{[B]} - \frac{\beta}{N}$$

Step 2: Following variable substitution: $y = \frac{1}{[B]}$. Then using the chain rule to express $\dot{y}$ in terms of $[B]$ and then to derive a simple expression for $\dot{y}$.

$$\frac{1}{[B]} = [B]^{-1}$$

Chain rule:

$$\frac{dy}{dt} = \frac{dy}{d[B]} * \frac{d[B]}{dt}$$

$$\frac{dy}{d[B]} = -[B]^{-2}$$

$$\frac{d[B]}{dt} = [\dot{B}] = [B](\beta - \frac{\beta[B]}{N} - \gamma)$$

$$\therefore \qquad \frac{dy}{dt} = -\frac{1}{[B]^2}[\dot{B}]$$

Substituting in $\dot{y} = -\frac{1}{[B]^2}[\dot{B}]$ and $y = \frac{1}{[B]}$:

$$\dot{y} = \frac{\beta}{N} - y(\beta - \gamma)$$

Full expression for $\dot{y}$:

$$\dot{y} = y(\gamma - \beta) + \frac{\beta}{N}$$

Step 3: Integrate this equation.

$y(t) = g(t)h(t)$ where in this equation $h(t)$ is the solution to the homogeneous equation $\dot{y} = y(\gamma - \beta)$.

Where $\lambda = (\gamma - \beta)$ and $I = \frac{\beta}{N}$

$$\frac{dh(t)}{dt} = \lambda h(t)$$

$$h(t) = ke^{\lambda(t)}$$

Apply the product rule:

$$\frac{dy(t)}{dt} = g(t)\frac{dh(t)}{dt} + h(t)\frac{dg(t)}{dt}$$

$$\frac{dy(t)}{dt} = \lambda g(t) + ke^{\lambda(t)}\frac{dg(t)}{dt}$$

$$I = ke^{\lambda(t)}\frac{dg(t)}{dt}$$

When rearranged: $$\frac{dg(t)}{dt} = \frac{I}{k}e^{-\lambda(t)}$$

$$g(t) = -\frac{I}{k\lambda}e^{-\lambda(t)} + c$$

So we can write:

$$y(t) = (-\frac{I}{k\lambda e^{\lambda(t)}}) + c)ke^{\lambda(t)}$$

$$y(t) = (-\frac{Ike^{\lambda(t)}}{k\lambda e^{\lambda(t)}}) + cke^{\lambda(t)}$$

$$y(t) = -\frac{I}{\lambda} + cke^{\lambda(t)}$$

Substituting in: $\lambda = (\gamma - \beta)$ and $I = \frac{\beta}{N}$

$$y(t) = -\frac{\beta}{N(\gamma - \beta)} + cke^{(\gamma-\beta)(t)}$$

ck is a constant and can be reduced to k.

$$y(t) = ke^{(\gamma-\beta)(t)} - \frac{\beta}{N(\gamma-\beta)}$$

Step 4: You can now produce a fully worked out expression for $[B](t)$ by remembering that $[B] = B0$ at time $t = 0$.

$y = \frac{1}{[B]}$, gives, $[B] = \frac{1}{y}$:

$$[B](t) = \frac{1}{ke^{(\gamma-\beta)(t)} - \frac{\beta}{N(\gamma-\beta)}}$$

$$[B](t) = \frac{1}{\frac{N(\gamma-\beta)ke^{(\gamma-\beta)(t)} - \beta}{N(\gamma-\beta)}}$$

$$[B](t) = \frac{N(\gamma-\beta)}{N(\gamma-\beta)ke^{(\gamma-\beta)(t)} - \beta}$$

Remembering that $[B] = B_0$ at time $t = 0$

$$B_0 = \frac{N(\gamma-\beta)}{kN(\gamma-\beta) - \beta}$$

$$B_0(kN(\gamma-\beta) - \beta) = N(\gamma-\beta)$$

$$B_0kN(\gamma-\beta) - B_0\beta = N(\gamma-\beta)$$

$$B_0kN(\gamma-\beta) = N(\gamma-\beta) + B_0\beta$$

$$k = \frac{N(\gamma-\beta) + B_0\beta}{B_0N(\gamma-\beta)}$$

Sub into $[B](t)$:

$$[B](t) = \frac{N(\gamma - \beta)}{N(\gamma - \beta)\frac{N(\gamma - \beta) + B_0\beta}{B_0 N(\gamma - \beta)}e^{(\gamma-\beta)(t)} - \beta}$$

$$[B](t) = \frac{N(\gamma - \beta)}{\frac{N(\gamma - \beta) + B_0\beta}{B_0}e^{(\gamma-\beta)(t)} - \beta}$$

$$[B](t) = \frac{N(\gamma - \beta)}{\frac{(N(\gamma - \beta) + B_0\beta)e^{(\gamma-\beta)(t)} - B_0\beta}{B_0}}$$

Full expression for $[B](t)$ when $\gamma \neq \beta$:

$$[B](t) = \frac{B_0 N(\gamma - \beta)}{(N(\gamma - \beta) + B_0\beta)e^{(\gamma-\beta)(t)} - B_0\beta}$$

Q.5 Verifying the $B(t)$ solution converges to $B*$ (Equilibrium) for various $R_0 s$ between 0.1 and 5.

The table below displays twenty $R_0 s$ and it's respective equilibria when the population, $N = 1000$ and $t = 200$. Followed by the corresponding analytical solution convergence for varying $B_0 s$. All converge to the equilibria when $R > 1$. In terms of $B_0 = 0$ at $t = 0$, it is a trivial equilibrium point. Plus, $(N{,}0)$ is stable when $\beta > \gamma$. Hence, no change will occur in the system. Analytically, the numerator of $B(t)$, $B_0 N(\gamma - \beta)$ will result in zero, thus creating a divide error.

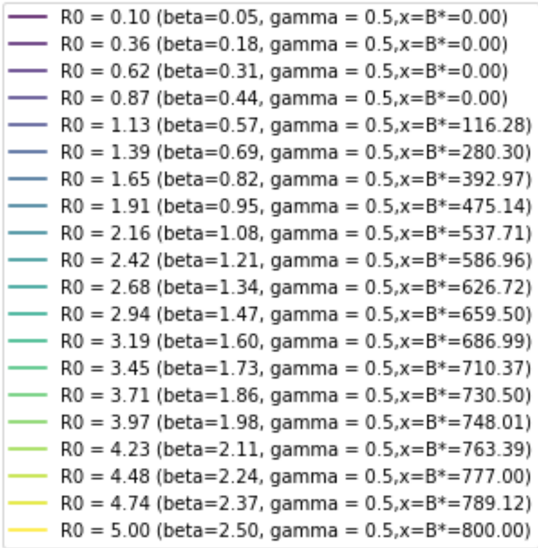
	R0	B*	B(t),B0=1	B(t),B0=150	B(t),B0=300
0	0.100000	-9000.000000	1.921734e-98	2.835661e-96	5.579848e-96
1	0.357895	-1794.117647	1.923379e-70	2.663953e-68	4.946271e-68
2	0.615789	-623.931624	1.923874e-42	2.330225e-40	3.903827e-40
3	0.873684	-144.578313	1.916213e-14	1.420467e-12	1.882407e-12
4	1.131579	116.279070	1.162791e+02	1.162791e+02	1.162791e+02
5	1.389474	280.303030	2.803030e+02	2.803030e+02	2.803030e+02
6	1.647368	392.971246	3.929712e+02	3.929712e+02	3.929712e+02
7	1.905263	475.138122	4.751381e+02	4.751381e+02	4.751381e+02
8	2.163158	537.712895	5.377129e+02	5.377129e+02	5.377129e+02
9	2.421053	586.956522	5.869565e+02	5.869565e+02	5.869565e+02
10	2.678947	626.719057	6.267191e+02	6.267191e+02	6.267191e+02
11	2.936842	659.498208	6.594982e+02	6.594982e+02	6.594982e+02
12	3.194737	686.985173	6.869852e+02	6.869852e+02	6.869852e+02
13	3.452632	710.365854	7.103659e+02	7.103659e+02	7.103659e+02
14	3.710526	730.496454	7.304965e+02	7.304965e+02	7.304965e+02
15	3.968421	748.010610	7.480106e+02	7.480106e+02	7.480106e+02
16	4.226316	763.387298	7.633873e+02	7.633873e+02	7.633873e+02
17	4.484211	776.995305	7.769953e+02	7.769953e+02	7.769953e+02
18	4.742105	789.123196	7.891232e+02	7.891232e+02	7.891232e+02
19	5.000000	800.000000	8.000000e+02	8.000000e+02	8.000000e+02

	B(t),B0=500	B(t),B0=850	B(t),B0=999
0	9.103963e-96	1.492680e-95	1.728196e-95
1	7.525097e-68	1.109931e-67	1.234906e-67
2	5.348589e-40	6.933471e-40	7.400727e-40
3	2.163889e-12	2.384078e-12	2.436914e-12
4	1.162791e+02	1.162791e+02	1.162791e+02
5	2.803030e+02	2.803030e+02	2.803030e+02
6	3.929712e+02	3.929712e+02	3.929712e+02
7	4.751381e+02	4.751381e+02	4.751381e+02
8	5.377129e+02	5.377129e+02	5.377129e+02
9	5.869565e+02	5.869565e+02	5.869565e+02
10	6.267191e+02	6.267191e+02	6.267191e+02
11	6.594982e+02	6.594982e+02	6.594982e+02
12	6.869852e+02	6.869852e+02	6.869852e+02
13	7.103659e+02	7.103659e+02	7.103659e+02
14	7.304965e+02	7.304965e+02	7.304965e+02
15	7.480106e+02	7.480106e+02	7.480106e+02
16	7.633873e+02	7.633873e+02	7.633873e+02
17	7.769953e+02	7.769953e+02	7.769953e+02
18	7.891232e+02	7.891232e+02	7.891232e+02
19	8.000000e+02	8.000000e+02	8.000000e+02

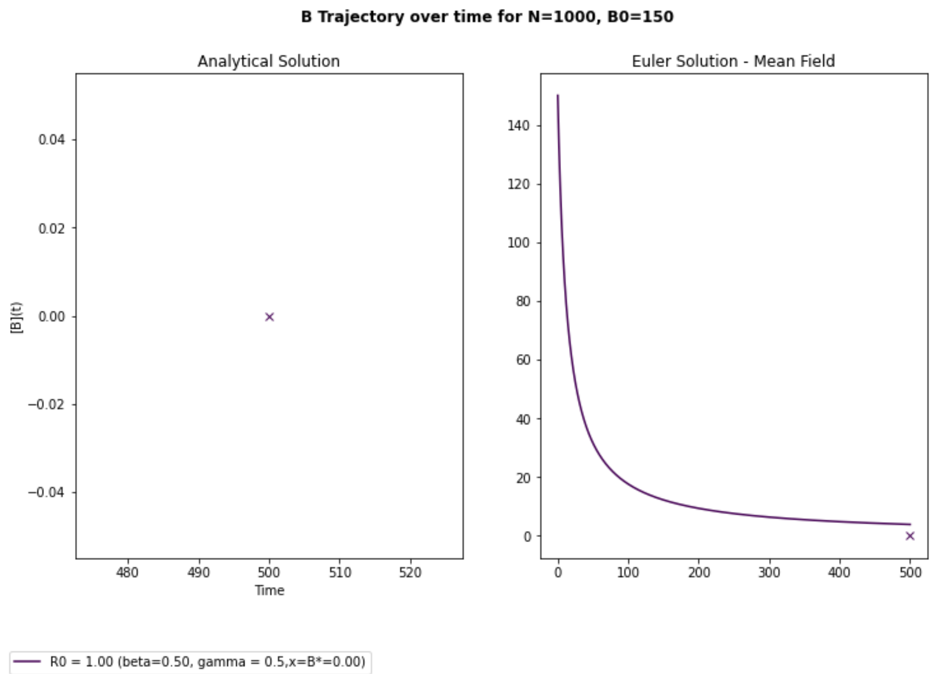
Euler Visual verification.

The ‘x’ points on the graph represents B^* from the table above.

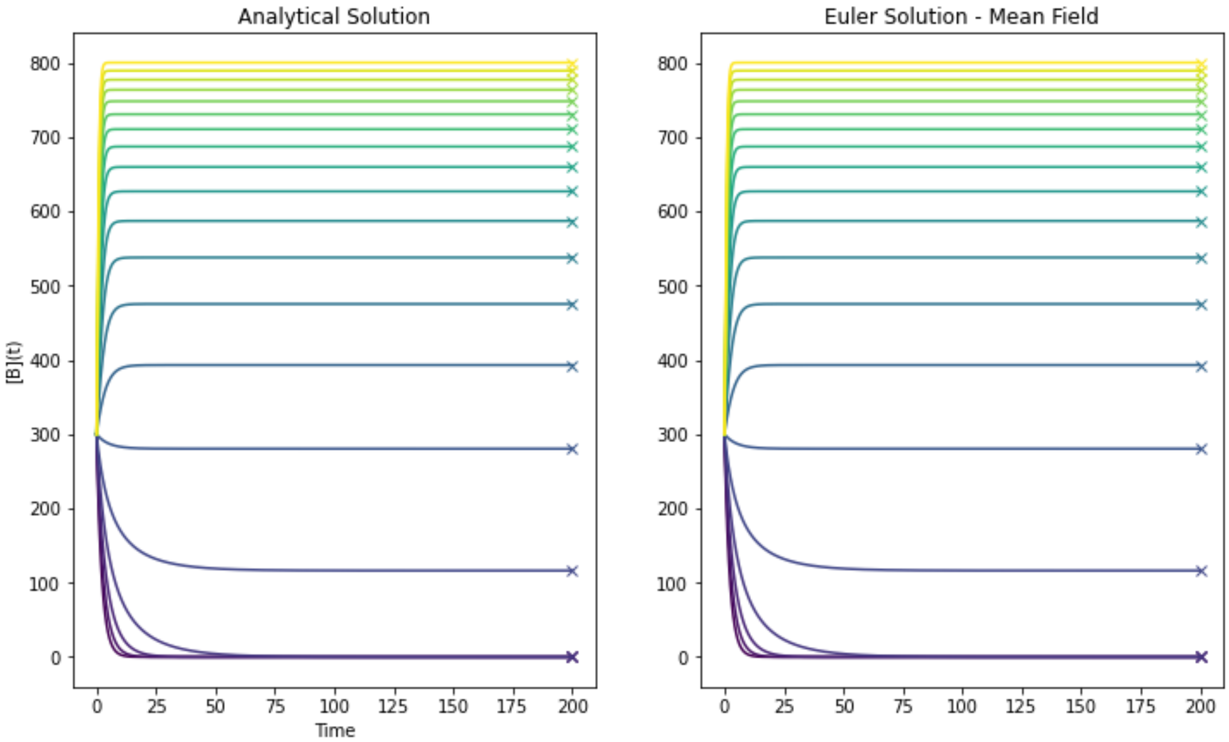
The legend for all graphs below:



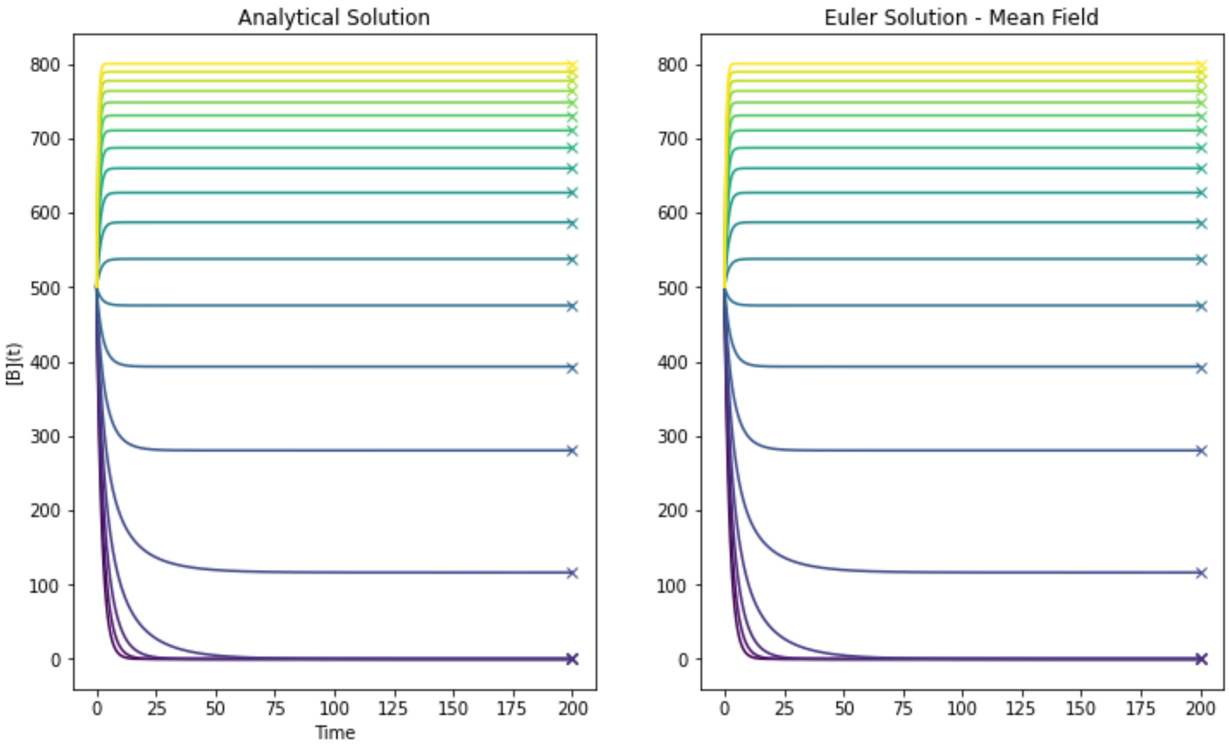
Briefly, $R_0 = 1$ $(\beta = \gamma)$, the creates zero on the numerator of the $B(t)$ solution. The Euler solution is showing asymptotic behaviour.



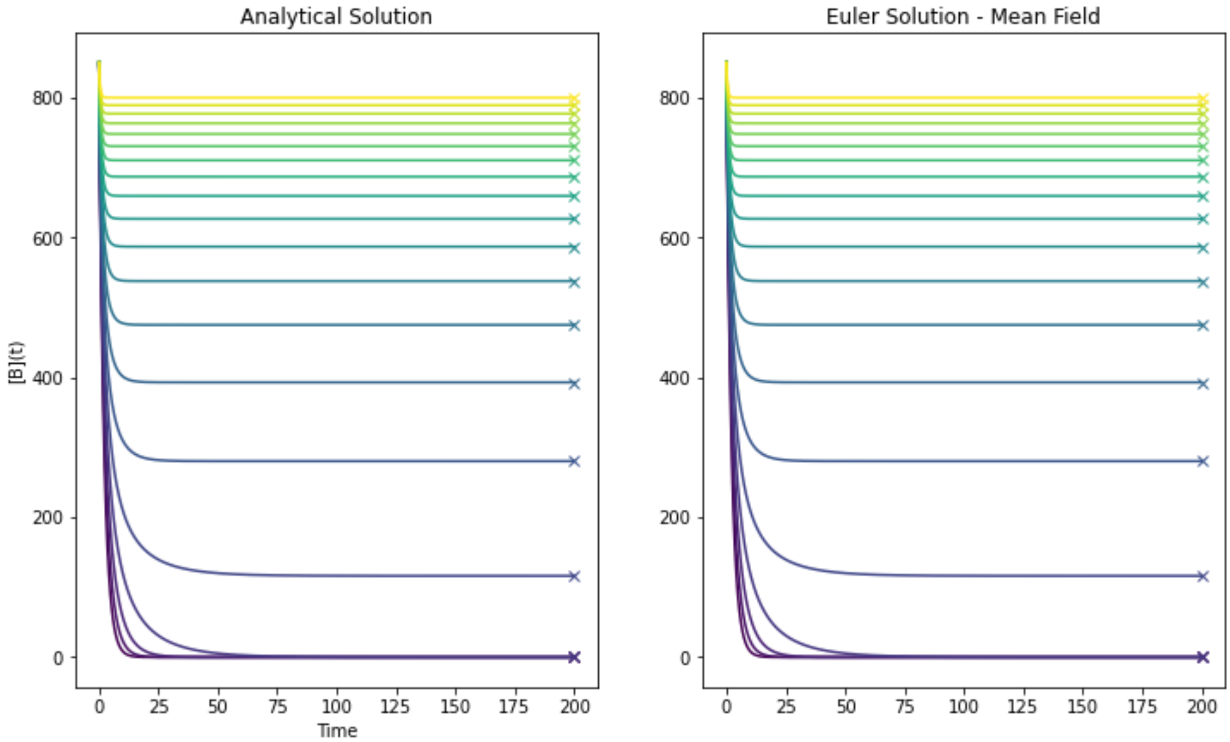
B Trajectory over time for N=1000, B0=300



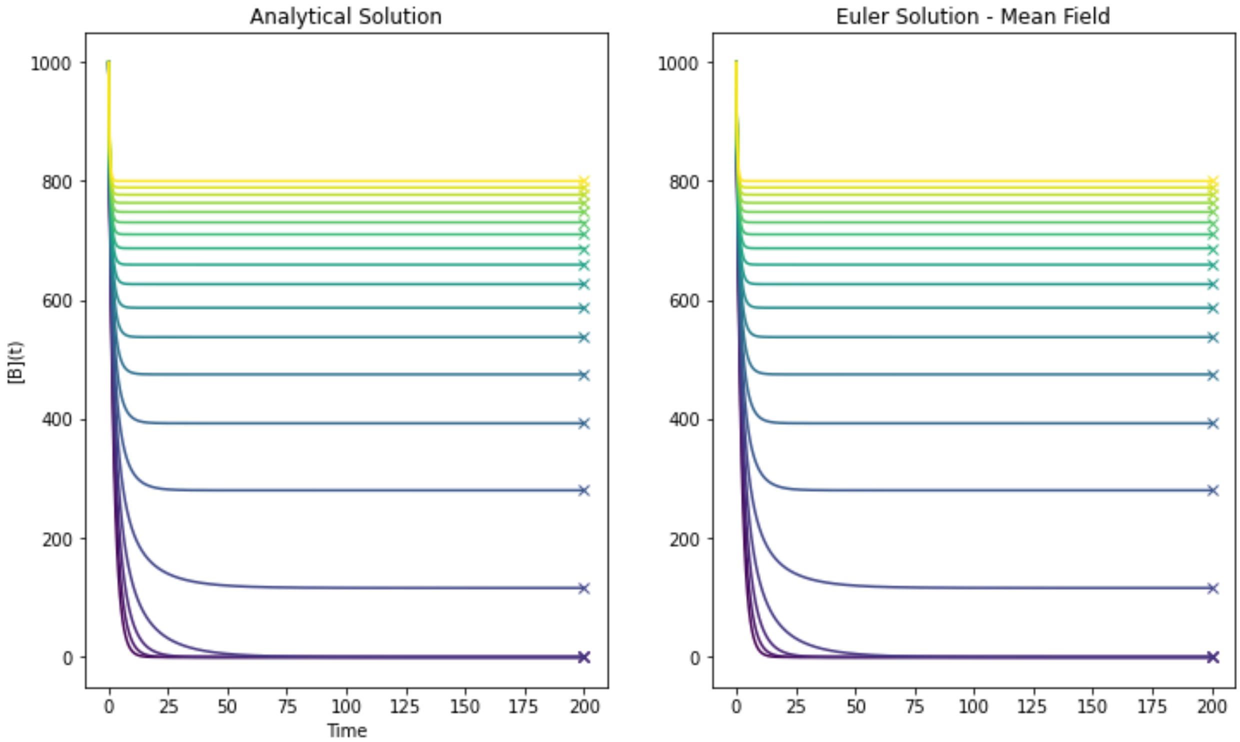
B Trajectory over time for N=1000, B0=500



B Trajectory over time for N=1000, B0=850



B Trajectory over time for N=1000, B0=999



The effects of varying the size of γ when the R_0 is constant can be seen in the graph below. Larger the γ, the faster the system reaches the equilibria. This is because β is growing larger in order to keep R_0 constant and leads to the difference between itself and γ growing. Also to note, γ effects directly the rate of $[B]$ changing into $[A]$. Unlike β, which has less of an effect because the rate of which $[A]$ changes to $[B]$ is also governed by the proportion of the population that is in state $[B]$ divided by N.